# The $(p, q)$-total Labeling Problem for Trees


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**Abstract.** A $(p, q)$-total labeling of a graph $G$ is an assignment $f$ from the vertex set $V(G)$ and the edge set $E(G)$ to the set of nonnegative integers such that $|f(x) - f(y)| \geq p$ if $x$ is a vertex and $y$ is an edge incident to $x$, and $|f(x) - f(y)| \geq q$ if $x$ and $y$ are a pair of adjacent vertices or a pair of adjacent edges, for all $x$ and $y$ in $V(G) \cup E(G)$. A $k$-$(p, q)$-total labeling is a $(p, q)$-total labeling $f : V(G) \cup E(G) \to \{0, \ldots, k\}$, and the $(p, q)$-total labeling problem asks the minimum $k$, which we denote by $\lambda^T_{p,q}(G)$, among all possible assignments. In this paper, we first give new upper and lower bounds on $\lambda^T_{p,q}(G)$ for some classes of graphs $G$, in particular, tight bounds on $\lambda^T_{p,q}(T)$ for trees $T$. We then show that if $p \leq 3q/2$, the problem for trees $T$ is linearly solvable, and give a complete characterization of trees achieving $\lambda^T_{p,q}(T)$ if in addition $\Delta \geq 4$ holds, where $\Delta$ is the maximum degree of $T$. It is contrasting to the fact that the $L(p, q)$-labeling problem, which is a generalization of the $(p, q)$-total labeling problem, is NP-hard for any two positive integers $p$ and $q$ such that $q$ is not a divisor of $p$.


## 1 Introduction

In the channel/frequency assignment problems, we need to assign different frequencies to 'close' transmitters so that they can avoid interference. The $L(p, q)$-labelings of a graph have been extensively studied as one of important graph theoretical models of this problem. An *$L(p, q)$-labeling* of a graph $G$ is an assignment $f$ from the vertex set $V(G)$ to the set of nonnegative integers such that $|f(x) - f(y)| \geq p$ if $x$ and $y$ are adjacent and $|f(x) - f(y)| \geq q$ if $x$ and $y$ are at distance 2, for all $x$ and $y$ in $V(G)$. A $k$-$L(p, q)$-labeling is an $L(p, q)$-labeling $f : V(G) \to \{0, \ldots, k\}$, and the *$L(p, q)$-labeling problem* asks the minimum $k$, which we denote by $\lambda_{p,q}(G)$, among all possible assignments. Notice that we can use $k + 1$ different labels when $\lambda_{p,q}(G) = k$ since we can use 0 as a label for conventional reasons. We can find a lot of related results on $L(p, q)$-labelings in comprehensive surveys by Calamoneri [3] and by Yeh [25]. From the applicational point of view, we assume that $p \geq q \geq 1$ unless otherwise stated. Also, we assume that $p$ and $q$ are relatively prime, since otherwise, an $L(p, q)$-labeling is equivalent to an $L(p/\ell, q/\ell)$-labeling, where $\ell = \gcd(p, q)$.


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**(*p*, *q*)-total labeling and a conjecture.** In [24], Whittlesey et al. studied the $L(2,1)$-labeling number of incidence graphs, where the *incidence graph* of a graph $G$ is the graph obtained from $G$ by replacing each edge $(v_i, v_j)$ with two edges $(v_i, v_{ij})$ and $(v_{ij}, v_j)$ after introducing one new vertex $v_{ij}$. Observe that an $L(p,q)$-labeling of the incidence graph of a given graph $G$ can be regarded as an assignment $f$ from $V(G) \cup E(G)$ to the set of nonnegative integers such that $|f(x) - f(y)| \geq p$ if $x$ is a vertex and $y$ is an edge incident to $x$, and $|f(x) - f(y)| \geq q$ if $x$ and $y$ are a pair of adjacent vertices or a pair of adjacent edges, for all $x$ and $y$ in $V(G) \cup E(G)$. Such a labeling of $G$ is called a $(p,q)$-*total labeling* of $G$, while the case of $q = 1$ was first introduced as a $(p,1)$-total labeling by Havet and Yu [15,16]. In particular, a $k$-$(p,q)$-total labeling is a $(p,q)$-total labeling $f : V(G) \cup E(G) \to \{0, \dots, k\}$, and the $(p,q)$-*total labeling problem* asks the minimum $k$ among all possible assignments. We call this invariant, the minimum $k$, the $(p,q)$-*total labeling number*, which is denoted by $\lambda^T_{p,q}(G)$.

We notice that a $(1,1)$-total labeling of $G$ is equivalent to a total coloring of $G$. Generalizing the Total Coloring Conjecture [2,22], Havet and Yu [15,16] conjectured that

$$\lambda^T_{p,1}(G) \leq \Delta + 2p - 1 \tag{1}$$

holds for any graph $G$, where $\Delta$ denotes the maximum degree of a vertex in $G$. They also investigated bounds on $\lambda^T_{p,1}(G)$ under various assumptions and some of their results are described as follows: (i) $\lambda^T_{p,1}(G) \geq \Delta + p - 1$, (ii) $\lambda^T_{p,1}(G) \geq \Delta + p$ if $p \geq \Delta$, (iii) $\lambda^T_{p,1}(G) \leq \min\{2\Delta + p - 1,\ \chi(G) + \chi'(G) + p - 2\}$ for any graph $G$ where $\chi(G)$ and $\chi'(G)$ denote the chromatic number and the chromatic index of $G$, respectively, and (iv) $\lambda^T_{p,1}(G) \leq n + 2p - 2$ if $G$ is the complete graph where $n = |V(G)|$. In particular, it follows by (iii) that if $G$ is bipartite, then $\lambda^T_{p,1}(G) \leq \Delta + p$ holds (by $\chi(G) \leq 2$ and König's theorem), and if in addition, $p \geq \Delta$, then $\lambda^T_{p,1}(G) = \Delta + p$ by (ii) [1,15,16]. Also, Bazzaro et al. [1] showed that $\lambda^T_{p,1}(G) \leq \Delta + p + s$ for any $s$-degenerated graph (by $\chi(G) \leq s + 1$ and $\chi'(G) \leq \Delta + 1$), where an *s-degenerated graph* $G$ is a graph which can be reduced to a trivial graph by successive removal of vertices with degree at most $s$, that $\lambda^T_{p,1}(G) \leq \Delta + p + 3$ for any planar graph (by the Four-Color Theorem), and that $\lambda^T_{p,1}(G) \leq \Delta + p + 1$ for any outerplanar graph other than an odd cycle (since any outerplanar graph is 2-degenerated, and any outerplanar graph other than an odd cycle satisfies $\chi'(G) = \Delta$ [9]). Also, there are many related works about bounds on $\lambda^T_{p,1}(G)$ [6,14,19,20]. From the algorithmic point of view, Havet and Thomassé [17] recently showed that for bipartite graphs, if (i) $p \geq \Delta$ or (ii) $\Delta = 3$ and $p = 2$, then the $(p,1)$-total labeling problem is polynomially solvable and otherwise it is NP-hard.

In [7,18,21], the $(r,s,t)$-coloring problem which is a generalization of the $(p,q)$-total labeling problem was studied, while results in the cases corresponding to the $(p,q)$-total labeling problem (actually, the cases of $t \geq r = s$) are limited to paths, cycles, stars or the complete graph with some $p$ and $q$. To our best knowledge, there are quite few studies about $(p,q)$-total labelings other than these. In this paper, we focus on the $(p,q)$-total labeling problem for some classes of graphs, especially for trees.

***L*(*p*, *q*)-labelings and (*p*, *q*)-total labelings of trees.** Let $T$ be a tree. As for the $L(2,1)$-labeling problem, Griggs and Yeh [12] showed that $\lambda_{2,1}(T) \in \{\Delta + 1, \Delta + 2\}$. Moreover, Chang and Kuo [5] showed that $\lambda_{2,1}(T)$ can be computed in polynomial time, and recently Hasunuma et al. [13] gave a linear time algorithm for this problem. However, a characterization of trees $T$ achieving $\lambda_{2,1}(T)$ is still open. Also, it was shown in [4] that $\lambda_{p,1}(T) \leq \min\{\Delta + 2p - 2, 2\Delta + p - 2\}$ and that $L(p, 1)$-labeling problem for trees can be solved in $O((p + \Delta)^{5.5} n) = O(\lambda_{p,1}(T)^{5.5} n)$ time by extending the algorithm in [5], where $n = |V(G)|$. On the other hand, Fiala et al. [8] showed that the $L(p, q)$-labeling problem for trees is NP-hard for any two positive integers $p$ and $q$ such that $q$ is not a divisor of $p$. Concerning bounds on $\lambda_{p,q}(T)$, Georges and Mauro [11] gave the exact value of $\lambda_{p,q}(T)$ for the infinite regular trees $T$, which gives tight upper bounds on $\lambda_{p,q}(T)$; following their results, we have $\lambda_{p,q}(T) \leq p + (2\Delta - 2)q$.

As for the $(p, q)$-total labeling problem, the above algorithms can also be applied to the case of $q = 1$, since the incidence graph of a tree is also a tree. Moreover, due to the structure of the incidence graph of a tree, it can be observed that $\lambda^T_{p,1}(T)$ becomes much smaller than $\lambda_{p,1}(T)$. By bounds for bipartite graphs in [1,15,16], it follows that $\lambda^T_{p,1}(T) \in \{p + \Delta - 1, p + \Delta\}$, and that if $p \geq \Delta$, then $\lambda^T_{p,1}(T) = p + \Delta$. Recently, Wang and Chen [23] gave a characterization of trees $T$ achieving $\lambda^T_{2,1}(T)$ in the case of $\Delta = 3$.

**Our contributions.** In this paper, we mainly focus on $(p, q)$-total labeling problem for trees for general $p$ and $q$, and obtain the following results:

- (Upper bounds on $\lambda^T_{p,q}(T)$) If $p = q + r$ for $r \in \{0, 1, \ldots, q - 1\}$ and $\Delta > 1$ (resp., $\Delta = 1$), then $\lambda^T_{p,q}(T) \leq p + (\Delta - 1)q + r$ holds and this bound is tight (resp., $\lambda^T_{p,q}(T) = p + q$). If $p \geq 2q$, then $\lambda^T_{p,q}(T) \leq p + \Delta q$ holds and this bound is tight. In particular, if $p \geq \Delta q$, then $\lambda^T_{p,q}(T) = p + \Delta q$.
- (Lower bounds on $\lambda^T_{p,q}(T)$) If $q \leq p < (\Delta - 1)q$, then $\lambda^T_{p,q}(T) \geq p + (\Delta - 1)q$ holds and this bound is tight. If $p = (\Delta - 1)q + r$ for $r \in \{0, 1, \ldots, q - 1\}$, then $\lambda^T_{p,q}(T) \geq p + (\Delta - 1)q + r$ holds and this bound is tight. If $p \geq \Delta q$, then $\lambda^T_{p,q}(T) = p + \Delta q$.
- The $(p, q)$-total labeling problem with $p \leq 3q/2$ for trees can be solved in linear time. In particular, if $\Delta \geq 2$, we have $\lambda^T_{p,q}(T) \in \{p + (\Delta - 1)q, p + (\Delta - 1)q + r\}$. If $p > q$ and $\Delta \geq 4$, then $\lambda^T_{p,q}(T) = p + (\Delta - 1)q$ holds if and only if no two vertices with degree $\Delta$ are adjacent.
- In the case of $p = 2q$, the condition that no two vertices with degree $\Delta$ are adjacent is sufficient for $\lambda^T_{p,q}(T) = p + (\Delta - 1)q$, while in the case of $p > 3q/2$ and $p \neq 2q$, this condition is not sufficient.
- For any two nonnegative integers $p$ and $q$, the $L(p, q)$-labeling problem for trees can be solved in polynomial time if $\Delta = O(\log^{1/3} |I|)$ where $|I| = \max\{|V(T)|, \log p\}$. Particularly, if $\Delta$ is a fixed constant, it is solved in linear time.

The first and second results provide tight upper and lower bounds on $\lambda^T_{p,q}(T)$ for all pairs $(p, q)$ with $p \geq q$. The first statement in the third result indicates that as for the $(p, q)$-total labeling problem for trees, there exists a tractable case even if $q$ is not a divisor of $p$, in contrast to the NP-hardness of the $L(p, q)$-labeling problem. The second and third statements in the third result completely characterize trees $T$ achieving $\lambda^T_{p,q}(T)$ in the case of $p \leq 3q/2$ and $\Delta \geq 4$ (note that if $p = q$, we have $\lambda^T_{p,q}(T) = p + (\Delta - 1)q$ by the first and second results). This is also contrasting to the fact that no simple characterization of trees $T$ achieving $\lambda_{2,1}(T)$ is known even for the $L(2, 1)$-labeling problem.

**Organization of the paper.** The rest of this paper is organized as follows. In Section 2, after giving some basic definitions, we show several properties about bounds on $\lambda^T_{p,q}(G)$. In Sections 3 and 4, we focus on the cases where a given graph is a tree. Section 3 provides tight upper and lower bounds on $\lambda^T_{p,q}(T)$ for trees $T$. In Section 4, we propose a linear time algorithm for solving the $(p,q)$-total labeling problem with $p \leq 3q/2$ for trees, and give a characterization of trees $T$ achieving $\lambda^T_{p,q}(T)$ in the case of $p > q$ and $\Delta \geq 4$. Also, we discuss the case of $p > 3q/2$. Finally, we give concluding remarks in Section 5. Some parts of the detailed analyses are omitted due to space limitation.

## 2 Bounds on $\lambda^T_{p,q}(G)$

In this section, we investigate several properties on $(p,q)$-total labelings of a graph $G$. For this, we first define some terminology. A graph $G$ is an ordered set of its vertex set $V(G)$ and edge set $E(G)$ and is denoted by $G = (V(G), E(G))$. We assume throughout this paper that a graph is undirected, simple and connected unless otherwise stated. Therefore, an edge $e \in E(G)$ is an unordered pair of vertices $u$ and $v$, which are *end vertices* of $e$, and we often denote it by $e = (u, v)$. Let $N_G(v)$ denote the set of neighbors of a vertex $v$ in $G$; $N_G(v) = \{u \in V \mid (u, v) \in E(G)\}$. The *degree* of a vertex $v$ is $|N_G(v)|$, and is denoted by $d_G(v)$. A vertex $v$ with $d_G(v) = k$ is called a *k-vertex*. We use $\Delta(G)$ (resp., $\delta(G)$) to denote the maximum (resp., minimum) degree of a vertex in a graph $G$. A $\Delta(G)$-vertex is called *major*. We often drop $G$ in these notations if there are no confusions. For a $(p,q)$-total labeling $f : V(G) \cup E(G) \to \{0, 1, \ldots, k\}$ of $G$ and an edge $e = (u, v) \in E(G)$, we may denote $f(e)$ by $f(u, v)$. Let $\overline{f}$ denote the labeling such that $\overline{f}(z) = k - f(z)$ for each $z \in V(G) \cup E(G)$. Note that $\overline{f}$ is also a $(p,q)$-total labeling of $G$.

We have the following lemmas about upper and lower bounds on $\lambda^T_{p,q}(G)$, some of which are extensions of those discussed in the case of $q = 1$ [16].

**Lemma 1.** (*i*) $\lambda^T_{p,q}(G) \geq p + (\Delta - 1)q$.
(*ii*) *If $G$ has a major vertex whose neighbors are all major, then $\lambda^T_{p,q}(G) \geq p + \Delta q$ holds for $p \geq 2q$, and $\lambda^T_{p,q}(G) \geq p + (\Delta - 1)q + r$ holds for $p = q + r$ $(r = 0, 1, \ldots, q - 1)$.*
(*iii*) $\lambda^T_{p,q}(G) \geq p + \min\{p, \Delta q\}$. *Hence, $\lambda^T_{p,q}(G) \geq p + (\Delta - 1)q + r$ holds where $r = q$ if $p \geq \Delta q$ and $r = p - (\Delta - 1)q$ otherwise.* □

**Lemma 2.** (*i*) $\lambda^T_{p,q}(G) \leq p + q(\chi(G) + \chi'(G) - 2)$.
(*ii*) $\lambda^T_{p,q}(G) \leq p + (2\Delta - 1)q$.
(*iii*) *Let $G$ be the complete graph. Then, $\lambda^T_{p,q}(G) \leq \min\{p + (2\Delta - 1)q, 2p + \Delta q - 1\}$. In particular, $\lambda^T_{p,q}(G) = p + (2\Delta - 1)q$ if $p \geq 2\Delta q + 1$ and $|V(G)| \geq 3$.* □

In the case where $G$ is a path (resp., cycle), the incidence graph of $G$ is also a path (resp., cycle). The following lemma is obtained directly from Georges and Mauro's results about $L(p,q)$-labeling of paths or cycles [10].

**Lemma 3.** (*i*) *Let $G$ be a path. We have $\lambda^T_{p,q}(G) = p + q$ (resp., $p + 2q$, resp., $2p$) if $|V| = 2$ (resp., $|V| \geq 3$ and $p \geq 2q$, resp., $|V| \geq 3$ and $p \leq 2q$).*
(*ii*) *Let $G$ be a cycle. We have $\lambda^T_{p,q}(G) = p + 2q$ if (a) $|V|$ is even and $p \geq 2q$ or (b) $2|V| \neq 0$ mod 3 and $p \leq 2q$, $\lambda^T_{p,q}(G) = p + 3q$ if $|V|$ is odd and $p \geq 3q$, and $\lambda^T_{p,q}(G) = 2p$ otherwise.* □

Similarly to the arguments in Section 1, we can see that the following properties hold, where a graph is called *a series-parallel graph* or *a partial 2-tree* if it contains no subgraph isomorphic to a subdivision of the complete graph with four vertices. Notice that an outerplanar graph is series-parallel.

**Corollary 1.** (*i*) *If $G$ is bipartite, then $\lambda^T_{p,q}(G) \le p + \Delta q$. In particular, if $p \ge \Delta q$, then $\lambda^T_{p,q}(G) = p + \Delta q$.*
(*ii*) *If $G$ is $s$-degenerated, then $\lambda^T_{p,q}(G) \le p + (\Delta + s)q$.*
(*iii*) *If $G$ is planar, then $\lambda^T_{p,q}(G) \le p + (\Delta + 3)q$.*
(*iv*) *If $G$ is series-parallel, then $\lambda^T_{p,q}(G) \le p + (\Delta + 1)q$.* □

## 3 Tight Bounds on $\lambda^T_{p,q}(G)$ for Trees

In this section, we show the following properties about tight upper and lower bounds on $\lambda^T_{p,q}(T)$ for trees $T$.

**Theorem 1.** *Let $T$ be a tree. Then the following properties hold.*
(*i*) *If $p \ge \Delta q$, then $\lambda^T_{p,q}(T) = p + \Delta q$.*
(*ii*) *If $p = (\Delta - 1)q + r$ $(r = 0, 1, \ldots, q - 1)$, then $\lambda^T_{p,q}(T) \ge p + (\Delta - 1)q + r$ and this bound is tight.*
(*iii*) *If $p \ge 2q$, then $\lambda^T_{p,q}(T) \le p + \Delta q$ and this bound is tight.*
(*iv*) *If $p = q+r$ $(r = 0, 1, \ldots, q-1)$ and $\Delta > 1$ (resp., $\Delta = 1$), then $\lambda^T_{p,q}(T) \le p+(\Delta-1)q+r$ and this bound is tight (resp., $\lambda^T_{p,q}(T) = p + q$).*

Since trees are bipartite, the statement (i) and the former part of the statement (iii) follow from Corollary 1 (i). The former part of the statement (ii) follows from Lemma 1 (iii), and it is not difficult to see that a star $T$ achieves $\lambda^T_{p,q}(T) = p+(\Delta-1)q+r$. Lemma 1 (ii) indicates that a tree $T$ which has a major vertex whose neighbors are all major achieves $\lambda^T_{p,q}(T) = p + \Delta q$ (resp., $p + (\Delta - 1)q + r$) if $p \ge 2q$ (resp., if $p = q + r$ $(< 2q)$, $\Delta > 1$, and the former part of the statement (iv) is true). The case of $\Delta = 1$ in the statement (iv) follows from Lemma 3.

In the rest of this section, we give a proof of the former part of the case of $\Delta > 1$ in the statement (iv) to complete the proof of this theorem. For this, we assume that $\Delta \ge 2$ and give an algorithm for finding a $(p+(\Delta-1)q+r)$-$(p, q)$-total labeling of $T$ if $p = q+r$ for $r \in [0, q - 1]$. For simplicity of description, assume that $T = (V, E)$ is a tree such that all non-leaves are major.

From [10, Lemma 2.1], it follows that there exists a $\lambda^T_{p,q}(T)$-$(p, q)$-total labeling of $T$ which consists of labels with form $\alpha p + \beta q$ where $\alpha, \beta \in \mathbb{Z}_+$. Here we can assume that $\lambda^T_{p,q}(T) \ge p+(\Delta-1)q+r$ by Lemma 1(ii) and the assumption on $T$. Considering these two properties, we will seek a $(p + (\Delta - 1)q + r)$-$(p, q)$-total labeling of $T$ with form $\alpha p + \beta q$. Then, it is not difficult to see that the candidates of labels of such a form to be assigned for each non-leaf vertex (i.e., major vertex) are $0$, $p + (\Delta - 1)q + r$ $(= 2p + (\Delta - 2)q)$, or $p + iq$ for some $i \in [0, \Delta - 2]$. In particular, if a major vertex $v$ has label $p + iq$ for $i \in [0, \Delta-2]$, then the set of labels for edges incident to $v$ is $\{jq \mid j \in [0, i]\} \cup \{p + jq + r \mid j \in [i + 1, \Delta - 1]\}$. Based on these observations, we regard $T$ as a rooted tree by choosing a major vertex $v_r$ as the root, and assign labels with form $\alpha p + \beta q$ to $V \cup E$ from the root $v_r$ in the breadth-first-search order, as shown in Algorithm $(p, q)$-LABEL. Actually, we use labels 0, $p$, $(\Delta - 1)q + r$ $(= p + (\Delta - 2)q)$, and $p + (\Delta - 1)q + r$ for vertices, and repeat applying essentially four types of labelings to each scanned vertex, its incident edges, and its children. In the description of the algorithm, $p(v)$ denotes the parent of $v$ (if exists) and $C(v)$ denotes the set of children of $v$ for each vertex $v$.

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**Algorithm 1.** Algorithm $(p, q)$-LABEL

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**Input:** A tree $T = (V, E)$ with $\Delta \geq 2$ such that all non-leaves are major, and two positive integers $p$ and $q$ with $p = q + r$ and $r \in [0, q - 1]$.

**Output:** A $(p, q)$-total labeling $f : V \cup E \to \{0, 1, \ldots, p + (\Delta - 1)q + r\}$ of $T$.

1: Assign label 0 to the root $v_r$; let $f(v_r) := 0$. For each $i \in [0, \Delta - 2]$, let $f(v_r, c_i(v_r)) := p + iq$ and $f(c_i(v_r)) := p + (\Delta - 1)q + r$, where $C(v_r) = \{c_i(v_r) \mid i = 0, 1, \ldots, \Delta - 1\}$ (i.e., assign labels $p + iq$ and $p + (\Delta - 1)q + r$ to the edge $(v_r, c_i(v_r))$ and the child $c_i(v_r)$ of $v_r$, respectively). Let $f(v_r, c_{\Delta-1}(v_r)) := p + (\Delta - 1)q + r$ and $f(c_{\Delta-1}(v_r)) := (\Delta - 1)q + r$.

2: **while** there exists a non-leaf $v \in V - \{v_r\}$ such that $f(v)$ has been determined but no label is assigned to any child of $v$ where $C(v) = \{c_i(v) \mid i = 0, 1, \ldots, \Delta - 2\}$ **do**

3: **if** (Case-1) $f(p(v), v) \in \{p + iq \mid i \in [0, \Delta - 2]\}$ and $f(v) = p + (\Delta - 1)q + r$ **then**

4: Let $f(c_i(v)) := 0$ for each $i \in [0, \Delta - 3]$, $f(c_{\Delta-2}(v)) := p$, and $f(v, c_{\Delta-2}(v)) := 0$. Assign labels in $\{p + iq \mid i \in [0, \Delta - 2]\} - \{f(p(v), v)\}$ injectively to edges $\{(v, c_i(v)) \mid i \in [0, \Delta - 3]\}$.

5: **else if** (Case-2) $f(p(v), v) = p + (\Delta - 1)q + r$ and $f(v) = (\Delta - 1)q + r$ **then**

6: Let $f(c_i(v)) := p + (\Delta - 1)q + r$ and $f(v, c_i(v)) := iq$ for each $i \in [0, \Delta - 2]$.

7: **else if** (Case-3) $f(p(v), v) \in \{iq \mid i \in [0, \Delta - 2]\}$ and $f(v) = p + (\Delta - 1)q + r$ **then**

8: Let $f(c_i(v)) := (\Delta - 1)q + r$ for each $i \in [0, \Delta - 3]$, $f(c_{\Delta-2}(v)) := 0$, and $f(v, c_{\Delta-2}(v)) := (\Delta - 1)q + r$. Assign labels in $\{iq \mid i \in [0, \Delta - 2]\} - \{f(p(v), v)\}$ injectively to edges $\{(v, c_i(v)) \mid i \in [0, \Delta - 3]\}$.

9: **else if** (Case-4) $f(p(v), v) \in \{iq \mid i \in [0, \Delta - 2]\}$ and $f(v) = (\Delta - 1)q + r$ **then**

10: Let $f(c_i(v)) := p + (\Delta - 1)q + r$ for each $i \in [0, \Delta - 3]$, $f(c_{\Delta-2}(v)) := 0$, and $f(v, c_{\Delta-2}(v)) := p + (\Delta - 1)q + r$. Assign labels in $\{iq \mid i \in [0, \Delta - 2]\} - \{f(p(v), v)\}$ injectively to edges $\{(v, c_i(v)) \mid i \in [0, \Delta - 3]\}$.

11: **else if** (Case-$j'$) $\overline{f}(p(v), v))$ and $\overline{f}(v)$ satisfy the conditions of Case-$j$ for $j \in [1, 4]$ **then**

12: After determining labels for $f(c_i(v))$ and $f(v, c_i(v))$ according to the above (Case-$j$) based on $\overline{f}(p(v), v))$ and $\overline{f}(v)$, let $f(c_i(v)) := p + (\Delta - 1)q + r - f(c_i(v))$ and $f(v, c_i(v)) := p + (\Delta - 1)q + r - f(v, c_i(v))$ for each $i \in [0, \Delta - 2]$.

13: **end if**

14: **end while**

15: Output $f$ as a $(p, q)$-total labeling of $T$.

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We prove the correctness of Algorithm $(p, q)$-LABEL. Note that whenever a vertex $v$ is chosen in line 2, $f(p(v), v)$ has also been already determined. Also note that the labels assigned in each step do not violate the feasibility. Hence, it suffices to show that as a result of line 1 (resp., each iteration of the while loop in lines 2–14), each $c_i(v_r) \in C(v_r)$ (resp., $c_i(v) \in C(v)$) satisfies the conditions of Case-$j$ or Case-$j'$ in lines 2–14 for some $j \in \{1, 2, 3, 4\}$. As for the children of $v_r$, $c_i(v_r)$ for $i \in [0, \Delta - 2]$ satisfies the conditions of Case-1 and $c_{\Delta-1}(v_r)$ satisfies those of Case-2. Also as for the children of $v$ in each case of lines 2–14, we can prove this as follows, where Case-$j'$, $j' \in \{1, 2, 3, 4\}$ is omitted by symmetry of labelings:

(Case-1) $c_i(v)$, $i \in [0, \Delta - 3]$ satisfies the conditions of Case-1′ and $c_{\Delta-2}(v)$ satisfies those of Case-2′.
(Case-2) $c_i(v)$, $i \in [0, \Delta - 2]$ satisfies the conditions of Case-3.
(Case-3) $c_i(v)$, $i \in [0, \Delta - 3]$ satisfies the conditions of Case-4 and $c_{\Delta-2}(v)$ satisfies those of Case-1′.
(Case-4) $c_i(v)$, $i \in [0, \Delta - 3]$ satisfies the conditions of Case-3 and $c_{\Delta-2}(v)$ satisfies those of Case-3′.

Notice that in Case-1, $c_i(v)$ for $i \in [0, \Delta - 3]$ satisfies the conditions of Case-1′ because $\{p + iq \mid i \in [0, \Delta - 2]\} = \{p + (\Delta - 1)q + r - (p + iq) \mid i \in [0, \Delta - 2]\}$ by $p = q + r$. Consequently, the correctness of the algorithm is proved and hence the proof of Theorem 1 is completed.

Also, we remark that Algorithm $(p, q)$-LABEL can be implemented to run in linear time.

## 4 Algorithms for $(p, q)$-total Labelings of Trees

In this section, we consider an algorithm for finding an optimal $(p, q)$-total labeling (i.e., a $\lambda^T_{p,q}(T)$-$(p, q)$-total labeling) of trees $T$. Here, we focus on the cases of $\Delta \geq 3$ and $p > q$ since the case of $\Delta \leq 2$ has been shown as Lemma 3, and in the case of $\Delta \geq 3$ and $p = q$, we have $\lambda^T_{p,q}(T) = p + (\Delta - 1)q$ by Lemma 1 (i) and Theorem 1 and such a labeling can be found in linear time by Algorithm $(p, q)$-LABEL. We discuss the case of $p \leq 3q/2$ in Subsection 4.1 and other cases in Subsection 4.2.

### 4.1 Case: $p \leq 3q/2$

Assume that $p \leq 3q/2$. We show that the problem can be solved in linear time, and we give a complete characterization of trees $T$ with $\Delta \geq 4$ achieving $\lambda^T_{p,q}(T)$; namely, we have the following theorem.

**Theorem 2.** *Let $T$ be a tree with $p \leq 3q/2$.*
*(i) An optimal $(p, q)$-total labeling (i.e., a $\lambda^T_{p,q}(T)$-$(p, q)$-total labeling) of $T$ can be found in linear time.*
*(ii) In the case of $\Delta \geq 2$, we have $\lambda^T_{p,q}(T) \in \{p+(\Delta-1)q, p+(\Delta-1)q+r\}$ where $r = p-q$.*
*(iii) In the case of $p > q$ and $\Delta \geq 4$, $\lambda^T_{p,q}(T) = p + (\Delta - 1)q$ if and only if*

$$\textit{no two major vertices are adjacent in } T. \tag{2}$$

First we consider the case of $\Delta \geq 4$. In this case, for proving Theorem 2, it suffices to show the following two lemmas. Note that the first lemma holds for an arbitrary graph.

**Lemma 4.** *Let $G$ be a graph. If $p \leq 3q/2$ and $\lambda^T_{p,q}(G) < p+(\Delta-1)q+r$ where $r = p-q$, then the condition (2) is satisfied.* □

**Lemma 5.** *If $p \leq 3q/2$, the condition (2) is satisfied, and $\Delta \geq 4$, then $\lambda^T_{p,q}(G) = p + (\Delta - 1)q$ holds, and such a labeling can be found in linear time.*

Recall that by Lemma 1 (i) and Theorem 1, we have $p+(\Delta-1)q \le \lambda^T_{p,q}(T) \le p+(\Delta-1)q+r$ where $r = p-q$. Hence, Lemmas 4 and 5 indicate that either $\lambda^T_{p,q}(T) = p+(\Delta-1)q$ or $\lambda^T_{p,q}(T) = p+(\Delta-1)q+r$ holds, and that the former case is characterized by the condition (2). Furthermore, in both cases, an optimal labeling can be found in linear time by Lemma 5 and Algorithm $(p,q)$-LABEL. Thus, these two lemmas show Theorem 2 in the case of $\Delta \ge 4$.

On the other hand, in the case of $\Delta = 3$, there exist instances $T$ with $\lambda^T_{p,q}(T) > p+2q$ even if the condition (2) holds. For example, consider a tree $T$ which contains the configuration (a) in Fig. 1 in which each major vertex is drawn by a black circle, and assume for contradiction that $T$ admits a $(p+2q)$-$(p,q)$-total labeling $f$. Without loss of generality, let $f(u) = 0$ and $f(u,v) = p$ (note that the set of labels for edges incident to $u$ is $\{p, p+q, p+2q\}$). By the feasibility of $f$, we have $f(v) \in [2p, p+2q]$. Since $w$ is major, it follows that $f(w) = 0$, however, we cannot assign any label to the edge $(v,w)$. Similarly, we can observe that any tree which contains the configuration (b) in Fig. 1 cannot admit a $(p+2q)$-$(p,q)$-total labeling. We can observe that there are many other such instances, and it seems difficult to characterize instances $T$ achieving $\lambda^T_{p,q}(T)$ in the case of $\Delta = 3$.

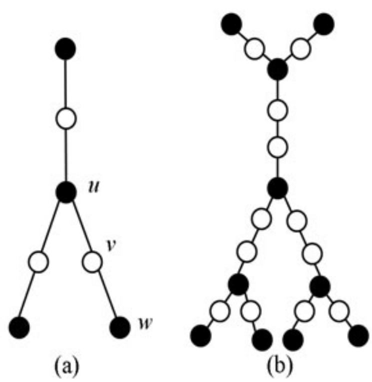


**Fig. 1.** Configurations that any tree $T$ with $\lambda_{p,q}(T) = p+2q$ does not contain in the case of $\Delta = 3$, where each major vertex is drawn by a black circle.

Nevertheless, we can prove that the case of $\Delta = 3$ is linearly solvable and $\lambda^T_{p,q}(T) \in \{p+2q, p+2q+r\}$ in another way. The following lemma shows Theorem 2 in the case of $\Delta = 3$.

**Lemma 6.** *If $\Delta = 3$, then $\lambda^T_{p,q}(G) \in \{p+2q, p+2q+r\}$ holds, and such a labeling can be found in linear time.* □

The latter part of Lemma 6 can be proved in a more general setting as the following theorem.

**Theorem 3.** *Let $|I| = \max\{|V(T)|, \log p\}$. For any nonnegative integers $p, q$, the $L(p, q)$-labeling problem for trees (hence, the $(p, q)$-total labeling problem for trees also) can be solved in polynomial time, if $\Delta = O(\log^{1/3}|I|)$ for general $p$, or if $\Delta = O(\log^{1/2}|I|)$ for $p = \Omega(\Delta q)$. In particular, it can be solved in linear time, if $\Delta$ is bounded by a constant.* □

Due to space limitation, we omit the proofs of Lemmas 4 and 6 and Theorem 3. We here give a proof of Lemma 5.

*Proof of Lemma 5.* Assume that a tree $T = (V, E)$ satisfies the condition (2) and $\Delta \geq 5$, while the case of $\Delta = 4$ is omitted due to space limitation. Also assume that the every non-major and non-leaf vertex in $T$ is a $(\Delta - 1)$-vertex for simplicity of description. Then, we prove this lemma by showing that we can find a $(p + (\Delta - 1)q)$-$(p, q)$-total labeling of $T$ according to Algorithm $(p, q)$-OPTLABEL$\Delta$5. The algorithm starts with choosing a major vertex $v_r$ as the root and assign labels to $V \cup E$ in the breadth-first-search order in a similar way to Algorithm $(p, q)$-TREE. Let $M$ denote the set of major vertices.

Observe that the labelings in each step do not violate the feasibility (note that $3q-p \geq p$ and $2q \geq p$ by $p \leq 3q/2$). Hence, for proving the correctness of the algorithm, we show that as a result of line 1 (resp., the while-loop in lines 2–19), each $c_i(v_r) \in C(v_r)$ (resp., $c_i(v) \in C(v)$) satisfies the conditions of Case-$j$ or Case-$j'$ of lines 2–19 for some $j \in \{1, 2, \ldots, 6\}$. Notice that by the condition (2), all children of each major vertex are non-major. Hence, $c_i(v_r)$, $i \in [1, \Delta - 1]$ satisfies the conditions of Case-2 and $c_0(v_r)$ satisfies those of Case-5. Similarly, we can observe that the children of $v$ in Case-1 of lines 2–19 satisfy the conditions of Case-2 or Case-5. As for children of $v$ in other cases, we can prove this as follows:

(Case-2) $c_i(v)$ satisfies the conditions of Case-1, Case-3, or Case-4.
(Case-3) $c_i(v)$ satisfies the conditions of Case-1, Case-1′, Case-2, or Case-2′.
(Case-4) $c_i(v)$ satisfies the conditions of Case-1, Case-1′, Case-2, Case-5, or Case-6.
(Case-5) $c_i(v)$ satisfies the conditions of Case-1, Case-1′, Case-2, or Case-6.
(Case-6) $c_i(v)$ satisfies the conditions of Case-1 or Case-2.

Also, it is not difficult to see that Algorithm $(p, q)$-OPTLABEL$\Delta$5 can be implemented to run in linear time. □

We remark that since the procedure of Case-4 needs the assumption of $\Delta \geq 5$, Algorithm $(p, q)$-OPTLABEL$\Delta$5 cannot be applied to the cases of $\Delta < 5$.

### 4.2 Case: $p > 3q/2$

First we consider the case of $p = 2$ and $q = 1$. In this case, the condition (2) is sufficient for $\lambda^T_{2,1}(T) = \Delta + 1$, as described in the following lemma.

**Lemma 7.** *Let $T$ be a tree. If $\Delta \geq 4$ and the condition (2) is satisfied, then $\lambda^T_{2,1}(T) = \Delta + 1$ and a $(\Delta + 1)$-$(2, 1)$-total labeling of $T$ can be found in linear time.* □

This lemma can be proved in a similar way to the proof of Lemma 5.

**Algorithm 2.** Algorithm $(p, q)$-OPTLABEL$\Delta$5

**Input:** A tree $T = (V, E)$ satisfying the condition (2) and $\Delta \geq 5$ such that the degree of all non-major and non-leaf vertices is $\Delta - 1$, and two integers $p$ and $q$ with $p \leq 3q/2$.

**Output:** A $(p, q)$-total labeling $f : V \cup E \to \{0, 1, \ldots, p + (\Delta - 1)q\}$ of $T$.

Assign label 0 to the root $v_r$; let $f(v_r) := 0$. For each $i \in [1, \Delta - 1]$, let $f(v_r, c_i(v_r)) := p + iq$ and $f(c_i(v_r)) := q$, where $C(v_r) = \{c_i(v_r) \mid i = 0, 1, \ldots, \Delta - 1\}$. Let $f(v_r, c_0(v_r)) := p$ and $f(c_0(v_r)) := 3q$. {Note that the root $v_r$ is major.}

**while** there exists a non-leaf $v \in V - \{v_r\}$ such that $f(v)$ has been determined but no label is assigned to any child of $v$ **do**

  {Let $M$ denote the set of major vertices. Denote $C(v)$ by $\{c_i(v) \mid i = 0, 1, \ldots, |C(v)| - 1\}$ such that $d(c_0(v)) \geq d(c_1(v)) \geq \cdots \geq d(c_{|C(v)|-1}(v))$ and let $j$ be an index such that $M \cap C(v) = \{c_i(v) \mid i \in [0, j]\}$ if $M \cap C(v) \neq \emptyset$, and $j = -1$ otherwise.}

  **if** (Case-1) $v$ is major, $f(p(v), v) \in \{p + iq \mid i \in [0, \Delta - 1]\}$ and $f(v) = 0$ **then**

    Assign labels in $\{p + iq \mid i \in [0, \Delta - 1]\} - \{f(p(v), v)\}$ injectively to edges $\{(v, c_i(v)) \mid i \in [0, \Delta - 2]\}$ so that $f(v, c_0(v)) < f(v, c_1(v)) < \cdots < f(v, c_{\Delta-2}(v))$, and let $f(c_i(v)) := q$ for each $i \in [0, \Delta - 2]$. Only if $f(v, c_0(v)) = p$, then relabel $c_0(v)$ as $f(c_0(v)) := 3q$.

  **else if** (Case-2) $v$ is non-major, $f(p(v), v) \in \{p + iq \mid i \in [1, \Delta - 1]\}$ and $f(v) = q$ **then**

    Assign labels in $\{p + iq \mid i \in [1, \Delta - 1]\} - \{f(p(v), v)\}$ injectively to edges $\{(v, c_i(v)) \mid i \in [0, \Delta - 3]\}$ so that $f(v, c_0(v)) < f(v, c_1(v)) < \cdots < f(v, c_{\Delta-3}(v))$, let $f(c_i(v)) := 0$ for each $i \in [0, j]$ and $f(c_i(v)) := 2q$ for each $i \in [j + 1, \Delta - 3]$. Only if $f(v, c_0(v)) = p + q$ and $M \cap C(v) = \emptyset$, then relabel $c_0(v)$ as $f(c_0(v)) := 4q$.

  **else if** (Case-3) $v$ is non-major, $f(p(v), v) \in \{p + iq \mid i \in [2, \Delta - 1]\}$ and $f(v) = 2q$ **then**

    Assign labels in $\{p + iq \mid i \in [2, \Delta - 1]\} - \{f(p(v), v)\}$ injectively to edges $\{(v, c_i(v)) \mid i \in [1, \Delta - 3]\}$, and let $f(c_i(v)) := 0$ for each $i \in [1, j]$, $f(c_i(v)) := q$ for each $i \in [\max\{1, j+1\}, \Delta - 3]$, and $f(v, c_0(v)) := 0$. If $M \cap C(v) \neq \emptyset$, then let $f(c_0(v)) := p + (\Delta - 1)q$ and otherwise let $f(c_0(v)) := p + (\Delta - 2)q$.

  **else if** (Case-4) $v$ is non-major, $f(p(v), v) = p + q$ and $f(v) = 4q$ **then**

    Let $f(v, c_0(v)) := 0$, $f(v, c_i(v)) := p + (i + 2)q$ for each $i \in [2, \Delta - 3]$, $f(c_i(v)) := 0$ for each $i \in [2, j]$ and $f(c_i(v)) := q$ for each $i \in [\max\{2, j + 1\}, \Delta - 3]$. If $|M \cap C(v)| \geq 2$, then let $f(c_0(v)) := f(c_1(v)) := p + (\Delta - 1)q$, and $f(v, c_1(v)) := q$. If $|M \cap C(v)| = 1$, then let $f(c_0(v)) := p + (\Delta - 1)q$, $f(c_1(v)) := 3q$, and $f(v, c_1(v)) := p$. If $M \cap C(v) = \emptyset$, then let $f(c_0(v)) := 2q$, $f(c_1(v)) := 3q$, and $f(v, c_1(v)) := p$.

  **else if** (Case-5) $v$ is non-major, $f(p(v), v) = p$ and $f(v) = 3q$ **then**

    Let $f(v, c_0(v)) := 0$, $f(v, c_i(v)) := p + (i + 2)q$ for each $i \in [1, \Delta - 3]$, $f(c_i(v)) := 0$ for each $i \in [1, j]$, and $f(c_i(v)) := q$ for each $i \in [\max\{1, j + 1\}, \Delta - 3]$. If $M \cap C(v) \neq \emptyset$, then let $f(c_0(v)) := p + (\Delta - 1)q$ and otherwise let $f(c_0(v)) := 2q$.

  **else if** (Case-6) $v$ is non-major, $f(p(v), v) = 0$ and $f(v) = 2q$ **then**

    Let $f(v, c_i(v)) := p + (i + 2)q$ for each $i \in [0, \Delta - 3]$, $f(c_i(v)) := 0$ for each $i \in [0, j]$, and $f(c_i(v)) := q$ for each $i \in [j + 1, \Delta - 3]$.

  **else if** (Case-$j'$) $\overline{f}(p(v), v))$ and $\overline{f}(v)$ satisfy the conditions of Case-$j$ for $j \in [1, 6]$ **then**

    After determining labels for $f(c_i(v))$ and $f(v, c_i(v))$ according to the above (Case-$j$) based on $\overline{f}(p(v), v))$ and $\overline{f}(v)$, let $f(c_i(v)) := p + (\Delta - 1)q - f(c_i(v))$ and $f(v, c_i(v)) := p + (\Delta - 1)q - f(v, c_i(v))$ for each $i$.

  **end if**

**end while**

Output $f$ as a $(p, q)$-total labeling of $T$.

On the other hand, the condition (2) is not necessary for $\lambda^T_{2,1}(T) = \Delta + 1$, in contrast to the case of $p \leq 3q/2$. For example, a tree $T$ which consists of two adjacent major vertices and $2(\Delta - 1)$ leaves satisfies $\lambda^T_{2,1}(T) = \Delta + 1$, and there are many other such instances.

We also remark that the case of $p = 2$ and $q = 1$ is linearly solvable by Hasunuma et al.'s algorithm for the $L(2, 1)$-labeling problem [13], while the algorithm proposed in the proof of this lemma is much simpler (though it can be applied only to some restricted cases). Here we omit the details about this algorithm.

Consider the case of $p > 3q/2$ and $p \neq 2q$. In this case, the condition (2) is not even sufficient for $\lambda^T_{p,q}(T) = p + (\Delta - 1)q$, i.e., if $p > 3q/2$ and $p \neq 2q$, then for an arbitrary $\Delta$, there exist instances $T$ with $\lambda^T_{p,q}(T) > p + (\Delta - 1)q$ even if the condition (2) holds. For example, a tree which contains the configuration (a′) is one of such instances, where the configuration (a′) denotes one obtained from (a) in Fig. 1 by replacing each vertex with degree 3 (resp., degree 2) with a vertex with degree $\Delta > 0$ (resp., degree $\Delta - 1$) (note that $v$ has $\Delta - 1$ major vertices as its neighbors).

## 5 Concluding Remarks

In this paper, we have discussed the $(p, q)$-total labeling problem for general $p$ and $q$. By extending known results about the case of $q = 1$, we have derived upper and lower bounds on $\lambda_{p,q}(G)$ for some classes of graphs $G$. In particular, we provided tight bounds on $\lambda_{p,q}(T)$ for trees $T$ for all possible $p$ and $q$. Also, in the case of $p \leq 3q/2$, we showed that the $(p, q)$-total labeling problem can be solved in linear time, and characterized trees $T$ achieving $\lambda^T_{p,q}(T)$ if $\Delta \geq 4$, in contrast to the counterparts of the $L(p, q)$-labeling problem. On the other hand, in the case of $3q/2 < p \leq \Delta q - 1$ and $p \neq 2q$, it is left open whether the $(p, q)$-total labeling problem for trees is polynomially solvable or not.

It is also challenging to derive a tight upper bound on $\lambda^T_{p,q}(G)$ for a general graph $G$, where even the case of $q = 1$ is open. We here give the following conjecture, which is a generalization of Havet and Yu's conjecture (1).

*Conjecture 1.* $\lambda^T_{p,q}(G) \leq 2p + \Delta q - 1$.

By Lemma 2, Corollary 1, and the fact that $\lambda^T_{1,1}(G) \leq \Delta + 1$ for any series-parallel graph $G$ [26], this conjecture is true if $p > (\Delta - 1)q$ holds or $G$ is the complete graph, a bipartite graph, or a series-parallel graph.

Another interesting issue might be to investigate the case $p < q$. We actually obtain tight bounds on $\lambda^T_{p,q}(T)$ for trees $T$ about the case, though we omit the details.

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