Cyclic Vertex Connectivity of Star Graphs*

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Abstract. For a connected graph G, a vertex subset $F \subset V(G)$ is a cyclic vertex-cut of G if $G - F$ is disconnected and at least two of its components contain cycles. The cardinality of a minimum cyclic vertex-cut of G, denoted by $\kappa_c(G)$, is the cyclic vertex-connectivity of G. In this paper, we show that for any integer $n \geq 4$, the n-dimensional star graph SG_n has $\kappa_c(SG_n) = 6(n-3)$.

Keywords: star graph; cyclic vertex-connectivity.

1 Introduction

Let $G = (V(G), E(G))$ be a simple connected graph, where $V(G)$, $E(G)$ are the vertex set and the edge set, respectively. A vertex subset $F \subseteq V(G)$ is a *cyclic vertex-cut* of G if $G - F$ has at least two connected components containing cycles. Vertices in F are called *faulty*, and vertices in $V(G) - F$ are said to be *good*. If G has a cyclic vertex-cut, then the *cyclic vertex-connectivity* of G, denoted by $\kappa_c(G)$, is the minimum cardinality over all cyclic vertex-cuts of G. When G has no cyclic vertex-cut, the definition of $\kappa_c(G)$ can be found in [15] using Betti number. The cyclic edge-connectivity $\lambda_c(G)$ can be defined similarly, changing 'vertex' to 'edge' (see for example [13,14]).

The concepts of cyclic vertex- and edge-connectivity date to Tait (1880) in attacking Four Color Conjecture [16]. Since then, they are used in many classic fields of graph theory such as integer flow conjectures [21], n-extendable graphs [9,12], etc.

In [18], the authors showed that $\lambda_c(G)$ coincides with $\lambda^2(G)$, where $\lambda^k(G)$ is a kind of conditional connectivity [7] defined as follows: for a connected graph G, an edge subset $F \subset V(G)$ is a R^k*-edge-cut* if $G - F$ is disconnected and each vertex in $V(G) - F$ has at least k good neighbors in $G - F$ (or equivalently, $\delta(G - F) \geq 2$, where δ is the minimum degree of the graph). The R^k*-edge connectivity* of G, denoted by $\lambda^k(G)$ is the cardinality of a minimum R^k-vertex-cut of G. Thus many results obtained for $\lambda^2(G)$ can be directly transformed to those of $\lambda_c(G)$, for example, results in [11,20].

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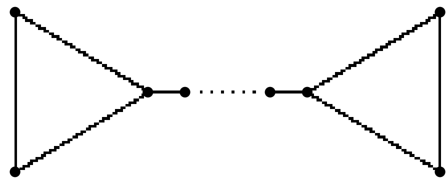

Fig. 1. $\kappa_c(G) = 1 < n - 6 = \kappa^2(G)$, where n is the number of vertices in G

However, the story is different for $\kappa_c(G)$. Changing 'edge' to 'vertex', we obtain the definition of *R^k-vertex-connectivity* $\kappa^k(G)$. Since every graph with minimum degree at least 2 has a cycle, we have $\kappa_c(G) \le \kappa^2(G)$ as long as both $\kappa_c(G)$ and $\kappa^2(G)$ exist. The following example shows that the strict inequality may hold and the gap between $\kappa_c(G)$ and $\kappa^2(G)$ can be arbitrarily large.

In this paper, we determine κ_c for star graphs. Let S_n be the *symmetric group* of order n, that is, the set of all permutations of $\{1, 2, ..., n\}$. The *n-dimensional star graph SG_n* is the graph with vertex set $V(SG_n) = S_n$, two vertices u, v are adjacent in SG_n if and only if $v = u(1i)$, for some $2 \le i \le n$. We say that the *label* on the edge uv is $(1i)$. Star graphs have been shown to have many desirable properties such as high connectivity, small diameter ect., which makes it favorable as a network topology (see for example [2,8]).

We will show in this paper that $\kappa_c(SG_n) = 6(n-3)$ for $n \ge 4$. In [17], Wan and Zhang proved that for any integer $n \ge 4$, $\kappa^2(S_n) = 6(n-3)$. We guess that this is not an accidental coincidence, which deserves further study.

2 Some Preliminaries

Terminologies not defined here are referred to [3].

For a graph G, a subgraph G_1 of G, and a vertex $u \in V(G)$, we use $N_{G_1}(u) = \{v \in V(G_1) \mid v \text{ is adjacent with } u \text{ in } G\}$ to denote the *neighbor set of u in G_1*. In particular, if $G_1 = G$, then $N_G(u)$ is the neighbor set of u in G, and $d_G(u) = |N_G(u)|$ is the *degree* of vertex u in G. The *minimum degree* of G is $\delta(G) = \min\{d_G(u) \mid u \in V(G)\}$. For a vertex subset $U \subseteq V(G)$, let $N_{G_1}(U) = (\bigcup_{u \in U} N_{G_1}(u)) - U$ be the neighbor set of U in G_1. For simplicity of notation, we sometimes use a subgraph and its vertex set interchangeably, for example, $N_G(G_1)$ is used to denote $N_G(V(G_1))$ where G_1 is a subgraph of G, and $N_A(U)$ is used to denote $N_{G[A]}(U)$ where A, U are two vertex sets and $G[A]$ is the *subgraph of G induced by A*.

It is known that SG_n is $(n-1)$-regular, bipartite, vertex transitive, and edge transitive [1]. We will also use the following result given by Cheng and Lipman.

Lemma 1 ([4]). *For $n \ge 4$, let T be a vertex subset of SG_n with $|T| \le 2n - 4$. Then one of the following occurs:*

(i) $SG_n - T$ is connected;

(ii) $SG_n - T$ has two connected components, one of which is a singleton;

(iii) $SG_n - T$ has two connected components, one of which is an edge uv, furthermore, $T = N_{SG_n}(uv)$.

As a corollary of Lemma 1, we have

Corollary 1. *For $n \geq 4$, $\kappa^1(SG_n) = 2n - 4$. Furthermore, if T is a minimum R^1-vertex-cut of SG_n, then $T = N_{SG_n}(uv)$ for some edge $uv \in E(SG_n)$.*

The *girth* of a graph G is the length of the shortest cycle in G. The following lemma characterizes the structure of shortest cycles of SG_n.

Lemma 2 ([17]). *The girth of SG_n is 6. Any 6-cycle in SG_n has the form $u_1u_2u_3u_4u_5u_6u_1$, where $u_2 = u_1(1i), u_3 = u_2(1j), u_4 = u_3(1i), u_5 = u_4(1j), u_6 = u_5(1i), u_1 = u_6(1j)$ for some i, j with $i \neq j$.*

Lemma 2 shows that any 6-cycle of SG_n has its edges labeled with $(1i)$ and $(1j)$ alternately for some $i, j \in \{2, ..., n\}$ and $i \neq j$. As a consequence, we see that

Corollary 2. *Any two 6-cycles of SG_n have at most one common edge.*

Proof. Suppose $C_1 = u_1u_2u_3u_4u_5u_6u_1$ and $C_2 = u_1u_2v_3v_4v_5v_6u_1$ are two 6-cycles of SG_n having a common edge u_1u_2, the label on u_1u_2 is $(1i)$, and the label on u_2u_3 is $(1j)$ for $j \neq i$. By Lemma 2, the label on u_2v_3 is $(1k)$ for some $k \neq i, j$. Then the common edges of C_1 and C_2 must have label $(1i)$. Notice that $v_3, v_6 \notin V(C_1)$ since the girth of SG_n is 6. Hence v_3v_4 and v_5v_6, which are the only two other edges on C_2 with label $(1i)$, do not belong to C_1. Thus u_1u_2 is the only common edge of C_1 and C_2. $\square$

Let S_n^i be the subset of S_n that consists of all permutations with element i in the rightmost position, and let SG_{n-1}^i be the subgraph of SG_n induced by S_n^i. Clearly SG_{n-1}^i is isomorphic to SG_{n-1}, and thus we call it a *copy* of SG_{n-1}. It is easy to see that SG_n can be decomposed into n copies of SG_{n-1}, namely $SG_{n-1}^1, SG_{n-1}^2, ..., SG_{n-1}^n$. For any copy SG_{n-1}^i and any vertex $u \in V(SG_{n-1}^i)$, there is exactly one neighbor of u outside of SG_{n-1}^i, namely the vertex $u(1n)$. We call it the *outside neighbor* of u and use u' to denote it.

The following property was proved in Lemma 3 of [17], though we state it in a different way to suit the needs of this paper.

Lemma 3 ([17]). *For any path $P = u_0u_1u_2$ which is contained in some copy, the outside neighbors u_0', u_1', u_2' are in three different copies. As a consequence, for any edge u_1u_2 in some copy, u_1' and u_2' are in different copies.*

The next result can also be found in [17].

Lemma 4 ([17]). *For any $i \in \{1, 2, ..., n\}$, $N_{SG_n}(SG_{n-1}^i)$ is an independent set of cardinality $(n-1)!$, and $|N_{SG_{n-1}^j}(SG_{n-1}^i)| = (n-2)!$ for any $j \neq i$.*

3 Main Result

In this section, we determine the value of $\kappa_c(SG_n)$ for $n \geq 4$.

Lemma 5. *Let C be a 6-cycle of SG_n ($n \geq 4$). Then $N_{SG_n}(C)$ is a cyclic vertex-cut of SG_n.*

Proof. Clearly, $SG_n - N_{SG_n}(C)$ is disconnected which contains cycle C as a connected component. Hence to prove the lemma, it suffices to show that the subgraph $\widetilde{G} = SG_n - N_{SG_n}(C) - C$ has a cycle. In fact, we can prove a stronger property $\delta(\widetilde{G}) \geq 2$ as follows.

Suppose $C = u_1u_2...u_6u_1$. By Lemma 2, there exist two indices $i, j \neq n$ such that the labels on the edges of C are $(1i)$ and $(1j)$ alternately. If $\delta(\widetilde{G}) \leq 1$, then there exists a vertex $v \in V(\widetilde{G})$ which has at least $n-2 \geq 2$ neighbors in $N_{SG_n}(C)$ (recall that SG_n is $(n-1)$-regular). Let v_1, v_2 be two distinct vertices in $N_{SG_n}(v) \cap N_{SG_n}(C)$. Suppose, without loss of generality, that v_1 is a neighbor of u_1. Since SG_n is bipartite, there is no odd cycle in SG_n. Hence v_2 can only be a neighbor of vertex u_3 or u_5, say u_3. But then $C' = vv_1u_1u_2u_3v_2v$ is a 6-cycle of SG_n which have two common edges u_1u_2, u_2u_3 with the 6-cycle C, contradicting Corollary 2. Thus $\delta(\widetilde{G}) \geq 2$.

Since every graph with minimum degree at least 2 has a cycle, the lemma is proved. □

Theorem 1. *For any integer $n \geq 4$, $\kappa_c(S_n) = 6(n-3)$.*

Proof. Let C be a 6-cycle in SG_n and $F = N_{SG_n}(C)$. Since the girth of SG_n is 6, no two vertices on C have a common neighbor in $N_{SG_n}(C)$. Thus $|F| = 6(n-3)$. By Lemma 5, F is a cyclic vertex-cut. Hence $\kappa_c(SG_n) \leq |F| \leq 6(n-3)$.

To prove the converse, let F be a minimum cyclic vertex-cut of SG_n. Suppose $|F| < 6(n-3)$, we are to derive a contradiction. For $i \in \{1, 2, ..., n\}$, denote $F_i = F \cap SG^i_{n-1}$, and U_i the set of isolated vertices of $SG^i_{n-1} - F_i$.

Claim 1. If $|F_i| \leq 2n-7$, then $|U_i| \leq 1$.

Otherwise, let u, v be two vertices in U_i. Since SG^i_{n-1} has girth 6, we see that u, v have at most one common neighbor. Hence $|N_{SG^i_{n-1}}(\{u,v\})| \geq 2(n-2)-1 > 2n-7$. Since U_i is an independent set, we see that $N_{SG^i_{n-1}}(U_i) \supseteq N_{SG^i_{n-1}}(\{u,v\})$. It follows that $|F_i| \geq |N_{SG^i_{n-1}}(U_i)| \geq |N_{SG^i_{n-1}}(\{u,v\})| > 2n-7$, contradicting that $|F_i| \leq 2n-7$.

Claim 2. If $|F_i| \leq 2n-7$, then $SG^i_{n-1} - (F_i \cup U_i)$ is connected.

Suppose this is not true, then $F_i \cup U_i$ is a R^1-vertex-cut of SG^i_{n-1}. By Corollary 1, $|F_i \cup U_i| \geq \kappa^1(SG^i_{n-1}) = 2n-6$. Combining this with $|U_i| \leq 1$ (by Claim 1) and $|F_i| \leq 2n-7$, we see that $|U_i| = 1$ and $|F_i \cup U_i| = \kappa^1(SG^i_{n-1})$ (thus $F_i \cup U_i$ is a minimum R^1-vertex-cut of SG^i_{n-1}). Again by Corollary 1, $F_i \cup U_i = N_{SG^i_{n-1}}(vw)$ for some edge $vw \in E(SG^i_{n-1})$. Let u be the unique vertex in U_i. Then u is adjacent with either v or w, contradicting that u is an isolated vertex in $SG^i_{n-1} - F_i$. Thus Claim 2 is proved.

Let $I = \{i : |F_i| \geq 2n-6\}$. Since $|F| < 6(n-3)$, we have $|I| \leq 2$.

Claim 3. Let G_1 be the subgraph of SG_n induced by $\bigcup_{i\notin I} V(SG^i_{n-1}-(F_i\cup U_i))$. Then G_1 is connected.

By Claim 2, $SG^i_{n-1}-(F_i\cup U_i)$ is connected for any $i\notin I$. Hence to prove Claim 3, it suffices to show that for any two indices $i,j\notin I$, there is a path in G_1 connecting $SG^i_{n-1}-(F_i\cup U_i)$ and $SG^j_{n-1}-(F_j\cup U_j)$. For such i,j, $|U_i|,|U_j|\le 1$ by Claim 1.

If there is an edge between $SG^i_{n-1}-(F_i\cup U_i)$ and $SG^j_{n-1}-(F_j\cup U_j)$, then we are done. Hence we suppose that there is no edge between $SG^i_{n-1}-(F_i\cup U_i)$ and $SG^j_{n-1}-(F_j\cup U_j)$. Then

$$N_{SG^j_{n-1}}(SG^i_{n-1}-(F_i\cup U_i))\subseteq F_j\cup U_j, \tag{1}$$

and thus $|N_{SG^j_{n-1}}(SG^i_{n-1}-(F_i\cup U_i))|\le |F_j|+1$. We will show that this inequality can be refined to

$$|N_{SG^j_{n-1}}(SG^i_{n-1}-(F_i\cup U_i))|\le |F_j|. \tag{2}$$

Suppose (2) is not true, then $N_{SG^j_{n-1}}(SG^i_{n-1}-(F_i\cup U_i))=F_j\cup U_j$ and $|U_j|=1$. Let u be the unique vertex in U_j. Since u is an isolated vertex in $SG^j_{n-1}-F_j$, it has a neighbor v in F_j. By $N_{SG^j_{n-1}}(SG^i_{n-1}-(F_i\cup U_i))=F_j\cup U_j$, we see that u and v have outside neighbors u' and v' in $SG^i_{n-1}-(F_i\cup U_i)$, respectively. Since the girth of SG_n is 6 and u and v are neighbors to each other, $u'\ne v'$. Since either u and v has another neighbor in SG^i_{n-1}, by the latter part of Lemma 3, we have a contradiction.

Next, we show that

$$|N_{SG^j_{n-1}}(F_i\cup U_i)|\le |F_i|. \tag{3}$$

Since each vertex has exactly one outside neighbor, we have $|N_{SG^j_{n-1}}(F_i\cup U_i)|\le |F_i\cup U_i|\le |F_i|+1$. If equality holds, then $|U_i|=1$ and every vertex in $F_i\cup U_i$ has its outside neighbor in SG^j_{n-1}. Similar to the above, the unique vertex $u\in U_i$ has a neighbor v in F_i, and thus the outside neighbors u',v' can not be both in the same copy. This contradiction establishes inequality (3).

By Lemma 4 and inequalities (2), (3),

$$\begin{aligned}(n-2)! &= |N_{SG^j_{n-1}}(SG^i_{n-1})|\\ &= |N_{SG^j_{n-1}}(SG^i_{n-1}-(F_i\cup U_i))|+|N_{SG^j_{n-1}}(F_i\cup U_i)|\\ &\le |F_i|+|F_j|\\ &\le 2(2n-7).\end{aligned}$$

This is impossible for $n\ge 6$. Thus $n=4$ or 5, in which case the above inequalities become equalities, and thus

$$|F_i|=|F_j|=2n-7, \tag{4}$$

$$|N_{SG^j_{n-1}}(SG^i_{n-1}-(F_i\cup U_i))|=|F_j|, \tag{5}$$

$$|N_{SG^i_{n-1}}(SG^j_{n-1}-(F_j\cup U_j))|=|F_i|. \tag{6}$$

We can show that

$$N_{SG^j_{n-1}}(SG^i_{n-1} - (F_i \cup U_i)) = F_j. \tag{7}$$

Suppose this is not true, then by (1) and (5), we see that

$$N_{SG^j_{n-1}}(SG^i_{n-1} - (F_i \cup U_i)) = \{u\} \cup (F_j \setminus \{v\}), \tag{8}$$

where u is the unique vertex in U_j and v is some vertex in F_j. Since u is an isolated vertex in $SG^j_{n-1} - F_j$, it has at least two neighbors in F_j (recall that SG_{n-1} is $(n-2)$-regular and $n \geq 4$). Thus there exists a vertex $w \in F_j$ such that w is adjacent with u and $w \neq v$. By Lemma 3, the outside neighbors u' and w' can not be both in SG^i_{n-1}, contradicting (8). Thus (7) is proved.

In the case that $n = 4$ and $|I| = 2$, suppose, without loss of generality, that $I = \{1, 2\}$. By (4), $|F_3| = |F_4| = 2n - 7$. Hence $|F| \geq 2(2n-7) + 2(2n-6) = 6$, contradicting that $|F| < 6(n-3) = 6$. Hence $n = 5$, or $n = 4$ and $|I| = 1$. In these cases, there exists an index $k \notin I$ and $k \neq i, j$ (recall that $|I| \leq 2$). Since (7) says that every faulty vertex in SG^j_{n-1} has its outside neighbor in SG^i_{n-1}, we see that vertices in $N_{SG^j_{n-1}}(SG^k_{n-1})$ are all good. Hence in the case that $n = 5$, by $|N_{SG^j_{n-1}}(SG^k_{n-1} - (F_k \cup U_k))| \geq (n-2)! - |F_k \cup U_k| \geq (n-2)! - (2n-7) - 1 = 2$ and $|U_j| \leq 1$, we see that $SG^k_{n-1} - (F_k \cup U_k)$ has a good neighbor in $SG^j_{n-1} - (F_j \cup U_j)$. In the case that $n = 4$, we must have $U_k = \emptyset$. Otherwise U_k has a unique vertex u by Claim 1. Since $N_{SG^k_{n-1}}(u) \subseteq F_k$, we have $n - 2 = |N_{SG^k_{n-1}}(u)| \leq |F_k| \leq 2n - 7$, and thus $n \geq 5$, contradicting $n = 4$. Similarly, $U_j = \emptyset$. Then by $|N_{SG^j_{n-1}}(SG^k_{n-1} - F_k)| \geq (n-2)! - |F_k| \geq (n-2)! - (2n-7) = 1$, we see that $SG^k_{n-1} - F_k$ has a good neighbor in $SG^j_{n-1} - F_j$. In any case, there is an edge between $SG^k_{n-1} - (F_k \cup U_k)$ and $SG^j_{n-1} - (F_j \cup U_j)$. Symmetrically, it can be shown that there is an edge between $SG^k_{n-1} - (F_k \cup U_k)$ and $SG^i_{n-1} - (F_i \cup U_i)$. Then $SG^i_{n-1} - (F_i \cup U_i)$ is connected to $SG^j_{n-1} - (F_j \cup U_j)$ through $SG^k_{n-1} - (F_k \cup U_k)$ (all the vertices on the path connecting $SG^i_{n-1} - (F_i \cup U_i)$ and $SG^j_{n-1} - (F_j \cup U_j)$ belong to G_1).

Claim 3 is proved.

By Claim 3, we may assume that G_1 is contained in a connected component $\widetilde{C}$ of $SG_n - F$. If $I = \emptyset$, then $V(SG_n - F - \widetilde{C}) \subseteq \bigcup_{i=1}^n U_i$. For each vertex $u \in \bigcup_{i=1}^n U_i$, if its outside neighbor is good, then $d_{SG_n - F}(u) = 1$, otherwise $d_{SG_n - F}(u) = 0$. It follows that $\delta(SG_n - F - \widetilde{C}) \leq 1$ and thus $SG_n - F - \widetilde{C}$ has no cycle, contradicting that F is a cyclic vertex-cut. Hence $1 \leq |I| \leq 2$. Let G_2 be the subgraph of SG_n induced by $\bigcup_{i \in I}(SG^i_{n-1} - F_i)$.

Claim 4. Let C be a connected component of G_2 which contains at least one cycle. Then there is an edge between C and G_1.

Suppose this is not true, then $N_{SG_n}(C) \subseteq F \cup U$, where $U = \bigcup_{j \notin I} U_j$. It follows that for any index $j \notin I$, $N_{SG^j_{n-1}}(C) \subseteq F_j \cup U_j$, and for any index $i \in I$, $N_{SG^i_{n-1}}(C) \subseteq F_i$. As a consequence, using Claim 1,

$$|N_{SG^j_{n-1}}(C)| \leq |F_j| + |U_j| \leq |F_j| + 1 \text{ for } j \notin I, \text{ and} \tag{9}$$

$$|N_{SG^i_{n-1}}(C)| \leq |F_i| \text{ for } i \in I. \tag{10}$$

We can further refine (9) to

$$|N_{SG^j_{n-1}}(C)| \leq |F_j| \text{ for } j \notin I. \tag{11}$$

Suppose (11) is not true, then $N_{SG^j_{n-1}}(C) = F_j \cup U_j$ and $|U_j| = 1$. Let u be the unique vertex in U_j, and v, w be two neighbors of u in F_j. By Lemma 3, the outside neighbors u', v', w' should be in three different copies. But this is impossible since C has non-empty intersection with at most two copies, namely the copies corresponding to I. Thus (11) is proved.

Combining inequalities (10) and (11), we have

$$|N_{SG_n}(C)| \leq |F|. \tag{12}$$

In the following, we count $|N_{SG_n}(C)|$ and derive contradictions to (12).

Case 1. $|I| = 1$.

Suppose, without loss of generality, that $I = \{1\}$. In this case,

$$C \text{ is contained in } SG^1_{n-1} - (F_1 \cup U_1). \tag{13}$$

Let D be a shortest cycle in C, and $u_1, ..., u_6$ be six sequential vertices on D. Since the girth of SG_n is 6 and there is no odd cycle in SG_n, we see that if u_1u_6 is an edge, then no vertices of $\{u_1, ..., u_6\}$ can have a common neighbor outside of D; if u_1u_6 is not an edge, then the only pairs of vertices of $\{u_1, ..., u_6\}$ that may have a common neighbor outside of D are $\{u_1, u_5\}$ and $\{u_2, u_6\}$. Furthermore, we see from Corollary 2 that if u_1, u_5 have a common neighbor outside of D, then u_2, u_6 cannot have common neighbor outside of D, and vice versa. Denote $Y = N_{SG^1_{n-1}}(D)$. By the above analysis, we see that $|Y| \geq 6(n-4) = 6n - 24$ if u_1u_6 is an edge, and $|Y| \geq 4(n-4) + 2(n-3) - 1 = 6n - 23 > 6n - 24$ otherwise.

Let $Y' = Y \cap V(C)$ and $Y'' = Y \setminus Y'$. Clearly, each vertex $y \in Y''$ is in $N_{SG_n}(C)$. For each vertex $y \in Y'$, since its outside neighbor $y' \notin V(C)$ (by (13)), we have $y' \in N_{SG_n}(C)$. Since the outside neighbors of vertices in a same copy are all different, we have

$$|N_{SG_n}(C)| \geq |Y''| + |\{y' \mid y \in Y'\}| + |\{u'_1, ..., u'_6\}| = |Y| + 6 \geq 6n - 18 > |F|,$$

contradicting (12).

Case 2. $|I| = 2$.

Suppose, without loss of generality, that $I = \{1, 2\}$. Then C is contained in $(SG^1_{n-1} - F_1) \cup (SG^2_{n-1} - F_2)$. If C is completely contained in SG^1_{n-1} or SG^2_{n-1}, then a contradiction can be obtained as in Case 1. Thus we assume $V(C) \cap V(SG^i_{n-1}) \neq \emptyset$ for $i = 1, 2$. In this case, there exists an edge of C

between SG^1_{n-1} and SG^2_{n-1}. Let uv be such an edge, and let P be a path of C on 6 vertices which passes through uv. Denote $X_1 = V(P) \cap V(SG^1_{n-1})$ and $X_2 = V(P) \cap V(SG^2_{n-1})$. Then $|X_1|, |X_2| \geq 1$, and thus $|X_1|, |X_2| \leq 5$ by $|X_1| + |X_2| = 6$.

For $i = 1, 2$, let $Y_i = N_{SG^i_{n-1}}(X_i)$. Since SG^i_{n-1} has girth 6, we have

$$|Y_1| + |Y_2| = \begin{cases} 6n - 21 \text{ if one of } X_1 \text{ and } X_2 \text{ is a path on five vertices} \\ \qquad\qquad \text{the ends of which have a common neighbor,} \\ 6n - 20 \quad \text{otherwise.} \end{cases} \tag{14}$$

For $i = 1, 2$, let $n_i = N_{SG^{3-i}_{n-1}}(X_i \cup Y_i) \cap V(C)$. We claim that

$$1 \leq n_i \leq \begin{cases} 1, & \text{for } |X_i| = 1, \\ |X_i| - 1, & \text{for } 2 \leq |X_i| \leq 5. \end{cases} \tag{15}$$

The left hand side $n_i \geq 1$ is obvious because of the edge uv. For the case that $|X_1| = 5$, assume X_1 induces a path $u_1u_2u_3u_4u_5$ in SG^1_{n-1}, where $u = u_5$. Then u_5 has its outside neighbor v in SG^2_{n-1}. By Lemma 3, vertices in $N_{SG^1_{n-1}}(\{u_5, u_4\})$ do not have their outside neighbors in SG^2_{n-1}; at most one vertex in $N_{SG^1_{n-1}}(\{u_3\})$ has its outside neighbor in SG^2_{n-1}; for $i = 1, 2$, at most one vertex in $N_{SG^1_{n-1}}(\{u_i\}) \cup \{u_i\}$ has its outside neighbor in SG^2_{n-1}. Thus $n_i \leq 4 = |X_i| - 1$. The other cases can be proved similarly.

Suppose $|Y_1| + |Y_2| - n_1 - n_2 \geq 6n - 24$. For $i = 1, 2$, denote $Y'_i = Y_i \cap V(C)$ and $Y''_i = Y_i \setminus Y'_i$. Then $Y''_i \subseteq N_{SG_n}(C)$, and each vertex y in $Y'_1 \cup X_1$ (resp. $Y'_2 \cup X_2$) whose outside neighbor y' is not in SG^2_{n-1} (resp. SG^1_{n-1}) has $y' \in N_{SG_n}(C)$. Hence

$$\begin{aligned} |N_{SG_n}(C)| &\geq |Y''_1| + |\{y' \mid y \in Y'_1 \cup X_1\}| - n_1 + |Y''_2| + |\{y' \mid y \in Y'_2 \cup X_2\}| - n_2 \\ &= |Y_1| + |Y_2| - n_1 - n_2 + 6 \geq 6n - 18 > |F| \end{aligned}$$

(observe that since the outside neighbors counted in the above inequality are not in $V(SG^1_{n-1} \cup SG^2_{n-1})$, no two of them can coincide), which contradicts (12).

Next we consider the case that

$$\text{for any edge } uv \text{ between } SG^1_{n-1} \text{ and } SG^2_{n-1} \text{ and any path } P \\ \text{taken as above, } |Y_1| + |Y_2| - n_1 - n_2 \leq 6n - 25. \tag{16}$$

Combining this with (14), we have

$$n_1 + n_2 \geq \begin{cases} 5, \text{ if } |Y_1| + |Y_2| = 6n - 20, \\ 4, \text{ if } |Y_1| + |Y_2| = 6n - 21. \end{cases} \tag{17}$$

If both $|X_1| \geq 2$ and $|X_2| \geq 2$, then by (15) and the fact $|X_1| + |X_2| = 6$, we have $n_1 + n_2 \leq 4$. Then by (17), we see that $n_1 + n_2 = 4$ and $|Y_1| + |Y_2| = 6n - 21$. But by (14), one of $|X_1|$ and $|X_2|$ must be 1, a contradiction. Hence suppose

$$\text{for any edge } uv \text{ between } SG^1_{n-1} \text{ and } SG^2_{n-1} \text{ and any path } P \\ \text{taken as above, } |X_1| = 5,\ |X_2| = 1, \text{ or vice versa.} \tag{18}$$

Suppose $P = u_1u_2u_3u_4u_5u_6$ is such a path, where u_6 is the only vertex in X_2. Then $n_2 = 1$ and $n_1 \geq 3$ by (17). By the deduction in proving (15), we see that besides u_6, the only outside neighbors which can contribute to n_1 are in $N_{SG^1_{n-1}}(u_3)$, $N_{SG^1_{n-1}}(\{u_2\}) \cup \{u_2\}$, $N_{SG^1_{n-1}}(\{u_1\}) \cup \{u_1\}$, and at most one from each of the three sets. If there is a vertex $u_7 \in N_{SG^1_{n-1}}(u_3)$ whose outside neighbor $u_7' \in SG^2_{n-1} \cap V(C)$, then $u_7'u_7u_3u_4u_5u_6$ is a path contradiction (18). If $u_2' \in SG^2_{n-1} \cap V(C)$, then $u_2'u_2u_3u_4u_5u_6$ is a path contradiction (18). Hence in order that $n_1 \geq 3$, there must be a vertex $u_7 \in N_{SG^1_{n-1}}(u_2)$ such that $u_7' \in SG^2_{n-1} \cap V(C)$. By Lemma 3, $u_1' \notin SG^2_{n-1}$. Hence in order that $n_1 \geq 3$, there must be a vertex $u_8 \in N_{SG^1_{n-1}}(u_1)$ such that $u_8' \in SG^2_{n-1} \cap V(C)$. But then $u_8'u_8u_1u_2u_7u_7'$ is a path contradiction (18).

Claim 4 is proved.

As a consequence of Claim 4, every connected component of G_2 which contains a cycle is in $\widetilde{C}$. Thus $SG_n - F - \widetilde{C}$ consists of some vertices in U and some acyclic connected components of G_2. Since every vertex in U has degree at most 1 in $SG_n - F$, we see that $SG_n - F - \widetilde{C}$ does not contain cycle, contradicting that F is a cyclic vertex-cut. The theorem is proved. □

4 Conclusion and Future Work

In this paper, we determined the cyclic vertex-connectivity κ_c of the n-dimensional star graph SG_n. Generally, κ_c is different from κ^2. For SG_n, these two parameters coincide. Is there something deeper under the coincidence? This is the focus of our future research.

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