# Label-Free Proof Systems for Intuitionistic Modal Logic IS5

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**Abstract.** In this paper we propose proof systems without labels for the intuitionistic modal logic IS5 that are based on a new multi-contextual sequent structure appropriate to deal with such a logic. We first give a label-free natural deduction system and thus derive natural deduction systems for the classical modal logic S5 and also for an intermediate logic IM5. Then we define a label-free sequent calculus for IS5 and prove its soundness and completeness. The study of this calculus leads to a decision procedure for IS5 and thus to an alternative syntactic proof of its decidability.

## 1 Introduction

Intuitionistic modal logics have important applications in computer science, for instance for the formal verification of computer hardware [7] and for the definition of programming languages [6,10]. Here, we focus on the intuitionistic modal logic IS5, introduced by Prior [13] and initially named MIPQ, that is the intuitionistic version of the modal logic S5. It satisfies the requirements given in [16] for the correct intuitionistic analogues of the modal logics. An algebraic semantics for IS5 has been introduced in [3] and the finite model property w.r.t. this semantics and consequently the decidability of this logic have been proved [11,15]. Moreover a translation of this logic into the monadic fragment of the intuitionistic predicate logic has been defined [4] and relations between some extensions of IS5 and intermediate predicate logics have been investigated [12]. In addition a Kripke semantics for IS5 was defined using frames where the accessibility relation is reflexive, transitive and symmetric [16]. As it is an equivalence relation, there exists an equivalent semantics with frames without accessibility relation like in the case of classical modal logic S5.

Here we mainly focus on proof theory for IS5 and on the design of new proof systems without labels for this logic. A Gentzen calculus was proposed in [11], but it does not satisfy the cut-elimination property. A natural deduction and cut-free Gentzen systems for IS5 have been proposed in [16], but a key point is that they are not considered as syntactically pure because of the presence of labels and relations between them corresponding to semantic information. In fact, they were introduced in order to support accessibility relations with arbitrary properties. There exist labelled systems without relations between labels

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but for the fragment without $\bot$ and $\vee$ [10]. Moreover a hybrid version of IS5 has been introduced in [5,9] in order to reason about places with assertions of the form $A@p$ meaning that $A$ is true at $p$. We observe that, by restricting the natural deduction system for this hybrid version, we can obtain a labelled natural deduction system for IS5.

In this paper, we aim at studying proof systems without labels for IS5. Then a first contribution is a label-free natural deduction system, called $\mathsf{ND}_{\mathsf{IS5}}$. It is based on a new sequent structure, called MC-sequent, that is multi-contextual and without labels, allowing the distribution of hypotheses in a multi-contextual environment. Compared to the hypersequent structure [1] that is adapted to classical logic the MC-sequent structure is more appropriate to deal with intuitionistic and modal operators. From this system we can deduce label-free natural deduction systems for S5 but also for IM5 [8] that is an intermediate logic between IS5 and S5. These natural deduction systems, without labels, illustrates the appropriateness of the MC-sequent structure for logics defined over IS5.

To complete these results another contribution is the definition of a sequent system for IS5 that is based on the MC-sequent structure and called $\mathsf{G}_{\mathsf{IS5}}$. Its soundness and completeness are proved from its equivalence with the natural deduction system. We also prove that $\mathsf{G}_{\mathsf{IS5}}$ satisfies the cut-elimination property. Moreover from the subformula property satisfied by the cut-free derivations, we introduce a notion of redundancy so that any valid MC-sequent has an irredundant derivation. Therefore we provide a new decision procedure for IS5 and thus obtain an alternative proof of the decidability of IS5 from our label-free sequent calculus.

## 2 The Intuitionistic Modal Logic IS5

The language of IS5 is obtained from the one of propositional intuitionistic logic IPL by adding the unary operators $\Box$ and $\Diamond$. Let Prop be a countably set of propositional symbols. We use $p, q, r, \ldots$ to range over Prop. The formulas of IS5 are given by the grammar:

$$\mathcal{F} ::= p \mid \bot \mid \mathcal{F} \wedge \mathcal{F} \mid \mathcal{F} \vee \mathcal{F} \mid \mathcal{F} \supset \mathcal{F} \mid \Box\mathcal{F} \mid \Diamond\mathcal{F}$$

The negation is defined by $\neg A \triangleq A \supset \bot$. A Hilbert axiomatic system for IS5 is given in Figure 1 (see [16]).

Note that the interdefinability between $\Box$ and $\Diamond$ given by $\Diamond A \triangleq \neg\Box\neg A$ breaks down in intuitionistic modal logics. That is similar to the fact that $\forall$ and $\exists$ are independent in intuitionistic first-order logic.

**Definition 1.** *A* Kripke model *is a tuple* $(W, \leqslant, \{D_w\}_{w\in W}, \{V_w\}_{w\in W})$ *where*
*- $W$ is a non-empty set (of 'worlds') partially ordered by $\leqslant$;*
*- for each $w \in W$, $D_w$ is a non-empty set such that $w \leqslant w'$ implies $D_w \subseteq D_{w'}$ and*
*- for each $w \in W$, $V_w$ is a function that assigns to each $p \in$ Prop a subset of $D_w$ such that $w \leqslant w'$ implies $V_w(p) \subseteq V_{w'}(p)$.*

0) All substitution instances of theorems of IPL.
1) $\Box(A \supset B) \supset (\Box A \supset \Box B)$.
2) $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$.
3) $\Diamond\bot \supset \bot$.
4) $\Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$.
5) $(\Diamond A \supset \Box B) \supset \Box(A \supset B)$.
6) $(\Box A \supset A) \wedge (A \supset \Diamond A)$.
7) $(\Diamond\Box A \supset \Box A) \wedge (\Diamond A \supset \Box\Diamond A)$.

$$\frac{A \supset B \qquad A}{B}\ [MP] \qquad\qquad \frac{A}{\Box A}\ [Nec]$$

**Fig. 1.** An Axiomatization of IS5

**Definition 2.** *Let $\mathcal{M} = (W, \leqslant, \{D_w\}_{w \in W}, \{V_w\}_{w \in W})$ be a Kripke model, $w \in W$, $d \in D_w$ and $A$ be a formula, we define $\mathcal{M}, w, d \vDash A$ inductively as follows:*

- $\mathcal{M}, w, d \vDash p$ *iff* $d \in V_w(p)$;
- $\mathcal{M}, w, d \vDash \bot$ *never;*
- $\mathcal{M}, w, d \vDash A \wedge B$ *iff* $\mathcal{M}, w, d \vDash A$ *and* $\mathcal{M}, w, d \vDash B$;
- $\mathcal{M}, w, d \vDash A \vee B$ *iff* $\mathcal{M}, w, d \vDash A$ *or* $\mathcal{M}, w, d \vDash B$;
- $\mathcal{M}, w, d \vDash A \supset B$ *iff for all* $v \geqslant w$, $\mathcal{M}, v, d \vDash A$ *implies* $\mathcal{M}, v, d \vDash B$;
- $\mathcal{M}, w, d \vDash \Box A$ *iff for all* $v \geqslant w$, $e \in D_v$, $\mathcal{M}, v, e \vDash A$;
- $\mathcal{M}, w, d \vDash \Diamond A$ *iff there exists* $e \in D_w$ *such that* $\mathcal{M}, w, e \vDash A$.

A formula $A$ is valid in $\mathcal{M} = (W, \leqslant, \{D_w\}_{w \in W}, \{V_w\}_{w \in W})$, written $\mathcal{M} \vDash A$, if and only if $\mathcal{M}, w, d \vDash A$ for every $w \in W$ and every $d \in D_w$. A formula is valid in IS5, written IS5 $\vDash A$, if and only if $\mathcal{M} \vDash A$ for every Kripke model $\mathcal{M}$.

IS5 has an equivalent Kripke semantics using frames where there is an accessibility relation which is reflexive, transitive and symmetric [16]. A simple way to prove the soundness and the completeness of the Kripke semantics defined here consists in the use of the translation of IS5 in the monadic fragment of the intuitionistic predicate logic [4]. This translation, denoted $(.)^*$, is defined by:

- $(\bot)^* = \bot$; $(p)^* = P(x)$;
- $(A \otimes B)^* = (A)^* \otimes (B)^*$, for $\otimes = \wedge, \vee, \supset$;
- $(\Box A)^* = \forall x.(A)^*$;
- $(\Diamond A)^* = \exists x.(A)^*$.

**Proposition 3 (Monotonicity).** *If we have $\mathcal{M}, w, d \vDash A$ and $w \leqslant w'$, then we have $\mathcal{M}, w', d \vDash A$.*

*Proof.* By structural induction on $A$.

## 3 Label-Free Natural Deduction for IS5

In this section, we introduce a natural deduction system for IS5, called $\mathsf{ND}_{\mathsf{IS5}}$, based on the definition of a particular sequent structure, called MC-sequent. The soundness of this system is proved using Kripke semantics and its completeness is proved *via* the axiomatization given in Figure 1.

### 3.1 The MC-Sequent Structure

Let us recall that a context, denoted by the letters $\Gamma$ and $\Delta$, is a finite multiset of formulae and that a sequent is a structure of the form $\Gamma \vdash C$ where $\Gamma$ is a context and $C$ is a formula.

**Definition 4 (MC-sequent).** *An MC-sequent is a structure* $\Gamma_1; \ldots; \Gamma_k \vdash \Gamma \vdash C$ *where* $\{\Gamma_1, \ldots, \Gamma_k\}$ *is a finite multiset of contexts, called LL-context, and* $\Gamma \vdash C$ *is a sequent, called contextual conclusion.*

Let $G \vdash \Gamma \vdash C$ be a MC-sequent. If $\Gamma$ is the empty context $\emptyset$, then we write $G \vdash \vdash C$ instead of $G \vdash \emptyset \vdash C$. Concerning the empty contexts in $G$ (LL-context), they are not omitted.

The MC-sequent structure simply captures the fact that all the assumptions are relative and not absolute in the sense that if a formula is true in a given context, it is not necessary true in the other contexts. Intuitively, this can be seen as a spatial distribution of the assumptions. Indeed, each context represents a world in Kripke semantics with the fact that two different contexts do not necessarily represent two different worlds. This fact is highlighted by the corresponding formula of any MC-sequent, namely the MC-sequent $\Gamma_1; \ldots; \Gamma_k \vdash \Gamma \vdash C$ corresponds to the formula $(\Diamond(\bigwedge \Gamma_1) \wedge \ldots \wedge \Diamond(\bigwedge \Gamma_k)) \supset ((\bigwedge \Gamma) \supset C)$. We use the notation $\bigwedge \Gamma$ as a shorthand for $A_1 \wedge \ldots \wedge A_k$ when $\Gamma = A_1, \ldots, A_k$. If $\Gamma$ is empty, we identify $\bigwedge \Gamma$ with $\top$.

This structure is similar to the hypersequent structure that is a multiset of sequents, called components, separated by a symbol denoting disjunction [1], in the sense that it is a multi-contextual structure. Since IS5 satisfies the disjunction property, namely if $A \vee B$ is a theorem then $A$ is a theorem or $B$ is a theorem, the hypersequent structure does not really enrich the sequent structure in this case and it appears that MC-sequent is more appropriate to deal with intuitionistic and modal operators. However, our approach is similar to the one in [14] where a hypersequent calculus for the classical modal logic S5 was introduced.

### 3.2 A Natural Deduction System for IS5

The rules of the natural deduction system $\mathsf{ND}_{\mathsf{IS5}}$ are given in Figure 2. Let us remark that if we consider the set of rules obtained from $\mathsf{ND}_{\mathsf{IS5}}$ by replacing any MC-sequent occurring in any rule by its contextual conclusion and by removing all the modal rules and the $[\vee_E^2]$ and $[\bot^2]$ rules then we obtain a set of rules corresponding to the known natural deduction system of IPL [17]. Hence, we obtain the following proposition:

**Proposition 5.** *If $A$ is a substitution instance of a theorem of* IPL *then* $\vdash \vdash A$ *has a proof in* $\mathsf{ND}_{\mathsf{IS5}}$.

$$\frac{}{G \vdash \Gamma, A \vdash A}\,[Id] \qquad \frac{G \vdash \Gamma \vdash \bot}{G \vdash \Gamma \vdash A}\,[\bot^1] \qquad \frac{G; \Gamma \vdash \Gamma' \vdash \bot}{G; \Gamma' \vdash \Gamma \vdash A}\,[\bot^2]$$

$$\frac{G \vdash \Gamma \vdash A_i}{G \vdash \Gamma \vdash A_1 \vee A_2}\,[\vee_I^i] \qquad \frac{G \vdash \Gamma \vdash A \vee B \quad G \vdash \Gamma, A \vdash C \quad G \vdash \Gamma, B \vdash C}{G \vdash \Gamma \vdash C}\,[\vee_E^1]$$

$$\frac{G; \Gamma' \vdash \Gamma \vdash A \vee B \quad G; \Gamma, A \vdash \Gamma' \vdash C \quad G; \Gamma, B \vdash \Gamma' \vdash C}{G; \Gamma \vdash \Gamma' \vdash C}\,[\vee_E^2]$$

$$\frac{G \vdash \Gamma \vdash A \quad G \vdash \Gamma \vdash B}{G \vdash \Gamma \vdash A \wedge B}\,[\wedge_I] \qquad \frac{G \vdash \Gamma \vdash A_1 \wedge A_2}{G \vdash \Gamma \vdash A_i}\,[\wedge_E^i]$$

$$\frac{G \vdash \Gamma, A \vdash B}{G \vdash \Gamma \vdash A \supset B}\,[\supset_I] \qquad \frac{G \vdash \Gamma \vdash A \supset B \quad G \vdash \Gamma \vdash A}{G \vdash \Gamma \vdash B}\,[\supset_E]$$

$$\frac{G \vdash \Gamma \vdash A}{G \vdash \Gamma \vdash \Diamond A}\,[\Diamond_I^1] \qquad \frac{G; \Gamma' \vdash \Gamma \vdash A}{G; \Gamma \vdash \Gamma' \vdash \Diamond A}\,[\Diamond_I^2] \qquad \frac{G \vdash \Gamma \vdash \Diamond A \quad G; A \vdash \Gamma \vdash C}{G \vdash \Gamma \vdash C}\,[\Diamond_E^1]$$

$$\frac{G; \Gamma' \vdash \Gamma \vdash \Diamond A \quad G; \Gamma; A \vdash \Gamma' \vdash C}{G; \Gamma \vdash \Gamma' \vdash C}\,[\Diamond_E^2]$$

$$\frac{G; \Gamma \vdash \vdash A}{G \vdash \Gamma \vdash \Box A}\,[\Box_I] \qquad \frac{G \vdash \Gamma \vdash \Box A}{G \vdash \Gamma \vdash A}\,[\Box_E^1] \qquad \frac{G; \Gamma' \vdash \Gamma \vdash \Box A}{G; \Gamma \vdash \Gamma' \vdash A}\,[\Box_E^2]$$

**Fig. 2.** The Natural Deduction System $\mathsf{ND_{IS5}}$

Let us comment now the modal rules of $\mathsf{ND_{IS5}}$. The rule $[\Box_I]$ internalizes the fact that if a formula $A$ is true in a given context without any assumption, then $\Box A$ is true in any context. $[\Box_E^1]$ internalizes the notion that if $\Box A$ is true in a given context then $A$ is true in this context. $[\Box_E^2]$ internalizes the notion that if $\Box A$ is true in a given context then $A$ is true in any other context. Indeed, this rule consists in an elimination of $\Box$ combined with a switch from the current context $\Gamma$ to an other context $\Gamma'$. So $[\Box_E^1]$ and $[\Box_E^2]$ both internalize the fact that if $\Box A$ is true then $A$ is true in any context. The rules of $\Diamond$ are dual to these of $\Box$. $[\Diamond_I^1]$ and $[\Diamond_I^2]$ both internalize the fact that if $A$ is true in a given context then $\Diamond A$ is true in any context. $[\Diamond_E^1]$ and $[\Diamond_E^2]$ both internalize the fact that the assumption "$A$ is true in a context without any other assumption" is equivalent to the assumption "$\Diamond A$ is true". This comes from the fact that if $\Diamond A$ is true then we may not necessary know in what context.

Let us illustrate our system by considering the formula $\Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$. A proof of this formula in $\mathsf{ND_{IS5}}$ is given by:

$$\frac{\dfrac{}{\vdash \Diamond(A \vee B) \vdash \Diamond(A \vee B)}\,[Id] \qquad \dfrac{\dfrac{}{\Diamond(A \vee B) \vdash A \vee B \vdash A \vee B}\,[Id] \quad \mathcal{D}_1 \quad \mathcal{D}_2}{A \vee B \vdash \Diamond(A \vee B) \vdash \Diamond A \vee \Diamond B}\,[\vee_E 2]}{\vdash \Diamond(A \vee B) \vdash \Diamond A \vee \Diamond B}\,[\Diamond_E 1]$$

with

$$\mathcal{D}_1 = \left\{ \dfrac{\dfrac{\overline{\Diamond(A\vee B)\vdash A\vee B, A\vdash A}\ [Id]}{A\vee B, A\vdash \Diamond(A\vee B)\vdash \Diamond A}\ [\Diamond_I 2]}{A\vee B, A\vdash \Diamond(A\vee B)\vdash \Diamond A\vee \Diamond B}\ [\vee_I 1] \right. \qquad \mathcal{D}_2 = \left\{ \dfrac{\dfrac{\overline{\Diamond(A\vee B)\vdash A\vee B, B\vdash B}\ [Id]}{A\vee B, B\vdash \Diamond(A\vee B)\vdash \Diamond B}\ [\Diamond_I 2]}{A\vee B, B\vdash \Diamond(A\vee B)\vdash \Diamond A\vee \Diamond B}\ [\vee_I 2] \right.$$

Now we give a proof of the soundness of $\mathsf{ND}_{\mathsf{IS5}}$ using Kripke semantics. It consists in showing that for every rule, if its premise(s) are valid, then its conclusion is valid.

**Theorem 6 (Soundness).** *$ND_{\mathsf{IS5}}$ is sound, i.e., if a MC-sequent of* IS5 *is provable in $ND_{\mathsf{IS5}}$ then it is valid in* IS5.

*Proof.* Proceeding contrapositively, for every rule, we suppose that its conclusion is not valid and prove that one of its premises is not valid. Here, we only show the cases of $[\Box_I]$, $[\Box_E^2]$ and $[\Diamond_E^1]$.

- Case $[\Box_I]$. Let $\mathcal{M} = (W, \leqslant, \{D_w\}_{w\in W}, \{V_w\}_{w\in W})$ be a countermodel of $G \vdash \Gamma \vdash \Box A$. Then there exist $w_0$ in $W$ and $d_0 \in D_{w_0}$ such that for all $\Gamma' \in G$, $w_0, d_0 \vDash \Diamond \bigwedge \Gamma'$ and $w_0, d_0 \nvDash \bigwedge \Gamma$ and $w_0, d_0 \nvDash \Box A$.

From $w_0, d_0 \nvDash \Box A$, there exist $w_1 \in W$ and $d_1 \in D_{w_1}$ such that $w_0 \leqslant w_1$ and $w_1, d_1 \nvDash A$. Using Kripke monotonicity (Proposition 3), for all $\Gamma' \in G \cup \{\Gamma\}$, $w_1, d_1 \vDash \Diamond \bigwedge \Gamma'$. Thus, we deduce that $\mathcal{M}$ is a countermodel of $G; \Gamma \vdash \vdash A$.

- Case $[\Box_E^2]$. Let $\mathcal{M} = (W, \leqslant, \{D_w\}_{w\in W}, \{V_w\}_{w\in W})$ be a countermodel of $G; \Gamma \vdash \Gamma' \vdash A$. Then there exists $w_0$ in $W$ and $d_0 \in D_{w_0}$ such that for all $\Gamma'' \in G \cup \{\Gamma\}$, $w_0, d_0 \vDash \Diamond \bigwedge \Gamma''$, $w_0, d_0 \vDash \bigwedge \Gamma'$ and $w_0, d_0 \nvDash A$.

Using $w_0, d_0 \vDash \Diamond \bigwedge \Gamma$, there exists $d_1$ in $D_{w_0}$ such that $w_0, d_1 \vDash \bigwedge \Gamma$. Using $w_0, d_0 \vDash \bigwedge \Gamma'$, $w_0, d_1 \nvDash \Diamond \bigwedge \Gamma'$ holds. Using $w_0, d_0 \nvDash A$, $w_0, d_1 \nvDash \Box A$ holds. Thus, we deduce that $\mathcal{M}$ is a countermodel of $G; \Gamma' \vdash \Gamma \vdash \Box A$.

- Case $[\Diamond_E^1]$. Let $\mathcal{M} = (W, \leqslant, \{D_w\}_{w\in W}, \{V_w\}_{w\in W})$ be a countermodel of $G \vdash \Gamma \vdash C$. Then there exist $w_0$ in $W$ and $d_0 \in D_{w_0}$ such that for all $\Gamma' \in G$, $w_0, d_0 \vDash \Diamond \bigwedge \Gamma'$ and $w_0, d_0 \vDash \bigwedge \Gamma$ and $w_0, d_0 \nvDash C$.

If $w_0, d_0 \nvDash \Diamond A$ then $\mathcal{M}$ is a countermodel of $G \vdash \Gamma \vdash \Diamond A$. Otherwise, $w_0, d_0 \vDash \Diamond A$ and then $\mathcal{M}$ is a countermodel of $G; A \vdash \Gamma \vdash C$.

**Proposition 7.** *The following MC-sequents are provable in $ND_{\mathsf{IS5}}$*

*1)* $\vdash\vdash \Box(A \supset B) \supset (\Box A \supset \Box B)$ *2)* $\vdash\vdash \Diamond\bot \supset \bot$
*3)* $\vdash\vdash \Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$ *4)* $\vdash\vdash \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$
*5)* $\vdash\vdash (\Diamond A \supset \Box B) \supset \Box(A \supset B)$ *6)* $\vdash\vdash (\Box A \supset A) \wedge (A \supset \Diamond A)$
*7)* $\vdash\vdash (\Diamond\Box A \supset \Box A) \wedge (\Diamond A \supset \Box\Diamond A)$

*Proof.* For 4) the proof is given as example before.

- For 1) we have

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\emptyset \vdash \Box(A \supset B), \Box A \vdash \Box(A \supset B)}\,[Id]}{\Box(A \supset B), \Box A \vdash \vdash A \supset B}\,[\Box_E^2] \quad \dfrac{\dfrac{}{\emptyset \vdash \Box(A \supset B), \Box A \vdash \Box A}\,[Id]}{\Box(A \supset B), \Box A \vdash \vdash A}\,[\Box_E^2]}{\Box(A \supset B), \Box A \vdash \vdash B}\,[\supset_E]}{\vdash \Box(A \supset B), \Box A \vdash \Box B}\,[\Box_I]}{\vdash \Box(A \supset B) \vdash \Box A \supset \Box B}\,[\supset_I]}{\vdash \vdash \Box(A \supset B) \supset (\Box A \supset \Box B)}\,[\supset_I]
$$

- For 2) we have

$$
\dfrac{\dfrac{\dfrac{}{\vdash \Diamond\bot \vdash \Diamond\bot}\,[Id] \quad \dfrac{\dfrac{}{\Diamond\bot \vdash \bot \vdash \bot}\,[Id]}{\bot \vdash \Diamond\bot \vdash \bot}\,[\bot^2]}{\vdash \Diamond\bot \vdash \bot}\,[\Diamond_E^1]}{\vdash \vdash \Diamond\bot \supset \bot}\,[\supset_I]
$$

- For 3) we have

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \Box(A \supset B), \Diamond A \vdash \Diamond A}\,[Id] \quad \dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash \Box(A \supset B), \Diamond A \vdash \Box(A \supset B)}\,[Id]}{\Box(A \supset B), \Diamond A \vdash A \vdash A \supset B}\,[\Box_E^2] \quad \dfrac{}{\Box(A \supset B), \Diamond A \vdash A \vdash A}\,[Id]}{\Box(A \supset B), \Diamond A \vdash A \vdash B}\,[\supset_E]}{A \vdash \Box(A \supset B), \Diamond A \vdash \Diamond B}\,[\Diamond_I^2]}{\vdash \Box(A \supset B), \Diamond A \vdash \Diamond B}\,[\Diamond_E^1]}{\vdash \Box(A \supset B) \vdash \Diamond A \supset \Diamond B}\,[\supset_I]}{\vdash \vdash \Box(A \supset B) \supset (\Diamond A \supset \Diamond B)}\,[\supset_I]
$$

- For 5) we have

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash \Diamond A \supset \Box B \vdash \Diamond A \supset \Box B}\,[Id] \quad \dfrac{\dfrac{}{\Diamond A \supset \Box B \vdash A \vdash A}\,[Id]}{A \vdash \Diamond A \supset \Box B \vdash \Diamond A}\,[\Diamond_I^2]}{A \vdash \Diamond A \supset \Box B \vdash \Box B}\,[\supset_E]}{\Diamond A \supset \Box B \vdash A \vdash B}\,[\Box_E^2]}{\Diamond A \supset \Box B \vdash \vdash A \supset B}\,[\supset_I]}{\vdash \Diamond A \supset \Box B \vdash \Box(A \supset B)}\,[\Box_I]}{\vdash \vdash (\Diamond A \supset \Box B) \supset \Box(A \supset B)}\,[\supset_I]
$$

- For 6) we have

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \Box A \vdash \Box A}\,[Id]}{\vdash \Box A \vdash A}\,[\Box_E^1]}{\vdash \vdash \Box A \supset A}\,[\supset_I] \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Diamond A \supset \Box B \vdash A \vdash A}\,[Id]}{A \vdash \Diamond A \supset \Box B \vdash \Diamond A}\,[\Diamond_I^2]}{\vdash A \vdash A}\,[Id]}{\vdash A \vdash \Diamond A}\,[\Diamond_I^1]}{\vdash \vdash A \supset \Diamond A}\,[\supset_I]}{\vdash \vdash (\Box A \supset A) \wedge (A \supset \Diamond A)}\,[\wedge_I]
$$

- For 7) we have

$$\dfrac{\dfrac{}{\vdash \Diamond\Box A \vdash \Diamond\Box A}[Id] \quad \dfrac{\dfrac{\dfrac{}{\emptyset; \Diamond\Box A \vdash \Box A \vdash \Box A}[Id]}{\Box A; \Diamond\Box A \vdash \vdash A}[\Box_E^2]}{\Box A \vdash \Diamond\Box A \vdash \Box A}[\Box_I]}{\dfrac{\vdash \Diamond\Box A \vdash \Box A}{\vdash \vdash \Diamond\Box A \supset \Box A}[\supset_I]}[\Diamond_E^1] \quad \dfrac{\dfrac{\dfrac{}{\emptyset; \vdash \Diamond A \vdash \Diamond A}[Id] \quad \dfrac{\dfrac{}{\emptyset; \Diamond A \vdash A \vdash A}[Id]}{\Diamond A; A \vdash \vdash \Diamond A}[\Diamond_I^2]}{\Diamond A \vdash \vdash \Diamond A}[\Diamond_E^2]}{\dfrac{\vdash \Diamond A \vdash \Box\Diamond A}{\vdash \vdash \Diamond A \supset \Box\Diamond A}[\supset_I]}[\Box_I]$$

$$\dfrac{\vdash \vdash \Diamond\Box A \supset \Box A \qquad \vdash \vdash \Diamond A \supset \Box\Diamond A}{\vdash \vdash (\Diamond\Box A \supset \Box A) \wedge (\Diamond A \supset \Box\Diamond A)}[\wedge_I]$$

**Proposition 8.** *The following properties are satisfied:*

1. *if* $G \vdash \Gamma \vdash C$ *has a proof in* $\mathsf{ND_{IS5}}$ *then* $G \vdash \Gamma, A \vdash C$ *has a proof in* $\mathsf{ND_{IS5}}$*;*
2. *if* $G; \Gamma' \vdash \Gamma \vdash C$ *has a proof in* $\mathsf{ND_{IS5}}$ *then* $G; \Gamma', A \vdash \Gamma \vdash C$ *has a proof in* $\mathsf{ND_{IS5}}$*;*
3. *if* $G \vdash \Gamma \vdash C$ *has a proof in* $\mathsf{ND_{IS5}}$ *then* $G; \Gamma' \vdash \Gamma \vdash C$ *has a proof in* $\mathsf{ND_{IS5}}$*.*

*Proof.* The first two properties are proved by mutual induction on the proof of their assumptions. The third one is simply proved by induction on the proof of its assumption.

**Theorem 9.** *If* $A$ *is valid in* $\mathsf{IS5}$ *then* $\vdash \vdash A$ *has a proof in* $\mathsf{ND_{IS5}}$*.*

*Proof.* We identify the validity in $\mathsf{IS5}$ through the axiomatization given in Figure 1 and consider an induction on the proof of $A$ in this axiomatization.

If $A$ is an axiom then $\vdash \vdash A$ is provable in $\mathsf{ND_{IS5}}$ (Proposition 5 and Proposition 7).

Now, let us consider the last rule applied.

- If it is $[MP]$ then by applying the induction hypothesis, we have $\vdash \vdash A \supset B$ and $\vdash \vdash A$ have proofs in $\mathsf{ND_{IS5}}$. Using the rule $[\supset_E]$, we show that $\vdash \vdash B$ has also a proof.
- Otherwise, if it is $[Nec]$ then by applying the induction hypothesis, $\vdash \vdash A$ has a proof in $\mathsf{ND_{IS5}}$. Using Proposition 8, $\emptyset \vdash \vdash A$ has a proof in $\mathsf{ND_{IS5}}$ and with the rule $[\Box_I]$, we show that $\vdash \vdash \Box A$ has also a proof.

The following two propositions allow us to state that for every MC-sequent, if its corresponding formula has a proof then it has also a proof.

**Proposition 10.** $G \vdash \Gamma \vdash A \supset B$ *has a proof if and only if* $G \vdash \Gamma, A \vdash B$ *has a proof.*

*Proof.* The *if part* comes from the rule $[\supset_I]$. For the *only if part*, using Proposition 8, $G \vdash \Gamma, A \vdash A \supset B$ has a proof. Then $G \vdash \Gamma, A \vdash B$ has a proof using the rule $[\supset_I]$ as follows:

$$\dfrac{G \vdash \Gamma, A \vdash A \supset B \quad \dfrac{}{G \vdash \Gamma, A \vdash A}[Id]}{G \vdash \Gamma, A \vdash B}[\supset_E]$$

**Proposition 11.** *The following properties are satisfied:*

1. *if* $G \vdash \Gamma, A \wedge B \vdash C$ *has a proof then* $G \vdash \Gamma, A, B \vdash C$ *has a proof;*
2. *if* $G; \Gamma, A \wedge B \vdash \Gamma' \vdash C$ *has a proof then* $G; \Gamma, A, B \vdash \Gamma' \vdash C$ *has a proof;*
3. *if* $G \vdash \Gamma, \Diamond A \vdash C$ *has a proof then* $G; A \vdash \Gamma \vdash C$ *has a proof;*
4. *if* $G; \Gamma, \Diamond A \vdash \Gamma' \vdash C$ *has a proof then* $G; \Gamma; A \vdash \Gamma' \vdash C$ *has a proof.*

*Proof.* By mutual induction on the proofs of their assumptions.

**Theorem 12 (Completeness).** $\mathsf{ND_{IS5}}$ *is complete, i.e., if a MC-sequent of* IS5 *is valid then it has a proof in* $\mathsf{ND_{IS5}}$*.*

*Proof.* Let $\mathcal{S} = \Gamma_1; \ldots; \Gamma_k \vdash \Gamma \vdash C$ be a valid MC-sequent. Then $\mathcal{F}_\mathcal{S} = (\Diamond \bigwedge \Gamma_1 \wedge \ldots \wedge \Diamond \bigwedge \Gamma_k \wedge \bigwedge \Gamma) \supset C$ is valid in IS5. Using Theorem 9 $\vdash \vdash \mathcal{F}_\mathcal{S}$ has a proof. Using Proposition 10, $\vdash \Diamond \bigwedge \Gamma_1 \wedge \ldots \wedge \Diamond \bigwedge \Gamma_k \wedge \bigwedge \Gamma \vdash C$ has a proof. Finally, using Proposition 11, we deduce that $\mathcal{S}$ has a proof.

**Proposition 13 (Admissibility of cut rules).**

- *If* $G \vdash \Gamma \vdash A$ *and* $G \vdash \Gamma, A \vdash B$ *have proofs, then* $G \vdash \Gamma \vdash B$ *has also a proof.*
- *If* $G; \Gamma \vdash \Gamma' \vdash A$ *and* $G; \Gamma', A \vdash \Gamma \vdash B$ *have proofs, then* $G; \Gamma' \vdash \Gamma \vdash B$ *has also a proof.*

*Proof.* By using Kripke semantics similarly to Theorem 6.

### 3.3 Natural Deduction Systems for S5 and IM5

In this section, we provide two natural deduction systems for S5 and one for IM5. We recall that IM5 is the logic obtained from IS5 by adding the axiom $\neg\Box\neg A \supset \Diamond A$ [8]. It is an intermediate logic in the sense that the set of formulas valid in this logic is between the sets of formulas valid in IS5 and S5 w.r.t. inclusion: IS5 $\subset$ IM5 $\subset$ S5.

A natural deduction system for the classical modal logic S5 is obtained by replacing $[\bot^1]$ and $[\bot^2]$ in $\mathsf{ND_{IS5}}$ by the following two rules:

$$\frac{G \vdash \Gamma, \neg A \vdash \bot}{G \vdash \Gamma \vdash A}\ [\bot_c^1] \qquad \frac{G; \Gamma, \neg A \vdash \Gamma' \vdash \bot}{G; \Gamma' \vdash \Gamma \vdash A}\ [\bot_c^2]$$

This comes from the fact that the addition of the axiom $A \vee \neg A$ to IS5 yields S5 [16]. As an example, we give a proof of $\neg\Diamond\neg A \supset \Box A$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A \vdash \neg\Diamond\neg A \vdash \neg\Diamond\neg A}\ [Id] \quad \dfrac{\dfrac{}{\neg\Diamond\neg A \vdash \neg A \vdash \neg A}\ [Id]}{\neg A \vdash \neg\Diamond\neg A \vdash \Diamond\neg A}\ [\Diamond_I^2]}{\neg A \vdash \neg\Diamond\neg A \vdash \bot}\ [\supset_E]}{\neg\Diamond\neg A \vdash \vdash A}\ [\bot_c^2]}{\vdash \neg\Diamond\neg A \vdash \Box A}\ [\Box_I]}{\vdash \vdash \neg\Diamond\neg A \supset \Box A}\ [\supset_I]$$

Another natural deduction system for S5 is obtained by replacing the same rules by the following rules

$$\frac{G \vdash \Gamma, \Diamond\neg A \vdash \bot}{G \vdash \Gamma \vdash \Box A}\ [\bot_c^1] \qquad \frac{G; \Gamma, \Diamond\neg A \vdash \Gamma' \vdash \bot}{G; \Gamma' \vdash \Gamma \vdash \Box A}\ [\bot_c^2]$$

This rule internalizes the axiom $\neg\Diamond\neg A \supset \Box A$ and we know that the addition of this axiom to IS5 yields S5 [2].

Now, we consider the intermediate logic IM5 to show that the MC-sequent structure is appropriate to deal with some logics defined over IS5. Similarly to the case of S5, a natural deduction system for this logic is obtained by replacing $[\bot^1]$ and $[\bot^2]$ in $\mathsf{ND}_{\mathsf{IS5}}$ by the following two rules:

$$\frac{G \vdash \Gamma, \Box\neg A \vdash \bot}{G \vdash \Gamma \vdash \Diamond A}\ [\bot_w^1] \qquad \frac{G; \Gamma, \Box\neg A \vdash \Gamma' \vdash \bot}{G; \Gamma' \vdash \Gamma \vdash \Diamond A}\ [\bot_w^2]$$

The soundness of our systems for IM5 and S5 is obtained using the soundness of the rules of $\mathsf{ND}_{\mathsf{IS5}}$ and the axiomatizations of these logics. To prove completeness, we just have to use the axiomatization similarly to the proof of completeness of $\mathsf{ND}_{\mathsf{IS5}}$.

## 4 A Label-Free Sequent Calculus for IS5

In this section, we introduce a Gentzen calculus, called $\mathsf{G}_{\mathsf{IS5}}$, using the MC-sequent structure. Its soundness and completeness are proved using the natural deduction system $\mathsf{ND}_{\mathsf{IS5}}$. We prove that our calculus satisfies the key property of cut-elimination. Finally, from the subformula property satisfied by the cut-free proofs, we provide a new decision procedure for IS5. The rules of $\mathsf{G}_{\mathsf{IS5}}$ are given in Figure 3.

Note that $\mathsf{G}_{\mathsf{IS5}}$ is sound, complete and satisfies the cut-elimination property without the restriction on $[Id]$ that $p \in \mathsf{Prop}$. However, without this restriction, $\mathsf{G}_{\mathsf{IS5}}$ fails an important property necessary in our approach to prove the cut-elimination property, namely the *depth-preserving admissibility of contraction property.*

**Proposition 14.** *The MC-sequent* $G \vdash \Gamma, A \vdash A$ *is provable in* $\mathsf{G}_{\mathsf{IS5}}$ *for any* $A$.

*Proof.* By structural induction on $A$.

Weakening and contraction rules are not in $\mathsf{G}_{\mathsf{IS5}}$ because they have been absorbed into the rules and axioms. This approach is similar to the one used to obtain the calculus $G3i$ for the intuitionistic logic [17]. For instance, the choice of the axioms $G \vdash \Gamma, p \vdash p$, $G \vdash \Gamma, \bot \vdash C$ and $G; \Gamma', \bot \vdash \Gamma \vdash C$ instead of respectively $\vdash p \vdash p$, $\vdash \bot \vdash C$ and $\bot \vdash \vdash C$ allows us to absorb weakening.

$$\frac{}{G \vdash \Gamma, p \vdash p}\,[Id](p \in \mathsf{Prop}) \qquad \frac{}{G \vdash \Gamma, \bot \vdash C}\,[\bot^1] \qquad \frac{}{G; \Gamma', \bot \vdash \Gamma \vdash C}\,[\bot^2]$$

$$\frac{G \vdash \Gamma, A, B \vdash C}{G \vdash \Gamma, A \wedge B \vdash C}\,[\wedge_L] \qquad \frac{G; \Gamma', A, B \vdash \Gamma \vdash C}{G; \Gamma', A \wedge B \vdash \Gamma \vdash C}\,[\wedge_{LL}] \qquad \frac{G \vdash \Gamma \vdash A \quad G \vdash \Gamma \vdash B}{G \vdash \Gamma \vdash A \wedge B}\,[\wedge_R]$$

$$\frac{G \vdash \Gamma, A \vdash C \quad G \vdash \Gamma, B \vdash C}{G \vdash \Gamma, A \vee B \vdash C}\,[\vee_L] \qquad \frac{G; \Gamma', A \vdash \Gamma \vdash C \quad G; \Gamma', B \vdash \Gamma \vdash C}{G; \Gamma', A \vee B \vdash \Gamma \vdash C}\,[\vee_{LL}]$$

$$\frac{G \vdash \Gamma \vdash A}{G \vdash \Gamma \vdash A \vee B}\,[\vee_R^1] \qquad \frac{G \vdash \Gamma \vdash B}{G \vdash \Gamma \vdash A \vee B}\,[\vee_R^2]$$

$$\frac{G \vdash \Gamma, A \supset B \vdash A \quad G \vdash \Gamma, B \vdash C}{G \vdash \Gamma, A \supset B \vdash C}\,[\supset_L] \qquad \frac{G; \Gamma \vdash \Gamma', A \supset B \vdash A \quad G; \Gamma', B \vdash \Gamma \vdash C}{G; \Gamma', A \supset B \vdash \Gamma \vdash C}\,[\supset_{LL}]$$

$$\frac{G \vdash \Gamma, A \vdash B}{G \vdash \Gamma \vdash A \supset B}\,[\supset_R]$$

$$\frac{G \vdash \Gamma, \Box A, A \vdash C}{G \vdash \Gamma, \Box A \vdash C}\,[\Box_L^1] \qquad \frac{G; \Gamma', A \vdash \Gamma, \Box A \vdash C}{G; \Gamma' \vdash \Gamma, \Box A \vdash C}\,[\Box_L^2]$$

$$\frac{G; \Gamma', \Box A \vdash \Gamma, A \vdash C}{G; \Gamma', \Box A \vdash \Gamma \vdash C}\,[\Box_{LL}^1] \qquad \frac{G; \Gamma', \Box A, A \vdash \Gamma \vdash C}{G; \Gamma', \Box A \vdash \Gamma \vdash C}\,[\Box_{LL}^{2a}] \qquad \frac{G; \Gamma'', A; \Gamma', \Box A \vdash \Gamma \vdash C}{G; \Gamma''; \Gamma', \Box A \vdash \Gamma \vdash C}\,[\Box_{LL}^{2b}]$$

$$\frac{G; \Gamma \vdash \vdash A}{G \vdash \Gamma \vdash \Box A}\,[\Box_R] \qquad \frac{G; A \vdash \Gamma \vdash C}{G \vdash \Gamma, \Diamond A \vdash C}\,[\Diamond_L] \qquad \frac{G; A; \Gamma' \vdash \Gamma \vdash C}{G; \Gamma', \Diamond A \vdash \Gamma \vdash C}\,[\Diamond_{LL}]$$

$$\frac{G \vdash \Gamma \vdash A}{G \vdash \Gamma \vdash \Diamond A}\,[\Diamond_R^1] \qquad \frac{G; \Gamma \vdash \Gamma' \vdash A}{G; \Gamma' \vdash \Gamma \vdash \Diamond A}\,[\Diamond_R^2]$$

$$\frac{G \vdash \Gamma \vdash A \quad G \vdash \Gamma, A \vdash C}{G \vdash \Gamma \vdash C}\,[Cut^1] \qquad \frac{G; \Gamma \vdash \Gamma' \vdash A \quad G; \Gamma', A \vdash \Gamma \vdash C}{G; \Gamma' \vdash \Gamma \vdash C}\,[Cut^2]$$

**Fig. 3.** The MC-sequent Calculus $\mathsf{G}_{\mathsf{IS5}}$

**Theorem 15 (Soundness).** *If a MC-sequent is provable in* $\mathsf{G}_{\mathsf{IS5}}$ *then it is provable in* $\mathsf{ND}_{\mathsf{IS5}}$.

*Proof.* By induction on the proof of the MC-sequent in $\mathsf{G}_{\mathsf{IS5}}$ using Proposition 13. We only have to consider the cases of the last rule of this proof. Here, we only develop the cases of $[\Box_L^1]$, $[\Box_L^2]$ and $[\Diamond_{LL}]$.

- Case of $[\Box_L^1]$: using the induction hypothesis $G \vdash \Gamma, \Box A, A \vdash C$ is provable in $\mathsf{ND}_{\mathsf{IS5}}$. A proof of $G \vdash \Gamma, \Box A \vdash C$ in $\mathsf{ND}_{\mathsf{IS5}}$ is given by:

$$\frac{\dfrac{\dfrac{}{G \vdash \Gamma, \Box A \vdash \Box A}\,[Id]}{G \vdash \Gamma, \Box A \vdash A}\,[\Box_E^1] \quad G \vdash \Gamma, \Box A, A \vdash C}{G \vdash \Gamma, \Box A \vdash C}\,[Cut^1]$$

- Case of $[\Box_L^2]$: using the induction hypothesis $G; \Gamma', A \vdash \Gamma, \Box A \vdash C$ is provable in $\mathsf{ND}_{\mathsf{IS5}}$. A proof of $G; \Gamma' \vdash \Gamma, \Box A \vdash C$ in $\mathsf{ND}_{\mathsf{IS5}}$ is given by:

$$\dfrac{\dfrac{\dfrac{}{G;\Gamma' \vdash \Gamma, \Box A \vdash \Box A}[Id]}{G;\Gamma;\Box A \vdash \Gamma' \vdash A}[\Box_E^2] \quad G;\Gamma', A \vdash \Gamma, \Box A \vdash C}{G;\Gamma' \vdash \Gamma, \Box A \vdash C}[Cut^2]$$

- Case of $[\Diamond_{LL}]$: using the induction hypothesis $G;A;\Gamma' \vdash \Gamma \vdash C$ is provable in $\mathsf{ND_{IS5}}$. Therefore, $G;A;\Gamma', \Diamond A \vdash \Gamma \vdash C$ is also provable in $\mathsf{ND_{IS5}}$. A proof of $G;\Gamma', \Diamond A \vdash \Gamma \vdash C$ in $\mathsf{ND_{IS5}}$ is given by:

$$\dfrac{\dfrac{}{G;\Gamma \vdash \Gamma', \Diamond A \vdash \Diamond A}[Id] \quad G;A;\Gamma', \Diamond A \vdash \Gamma \vdash C}{G;\Gamma', \Diamond A \vdash \Gamma \vdash C}[\Diamond_E^2]$$

**Theorem 16 (Completeness).** *if a MC-sequent is provable in* $\mathsf{ND_{IS5}}$ *then it is provable in* $\mathsf{G_{IS5}}$*.*

*Proof.* We proceed by induction on the proof of the MC-sequent in $\mathsf{ND_{IS5}}$. We only have to consider the cases of the last rule applied in this proof. Here, we only develop the cases of $[\Box_E^2]$, $[\Diamond_E^1]$ and $[\Diamond_E^2]$.

- Case of $[\Box_E^2]$: using the induction hypothesis, $G;\Gamma' \vdash \Gamma \vdash \Box A$ is provable in $\mathsf{G_{IS5}}$. Then, a proof of $G;\Gamma \vdash \Gamma' \vdash A$ in $\mathsf{G_{IS5}}$ is given by:

$$\dfrac{G;\Gamma' \vdash \Gamma \vdash \Box A \quad \dfrac{\dfrac{}{G;\Gamma, \Box A \vdash \Gamma', A \vdash A}[Id]}{G;\Gamma, \Box A \vdash \Gamma' \vdash A}[\Box_{LL}^1]}{G;\Gamma \vdash \Gamma' \vdash A}[Cut^2]$$

- Case of $[\Diamond_E^1]$: using the induction hypothesis, $G \vdash \Gamma \vdash \Diamond A$ and $G;A \vdash \Gamma \vdash C$ are provable in $\mathsf{G_{IS5}}$. Then a proof of $G \vdash \Gamma \vdash C$ is given by:

$$\dfrac{G \vdash \Gamma \vdash \Diamond A \quad \dfrac{G;A \vdash \Gamma \vdash C}{G \vdash \Gamma, \Diamond A \vdash C}[\Diamond_L]}{G \vdash \Gamma \vdash C}[Cut^1]$$

- Case of $[\Diamond_E^2]$: using the induction hypothesis, $G;\Gamma' \vdash \Gamma \vdash \Diamond A$ and $G;A;\Gamma \vdash \Gamma' \vdash C$ are provable in $\mathsf{G_{IS5}}$. Then, a proof of $G;\Gamma \vdash \Gamma' \vdash C$ is given by:

$$\dfrac{G;\Gamma' \vdash \Gamma \vdash \Diamond A \quad \dfrac{G;A;\Gamma \vdash \Gamma' \vdash C}{G;\Gamma, \Diamond A \vdash \Gamma' \vdash C}[\Diamond_{LL}]}{G;\Gamma \vdash \Gamma' \vdash C}[Cut^2]$$

Let us illustrate $\mathsf{G_{IS5}}$ by considering the MC-sequent $\vdash\vdash (\Diamond\Box A \supset \Box A) \wedge (\Diamond A \supset \Box\Diamond A)$. A proof of this sequent is given by:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Box A;\emptyset \vdash A \vdash A}[Id]}{\Box A;\emptyset \vdash\vdash A}[\Box_{LL}^1]}{\Diamond\Box A \vdash\vdash A}[\Diamond_{LL}]}{\vdash \Diamond\Box A \vdash \Box A}[\Box_R]}{\vdash\vdash \Diamond\Box A \supset \Box A}[\supset_R] \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\emptyset \vdash A \vdash A}[Id]}{A;\emptyset \vdash\vdash \Diamond A}[\Diamond_R^2]}{\Diamond A \vdash\vdash \Diamond A}[\Diamond_{LL}]}{\vdash \Diamond A \vdash \Box\Diamond A}[\Box_R]}{\vdash\vdash \Diamond A \supset \Box\Diamond A}[\supset_R]}{\vdash\vdash (\Diamond\Box A \supset \Box A) \wedge (\Diamond A \supset \Box\Diamond A)}[\wedge_R]$$

### 4.1 Depth-Preserving Admissibility of Weakening and Contraction

We write $\triangleright_G \mathcal{S}$ if the MC-sequent $\mathcal{S}$ has a proof in a calculus $G$. Moreover, we write $\triangleright^n_G \mathcal{S}$ if $\mathcal{S}$ has a proof in $G$ of depth smaller or equal to $n$. Let us recall the notion of *depth-preserving admissibility.*

**Definition 17.** *A rule* $[R]$ *is said to be* admissible *for a calculus* $G$*, if for all instances* $\dfrac{H_1 \ \ldots \ H_k}{C}\,[R]$ *of* $[R]$*, if for all* $i \in [1,k]$ $\triangleright_G H_i$ *then* $\triangleright_G C$*.*
*A rule* $[R]$ *is said to be* depth-preserving admissible *for* $G$*, if for all* $n$*, if for all* $i \in [1,k]$ $\triangleright^n_G H_i$ *then* $\triangleright^n_G C$*.*

We note $\mathsf{G}^-_{\mathsf{IS5}}$ the sequent calculus $\mathsf{G}_{\mathsf{IS5}}$ without cut rules. The following proposition corresponds to the depth-preserving admissibility property of weakening.

**Proposition 18.**

1. *If* $\triangleright^n_{\mathsf{G}^-_{\mathsf{IS5}}} G \vdash \Gamma \vdash C$ *then* $\triangleright^n_{\mathsf{G}^-_{\mathsf{IS5}}} G \vdash \Gamma, A \vdash C$*.*
2. *If* $\triangleright^n_{\mathsf{G}^-_{\mathsf{IS5}}} G; \Gamma' \vdash \Gamma \vdash C$ *then* $\triangleright^n_{\mathsf{G}^-_{\mathsf{IS5}}} G; \Gamma', A \vdash \Gamma \vdash C$*.*
3. *If* $\triangleright^n_{\mathsf{G}^-_{\mathsf{IS5}}} G \vdash \Gamma \vdash C$ *then* $\triangleright^n_{\mathsf{G}^-_{\mathsf{IS5}}} G; \Gamma' \vdash \Gamma \vdash C$*.*

*Proof.* 1. and 2. are proved by mutual induction on $n$ and 3. by induction on $n$.

The following proposition is used to prove the depth-preserving admissibility of contraction. It is similar to the inversion lemma given in [17]. For some rules of $\mathsf{G}^-_{\mathsf{IS5}}$, if the conclusion has a proof of depth $n$, then some of its premises has proofs of depth smaller or equal to $n$.

**Proposition 19.**

1. *a) If* $\triangleright^n_{\mathsf{G}^-_{\mathsf{IS5}}} G \vdash \Gamma, A \wedge B \vdash C$ *then* $\triangleright^n_{\mathsf{G}^-_{\mathsf{IS5}}} G \vdash \Gamma, A, B \vdash C$*.*
   *b) If* $\triangleright^n_{\mathsf{G}^-_{\mathsf{IS5}}} G; \Gamma', A \wedge B \vdash \Gamma \vdash C$ *then* $\triangleright^n_{\mathsf{G}^-_{\mathsf{IS5}}} G; \Gamma', A, B \vdash \Gamma \vdash C$*.*
2. *a) If* $\triangleright^n_{\mathsf{G}^-_{\mathsf{IS5}}} G \vdash \Gamma, A_1 \vee A_2 \vdash C$ *then* $\triangleright^n_{S\mathsf{G}^-_{\mathsf{IS5}}} G \vdash \Gamma, A_i \vdash C$ *for* $i \in \{1,2\}$*.*
   *b) If* $\triangleright^n_{\mathsf{G}^-_{\mathsf{IS5}}} G; \Gamma', A_1 \vee A_2 \vdash \Gamma \vdash C$ *then* $\triangleright^n_{S\mathsf{G}^-_{\mathsf{IS5}}} G; \Gamma', A_i \vdash \Gamma \vdash C$ *for* $i \in \{1,2\}$*.*
3. *If* $\triangleright^n_{\mathsf{G}^-_{\mathsf{IS5}}} G \vdash \Gamma \vdash A_1 \wedge A_2$ *then* $\triangleright^n_{\mathsf{G}^-_{\mathsf{IS5}}} G \vdash \Gamma \vdash A_i$ *for* $i \in \{1,2\}$*.*
4. *If* $\triangleright^n_{\mathsf{G}^-_{\mathsf{IS5}}} G \vdash \Gamma \vdash A \supset B$ *then* $\triangleright^n_{\mathsf{G}^-_{\mathsf{IS5}}} G \vdash \Gamma, A \vdash B$*.*
5. *a) If* $\triangleright^n_{\mathsf{G}^-_{\mathsf{IS5}}} G \vdash \Gamma, A \supset B \vdash C$ *then* $\triangleright^n_{\mathsf{G}^-_{\mathsf{IS5}}} G \vdash \Gamma, B \vdash C$*.*
   *b) If* $\triangleright^n_{\mathsf{G}^-_{\mathsf{IS5}}} G; \Gamma', A \supset B \vdash \Gamma \vdash C$ *then* $\triangleright^n_{\mathsf{G}^-_{\mathsf{IS5}}} G; \Gamma, B \vdash \Gamma \vdash C$*.*
6. *If* $\triangleright^n_{\mathsf{G}^-_{\mathsf{IS5}}} G \vdash \Gamma \vdash \Box A$ *then* $\triangleright^n_{\mathsf{G}^-_{\mathsf{IS5}}} G; \Gamma \vdash \vdash A$*.*
7. *a) If* $\triangleright^n_{\mathsf{G}^-_{\mathsf{IS5}}} G \vdash \Gamma, \Diamond A \vdash C$ *then* $\triangleright^n_{\mathsf{G}^-_{\mathsf{IS5}}} G; A \vdash \Gamma \vdash C$*.*
   *b) If* $\triangleright^n_{\mathsf{G}^-_{\mathsf{IS5}}} G; \Gamma', \Diamond A \vdash \Gamma \vdash C$ *then* $\triangleright^n_{\mathsf{G}^-_{\mathsf{IS5}}} G; \Gamma'; A \vdash \Gamma \vdash C$*.*

*Proof.* 3., 4. and 6. are proved by induction on $n$. The other cases are proved by mutual induction. Here we only develop the proof of 6.

- If $n = 0$ then $G \vdash \Gamma \vdash \Box A$ is an instance of $[\bot^1]$ or $[\bot^2]$. Indeed, this sequent is not an instance of $[Id]$ because of $\Box A \notin \mathsf{Prop}$. Therefore, $\rhd^0_{\mathsf{G}^-_{\mathsf{IS5}}} G; \Gamma \vdash \vdash A$ holds.
- We assume that $\rhd^{n+1}_{\mathsf{G}^-_{\mathsf{IS5}}} G \vdash \Gamma \vdash \Box A$ by a proof $\mathcal{D}$. If $\Box A$ is not principal in the last rule applied in $\mathcal{D}$, then by applying induction hypothesis to the premise(s) and using the same rule, $\rhd^{n+1}_{\mathsf{G}^-_{\mathsf{IS5}}} G; \Gamma \vdash \vdash A$ holds. Otherwise, $\Box A$ is principal and $\mathcal{D}$ ends with

$$\frac{G; \Gamma \vdash \vdash A}{G \vdash \Gamma \vdash \Box A}\ [\Box_R]$$

By taking the immediate subdeduction of the premise, $\rhd^{n+1}_{\mathsf{G}^-_{\mathsf{IS5}}} G; \Gamma \vdash \vdash A$ holds.

The following proposition corresponds to the depth-preserving admissibility property of contraction.

**Proposition 20.**

1. *If* $\rhd^n_{\mathsf{G}^-_{\mathsf{IS5}}} G \vdash \Gamma, A, A \vdash C$ *then* $\rhd^n_{\mathsf{G}^-_{\mathsf{IS5}}} G \vdash \Gamma, A \vdash C$.
2. *If* $\rhd^n_{\mathsf{G}^-_{\mathsf{IS5}}} G; \Gamma', A, A \vdash \Gamma \vdash C$ *then* $\rhd^n_{\mathsf{G}^-_{\mathsf{IS5}}} G; \Gamma', A \vdash \Gamma \vdash C$.
3. *If* $\rhd^n_{\mathsf{G}^-_{\mathsf{IS5}}} G; \Gamma \vdash \Gamma \vdash C$ *then* $\rhd^n_{\mathsf{G}^-_{\mathsf{IS5}}} G \vdash \Gamma \vdash C$.
4. *If* $\rhd^n_{\mathsf{G}^-_{\mathsf{IS5}}} G; \Gamma'; \Gamma' \vdash \Gamma \vdash C$ *then* $\rhd^n_{\mathsf{G}^-_{\mathsf{IS5}}} G; \Gamma' \vdash \Gamma \vdash C$.

*Proof.* By mutual induction on $n$ using Proposition 19

### 4.2 Cut-Elimination in $\mathsf{G}_{\mathsf{IS5}}$

In order to prove the cut-elimination property, we use a variant of Gentzen's original proof of this property for classical and intuitionistic logic [17].

**Theorem 21 (Cut-elimination).** *The cut-elimination property holds for* $\mathsf{G}_{\mathsf{IS5}}$.

*Proof.* It consists in transforming the applications of cut rules to applications of cut rules on smaller formulae or applications of less height. For our calculus, because of the presence of two cut rules, the cut-elimination is proved by mutual induction. Here we only consider some cases.

If we have

$$\frac{\dfrac{G; \Gamma \vdash \vdash A}{G \vdash \Gamma \vdash \Box A}\ [\Box_R] \qquad \dfrac{G \vdash \Gamma, \Box A, A \vdash C}{G \vdash \Gamma, \Box A \vdash C}\ [\Box^1_L]}{G \vdash \Gamma \vdash C}\ [Cut^1]$$

Then, to apply the induction hypothesis, we transform this derivation as follows:

$$\dfrac{\dfrac{\dfrac{G; \Gamma \vdash \vdash A}{G; \Gamma \vdash \Gamma \vdash A}\ [Prop\ 18] \qquad \dfrac{\dfrac{G \vdash \Gamma \vdash \Box A}{G; \Gamma \vdash \Gamma, A \vdash \Box A}\ [Prop\ 18] \qquad \dfrac{G \vdash \Gamma, \Box A, A \vdash C}{G; \Gamma \vdash \Gamma, \Box A, A \vdash C}\ [Prop\ 18]}{G; \Gamma \vdash \Gamma, A \vdash C}\ [Cut^1]}{G; \Gamma \vdash \Gamma \vdash C}\ [Cut^1]}{G \vdash \Gamma \vdash C}\ [Prop\ 20]$$

If we have

$$\dfrac{\dfrac{G';\Gamma';\Gamma\vdash\vdash A}{G';\Gamma'\vdash\Gamma\vdash\Box A}[\Box_R]\quad\dfrac{G';\Gamma',A\vdash\Gamma,\Box A\vdash C}{G';\Gamma'\vdash\Gamma,\Box A\vdash C}[\Box_L^2]}{G';\Gamma'\vdash\Gamma\vdash C}[cut^1]$$

Then, to apply the induction hypothesis, we transform this derivation as follows:

$$\dfrac{\dfrac{\dfrac{G';\Gamma';\Gamma\vdash\vdash A}{G';\Gamma;\Gamma'\vdash\Gamma'\vdash A}[Prop\ 18]\quad\dfrac{\dfrac{G';\Gamma'\vdash\Gamma\vdash\Box A}{G';\Gamma';\Gamma',A\vdash\Gamma\vdash\Box A}[Prop\ 18]\quad\dfrac{G';\Gamma',A\vdash\Gamma,\Box A\vdash C}{G';\Gamma';\Gamma',A\vdash\Gamma,\Box A\vdash C}[Prop\ 18]}{G;\Gamma';\Gamma',A\vdash\Gamma\vdash C}[cut^1]}{G;\Gamma';\Gamma'\vdash\Gamma\vdash C}[cut^2]}{G;\Gamma'\vdash\Gamma\vdash C}[Prop\ 20]$$

**Corollary 22 (Subformula Property).** *Any formula in any cut-free proof in* $\mathsf{G_{IS5}}$ *of a MC-sequent* $\mathcal{S}$ *is a subformula of a formula appearing in* $\mathcal{S}$.

*Proof.* Each rule of $\mathsf{G_{IS5}}$ except $[Cut^1]$ and $[Cut^2]$ has the property that every subformula of the formulae in the premise(s) is also a subformula of a formula in the conclusion.

## 5 A New Decision Procedure for IS5

In this section we provide a decision procedure for IS5 based on the use of $\mathsf{G_{IS5}}$. The key point is the introduction of a notion of redundancy on the cut-free proof in $\mathsf{G_{IS5}}$ satisfying the fact that any MC-sequent valid has an irredundant proof. And then using the subformula property, we prove that there is no infinite proof which is not irredundant. Finally, by an exhaustive search for an irredundant proof, we can decide any sequent.

We are interested in the size of proofs, i.e, the number of nodes. Previously, we proved that weakening and contraction are depth-preserving admissible for $\mathsf{G^-_{IS5}}$. Weakening and contraction are also size-preserving admissible for $\mathsf{G^-_{IS5}}$. We can prove this similarly to the proofs of Proposition 18 and Proposition 20. We use $set(\Gamma)$ to denote the set underlying the multiset $\Gamma$ (the set of the formulas of $\Gamma$). We define a preorder, denoted $\lesssim$, on MC-sequent as follows: $\Gamma_1;\ldots;\Gamma_k\vdash\Gamma\vdash A\lesssim\Delta_1;\ldots;\Delta_l\vdash\Delta\vdash B$ iff $A=B$, $set(\Gamma)\subseteq set(\Delta)$ and for all $i\in[1,k]$ there exists $j\in[1,l]$ such that $set(\Gamma_i)\subseteq set(\Delta_j)$.

**Proposition 23.** *Let* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *be two MC-sequents. If* $\mathcal{S}_1\lesssim\mathcal{S}_2$ *then if* $\mathcal{S}_1$ *has a proof of size* $n$ *then* $\mathcal{S}_2$ *has a proof of size smaller or equal to* $n$.

*Proof.* This follows directly from the size-preserving admissibility of weakening and contraction.

**Definition 24.** *A derivation is said to be* redundant *if it contains two MC-sequents* $\mathcal{S}_1$ *and* $\mathcal{S}_2$*, with* $\mathcal{S}_1$ *occurring strictly above* $\mathcal{S}_2$ *in the same branch, such that* $\mathcal{S}_1\lesssim\mathcal{S}_2$*. A derivation is irredundant if it is not redundant.*

Now, let us give our decision procedure for the MC-sequents in IS5.

Let $\mathcal{S}$ be a MC-sequent.

**- Step** 1. We start with the derivation containing only $\mathcal{S}$ which is the unique irredundant derivation of size 1. If this derivation is a proof then we return it. Otherwise we move to the next step.

**- Step** $i+1$. We construct the set of all the irredundant derivations of size $i+1$. If this set contains a proof of $\mathcal{S}$ then we return it. Otherwise if this set is empty then the decision algorithm fails, else we move to the next step.

There are only a finite number of possible rule applications. Thus, the set of the irredundant derivations of size $i+1$ is finite. Moreover, this set can be built in a finite time because the $\lesssim$ relation is decidable.

**Theorem 25.** IS5 *is decidable.*

*Proof.* Using Corollary 22, we know that there is no infinite irredundant derivation. Thus, we deduce that our algorithm terminates. Therefore, IS5 is decidable.

## 6 Conclusion and Perspectives

In this work, we introduce a new multi-contextual structure in order to deal with the intuitionistic modal logic IS5. An important contribution is the definition of a label-free natural deduction system for IS5 based on this structure. Then we deduce natural deduction systems for the modal logic S5 and the intermediate logic IM5. Another important contribution is the definition of a label-free sequent calculus satisfying the cut-elimination property. Then we define a new decision procedure from the subformula property satisfied by the cut-free derivation in this calculus. In further works, we will define natural deduction systems and sequent calculi for logics defined over IS5, for instance the ones in [12].

## References


1. Avron, A.: The method of hypersequents in the proof theory of propositional non-classical logics. In: Hodges, W., Hyland, M., Steinhorn, C., Truss, J. (eds.) Logic: From Foundations to Applications, pp. 1–32. Oxford University Press, Oxford (1996)
2. Bezhanishvili, G., Zakharyaschev, M.: Logics over MIPC. In: Proceedings of Sequent Calculi and Kripke Semantics for Non-Classical Logics, pp. 86–95. Kyoto University (1997)
3. Bull, R.A.: A modal extension of intuitionistic logic. Notre Dame Journal of Formal Logic 6, 142–145 (1965)
4. Bull, R.A.: MIPC as the formalization of an intuitionistic concept of modality. Journal of Symbolic Logic 31, 609–616 (1966)
5. Chadha, R., Macedonio, D., Sassone, V.: A hybrid intuitionistic logic: Semantics and decidability. Journal of Logic and Computation 16(1), 27–59 (2006)
6. Davies, R., Pfenning, F.: A modal analysis of staged computation. Journal of ACM 48(3), 555–604 (2001)

7. Fairtlough, M., Mendler, M.: An intuitionistic modal logic with applications to the formal verification of hardware. In: Kleine Büning, H. (ed.) CSL 1995. LNCS, vol. 1092, pp. 354–368. Springer, Heidelberg (1996)
8. Font, J.M.: Modality and possibility in some intuitionistic modal logics. Notre Dame Journal of Formal Logic 27, 533–546 (1986)
9. Jia, L., Walker, D.: Modal proofs as distributed programs (extended abstract). In: Schmidt, D. (ed.) ESOP 2004. LNCS, vol. 2986, pp. 219–233. Springer, Heidelberg (2004)
10. Murphy VII, T., Crary, K., Harper, R., Pfenning, F.: A symmetric modal lambda calculus for distributed computing. In: Proceedings of the 19th IEEE Symposium on Logic in Computer Science (LICS), pp. 286–295. IEEE Press, Los Alamitos (2004)
11. Ono, H.: On some intuitionistic modal logics. Publications of the Research Institute for Mathematical Science 13, 55–67 (1977)
12. Ono, H., Suzuki, N.: Relations between intuitionistic modal logics and intermediate predicate logics. Reports on Mathematical Logic 22, 65–87 (1988)
13. Prior, A.: Time and Modality. Clarendon Press, Oxford (1957)
14. Restall, G.: Proofnets for S5: sequents and circuits for modal logic. In: Dimitracopoulos, C., Newelski, L., Normann, D. (eds.) Logic Colloquium 2005. Lecture Notes in Logic, vol. 28, pp. 151–172. Cambridge University Press, Cambridge (2007)
15. Fischer Servi, G.: The finite model property for MIPQ and some consequences. Notre Dame Journal of Formal Logic XIX, 687–692 (1978)
16. Simpson, A.: The proof theory and semantics of intuitionistic modal logic. PhD thesis, University of Edinburgh (1994)
17. Troelstra, A.S., Schwichtenberg, H.: Basic Proof Theory. Cambridge Tracts in Theoretical Computer Science, vol. 43. Cambridge University Press, Cambridge (1996)