A Characterisation of Stable Sets in Games with Transitive Preference

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Abstract. This article characterises stable sets in an abstract game. We show that every stable subset of the pure strategies for the game is characterised as a fixed point of the mapping assigning to each upper boundedly preordered subset of the strategies the set of all its maximal elements.

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1 Introduction

This article investigates a stable property in strategies of abstract games. In cooperative game theory the central solution concept is stable sets, which are sets of outcomes on which a preference relation $\precsim$ satisfies the two property: Reflexivity ($x \precsim x$) and Transitivity (If $x \precsim y$ and $y \precsim z$ then $x \precsim z$.) A stable set consists of outcomes satisfying (i) the *internal stability* (for every outcomes being not stable, some coalition has an objection), and (ii) the *external stability* (no coalition has an objection to any stable outcome.)

This solution concept is introduced as a 'standard of behaviour' by von Neumann and Morgenstern [5]. The stable sets can be treated in the abstract game framework. Many mathematical difficulties still arise in the stable sets when the preference is not transitive.

Jiang [3] treats abstract games with *transitive* preferences, which arise from strategic games. He addresses the existence problem of the stable sets in the set of Nash equilibria of a strategic game. Regarding to the original intents of von Neumann and Morgenstern [5], it is unpleasant to restrict stable sets to subsets of the Nash equilibria set for the strategic game a priori.

This article aims to improve the point: Removing out the restriction of stable sets to subsets of Nash equilibria we treat stable sets in the framework of abstract games; we address a class of abstract games having stable sets, and characterise

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the stable sets as the maximal sets in an upper bounded set of the outcomes. Our main result is as follows:

Characterisation theorem. *Every stable subset of the pure strategies for an abstract game is characterised as a fixed point of the mapping assigning to each upper boundedly preordered preference subset of the strategies the set of all its maximal elements.*

After reviewing basic notions and terminology, Section 2 presents the extended notion of stable sets (Definition 2) in an abstract game. The notion of upper bounded game (Definition 3) is also introduced, which plays crucial role in determining the existence of stable sets. Section 3 introduces the Jiang mapping for an abstract game and presents the main theorem (Theorem 1) and the characterisation theorem (Theorem 2). Section 4 establishes the main theorem, from which the theorem of Jiang [3] follows as a corollary. Finally I conclude with some remarks on the assumptions in the theorems.

2 Model

2.1 Preference

A binary relation R on a non-empty set X is a subset of $X \times X$ with (x, y) denoted by xRy. A relation R may satisfy one and more properties:

Ref (Reflexivity) For all $x \in X, xRx$;

Trn (Transitivity) For all $x, y, z \in X$, if xRy and yRz then xRz;

Sym (Symmetry) For all $x, y \in X, xRy$ implies yRx;

Asym (Antisymmetry) For all $x, y \in X$, if xRy and yRx then $x = y$;

Cmp (Completeness) For all $x, y \in X$, we have xRy or yRx (or both).

Definition 1. A *preference* relation on a non-empty X is a binary relation $\precsim$ on X. For $x, y \in X$ we will read '$x \precsim y$ as 'y is at least as preferable as x.' The *strict* preference relation $\prec$ is defined by

$$x \prec y \iff x \precsim y \text{ but not } y \precsim x.$$

The *indifference* relation $\sim$ is defined by

$$x \sim y \iff x \precsim y \text{ and } y \precsim x.$$

A set X together with a definite preference relation $\precsim$ will be called a *preference set* denoted by $(X, \precsim)$. A preference relation on X is called *rational* if it satisfies the properties **Trn** and **Cmp**. A *preorder* on X is a binary relation $\precsim$ on X satisfying the properties **Ref** and **Trn**. A preorder with **Asym** is called *partial order*.

Remark 1. Let $(X, \precsim)$ be a rational preference set. The following properties are true:

(i) $\precsim$ is a preorder.
(ii) $\prec$ is irreflexive ($x \prec x$ is never true) and transitive;
(iii) $\sim$ is an equivalence relation.

Let $(X, \precsim)$ be a preordered set and Y a subset of X. A *maximal* element of Y is an element a such that Y contains no element b with $a \prec b$. An element a is an *upper bound* of Y in case $x \precsim a$ for every x in Y. A subset Y of X is called *upper bounded* in X if it has at least one upper bound in it. A subset Y is called a *chain* if for every $x, y \in Y$, either $x \precsim y$ or $y \precsim y$ has to hold.

It is worthy noting that

Lemma 1. *Each of the following two statements is equivalent to the axiom of choice:*[1]

(i) (Zorn's Lemma) *Let* $(X, \precsim)$ *be a preordered set. If each chain in* X *has an upper bound then* X *has at least one maximal element.*
(ii) (Maximal Principle) *Let* X *be a partially ordered set. Each chain in* X *is contained in a maximal chain.*

2.2 Stable Sets and Bounded Games

An n-person *abstract game* is a tuple $\Gamma = \langle N, (A_i)_{i \in N}, (\precsim) \rangle$ consisting of

1. N is a set of n *players* with $n \geqq 2$ and i denotes a player;
2. A_i is a non-empty set of player i's *pure strategies* and $A = \prod_{i \in N} A_i$ is the set of *strategies*;
3. $\precsim$ is a binary relation on $A = \prod_{i \in N} A_i$, called *preference.*

Definition 2. Let $\Gamma = \langle N, (A_i)_{i \in N}, \precsim \rangle$ be an n-person abstract game and Y a non empty subset of $A = \prod_{i \in N} A_i$. A non-empty subset V of Y is said to be *von Neumann-Morgenstern stable in* Y or simply, Y*-stable* if the two conditions hold:

IS (Internal stability) For any $a, b \in V$, neither $a \prec b$ nor $b \prec a$ holds;
ES (External stability) For any $b \in Y \setminus V$ there exist an $a \in V$ such that $b \prec a$.

If a subset of A is an A-stable set then it will be simply called *N-M stable.*

We denote by $\text{NMS}(\Gamma; Y)$ the set of all stable sets in Y of $A = \prod_{i \in N} A_i$, and denote by $\text{NMS}(\Gamma)$ the set of all stable subsets in some non-empty set of $A = \prod_{i \in N} A_i$; i.e.,

$$\text{NMS}(\Gamma) = \cup_{\emptyset \neq Y \subseteq A} \text{NMS}(\Gamma; Y).$$

[1] See pp. 31-32 and p.58 in [1].

Definition 3. Let $\Gamma = \langle N, (A_i)_{i\in N}, \precsim \rangle$ be an n-person abstract game and Y a non empty subset of $A = \prod_{i\in N} A_i$. The game Γ is called Y*-upper bounded* if the following two conditions are true:

(i) $(Y, \precsim)$ is a preordred set, and
(ii) Each chain in Y has an upper bound in Y.

The game Γ will be called simply *upper bounded* if it is A-upper bounded.

Denote by TUB(Γ) the set of all non-empty upper bounded subsets of $A = \prod_{i\in N} A_i$. We will write by NMS$^*(\Gamma)$ the set of all stable subsets in some upper bounded set of $A = \prod_{i\in N} A_i$; i.e.,

$$\mathrm{NMS}^*(\Gamma) = \cup_{Y\in \mathrm{TUB}(\Gamma)} \mathrm{NMS}(\Gamma; Y).$$

2.3 Classical Case

An n-person *strategic game* is a tuple $\Gamma = \langle N, (A_i)_{i\in N}, (\precsim_i)_{i\in N} \rangle$ consisting of

1. N and A_i are the same as above;
2. $\precsim_i$ is i's *rational* preference relation.

The *uniform preference* relation $\precsim$ on $A = \prod_{i=1}^n A_i$ is a binary relation on A defined by

$$a \precsim b \iff a \precsim_i b \text{ for any } i \in N.$$

The *strict* preference relation $\prec$ is defined by

$$a \prec b \iff a \precsim b \text{ but not } b \precsim a.$$

The *indifference* relation $\sim$ is defined by

$$x \sim y \iff x \precsim y \text{ and } y \precsim x.$$

Remark 2. The game $\Gamma = \langle N, (A_i)_{i\in N}, \precsim \rangle$ with the uniform preference is an abstract game. The preference $\precsim$ is a preorder on A, but it is not always rational; i.e., it satisfies **Ref** and **Trn**, but not **Cmp** in general.

A profile $a^* = (a_1^*, \cdots, a_i^*, \cdots, a_n^*)$ is a *pure* Nash equilibrium for a strategic game $\Gamma = \langle N, (A_i)_{i\in N}, (\precsim_i)_{i\in N} \rangle$ provided that for each $i \in N$ and for every $a_i \in A_i$, $(a_{-i}^*, a_i) \precsim_i a^*$. We denote by PNE($\Gamma$) the set of all pure Nash equilibria for Γ.

Remark 3. In his paper [3], Jiang calls a strategic game Γ *regular* if Γ is PNE(G)-upper bounded. By N-M stable set Jiang [3] means a stable set in PNE(Γ).

Example 1 (Jiang [3]). The *vagabonds game*, in which the preference relation is derived from individual utility functions, is the tuple $\langle \Gamma, (A_i), (\precsim_i) \rangle$ consisting of

1. $N = \{1, 2, \cdots, n\}$ is a set of n players called *vagabonds* ($n \in \mathbb{N}$) and i denotes a vagabond;
2. $A_i = \mathbb{R}_+$

3. i's utility function $u_i : \mathbb{R}^n_+ \to \mathbb{R}$ is given by

$$u_i(x_1, \cdots, x_i, \cdots, x_n) = \begin{cases} x_i & \text{if } \sum_{j \in N} x_j \in [0,1] \\ 0 & \text{if } \sum_{j \in N} x_j \in (1, +\infty) \end{cases}$$

4. $\precsim_i$ is i's preference relation represented by the function u_i as follows: For any $x, x' \in A = \prod_{i=1}^n A_i$, $x \precsim_i x' \iff u_i(x) \leqq u_i(x')$.

Set

$$S_n = \{x = (x_1, \cdots, x_i, \cdots, x_n) \in \mathbb{R}^n_+ | \sum_{i \in N} x_i = 1\}$$

Then the game Γ is upper bounded with S_n a stable set in PNE(Γ). Moreover, it is 'regular' in the sense of Jiang [3].

Remark 4. The game Γ actually contains the unique stable set S_n in PNE(Γ). This can be verified by Corollary 1 that will be shown later in the next section.

3 Main Theorem

Let $\Gamma = \langle N, (A_i)_{i \in N}, \precsim \rangle$ be an n-person abstract game.

Definition 4. By the *Jiang mapping* for the abstract game Γ, we mean the mapping J_Γ : TUB(Γ) $\to$ NMS(Γ) which assigns to each Y of TUB(Γ) the set V_Y of all maximal elements in Y: For each $Y \in$ TUB(Γ),

$$J_\Gamma(Y) = V_Y = \{y \in Y \mid y \text{ is maximal in } Y\} \quad \text{if } Y \neq \emptyset;$$
$$J_\Gamma(\emptyset) = \emptyset \quad \text{otherwise.}$$

We can now state our main result.

Theorem 1 (Main theorem). *Let $\Gamma = \langle N, (A_i)_{i \in N}, \precsim \rangle$ be an n-person abstract game. The Jiang mapping J_Γ is well-defined mapping with the property: $J_\Gamma \circ J_\Gamma = J_\Gamma$. Furthermore, it is a surjective map onto the set NMS*(Γ) of all stable sets in some upper bounded subset of strategies in the game Γ.*

Before proceeding with the proof we will establish the characterisation theorem for stable sets mentioned in Section 1, and we state the theorem explicitly: Let Fix(J_Γ) denote the set of all fixed members of TUB(Γ) for J_Γ:

$$\text{Fix}(J_\Gamma) = \{Y \in \text{TUB}(\Gamma) \mid J_\Gamma(Y) = Y\}.$$

Theorem 2 (Characterisation theorem). *Let J_Γ be the Jiang mapping for an n-person abstract game Γ with the preorder preference $\precsim$. Then the set NMS*(Γ) of all stable sets in some upper bounded subset of strategies in the game Γ coincides with the set of all fixed points of the Jiang mapping J_Γ in TUB(Γ); i.e.,*

$$\text{NMS}^*(\Gamma) = \text{Fix}(J_\Gamma).$$

In particular, every $W \in$ NMS(Γ) can be uniquely expressed by the form $W = J_\Gamma(Y)$ for some $Y \in$ TUB(Γ).*

Proof. For any $Y \in \mathrm{Fix}(J_\Gamma)$, it immediately follows that $Y = J_\Gamma(Y) \in \mathrm{TUB}(\Gamma) \cap \mathrm{NMS}(\Gamma) \subseteq \mathrm{NMS}^*(\Gamma)$, and hence $\mathrm{Fix}(J_\Gamma) \subseteq \mathrm{NMS}^*(\Gamma)$. The converse will be shown as follows: Let us take any $W \in \mathrm{NMS}^*(\Gamma)$. By the surjectivity of J_Γ it follows that there is a $Y \in \mathrm{TUB}(\Gamma)$ such that $W = J_\Gamma(Y)$, and thus it can be plainly seen by the property for J_Γ in Theorem 1 that $W \in \mathrm{Fix}(J_\Gamma)$ because $J_\Gamma(W) = J_\Gamma(J_\Gamma(Y)) = J_\Gamma(W) = W$. Therefore we obtain that $\mathrm{NMS}^*(\Gamma) \subseteq \mathrm{Fix}(J_\Gamma)$, in completing the proof. □

4 Proof of Theorem 1

We shall proceed with the proof by the following steps:

J_Γ is a well-defined mapping: This follows immediately from the below theorem:

Theorem 3. *Let $\Gamma = \langle N, (A_i)_{i\in N}, \precsim\rangle$ be an n-person abstract game with the preorder preference and Y a non-empty subset of $A = \prod_{i\in N} A_i$. If Γ is Y-upper bounded then it has the unique stable set in Y.*

Proof. **Existence**: Let V denote the set $J_\Gamma(Y)$ of all maximal elements in a preordered set $(Y, \precsim)$. By Lemma 1(i) we can observe that V is a non-empty set. We shall show that V is a stable set in Y.

For **IS**: On noting that each element in V is maximal, **IS** follows immediately.

For **ES**: Suppose to the contrary that there exists a $y_0 \in Y \setminus V$ such that for every $x \in V$, it is not true that $y_0 \prec x$. However, since y_0 is not maximal in Y, there is a $y_1 \in Y$ such that $y_0 \prec y_1$. Let $\mathcal{T}$ be the set of all the chains C satisfying the two conditions: (1) C consists of elements $x \in Y$ strictly prefereable than y_0 (i.e.; $x \succ y_0$), and (2) C contains the chain $T_0 = \{y_0 \prec y_1\}$. It is plainly seen that $T_0 \in \mathcal{T} \neq \emptyset$ and that $\mathcal{T}$ is a partially ordered set equipped with the set theoretical inclusion. Hence it follows by Lemma 1(ii) that the chain $T_0 \in \mathcal{T}$ is contained in a maximal chain T^* in Y. Since Γ is upper-bounded, the chain T^* has an upper bound $y^* \in Y$, and so it immediately follows that $T^* \cup \{y^*\} \in \mathcal{T}$ because $y_0 \prec y^*$ and $x \precsim y^*$ for all $x \in \mathcal{T}$ except y_0. This means that $T^* \cup \{y^*\} \in \mathcal{T}$ which properly contains T^*, in contradiction to the maximality of T^* in $\mathcal{T}$, as required.

Uniqueness: Suppose V and W are stable sets in Y with $V \neq W$. Without loss of generality we may assume $V \setminus W \neq \emptyset$. Take $a \in V \setminus W$. It follows from **ES** for W that there exists $b \in W$ such that $a \prec b$. By **IS** for V we obtain $b \in Y \setminus V$. From **ES** for V it follows that there exists $c \in V$ such that $b \prec c$, and thus $a \prec b \prec c$. By **Trn** on Y we obtain that $a \prec c$ for $a, c \in V$, in contradiction to **IS** for V, in completing the proof of Theorem 3. □

$J_\Gamma \circ J_\Gamma = J_\Gamma$: It is easily seen that $\mathrm{NMS}^*(\Gamma; Y) \subseteq \mathrm{TUB}(\Gamma)$ for any $Y \in \mathrm{TUB}(\Gamma)$, and so $\mathrm{NMS}^*(\Gamma) \subseteq \mathrm{TUB}(\Gamma)$. It follows that the composite mapping

$J_\Gamma \circ J_\Gamma$ is well-defined. For each $Y \in \mathrm{TUB}(\Gamma)$, in viewing of the definition of J_Γ it can be plainly observed that $J_\Gamma(Y)$ is the set of all maximal elements in Y, and so $J_\Gamma(Y) \subseteq Y$. Therefore it follows from **IS** that $Y \subseteq J_\Gamma(W)$, and $J_\Gamma \circ J_\Gamma(Y) = J_\Gamma(Y)$, as required. □

J_Γ is a surjection onto $\mathrm{NMS}^*(\Gamma)$: For any $W \in \mathrm{NMS}^*(\Gamma)$, we can take $Y \in \mathrm{TUB}(\Gamma)$ such that $W \in \mathrm{NMS}^*(\Gamma; Y)$. By the same argument as above we can obtain that $W = J_\Gamma(W)$, as required. This completes the proof of Theorem 1. □

As consequence of Theorem 3 we obtain the Jiang' s theorem:

Corollary 1 (Jiang [3]). *Every PNE(Γ)-upper bounded strategic game Γ has the unique stable set in PNE(Γ).*

5 Concluding Remarks

It well ends this article by giving remarks on the assumptions on Theorem 3: Transitivity on preference and upper boundedness for a game. These assumptions play crucial role in the theorem.

Game with non-transitive preference having no stable set: We can easily construct such game: See the game in Figure 1 in Lucas [4] (p.545) has no stable set at all. □

Non-upper bounded game having no stable set: Let $\langle \mathbb{R}_+, \leqq \rangle$ be the real line equipped with the usual inequality, and we will consider it as one player strategic game. Then we can easily observe that the game is neither $\mathbb{R}_+$-upper bounded nor has stable set in $\mathbb{R}_+$. □

Conclusion: This article treats the notion of stable sets in an abstract form game with transitive preference. We investigate conditions under which the stable sets are guaranteed. The main theorem shows that the stable sets for the abstract game is characterised as the fixed point of the mapping assigning to each upper bounded subset in the pure strategies the subset of the maximal elements of it. The key is to establish that the stable set uniquely exists in each inductive set of pure strategies for the abstract game. In the classical case of strategic game, we obtain Jiang's result as a consequence, which guarantees the unique stable set in the set of Nash equilibria for the game in the case the Nash equilibrium set is inductive.

The emphasis is that the continuity on the preferences is not assumed in the theorems as we can view in Example 1. However the two assumptions, transitivity on preference and upper boundedness for a game, play crucial role in guaranteeing existence of the stable set in the game. These comments show that the two assumptions are necessary to the theorems.

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