Strong Connectivity in Sensor Networks with Given Number of Directional Antennae of Bounded Angle

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Abstract. Given a set S of n sensors in the plane we consider the problem of establishing an ad hoc network from these sensors using directional antennae. We prove that for each given integer $1 \leq k \leq 5$ there is a strongly connected spanner on the set of points so that each sensor uses at most k such directional antennae whose range differs from the optimal range by a multiplicative factor of at most $2 \cdot \sin(\frac{\pi}{k+1})$. Moreover, given a minimum spanning tree on the set of points the spanner can be constructed in additional $O(n)$ time. In addition, we prove NP completeness results for $k = 2$ antennae.

Keywords: Antenna, Directional Antenna, Minimum Spanning Tree, Sensors, Spanner, Strongly Connected.

1 Introduction

The nodes of a wireless network can be connected using either omnidirectional antennae that transmit in all directions around the source or directional antennae that transmit only along a limited predefined angle. The energy usage of an antenna is proportional to its coverage area (for directional antennae, this is usually taken as the area of the sector delimited by the angle of the antenna). Therefore directional antennae can often perform more efficiently than omnidirectional ones in order to attain overall network connectivity. Given that the sensor range for a set S of sensors cannot be less than the length of the maximum edge of a minimum spanning tree on the set S, a reasonable way to lower energy consumption is by reducing the breadth (or angle or spread) of the

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antenna being used. However, by reducing antenna angles connectivity may be lost since communication between sensors can no longer be assumed to be bidirectional. Therefore an interesting question arising is how to maintain network connectivity when antenna angles are being reduced while at the same time the transmission range of the sensors is being kept as low as possible.

Formally, consider a set S of n sensors in the plane with identical range. Let $0 \leq \varphi \leq 2\pi$ be a given angle. Each sensor is allowed to use at most k directional antennae each of angle at most φ, for some integer value k. By directing an antenna at a sensor u towards another sensor v a directed edge (u, v) from u to v is formed provided that v is within u's range and lies inside the sector of angle φ formed by the antenna at u. By appropriately orienting such antennae at all the sensors we would like to form a strongly connected graph which spans all the sensors.

1.1 Preliminaries and Notation

Given spread φ and number of antennae k per sensor let $r_k(S, \varphi)$ denote the minimum range of directed antennae of angular spread at most φ so that if every sensor in S uses at most k such antennae then it is possible to direct them so that a strongly connected network (or spanner) on S is formed. A special case of this is to have angle $\varphi = 0$ i.e. a direct line connection, in which case we use the simpler notation $r_k(S) := r_k(S, 0)$. Let $\mathcal{D}_k(S)$ be the set of all strongly connected graphs on S which have out degree at most k. For any graph $G \in \mathcal{D}_k(S)$ let $r_k(G)$ be the length of the maximum length edge of G. It is easy to see that $r_k(S) := \min_{G \in \mathcal{D}_k(S)} r_k(G)$, i.e., $r_k(S)$ is the minimum length of a directed edge among all edges of a strongly connected graph with out degree k, for all such graphs in $\mathcal{D}_k(S)$.

It is useful to relate $r_k(S)$ to another quantity which arises from the Minimum Spanning Tree (MST) on S. Let $MST(S)$ denote the set of all MSTs on S. For $T \in MST(S)$ let $r(T)$ denote the length of longest edge of T, and let $r_{MST}(S) := \min\{r(T) : T \in MST(S)\}$. Further, for any angle $\varphi \geq 0$, it is clear that $r_{MST}(S) \leq r_k(S, \varphi)$ since every strongly connected, directed graph on S has an underlying spanning tree.

1.2 Related Work

The first paper to address the problem of converting a connected unidirectional graph consisting of omnidirectional sensors to a strongly connected graph of directional sensors having only one directional antenna each is [4]. In that paper the authors present polynomial time algorithms for the case when the sector angle of the antennae is at least $8\pi/5$. For smaller sector angles, they present algorithms that approximate the minimum radius. When the sector angle is smaller than $2\pi/3$, they show that the problem of determining the minimum radius in order to achieve strong connectivity is NP-hard. A different problem is considered in a subsequent paper [2]. Each sensor has multiple (fixed number of) directional antennae and the strong connectivity problem is considered under the assumption that the maximum (taken over all sensors) sum of angles is minimized. The authors present trade-offs between antennae range and specified sums of antennae per sensor.

When each sensor has one antenna and the angle $\varphi = 0$ then our problem is equivalent to finding a Hamiltonian cycle that minimizes the maximum length of an edge. For a set of n points $1, 2, \ldots, n$ with associated weights $c(i, j)$ satisfying the triangle inequality, the *Bottleneck Traveling Salesman Problem (BTSP)* is the min-max Hamiltonian cycle problem concerned with finding a Hamiltonian cycle for the complete graph which minimizes the maximum weight of an edge, i.e., $\min\{\max_{(i,j)\in H} c(i, j) : H \text{ is a Hamiltonian cycle}\}$. [10] shows that no polynomial time $(2 - \epsilon)$-approximation algorithm is possible for BTSP unless $P = NP$, and also gives a 2-approximation algorithm for this problem.

No literature is known on the connection between the MST of a set of points and strongly connected geometric spanners with given out-degree. Two papers relating somewhat these two concepts are the following. First, [5] shows that it is an NP-hard problem to decide for a given set S of n points in the Euclidean plane and a given real parameter k, whether S admits a spanning tree of maximum vertex degree four whose sum of edge lengths does not exceed k. Second, [7] gives a simple algorithm to find a spanning tree that simultaneously approximates a shortest-path tree and a minimum spanning tree.

Directional antennae are known to enhance ad hoc network capacity and performance and when replacing omnidirectional antennae can reduce the total energy consumption on the network. A theoretical model to this effect is presented in [6] showing that when n omnidirectional antennae are optimally placed and assigned optimally chosen traffic patterns the transport capacity is $\Theta(\sqrt{W/n})$, where each antenna can transmit W bits per second over the common channel(s). When both transmission and reception is directional [14] proves an $\sqrt{2\pi/\alpha\beta}$ capacity gain as well as corresponding throughput improvement factors, where α is the transmission angle and β is a parameter indicating that $\beta/2\pi$ is the average proportion of the number of receivers inside the transmission zone that will get interfered with. Additional experimental studies confirm the importance of using directional antennae in ad hoc networking (see, for example, [1,9,8,11,12,13]).

1.3 Results of the Paper

We are interested in the problem of providing an algorithm for orienting the antennae and ultimately for estimating the value of $r_k(S, \varphi)$. Without loss of generality antennae ranges will be normalized, i.e., $r_{MST}(S) = 1$. The two main results are the following.

Theorem 1. *Consider a set S of n sensors in the plane and suppose each sensor has k, $1 \leq k \leq 5$, directional antennae with any angle $\varphi \geq 0$. Then the antennae can be oriented at each sensor so that the resulting spanning graph is strongly connected and the range of each antenna is at most $2 \cdot \sin(\frac{\pi}{k+1})$ times the optimal. Moreover, given a MST on the set of points the spanner can be constructed with additional $O(n)$ overhead.*

Note that the case $k = 1$ was derived in [10].

Theorem 2. *For $k = 2$ antennae and angular sum of the antennae at most α, it is NP-hard to approximate the optimal radius to within a factor of x. where x and α are the solutions of equations $x = 2\sin(\alpha) = 1 + 2\cos(2\alpha)$.*

Using the identity $\cos(2\alpha) = 1 - 2\sin^2\alpha$ and solving the resulting quadratic equation with unknown $\sin\alpha$ we obtain numerical solutions $x \approx 1.30, \alpha \approx 0.45\pi$. Figure 9 depicts the geometric relation between α and x.

2 Upper Bound Result on Strongly Connected Spanners

The proof given in the sequel is in three parts. Due to space constraints only the proof for $k = 2$ is presented in detail in Subsection 2.1 and part of the pseudocode in Subsection 2.2, while in the full paper we prove the cases for $k = 3$ and $k = 4$ and present the remaining algorithm.

Preliminary Definitions. $D(u; r)$ is the open disk with radius r, centered at u and $C(u, r)$ is the circle with radius r and centered at u. $d(\cdot, \cdot)$ denotes the usual Euclidean distance between two points. In addition, we define the concept of *Antenna-Tree* (*A-Tree*, for short) which isolates the particular properties of a MST that we need in the course of the proof.

Definition 1. *An A-Tree is a tree T embedded in the plane satisfying the following three rules:*

1. *Its maximum degree is five.*
2. *The minimum angle among nodes with a common parent is at least $\pi/3$.*
3. *For any point u and any edge $\{u, v\}$ of T, the open disk $D(v; d(u, v))$ does not have a point $w \neq v$ which is also a neighbor of u in T.*

It is well known and easy to prove that for any set of points there is an MST on the set of points which satisfies Definition 1. We also recall that we consider normalized ranges (i.e. we assume $r(T) = 1$).

Definition 2. *For each real $r > 0$, we define the geometric r-th power of a A-Tree T, denoted by T^r, as the graph obtained from T by adding all edges between vertices of (Euclidean) distance at most r.*

For simplicity, in the sequel we slightly abuse terminology and refer to *geometric r-th power* as *r-th power*.

Definition 3. *Let G be a graph. An orientation $\overrightarrow{G}$ of G is a digraph obtained from G by orienting every edge of G in at least one direction.*

As usual, we denote with (u, v) a directed edge from u to v, whereas $\{u, v\}$ denotes an undirected edge between u and v. Let $d^+_{\overrightarrow{G}}(u)$ be the out-degree of u in $\overrightarrow{G}$ and $\Delta^+(\overrightarrow{G})$ the maximum out-degree of a vertex in $\overrightarrow{G}$.

2.1 Maximum Out-Degree 2

Theorem 3. *Given an A-Tree T, there exists a spanning subgraph $G \subseteq T^{\sqrt{3}}$ such that $\overrightarrow{G}$ is strongly connected and $\Delta^+(\overrightarrow{G}) \leq 2$. Moreover, $d^+_{\overrightarrow{G}}(u) \leq 1$ for each leaf u of T and either every edge of T which is incident to a leaf is contained in G or a leaf is connected to its two consecutive siblings in G.*

Before proving Theorem 3, we need to introduce a definition and a lemma which provides information on the proximity among the neighbors of two adjacent vertices in the tree depending on their degree. The proof of the lemma is technical and given in the full paper.

Definition 4. *We say that two consecutive neighbors of a vertex are* close *if the distance between them is at most $\sqrt{3}$. Otherwise we say that they are* far.

Lemma 1. *Let u, v and w be three consecutive siblings with parent p of an A-Tree T such that $\widehat{upv} + \widehat{vpw} \leq \pi$. If $d(v) = 3$ and the only two children of v are far, then at least one of them is close to either u or w.*

If $d(v) = 4$ and each pair of consecutive children of v are close, then at least one of them is close to either u or w.

If $d(v) = 4$, two consecutive children of v are far and all children of v are at distance at least $\sqrt{3} - 1$ of v, then one child of v is close to u and another child of v is close to w.

If $d(v) = 4$, two consecutive children of v are far and one child x of v is at distance at most $\sqrt{3} - 1$ of v, then at most one child of v different from x are far from u and w.

If $d(v) = 5$, then at least one child of v is close to either u or w.

*Proof (***Theorem 3***).* The proof is by induction on the diameter of the tree. Firstly, we do the base case. Let l be the diameter of T. If $l \leq 1$, let $G = T$ and the result follows trivially.

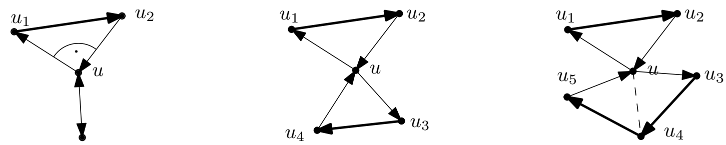
If $l = 2$, then T is an A-Tree which is a star with $2 \leq d \leq 5$ leaves, respectively. Four cases can occur:

$d = 2$. Let $G = T$ and orient every edge in both directions. This results in a strongly connected digraph which trivially satisfies the hypothesis of the theorem.

$d = 3$. Let u be the center of T. Since T is a star, two consecutive neighbors, say u_1 and u_2 are close. Let $G = T \cup \{\{u_1, u_2\}\}$ and orient edges of G as depicted in Figure 1a[1]. It is easy to check that G satisfies the hypothesis of the Theorem.

$d = 4$. Let u be the center of T and u_1, u_2, u_3, u_4 be the four neighbors of u in clockwise order around u starting at any arbitrary neighbor of u. Observe that at most two consecutive neighbors of u are far since T is a star and the angle between two nodes with a common parent is at least $\pi/3$. Assume without loss of generality that u_4 and u_1 are far. Let $G = T \cup \{\{u_1, u_2\}, \{u_3, u_4\}\}$ and orient edges of G as depicted in Figure 1b. Thus, G satisfies trivially the hypothesis of the Theorem.

[1] In all figures boldface arrows represent the newly added adges.



(a) T has three leaves

(b) T has four leaves

(c) T has five leaves

Fig. 1. T is a tree with diameter $l = 2$ (The angular sign with a dot depicts an angle of size at most $2\pi/3$ at vertex u and the dashed edge indicates that it exists in T but not in G.)

$d = 5$. Let u be the center of T and u_1, u_2, u_3, u_4, u_5 be the five neighbors of u in clockwise order around u starting at any arbitrary neighbor of u. Observe that all consecutive neighbors are close since T is a star and the angle between two nodes with a common parent is at least $\pi/3$. Let $G = T \setminus \{u, u_4\} \cup \{\{u_1, u_2\}, \{u_3, u_4\}\}, \{u_4, u_5\}$ and orient edges of G as depicted in Figure 1c. Observe that $\widehat{u_3 u u_5} \leq \pi$. Thus, $\overrightarrow{G}$ is strongly connected and $\Delta^+(\overrightarrow{G}) \leq 2$. Moreover, $d^+_{\overrightarrow{G}}(u) \leq 1$, all edges of T except $\{u, u_4\}$ are contained in G and $\{u_3, u_4\}$ and $\{u_4, u_5\}$ are contained in G.

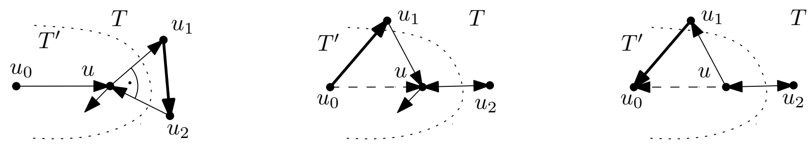
Next we continue with the inductive step. Let T' be the tree obtained from T by removing all leaves. Since removal of leaves does not violate the property of being an A-Tree, T' is also an A-Tree and has diameter less than the diameter of T. Thus, by inductive hypothesis there exists $G' \subseteq T'^{\sqrt{3}}$ such that $\overrightarrow{G'}$ is strongly connected, $\Delta^+(\overrightarrow{G'}) \leq 2$. Moreover, $d^+_{\overrightarrow{G}}(u) \leq 1$ for each leaf u of T and either every edge of T which is incident to a leaf is contained in G or a leaf is connected to its two consecutive siblings in G.

Let u be a leaf of T', u_0 be the neighbor of u in T' and $u_1, \ldots, u_c$ be the c neighbors of u in $T \setminus T'$ in clockwise order around u starting from u_0. Four cases can occur:

u has one neighbor in $T \setminus T'$. Let $G = G' \cup \{\{u, u_1\}\}$ and orient it in both directions. It is easy to see that $\overrightarrow{G}$ satisfies the inductive hypothesis.

u has two neighbors in $T \setminus T'$. We consider two cases. In the first case suppose that u_1 and u_2 are close. Let $G = G' \cup \{\{u, u_1\}, \{u, u_2\}, \{u_1, u_2\}\}$ and orient edges of G as depicted in Figure 2a. In the second case, u_1 and u_2 are far. Again we need to consider two subcases:

Subcase 1 ($\{u_0, u\}$ is in G'.) Either u_0 and u_1 are close or u_2 and u_0 are close. Without loss of generality, lets assume that u_1 and u_0 are close. Let $G = \{G' \setminus \{u_0, u\}\} \cup \{\{u, u_1\}, \{u, u_2\}, \{u_0, u_1\}\}$. The orientation of G will depend on the orientation of $\{u, u_0\}$ in G'. If (u_0, u) is in $\overrightarrow{G'}$, then orient edges of G as depicted in Figure 2b. Otherwise if (u, u_0) is in $\overrightarrow{G'}$, then orient edges of G as depicted in Figure 2c. Thus, $\overrightarrow{G}$ is strongly connected and $\Delta^+(\overrightarrow{G}) \leq 2$. Moreover, the leaves u_1 and u_2 of T have degree one and the edges of T incident to them are contained in G.



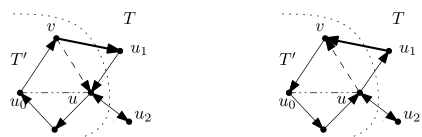
(a) u_1 and u_2 are close $\{u_0, u\}$

(b) u_0 and u_1 are far and (u, u_0) is in the orientation of G'

(c) u_0 and u_1 are far and (u, u_0) is in the orientation of G'

Fig. 2. Depicting the inductive step when u has two neighbors in $T \setminus T'$ (The dashed edge $\{u_0, u\}$ indicates that it does not exist in G but exists in G' and the dotted curve is used to separate T' from T.)

Subcase ($\{u_0, u\}$ is not in G'.) By inductive hypothesis, u is connected to its two siblings v and w in G'. Thus, by Lemma 1, either u_1 or u_2 are close to v or w. Without loss of generality, assume that u_1 and v are close. Let $G = (G' \setminus \{v, u\}) \cup \{\{u_1, u\}, \{u_2, u\}, \{v, u_1\}\}$. The orientation of G will depend on the orientation of $\{v, u\}$ in G'. If (v, u) is in $\overrightarrow{G'}$, then orient edges of G as depicted in Figure 3a. Otherwise if (u, v) is in $\overrightarrow{G'}$, then orient edges of G as depicted in Figure 3b. Thus, $\overrightarrow{G}$ is strongly connected and $\Delta^+(\overrightarrow{G}) \leq 2$. Moreover, the leaves u_1 and u_2 of T have degree one and the edges of T incident to them are contained in G.



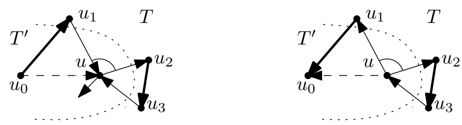
(a) (u_0, u) is in the orientation of G'

(b) (u, u_0) is in the orientation of G'

Fig. 3. Depicting the inductive step when u has two neighbors in $T \setminus T'$, u_0 and u_1 are far and $\{u_0, u\}$ is not in G' (The dashed edge $\{v, u\}$ indicates that it does not exist in G but exists in G', the dash dotted edge $\{u_0, u\}$ indicates that it exists in T' but not in G' and the dotted curve is used to separate T' from T.)

u has three neighbors in $T \setminus T'$. Two subcases can occur:

Subcase 1 ($\{u_0, u\}$ is in G'). At most two neighbors of u are far. Firstly, suppose that u_0 and u_3 are far (This case is equivalent to the case when u_1 and u_2 are far.) Let $G = \{G' \setminus \{u_0, u\}\} \cup \{\{u_1, u\}, \{u_2, u\}, \{u_3, u\}, \{u_1, u_0\}, \{u_2, u_3\}\}$. If (u_0, u) is in $\overrightarrow{G'}$, then orient edges of G as depicted in Figure 4a. Otherwise if (u, u_0) is in $\overrightarrow{G'}$, then orient edges of G as depicted in Figure 4b. Thus, $\overrightarrow{G}$ is strongly connected and $\Delta^+(\overrightarrow{G}) \leq 2$. Moreover, the leaves u_1, u_2 and u_3 of T



(a) (u_0, u) is in the orientation of G'

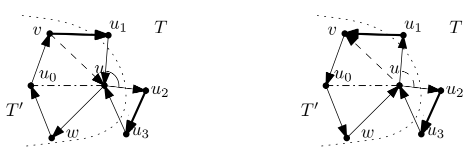
(b) (u, u_0) is in the orientation of G'

Fig. 4. Depicting the inductive step when u has three neighbors in $T \setminus T'$, u_1 and u_2 are far and $\{u_0, u\}$ is in G' (The dashed edge $\{u_0, u\}$ indicates that it does not exist in G but exists in G', the dotted curve is used to separate T' from T and the angular sign depicts an angle of size greater than $2\pi/3$ at vertex u.)

have degree one and the edges of T incident to them are contained in G. The case when u_1 and u_0 are far or u_2 and u_3 are far can be solved analogously by symmetry.

Subcase 2 ($\{u_0, u\}$ is not in G'). By inductive hypothesis u is connected to its two siblings v and w in G'. Three cases can occur.

Subcase 2.1 (u_1 is close to u_2 and u_2 is close to u_3.) By Lemma 1, either u_1 or u_3 are close to v or w. Without loss of generality, we assume that v and u_1 are close. Let $G = \{G' \setminus \{v, u\}\} \cup \{\{u_1, u\}, \{u_2, u\}, \{u_3, u\}, \{v, u_1\}, \{u_2, u_3\}\}$. The orientation of G will depend on the orientation of $\{v, u\}$ in G'. If (v, u) is in $\overrightarrow{G'}$, then orient edges of G as depicted in Figure 5a. Otherwise if (u, v) is in $\overrightarrow{G'}$, then orient edges of G as depicted in Figure 5b. Thus, $\overrightarrow{G}$ is strongly connected and $\Delta^+(\overrightarrow{G}) \leq 2$. Moreover, the leaves u_1, u_2 and u_3 of T have degree one and the edges of T incident to them are contained in G.



(a) (v, u) is in the orientation of G'

(b) (u, v) is in the orientation of G'

Fig. 5. Depicting the inductive step when u has three neighbors in $T \setminus T'$, u_1 and u_2 are far and $\{u_0, u\}$ is not in G' (The dashed edge $\{v, u\}$ indicates that it does not exist in G but exists in G', the dash dotted edge $\{u_0, u\}$ indicates that it exists in T' but not in G' and the dotted curve is used to separate T' from T.)

Subcase 2.2 (Either u_1 is far from u_2 or u_2 is far from u_3 and u_1, u_2 and u_3 are at distance greater than $\sqrt{3} - 1$ from u.) By Lemma 1 u_1 is close to one sibling of u, say v and u_3 is close to another sibling of u, say w. Notice that if u_1 is far from u_2, it is exactly the Case one. However, if u_2 are far from u_3, let $u'_i = u_{3-i+1}$, and Case 1 applies.

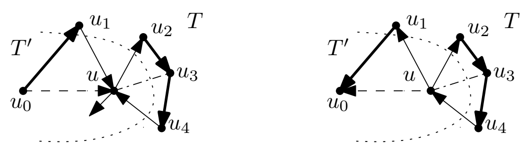
Subcase 2.3 (Either u_1 is far from u_2 or u_2 is far from u_3 and at least one child of u is at distance less than $\sqrt{3}-1$.) Without loss of generality, assume that u_1 is far from u_2. Therefore, $d(u,u_1) > \sqrt{3}-1$ and $d(u,u_3) \leq \sqrt{3}-1$. Observe that u_3 is close to u_1 and u_2. By Lemma 1 either u_1 or u_2 are close to v or w. Thus, if v is close to u_1, then apply the Case one. If w is close to u_2, then let $u'_1 = u_2$, $u'_2 = u_1$ and $u'_3 = u_3$ and apply Case 1.

u has four neighbors in $T \setminus T'$. Two subcases can occur:

Subcase 1 ($\{u_0, u\}$ is in G'). Let

$$G = \{G' \setminus \{u_0, u\}\} \cup \{\{u_1, u\}, \{u_2, u\}, \{u_4, u\}, \{u_1, u_0\}, \{u_2, u_3\}, \{u_3, u_4\}\}.$$

The orientation of G will depend on the orientation of $\{u_0, u\}$ in G'. If (u_0, u) is in $\overrightarrow{G'}$, then orient edges of G as depicted in Figure 6a. Otherwise if (u, u_0) is in $\overrightarrow{G'}$, then orient edges of G as depicted in Figure 6b. Thus, $\overrightarrow{G}$ is strongly connected and $\Delta^+(\overrightarrow{G}) \leq 2$. Moreover, the leaves u_1, u_2, u_3 and u_4 of T have degree one, the edges of T incident to u_1, u_2 and u_4 are contained in G and u_3 is connected to u_2 and u_4 in G. Observe that $\widehat{u_2 u u_4} \leq \pi/2$.



(a) (u_0, u) is in the orientation of G'

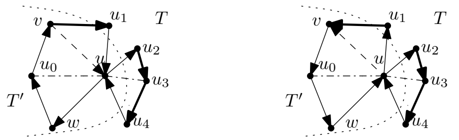
(b) (u, u_0) is in the orientation of G'

Fig. 6. Depicting the inductive step when u has four neighbors in $T \setminus T'$, $\{u_0, u\}$ is in G' (The dashed edge $\{u_0, u\}$ indicates that it does not exist in G but exists in G', the dotted curve is used to separate T' from T and the dash dotted edge $\{u, u_3\}$ indicates that it exists in T but not in G.)

Subcase 2 ($\{u_0, u\}$ is not in G'). By inductive hypothesis u is connected to its two siblings v and w in G'. By Lemma 1 either u_1 or u_4 is close to v or w. Without loss of generality, assume that u_1 and v are close. Let $G = \{G' \setminus \{v, u\}\} \cup \{\{u_1, u\}, \{u_2, u\}, \{u_4, u\}, \{v, u_1\}, \{u_2, u_3\}, \{u_3, u_4\}\}$. The orientation of G will depend on the orientation of $\{v, u\}$ in G'. If (v, u) is in $\overrightarrow{G'}$, then orient edges of G as depicted in Figure 7a. Otherwise if (u, v) is in $\overrightarrow{G'}$, then orient edges of G as depicted in Figure 7b. Thus, $\overrightarrow{G}$ is strongly connected and $\Delta^+(\overrightarrow{G}) \leq 2$. Moreover, u_1, u_2, u_3 and u_4 have degree one, the edges of T incident to u_1, u_2 and u_4 are contained in G and u_3 is connected to u_2 and u_4 in G. Observe that $\widehat{u_2 u u_4} \leq \pi/2$. This completes the proof of Theorem 3.

2.2 Algorithm

In this section we present the pseudocode for Algorithm 1 that constructs a strongly connected spanner with max out-degree $2 \leq k \leq 5$ and range bounded



(a) (v, u) is in the orientation of G'

(b) (u, v) is in the orientation of G'

Fig. 7. Depicting the inductive step when u has four neighbors in $T \setminus T'$ and $\{u_0, u\}$ is not in G' (The dashed edge $\{v, u\}$ indicates that it does not exist in G but exists in T', the dotted curve is used to separate T' from T, the dash dotted edge $\{u_0, u\}$ idicates that it exists in T' but not in G' and the dash dotted edge $\{u, u_3\}$ indicates that it exist in T but not in G.)

by $2 \cdot \sin\left(\frac{\pi}{k+1}\right)$ times the optimal. It uses the recursive Procedure kAntennae when $3 \le k \le 5$ and the recursive Procedure TwoAntennae when $k = 2$ which is presented in the full paper. The correctness of TwoAntennae procedure is derived from Theorem 3 and the correctness of kAntennae procedure is derived in the full paper. It is not difficult to see that Algorithm 1 runs in linear time.

Algorithm 1. Strongly connected spanner with max out-degree $2 \ge k \ge 5$ and edge length bounded by $2 \cdot \sin\left(\frac{\pi}{k+1}\right)$

input : T, k; where T is a MST with max length 1 and k an integer in $[2, 5]$.
output: Strongly connected spanner G with max out-degree k and range bounded by $2 \cdot \sin\left(\frac{\pi}{k+1}\right)$

Let u be any leaf of T and v its neighbor in T;
Let $G \leftarrow \{(v, u), (u, v)\}$;
if $k = 2$ **then** TwoAntennae(G, T, v, u);
if $3 \le k < 5$ **then** kAntennae(G, T, v, u, k);

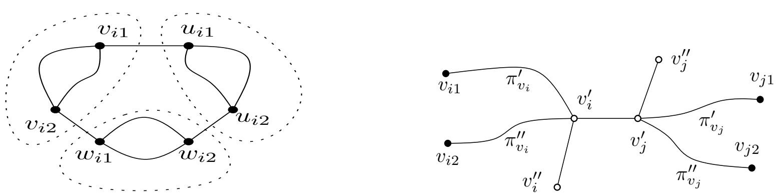
3 NP Completeness

*Proof (***Theorem 2).** By reduction from the well-known NP-hard problem for finding Hamiltonian cycles in degree three planar graphs. Take a degree three planar graph $G = (V, E)$ and replace each vertex v_i by a vertex-graph (meta-vertex) G_{v_i} shown in Figure 8a. Furthermore, replace each edge $e = \langle v_i, v_j \rangle$ of G by an edge-graph (meta-edge) G_e shown in Figure 8b.

Each meta-vertex has three parts connected in a cycle, with each part consisting of a pair of vertices (called *connecting vertices*) connected by two paths. Each meta-edge G_e has a pair of connecting vertices at each endpoint – these vertices coincide with the connecting vertices in the corresponding parts of the meta-vertices G_{v_i} and G_{v_j}. This means that after each vertex and each edge is replaced, each connecting vertex is of degree 4.

Procedure kAntennae(G, T, u, w, k)

Let $u_0 = w, u_1, \cdots, u_{d(u)-1}$ be the neighbors of $u \in T$ in clockwise order around u;
if $d(u) \leq k$ **then** Add to G a bidirectional arc for each u_i such that $i > 0$;
else if $d(u) = k + 1$ **then**
  Let u_i, u_{i+1} be the consecutive neighbor of u with smallest angle;
  if $i = 0$ **or** $i + 1 = 0$ **then**
    if $i = 0$ **then** Let $i \leftarrow 1$;
    if $(u, u_0) \in G$ **then** Let $G \leftarrow \{G \setminus \{(u, u_0)\}\} \cup \{(u, u_i), (u_i, u_0)\}$;
    else Let $G \leftarrow \{G \setminus \{(u_0, u)\}\} \cup \{(u_0, u_i), (u_i, u)\}$;
  end
  else Let $G \leftarrow G \cup \{(u, u_i), (u_i, u_{i+1}), (u_{i+1}, u)\}$;
  Add to G a bidirectional arc for each u_j such that $j \notin \{0, i, i + 1\}$;
end
else if $d(u) = k + 2$ **then**
  Let u_i, u_{i+1} be the consecutive neighbors of u with longest angle;
  if $i = 0$ **or** $i = 2$ **or** $i = 4$ **then** Let $G \leftarrow G \cup \{(u, u_1), (u_1, u_2), (u_2, u), (u, u_3), (u_3, u_4), (u_4, u)\}$;
  else
    if $(u, u_0) \in G$ **then** Let $G \leftarrow \{G \setminus \{(u, u_0)\}\} \cup \{(u, u_1), (u_1, u_0)\}$;
    else Let $G \leftarrow \{G \setminus \{(u_0, u)\}\} \cup \{(u_0, u_1), (u_1, u)\}$;
    Let $G \leftarrow G \cup \{(u, u_2), (u_2, u_3), (u_3, u), (u, u_4), (u_4, u)\}$;
  end
end
for $i \leftarrow 1$ **to** $d(u) - 1$ **do if** $d(u_i) > 1$ **then** $G \leftarrow$kAntennae(G, T, u_i, u, k) ;



(a) Vertex graph (The dotted ovals delimit the three parts.)

(b) Edge graph (The connecting vertices are black.)

Fig. 8. Meta-vertex and meta-edge for the NP completeness proof

Take the resulting graph G' and embed it in the plane in such a way that:

- the distance (in the embedding) between neighbours in G' is at most 1,
- the distance between non-neighbours in G' is at least x, and
- the smallest angle between incident edges in G' is at least α.

Let us call the resulting embedded graph G''. Note that such an embedding always exists [3]: We have a freedom to choose the length of the paths in the meta-graphs the way we need as we can stretch the configurations apart to fit everything in without violating the embedding requirements. The only constraining

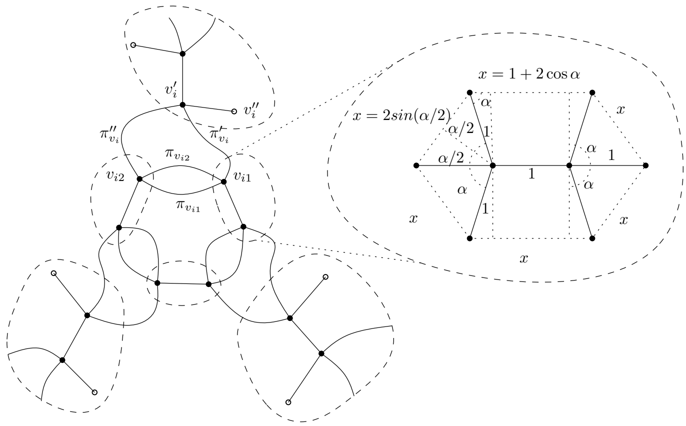


Fig. 9. Connecting meta-edges with meta-vertices (The dashed ovals show the places where embedding is constrained.)

places are the midpoints of the meta-edges and the three places in each meta-vertex where the parts are connected to each other. These can be embedded as shown in the right part of Figure 9. Note that the need to embed these parts without violating embedding requirements gives rise to the equations defining x and α (see Figure 9). This completes details of the main construction.

The proof of the Theorem is based on the following claim:

Claim. There is a Hamiltonian cycle in G if and only if there exists an assignment of two antennae with sum of angles less than α and radius less than x to the vertices of G'' such that the resulting connectivity graph is strongly connected.

*Proof (***Claim).** First we show that if G has a Hamiltonian cycle then there exists the assignment of such antennae that makes the resulting connectivity graph of G'' strongly connected. Figure 10 shows antenna assignments in the meta-edges corresponding to edges used and not used by the Hamiltonian cycle, respectively. Figure 11 shows the antenna assignments in a meta-vertex. Since each vertex of G has one incoming, one outgoing and one unused incident edge, and each edge is either used in one direction, or not used at all, this provides the full description of antenna assignments in G''.

Observe that the connecting pair of vertices at the meta-vertex uses two antennae towards the meta-edge it is connected to if and only if this meta-edge is outgoing; otherwise only one antenna is used towards the meta-edge and another is used towards the next part of the meta-vertex. It is easy to verify that the resulting connectivity graph is strongly connected:

– if the edge $e = \langle v_i, v_j \rangle$ is not used in the Hamiltonian path in the direction from v_i to v_j, then the near half of the meta-edge G_e (i.e. v_j', v_j'', π_{v_j}' and π_{v_j}'') together with the connecting part of the meta-vertex G_{v_j} form a strongly connected subgraph,

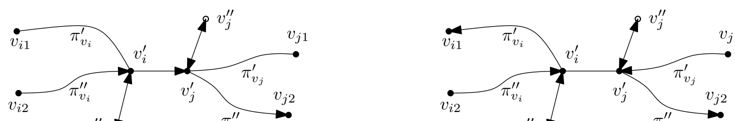
– in each meta-vertex the part corresponding to the outgoing edge is reachable from the part corresponding to the unused edge, which is in turn reachable from the part corresponding to the incoming edge, and
– all vertices of a meta-edge corresponding to an outgoing edge $\langle v_i, v_j \rangle$ are reachable from either v_{i1} or v_{i2}; furthermore the destination vertices v_{j_1} and v_{j_2} are reachable from all these vertices.

Combining these observations with the fact that the Hamiltonian cycle spans all vertices yields that the resulting graph is strongly connected.

Next we show that if it is possible to assign the antennae in G'' such that the resulting graph is strongly connected then there exists a Hamiltonian cycle in G. Recall that G'' is constructed in such a manner that no antenna of radius less than x and angle less than α can reach two neighbouring vertices, and that no antenna can reach a vertex that is not a neighbor in G''.

Assume an assignment of antennae such that the resulting graph is strongly connected. First, consider a pair of connecting vertices v_{i1} and v_{i2}. Since both path $\pi_{v_{i1}}$ and $\pi_{v_{i2}}$ are connected only to them, v_{i1} and v_{i2} must together use at least two antennae towards these two paths.

Let us call a meta-edge corresponding to edge $\langle v_i, v_j \rangle$ *directed* if in the connectivity graph there is an edge $\langle v_i', v_j' \rangle$. Without loss of generality assume the direction is from v_i' to v_j', i.e. v_i' used an antenna to reach v_j'. Since v_i'' is reachable only from v_i' (and hence v_i' used its second antenna on v_i''), this means that there is no antenna pointing from v_i' towards the paths π'_{v_i} and π''_{v_i}. Therefore, the only way for the vertices of these two paths to be reachable is to have both connecting vertices (which for simplicity we call v_{i1} and v_{i2}, respectively) use an antenna towards these paths. Since they already used two antennae to ensure reachability of $\pi_{v_i 1}$ and $\pi_{v_i 2}$ are reachable, they have no antenna left to connect to another part of the meta-vertex.



(a) Edge used in the Hamiltonian cycle.

(b) Edge not used in the Hamiltonian cycle.

Fig. 10. Antenna assignments in a meta-edges corresponding to an edge v_i to v_j

Consider now the other half of the meta-edge. Observe that since v_j' must use one antenna on v_j'', it can use at most one antenna towards the paths π'_{v_j} and π''_{v_j}. Hence, either v_{j1} or v_{j2} must use an antenna towards one of these paths. Since these vertices must use two more antennae to ensure that the paths $\pi_{v_j 1}$ and $\pi_{v_j 2}$ are reachable, only one antenna is left for connecting to other parts of the meta vertex. Note that this argument holds both for receiving ends of directed meta-edges, as well as for non-directed meta-edges.

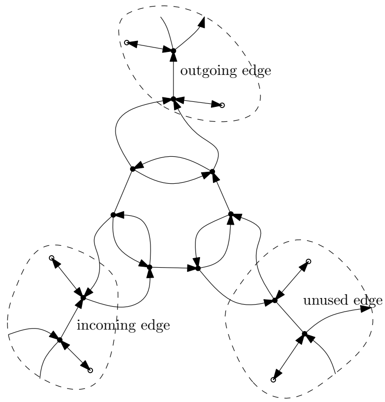


Fig. 11. Antenna assignments at the meta-vertex and incident meta-vertices

However, this means that in a meta-vertex there can be at most one outgoing directed meta-edge – otherwise there is no way to make the meta-vertex connected. Since each meta-vertex must have at least one outgoing directed meta-edge (otherwise the rest of the graph would be unreachable) and at least one incoming directed meta-edge (otherwise it would not be reachable from the rest), from the fact that the whole graph is strongly connected it follows that each meta-vertex must have exactly one undirected meta-edge, one directed incoming meta-edge and one directed outgoing meta-edge. Obviously, these correspond to unused/incoming/outgoing edges in the original graph G, with the directed edges forming the Hamiltonian cycle.

4 Conclusion

We have provided an algorithm which when given as input a set of n points (modeling sensors) in the plane and an integer $1 \leq k \leq 5$ produces a strongly connected spanner so that each sensor uses at most k directional antennae of angle 0 and range at most $2 \cdot \sin\left(\frac{\pi}{k+1}\right)$ times the optimal. Interesting open problems include looking at tradeoffs when the angle of the antennae is $\varphi > 0$ as well as deriving better lower bounds.

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