Drawing a Tree as a Minimum Spanning Tree Approximation

Emilio Di Giacomo[1], Walter Didimo[1], Giuseppe Liotta[1], and Henk Meijer[2]

[1] Dip. di Ingegneria Elettronica e dell'Informazione, Università degli Studi di Perugia
{digiacomo,didimo,liotta}@diei.unipg.it
[2] Roosevelt Academy, The Netherlands
h.meijer@roac.nl

Abstract. We introduce and study $(1+\varepsilon)$-*EMST drawings*, i.e. planar straight-line drawings of trees such that, for any fixed $\varepsilon > 0$, the distance between any two vertices is at least $\frac{1}{1+\varepsilon}$ the length of the longest edge in the path connecting them. $(1+\varepsilon)$-EMST drawings are good approximations of Euclidean minimum spanning trees. While it is known that only trees with bounded degree have a Euclidean minimum spanning tree realization, we show that every tree T has a $(1+\varepsilon)$-EMST drawing for any given $\varepsilon > 0$. We also present drawing algorithms that compute $(1+\varepsilon)$-EMST drawings of trees with bounded degree in polynomial area. As a byproduct of one of our techniques, we improve the best known area upper bound for Euclidean minimum spanning tree realizations of complete binary trees.

1 Introduction

The Euclidean minimum spanning tree of a set points in $2D$ and in $3D$ is among the most fundamental and hence most studied geometric structures (see, e.g. [14]). In their seminal paper, Monma and Suri [13] initiated the investigation of the combinatorial properties of the Euclidean minimum spanning trees in the plane. This investigation naturally leads to the following question: Which are those trees that have an *EMST drawing*, i.e. a straight-line drawing that is also a Euclidean minimum spanning tree of the set of its vertices?

Besides their relevance in geometric graph theory, EMST drawings are also interesting for graph drawing applications. Namely, an EMST drawing satisfies some aesthetic requirements that are fundamental for the readability of a tree visualization: No two edges cross each other, groups of closely related vertices visually cluster together, and less related vertices are relatively far apart from each other [1,4,7,11].

Unfortunately, not all trees have an EMST drawing. Monma and Suri [13] proved that every tree with maximum vertex degree at most five admits an EMST drawing, while no tree with a vertex of degree greater than six admits this type of representation. As for trees having maximum degree equal to six, Eades and Whitesides [5] showed that it is NP-hard to decide whether such trees admit an EMST drawing. In order to enlarge the family of representable trees, the computation of EMST drawings in three-dimensional space was initiated in [12]. The authors of [12] proved that all trees with maximum vertex degree nine admit an EMST drawing in 3D-space, while no tree with

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vertex degree larger than twelve has an EMST drawing. King [10] reduced the gap between upper and lower bound by showing that all trees with vertex degree up to ten admit an EMST drawing in 3D-space.

In this paper we want to compute planar straight-line drawings of trees where groups of adjacent vertices be relatively close to each other while non-adjacent vertices be relatively far apart from one another. In order to overcome the vertex degree limitations imposed by EMST drawings, we define a new type of drawing that "approximates" an EMST drawing. Given a constant $\varepsilon > 0$ and a tree T, a $(1+\varepsilon)$-EMST drawing of T is a planar straight-line drawing Γ of T that satisfies the following *proximity constraint*: For any two vertices u and v, $d(u,v) \geq \frac{1}{1+\varepsilon}|e_T(u,v)|$, where $d(u,v)$ is the Euclidean distance between u and v in Γ, and $|e_T(u,v)|$ is the maximum length of an edge of Γ in the path from u to v in T.

One of the leading motivations behind our study is to investigate the area requirements of "good approximations" of EMST-drawings. Perhaps the most longstanding open problem about EMST-drawings is due to Monma and Suri [13], who conjecture that there exists a tree T of maximum degree five and a constant $c > 1$ such that any two-dimensional EMST drawing of T requires $\Omega(c^n \times c^n)$ area. Recently, Kaufmann [8] and Frati and Kaufmann [6] have made some significant progress on this problem disproving the conjecture of Monma and Suri for vertex degree up to four. In [6,8] an area bound of $O(n^{21.252})$ is proved for trees having vertex degree at most four and of $O(n^{11.387})$ for those having vertex degree at most three. In the same papers it is also shown that these bounds can be significantly improved when the input tree has logarithmic height: For example, an area bound of $O(n^{4.3})$ is proved for EMST drawings of complete binary trees. However, the question whether all trees having vertex degree at most five admit an EMST drawing of polynomial area remains to date unanswered.

An overview the results in this paper is as follows.

- We study the relationships between $(1+\varepsilon)$-EMST drawings and Euclidean minimum spanning trees. We show that the sum of the lengths of the edges in a $(1+\varepsilon)$-EMST drawing is within a $(1+\varepsilon)$-factor of the sum of the lengths of the edges of a Euclidean minimum spanning tree of the points representing the vertices (Section 2).
- We present a drawing algorithm that, for any given $\varepsilon > 0$ and any tree T, computes a two-dimensional $(1+\varepsilon)$-EMST drawing of T (Section 3).
- We describe polynomial area approximation schemes for $(1+\varepsilon)$-EMST drawings: Any tree with n vertices and degree bounded by a constant admits a $(1+\varepsilon)$-EMST drawing whose area is $O(n^{c+f(\varepsilon)})$, where c is a positive constant and $f(\varepsilon)$ is a polylogarithmic function of ε that tends to infinity as ε tends to zero (Section 4).
- We study ad-hoc polynomial area approximation schemes for families of trees that have logarithmic height. We obtain area bounds that significantly improve the general case. These techniques are also extended to the 3D-space (Section 5.1).
- Finally, as a variant of our techniques, we are able to compute an EMST drawing of a complete binary tree of n vertices in $O(n^{3.802})$ area. This result improves the best previously known upper bound of $O(n^{4.3})$ proved by Frati and Kaufmann [6] (Section 5.2).

For reasons of space, some proofs are sketched or omitted.

2 $(1 + \varepsilon)$-EMST Drawings

Let $T = (V, E)$ be a tree and let Γ be a straight-line drawing of T. We denote by $|e|$ the length of edge e in Γ and we call $|\Gamma| = \sum_{e \in E} |e|$ the *weight* of Γ. For any pair of vertices $u, v \in V$, $d(u, v)$ is the Euclidean distance between u and v in Γ, $\pi_T(u, v)$ is the path from u to v in T, and $e_T(u, v)$ is the longest edge of Γ along the path $\pi_T(u, v)$.

Given a set of points P, a *Euclidean minimum spanning tree* of P is a geometric tree spanning all points of P and having minimum total weight. In this paper we denote by $EMST(P)$ a Euclidean minimum spanning tree of P, and by $|EMST(P)|$ its weight. Let Γ be a straight-line drawing of a tree T and let P be the set of points corresponding to the vertices of T in Γ. If Γ is a Euclidean minimum spanning tree of P, we say that Γ is an *EMST drawing*. We recall that a drawing Γ is an EMST drawing if and only if it satisfies the following condition: $\forall u, v \in V, d(u, v) \geq |e_T(u, v)|$. Also, every EMST drawing is planar, i.e., it does not contain edge crossings (see, e.g., [14]).

Let $\varepsilon > 0$ be a given constant and let T be a tree. A $(1 + \varepsilon)$-*EMST drawing* of T is a planar straight-line drawing of T such that for any two vertices u and v, $d(u, v) \geq \frac{1}{1+\varepsilon}|e_T(u, v)|$. This last condition will be referred to as the *proximity constraint* of Γ.

The next theorem establishes a relationship between $(1 + \varepsilon)$-EMST drawings and Euclidean minimum spanning trees. Its proof is omitte dfor reasons of space.

Theorem 1. *Let $\varepsilon > 0$ be a given constant. Let T be a tree, let Γ be a straight-line drawing of T, and let P be the set of points corresponding to the vertices of T in Γ. If Γ is a $(1 + \varepsilon)$-EMST drawing of T, then $|\Gamma| \leq (1 + \varepsilon)|EMST(P)|$.*

3 Computing $(1 + \varepsilon)$-EMST Drawings of General Trees

In this section we consider the problem of computing a $(1+\varepsilon)$-EMST drawing of a tree. We start by remarking that the converse of Theorem 1 does not hold, which is a major difference to take into account between the problem of computing an EMST drawing and the one of computing a $(1 + \varepsilon)$-EMST drawing. Namely, every planar straight-line drawing of a tree having minimum weight satisfies the property that $\forall u, v \in V, d(u, v) \geq |e_T(u, v)|$; for a contrast, it is not true that every planar straight-line drawing of a tree whose weight is $(1 + \varepsilon)$ times the weight of a Euclidean minimum spanning tree satisfies the proximity constraint. Hence, a simple construction like the one illustrated in Fig. 1 (suitably fix the length of and edge according to the desired approximation factor and make all other edge lengths negligible) computes a drawing whose weight can be made arbitrarily close to the one of a minimum spanning tree, but it does not guarantee that for any two vertices u and v, $d(u, v) \geq \frac{1}{1+\varepsilon}|e_T(u, v)|$. The proof of the next theorem is in fact based on a different technique.

Theorem 2. *Let $\varepsilon > 0$ be a given constant. Any tree admits a $(1 + \varepsilon)$-EMST drawing that can be computed in $O(n)$ time in the real RAM model of computation.*

Sketch of Proof: Let T be a tree. Root T at an arbitrary vertex v. We describe a recursive algorithm that computes a drawing of T contained in a disc C_v of radius r, for any given value of $r > 0$. A high-level description of the algorithm is as follows. Vertex

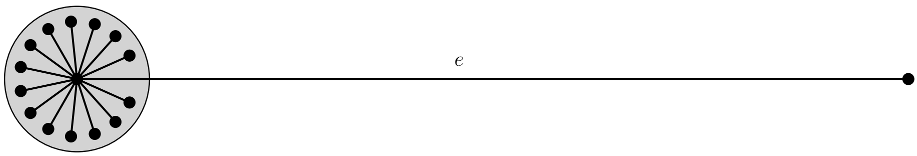

Fig. 1. By making edge e sufficiently long, the weight of the drawing can arbitrarily approximate the weight of a Euclidean minimum spanning tree of its vertex set. However, the resulting drawing may not be a $(1+\varepsilon)$-EMST drawing.

v is drawn at the center of C_v; each neighbor u of v is drawn at an arbitrary point of a distinct circle centered at v and of radius smaller than r. The subtree rooted at each neighbor u of v is recursively drawn inside a sufficiently small disc C_u centered at u and such that $C_u \subset C_v$. The radii of the concentric circles hosting the neighbors of v and the radius of each C_u are chosen in such a way that the resulting drawing satisfies the statement. See also Fig. 2.

More formally, let n be the number of vertices of T, let $r > 0$ be a given value and assume that any tree with at most $n-1$ vertices admits a planar drawing contained in a disc of radius r' for any $r' > 0$ and satisfying the proximity constraint (the base case with $n = 1$ is trivially true by representing T as the center of this disc). Denote with $\deg(v)$ the degree of the root v of T. We prove that T admits a planar drawing Γ that satisfies the proximity constraint and that is contained in a disc of radius r centered at v. Compute a real number c such that $c \geq \max\{\varepsilon, \frac{1+\varepsilon}{\varepsilon}\}$; note that, this also implies that $\frac{c-1}{c} \geq \frac{1}{1+\varepsilon}$ and $c > 1$. Choose λ to be a real number such that $\lambda c^{deg(v)+1} = r$. Let $u_1, u_2, \ldots, u_{deg(v)}$ be the neighbors of v. Draw v at the origin of the plane and draw u_i at polar coordinates $(\lambda c^i, (i-1)\cdot\theta)$, where $\theta = \frac{\pi}{deg(v)-1}$. Clearly, no two edges (v, u_i) and (v, u_j) of T overlap. The subtree rooted at each u_h $(h = 1, 2, \ldots, deg(v))$ is recursively drawn in the disc centered at u_h and having radius $r' = \min\{\frac{\varepsilon}{1+\varepsilon}\lambda c, (d(u_2, u_1) - \lambda(c^2 - c^1))/2\}$. This drawing exists by inductive hypothesis. Also, it is easy to see that $(d(u_2, u_1) - \lambda(c^2 - c))/2 \leq (d(u_i, u_j) - \lambda(c^i - c^j))/2$, for $1 \leq j < i \leq deg(v)$.

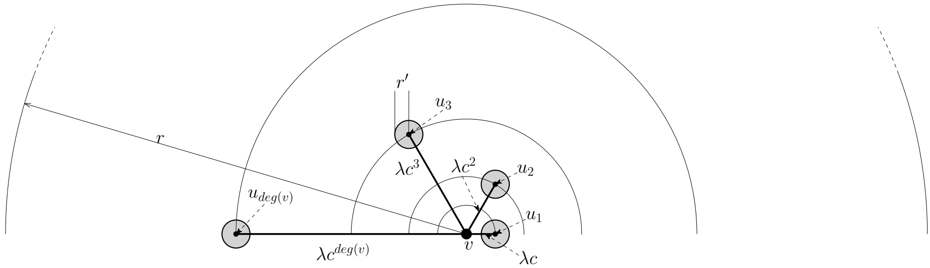

Fig. 2. The drawing construction described in the proof of Theorem 2

We prove that the computed drawing Γ of T is contained in the disc of radius r centered at v. The distance between v and the vertex that is farthest from v is at most $\lambda c^{deg(v)} + r'$. We need to prove that such a distance is at most $\lambda c^{deg(v)+1} = r$, i.e. that $\lambda c^{deg(v)+1} - \lambda c^{deg(v)} \geq r'$. We have $\lambda c^{deg(v)+1} - \lambda c^{deg(v)} = \lambda c^{deg(v)}(c-1)$ and $(c-1) \geq \frac{c}{1+\varepsilon}$. Thus, $\lambda c^{deg(v)+1} - \lambda c^{deg(v)} = \lambda c^{deg(v)}(c-1) \geq \lambda c \frac{c^{deg(v)}}{1+\varepsilon}$. By definition of c, $c^{deg(v)} > c \geq \varepsilon$; since $r' \leq \frac{\varepsilon}{1+\varepsilon}\lambda c$, we have $\lambda c^{deg(v)+1} - \lambda c^{deg(v)} \geq \lambda r \frac{c^{deg(v)}}{1+\varepsilon} \geq \lambda c \frac{\varepsilon}{1+\varepsilon} \geq r'$.

We now prove that Γ satisfies the proximity constraint. Let u and w be any two vertices of Γ and assume they are not adjacent, because otherwise the statement is trivially satisfied. If u and w are in the same subtree rooted at u_i, then they satisfy the statement by induction. If u is in a subtree rooted at u_h and w is in a subtree rooted at u_k with $h > k$ we have $d(u,w) \geq d(u_h,u_k) - 2r' \geq d(u_h,u_k) - (d(u_2,u_1) - \lambda(c^2-c)) \geq d(u_h,u_k) - (d(u_h,u_k) - \lambda(c^h - c^k)) = \lambda(c^h - c^k) \geq \lambda(c^h - c^{h-1}) = \lambda c^h \frac{c-1}{c} \geq \frac{1}{1+\varepsilon}|e_T(u,w)|$. If u is in a subtree rooted at u_k and $w = v$ we have $d(u,w) = d(u,v) \geq d(v,u_k) - r' = \lambda c^k - r' \geq \lambda c^k - \frac{\varepsilon}{1+\varepsilon}\lambda c \geq \lambda c^k - \frac{\varepsilon}{1+\varepsilon}\lambda c^k = \frac{1}{1+\varepsilon}\lambda c^k = \frac{1}{1+\varepsilon}|e_T(u,w)|$. Since Γ is a planar drawing by construction, we conclude that Γ is a $(1+\varepsilon)$-EMST drawing.

The algorithm spends $O(deg(v))$ time for each vertex v, which implies an $O(n)$ time complexity. □

It may be worth recalling that no tree having vertex degree higher than six can be represented as a Euclidean minimum spanning tree of a set of points and that it is NP-hard deciding whether a tree of degree six admits an EMST drawing [5,13]. Theorem 2 provides a tool to construct a drawing that is as close as possible to an EMST drawing for those trees that have degree larger than five. We observe however that the drawing algorithm of Theorem 2 may lead to drawings whose area[1] is exponential in n. For example, let T be a star-tree with n vertices (i.e., T consists of a vertex connected to $n-1$ leaves). The algorithm of Theorem 2 computes a drawing of T inside a disc whose radius is $r = \lambda c^n$, and therefore the area of the smallest axis-parallel rectangle including this disc is $O(c^{2n})$. Computing $(1+\varepsilon)$-EMST drawings of polynomial area is the subject of the next two sections.

4 Polynomial Area Approximation Schemes for Bounded Degree Trees

In this section we show that a tree with n vertices and bounded degree admits a $(1+\varepsilon)$-EMST drawing whose area is $O(n^{c+f(\varepsilon)})$, where c is a positive constant and $f(\varepsilon)$ is a polylogarithmic function of ε that tends to infinity as ε tends to zero.

The very general idea of our approach is similar to that used in many papers that compute compact drawings of trees: Recursively compute the drawing by composing subdrawings of subtrees; if each composition increases the area of the current drawing

[1] The area of a drawing Γ is the area of the smallest axis-aligned rectangle enclosing Γ, for a given vertex resolution rule. A vertex resolution rule defines the minimum distance between any two vertices.

by a constant factor and if the number of recursive steps is a logarithmic function of the input size, then the total area is polynomial (see, e.g. [3,9]).

Based on this idea one could think of approaching the construction of $(1+\varepsilon)$-EMST drawings in polynomial area by using the edge-separator theorem of Valiant [15]. Namely, every tree T with n vertices and vertex degree at most Δ has an edge (called edge-separator) whose removal leaves two components, each containing at most $\frac{\Delta-1}{\Delta} \cdot n$ vertices. Therefore, one might think of drawing each of the components recursively, and add the removed edge back with a sufficient length that guarantees the proximity constraint. Because of the size of each component, the number of levels in the recursion is $O(\log_b n)$, with $b = \frac{\Delta}{\Delta-1}$ and hence the area of the resulting drawing is polynomial in n. Unfortunately, it is not clear how this simple approach could lead to drawings without edge crossings. We therefore follow a different approach.

In order to guarantee a logarithmic number of recursive steps, we decompose the tree into subtrees of smaller size by means of a *greedy path decomposition*. Let T be a rooted tree such that each vertex has at most k children, and let v_0 be the root of T. A *greedy path of* T is a path $v_0, v_1, \ldots, v_k$ connecting the root v_0 to a leaf v_k and such that v_i is the root of the largest subtree rooted at v_{i-1} ($1 \leq i \leq k$). A greedy path decomposition of a rooted tree T consists of recursively identifying greedy paths and on removing them so to decompose the tree into rooted subtrees of smaller size. The decomposition ends when the tree is a path (possibly consisting of a single vertex). Greedy paths decompositions of rooted trees have, for example, been used by Chan [2] to compute compact drawings of binary trees, and by Kaufmann [8] to prove polynomial area bounds for EMST drawings of ternary trees.

Let T be a tree with a given greedy path decomposition and let T' be a subtree of T. The *greedy depth* of T' (with respect to the given decomposition) is denoted as $\gamma(T')$, and defined as follows: (i) If T' is a path, $\gamma(T') = 1$; (ii) otherwise, $\gamma(T') = \max_i\{\gamma(T_i)\} + 1$, where each T_i is a tree obtained from T' by removing its greedy path for the given decomposition. Intuitively, the greedy depth of a tree for a given greedy path decomposition is the depth of the recursion in the decomposition process. If T has n vertices, the size of a subtree of T having greedy depth i is at most $\frac{n}{2^i}$. This immediately implies the following property.

Property 1. Let T be a tree with n vertices and a given greedy path decomposition. Then, $\gamma(T) \leq \lceil \log_2 n \rceil$.

Theorem 3. *Let $\varepsilon > 0$ be a given constant. Let $\Delta > 2$ be a positive integer. Let T be a tree with n vertices and vertex degree at most Δ. T admits a $(1+\varepsilon)$-EMST drawing whose area is $O(n^{4+2\log_2(\frac{c^{\Delta+1}+c^{\Delta}+c^2-3c}{c-1})})$, where c is a constant such that $c \geq \max\{\frac{1+\varepsilon}{\varepsilon}, \frac{1}{\sin\frac{\pi}{2(\Delta-1)}}\}$. Furthermore, such a drawing can be computed in $O(n)$ time in the real RAM model of computation.*

Sketch of Proof: We describe a drawing algorithm assuming that T is rooted and each internal vertex of T has exactly $\Delta - 1$ children. This assumption is not restrictive. Indeed, if T is a tree of degree at most Δ, we can always root T at a leaf and add to any internal vertex of degree $k < \Delta - 1$ a set of $\Delta - 1 - k$ dummy children. In this way, the number of vertices of the augmented tree is at most $(\Delta-1)n$, and hence, still linear in n.

The drawing algorithm applies a recursive construction based on a greedy path decomposition. Let $T' = (V', E')$ be a subtree of T such that $\gamma(T') = i$ $(1 \leq i \leq \lceil \log_2 n \rceil)$. Let Π be the greedy path of T'. The algorithm constructs a drawing Γ' of T' by composing the drawings of all trees obtained from T' by removing Π. Denote by $n(T')$ the number of vertices of T'. The algorithm will maintain the following invariants for drawing Γ' (see Fig. 3(a) for an illustration):

(I1) $\forall u, v \in V'$, $d(u,v) \geq \frac{1}{1+\varepsilon}|e_{T'}(u,v)|$.
(I2) Γ' is completely contained in the north-east quadrant of a disc C' such that C' is centered at the root v' of T' and the radius of C' is $r' = n(T')(2c(b+1))^{\log_2 n(T')+1}$ where $b = (c^{\Delta-1} + 2\frac{c^{\Delta-1}-1}{c-1})$.
(I3) Γ' is planar.

We now prove that Γ' exists. The proof is by induction on the greedy depth i of T'. The base case is for $i = 1$. Since $\Delta > 2$, each internal vertex of T' has $\Delta - 1 \geq 2$ children, and therefore in the base case T' consists of a single vertex. T' is drawn as a single point centered at a disc of radius $2c(b+1)$ and thus Γ' satisfies all invariants.

By inductive hypothesis, each subtree with greedy depth $i-1$ admits a drawing satisfying Invariants (I1), (I2), and (I3). We construct Γ' as follows (see also Fig. 3(b)).

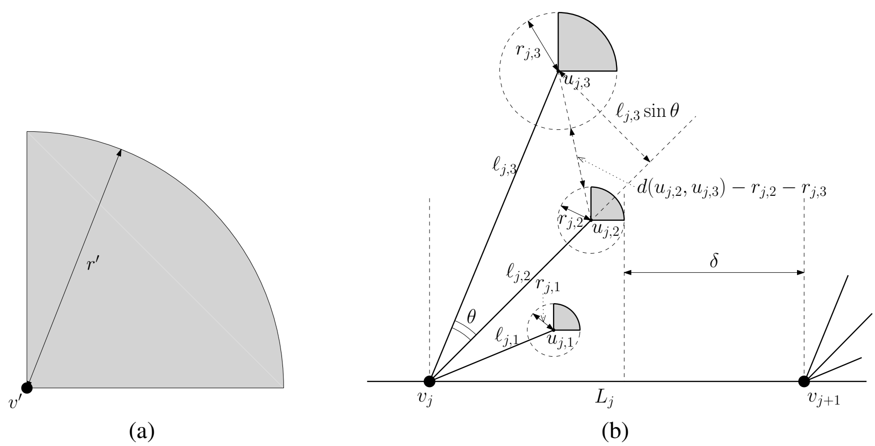

Fig. 3. The drawing construction in the proof of Theorem 3

Denote by $v_1, v_2, \ldots, v_h$ the vertices of Π, and let $u_{j,1}, u_{j,2}, \ldots, u_{j,\Delta-2}$ the children of v_j that are not in Π $(1 \leq j \leq h-1)$. The vertices $v_1, v_2, \ldots, v_h$ are drawn on a horizontal line, in this order from left to right. The distance between v_j and v_{j+1} $(1 \leq j \leq h-1)$ is denoted by L_j and its value will be specified later. Denote by $\Gamma_{u_{j,k}}$ the drawing of each subtree rooted at $u_{j,k}$ $(1 \leq j \leq h-1, 1 \leq k \leq \Delta-2)$. By Invariant (I2) each $\Gamma_{u_{j,k}}$ is contained in a disc of suitable radius; we denote this radius as $r_{j,k}$ and we assume that the children of v_j are ordered so that $r_{j,k} < r_{j,k+1}$ $(1 \leq k \leq \Delta-3)$.

For $1 \leq k \leq \Delta - 2$, drawing $\Gamma_{u_{j,k}}$ is placed in such a way that the polar coordinates of $u_{j,k}$ with respect to the position of v_j are $(\ell_{j,k}, (k+1)\theta)$, where $\theta = \frac{\pi}{2(\Delta-1)}$ and $\ell_{j,k}$ is defined as follows:

$$\ell_{j,k} = \begin{cases} cr_{j,0} & k = 0 \\ c(\ell_{j,k-1} + r_{j,k-1} + r_{j,k}) & k > 1 \end{cases}$$

The value of L_j is set to $L_j = c(\ell_{j,\Delta-2} + \ell_{j+1,\Delta-2} + r_{j,\Delta-2} + r_{j+1,\Delta-2})$.

For reasons of space, the proof that the Invariants `(I1)`, `(I2)` and `(I3)` are maintained is omitted. From Invariants `(I1)` and `(I3)` it follows that Γ is a $(1+\varepsilon)$-EMST drawing of T. Also, by Invariant `(I2)`, Γ is contained in a disc of radius $n(2c(b+1))^{\log_2 n+1} = (2c(b+1))n^{2+\log_2(c(b+1))}$. Hence, the area of the drawing is $O(n^{2+\log_2(c(b+1))})$ which, by the definition of b, is $O(n^{4+2\log_2(\frac{c^{\Delta+1}+c^{\Delta}+c^2-3c}{c-1})})$.

The algorithm spends $O(deg(v))$ time for each vertex v, i.e., $O(n)$ time in total. □

We recall that it is not known how to draw in polynomial area a tree of degree five as a Euclidean minimum spanning tree in the plane [13]. Theorem 3 implies that, for any given constant $\varepsilon > 0$, an approximation of an EMST drawing with polynomial area exists for trees with degree five vertices.

5 Trees with Logarithmic Height

We devote this section to trees having small vertex degree and logarithmic height. We describe ad-hoc algorithms that compute $(1+\varepsilon)$-EMST drawings of these trees by using significantly less area than the one given by Theorem 3. As a byproduct of this study, we show how to realize a complete binary tree with n vertices as a Euclidean minimum spanning tree in area $O(n^{3.802})$, which improves the best previously known upper bound of $O(n^{4.3})$ proved by Frati and Kaufmann [6].

5.1 Trees with Vertex Degree at Most Six

In the next theorem we exploit the maximum vertex degree six and the logarithmic height of the input tree to design a recursive algorithm that does not use the greedy path decomposition. In the statement, if a rooted tree T has degree Δ, each internal vertex of T has at most $\Delta - 1$ children.

Theorem 4. *Let $\varepsilon > 0$ and $h > 0$ be given constants. Let Δ be a positive integer such that $3 \leq \Delta \leq 6$. Let T be a rooted tree with n vertices, vertex degree at most Δ, and height at most $h \log_{\Delta-1} n$. T admits a $(1+\varepsilon)$-EMST drawing whose area is $O(n^{2h\log_{\Delta-1}(c+2)})$, where $c = \frac{2+\varepsilon}{\varepsilon}$. Furthermore, such a drawing can be computed in $O(n)$ time in the real RAM model of computation.*

Sketch of Proof: We describe an algorithm that constructs a drawing of T and prove that this drawing satisfies the properties in the statement. For any vertex v of T, denote by T_v the subtree of T rooted at v. If i is the level of v ($0 \leq i \leq h\log_{\Delta-1} n$), the algorithm recursively constructs a drawing Γ_v of T_v inside a disc C_v centered at v, with a suitable radius r_i. Γ_v will be such that the following invariants hold:

(I1) $\forall u, w \in T_v,\;\; d(u,w) \geq \frac{1}{1+\varepsilon}|e_{T_v}(u,w)|$;
(I2) There exists a ray of C_v departing from v that does not cross any edge of Γ_v; in the following we call this ray the *free ray* of Γ_v.
(I3) Γ_v is planar.

We show how to construct Γ_v by induction on the level of v, going from the highest level of T to level 0. Level 0 is the level of the root of T.

A vertex v at the deepest level of T is a leaf; in this case T_v is drawn as a single point centered at a disc of radius 1, so that Invariants (I1)-(I3) trivially hold for Γ_v.

Suppose by induction that for any vertex v' at level $i > 0$, a drawing $\Gamma_{v'}$ of $T_{v'}$ that satisfies Invariants (I1)-(I3) exists. Let v be a vertex at level $i-1$ and let $u_1, u_2, \ldots, u_d$ be its children. Note that, if v is not the root of T then $d \leq \Delta - 1$; if v is the root of T then $d \leq \Delta$. Drawing Γ_v is constructed by combining the drawings Γ_{u_j} of T_{u_j} $(1 \leq j \leq d)$. Namely, Γ_v is drawn inside a disc C_v of radius r_{i-1}, such that v is placed at the center of C_v and the drawings Γ_{u_j} are distributed around v. More precisely, if we assume (without loss of generality) that v is placed at the origin of the plane, then each u_j is placed at a point of polar coordinates $((r_{i-1} - r_i), j\frac{2\pi}{k})$, and Γ_{u_j} is rotated so that its free rays has the direction of the segment connecting v to u_j. Finally, the radius of C_v is set to $r_{i-1} = (c+2)r_i$, where $c = \frac{2+\varepsilon}{\varepsilon}$. See also Fig. 4(a).

We first prove that Invariant (I1) holds for Γ_v. Let u and w be two arbitrary vertices of T_v. Three cases are possible. If u and w are both in the same T_{u_j} $(1 \leq j \leq d)$ then $d(u,w) \geq \frac{1}{1+\varepsilon}|e_T(u,w)|$ by the inductive hypothesis.

If $u \in T_{u_j}$ and $w \in T_{u_l}$, with $1 \leq j < l \leq d$, we have that $d(u,w) \geq d(u_j, u_l) - 2r_i$. Denote as δ be the distance between the discs containing two consecutive drawings Γ_{u_j} and $\Gamma_{u_{j+1}}$ around v (see also Fig. 4(a)). We have that $d(u,w) \geq d(u_j,u_l) - 2r_i \geq d(u_j, u_{j+1}) - 2r_i = \delta$; also $e_{T_v}(u,w) = (v, u_j)$ and therefore $|e_{T_v}(u,w)| = r_{i-1} - r_i$. Hence, it suffices to show that $\delta \geq \frac{1}{1+\varepsilon}(r_{i-1} - r_i)$. By simple trigonometry, $\frac{\delta}{2} + r_i = (c+1)r_i \sin(\pi/\Delta)$. It follows that $\delta \geq \frac{1}{1+\varepsilon}(r_{i-1} - r_i)$ can be rewritten as $2((c+1)\sin(\pi/\Delta) - 1)r_i \geq \frac{1}{1+\varepsilon}(c+1)r_i$, which is verified for $c \geq \frac{1+2(1+\varepsilon)(1-\sin(\pi/\Delta))}{2(1+\varepsilon)\sin(\pi/\Delta)-1}$ and $2(1+\varepsilon)\sin(\pi/\Delta) - 1 > 0$. Note that for $3 \geq \Delta \geq 6$ we have both $\frac{2+\varepsilon}{\varepsilon} \geq \frac{1+2(1+\varepsilon)(1-\sin(\pi/\Delta))}{2(1+\varepsilon)\sin(\pi/\Delta)-1}$ and $2(1+\varepsilon)\sin(\pi/\Delta) - 1 > 0$.

If u coincides with v and w is in T_{u_j} $(1 \leq j \leq d)$, we have that $d(u,w) \geq r_{i-1} - 2r_i$; also $e_{T_v}(u,w) = (v,u_j)$ and therefore $|e_{T_v}(u,w)| = (r_{i-1} - r_i)$. To show Invariant (I1) it suffices to prove that $r_{i-1} - 2r_i \geq \frac{1}{1+\varepsilon}r_{i-1} - r_i$. By construction, $r_{i-1} - 2r_i = cr_i$ and $r_{i-1} - r_i = (c+1)r_i$. It follows that the previous inequality can be rewritten as $cr_i \geq \frac{1}{1+\varepsilon}(c+1)r_i$, which is verified since $c = \frac{2+\varepsilon}{\varepsilon}$.

We now prove that Invariant (I2) also holds for Γ_v. Since the distance δ between the discs containing two consecutive drawings Γ_{u_j} and $\Gamma_{u_{j+1}}$ around v is at least $\frac{1}{1+\varepsilon}(r_{i-1} - r_i)$, it follows that δ is positive. Hence, a free ray of Γ_v is any ray from v passing between the discs containing Γ_{u_j} and $\Gamma_{u_{j+1}}$.

Finally, by construction, it is easy to see that Invariant (I3) holds.

It follows that, if v is the root of T, drawing $\Gamma = \Gamma_v$ is a $(1+\varepsilon)$-EMST drawing. The bound on the area of the drawing is proved by observing that the radius r_0 of the disc containing Γ is related to the radius r_i by the following equation: $r_0 = (c+2)^i r_i$. Since the height of T is at most $h \log_{\Delta-1} n$ we have $r_0 \leq (c+2)^{h \log_{\Delta-1} n} = n^{h \log_{\Delta-1}(c+2)}$, which implies an area bound of $O(n^{2h \log_{\Delta-1}(c+2)})$.

The algorithm spends $O(deg(v))$ time for each vertex v, i.e., $O(n)$ time in total. □

It could be interesting to compare the bounds of Theorem 3 with those of Theorem 4. Suppose that T is a complete rooted tree with n vertices such that every internal vertex has five children. Assume that we wish to be within a factor $\varepsilon = 0.5$ of having an EMST drawing of T, i.e. we want to compute a 1.5-EMST drawing of T. By using the construction of Theorem 3 the area of the drawing is $O(n^{26.17})$; by using Theorem 4, we can compute a 1.5-EMST drawing of T in $O(n^{2.42})$ area.

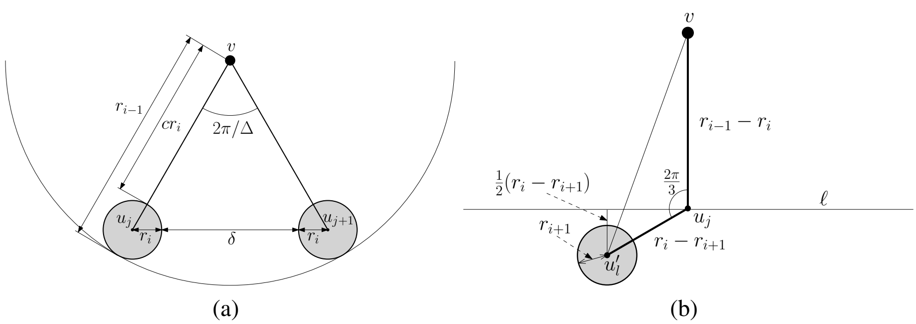

Fig. 4. (a) The drawing construction in the proof of (a) Theorem 4 (b) Theorem 6

One may wonder whether Theorem 4 can be extended to trees of degree larger than six. Notice however that the function $\frac{1+2(1+\varepsilon)(1-\sin(\frac{\pi}{\Delta}))}{2(1+\varepsilon)\sin(\frac{\pi}{\Delta})-1}$ used in Theorem 4 is finite and positive for every $\varepsilon > 0$ only when $3 \leq \Delta \leq 6$. If $\Delta > 6$ the argument in the proof of Theorem 4 cannot be applied. This motivates us to look at $(1+\varepsilon)$-EMST drawings in three dimensions. The proof of the next theorem is omitted.

Theorem 5. *Let $\varepsilon > 0$ and $h > 0$ be given constants. Let Δ be a positive integer such that $3 \leq \Delta \leq 12$. Let T be a rooted tree with n vertices, vertex degree at most Δ, and height at most $h \log_{\Delta-1} n$. T admits a $(1+\varepsilon)$-EMST in three dimensional space whose volume is $O(n^{3h \log_{\Delta-1}(c+2)})$, where $c = \frac{2+\varepsilon}{\varepsilon}$. Furthermore, such a drawing can be computed in $O(n)$ time in the real RAM model of computation.*

Finally, we observe that it is possible to use the drawing technique of Theorem 4 to compute $(1+\varepsilon)$-EMST drawings of trees of degree higher than six, provided that $(1+\varepsilon)$ approximation factor is not required to be arbitrarily close to 1, but it depends on the vertex degree.

5.2 *EMST* Drawings of Complete Binary Trees

Frati and Kaufmann [6] prove that a complete binary tree can be drawn as a Euclidean minimum spanning tree in area $O(n^{4.3})$. In this section we improve this bound as an application of the techniques in the proof of Theorem 4. Namely, we show that if $\Delta = 3$ the condition $cr_i \geq \frac{1}{1+\varepsilon}(c+1)r_i$ does not need to be verified for proving the correctness of the geometric construction of Theorem 4. An implication of this observation is that for $\Delta = 3$ the proof of Theorem 4 also works by setting $\varepsilon = 0$.

Theorem 6. *Let T be a rooted complete binary tree with n vertices. T admits an EMST drawing in area $O(n^{3.802})$. Furthermore, such a drawing can be computed in $O(n)$ time in the real RAM model of computation.*

Sketch of Proof: As already observed, a straight-line two-dimensional drawing of a tree T is a Euclidean minimum spanning tree of the points representing its vertices if and only if that the following property holds:

$$\forall u, w \in V, d(u,w) \geq |e_T(u,w)| \tag{1}$$

Let z be the root of T. We draw the two edges incident to z as segments of the same length and forming an angle of $\frac{2\pi}{3}$. By applying the drawing technique described in the proof of Theorem 4, we recursively construct a drawing where for each internal vertex v different from z any two consecutive edges around v form an angle of $\frac{2\pi}{3}$. Let v be an internal vertex of height $i-1>0$ and let u_1 and u_2 be its children. According to our drawing technique the segment connecting v to u_1 forms an angle of $\frac{2\pi}{3}$ with the positive x-axis, and the segment connecting v to u_2 forms an angle of $-\frac{2\pi}{3}$ with the positive x-axis. Hence the positive x-axis is a free ray for T_v; this free ray is used to connect the drawing of T_v to the parent of v after suitable rotation.

It remains to prove that the computed drawing satisfies Property (1). The proof is by induction. Let u and w be any two vertices of T_v. If u and w are vertices of the same subtree T_{u_j} ($1 \leq j \leq 2$), then Property (1) holds by induction. If u is in the subtree T_{u_1} and w is in T_{u_2} then $d(u,w) \geq d(u_1,u_2) - 2r_i = \delta$ and therefore it is sufficient to guarantee $\delta \geq (r_{i-1} - r_i)$. We have that $\frac{\delta}{2} + r_i = (c+1)r_i \sin(\pi/3) = (c+1)\frac{\sqrt{3}}{2}r_i$. Thus, it must be $\sqrt{3}(c+1)r_i - 2r_i \geq (c+1)r_i$, from which we obtain $c \geq \frac{3-\sqrt{3}}{\sqrt{3}-1} > 1.732$. If u coincides with v and if w is in the subtree T_{u_j} ($1 \leq j \leq 2$), then assume that w does not coincide with u_j (otherwise Property (1) is trivially true) and let u'_1 and u'_2 be the children of u_j. Vertex w is a vertex of $T_{u'_l}$ ($1 \leq l \leq 2$). We have that $d(u,w) \geq d(v,u'_l) - r_{i+1}$ (see Fig. 4(b)). Let ℓ be the straight-line orthogonal to the segment representing edge (v,u_j) passing through u_j. The distance from u'_l to ℓ is $(r_i - r_{i+1})\sin(\frac{\pi}{6}) = \frac{1}{2}(r_i - r_{i+1}) = \frac{1}{2}(c+1)r_{i+1}$, that is larger than r_{i+1} for any $c>1$. Thus, for every $c>1$, the disc containing $T_{u'_l}$ and v are on opposite sides of ℓ, which means that $d(u,w) > |e_T(u,w)| = |(v,u_j)| = r_{i-1} - r_i$.

Therefore we can choose $r_{i-1} = (c+2)r_i$ with $c = 1.733$, which implies that the radius of the disc enclosing the whole drawing is $r_0 = (c+2)^{\log_2 n+1} = (c+2)n^{\log_2(c+2)} = (c+2)n^{\log_2(3.733)} < (c+2)n^{1.901}$ and the area of the drawing is $O(n^{3.802})$.

The algorithm spends $O(deg(v))$ time for each vertex v, i.e., $O(n)$ time in total. □

6 Open Problems

The study of $(1+\varepsilon)$-EMST drawings and of their variants opens interesting perspectives both from the theoretical and from the practical point of view. For example, one can observe that drawings of trees computed by using spring embedder heuristics seem to be $(1+\varepsilon)$-EMST drawings in many cases (see, e.g., [4]). It would be nice to experimentally study this possible correlation. Also, extending the concept of "approximated drawing" to other types of proximity rules and/or to other families of graphs is a promising research subject. For example, does every triangulated planar graph admit a straight-line drawing whose weight is at most $(1+\varepsilon)$ times the weight of the Delaunay triangulation of its vertex set?

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