Bases of Primitive Nonpowerful Sign Patterns

Guanglong Yu[1,2], Zhengke Miao[2,⋆], and Jinlong Shu[1,⋆⋆]

[1] Department of Mathematics, East China Normal University, Shanghai, 200241, China
[2] Department of Mathematics, Xuzhou Normal University, Xuzhou, 221116, China
yglong01@163.com, zkmiao@xznu.edu.cn, jlshu@math.ecnu.edu.cn

Abstract. For a square primitive nonpowerful sign pattern A, the base of A, denoted by $l(A)$, is the least positive integer l such that every entry of A^l is #. In this paper, we consider the base set of the primitive nonpowerful sign pattern matrices. Some bounds on the bases for the sign pattern matrices with base at least $\frac{3}{2}n^2 - 2n + 4$ is given. Some sign pattern matrices with given bases is characterized and some "*gaps*" in the base set are shown.

AMS Classification: 05C50

Keywords: Sign pattern; Primitive; Nonpowerful; Base.

1 Introduction

We adopt the standard conventions, notations and definitions for sign patterns and generalized sign patterns, their entries, arithmetics and powers. The reader who is not familiar with these matters is referred to [5], [11].

The sign pattern of a real matrix A, denoted by $\mathrm{sgn}(A)$, is the $(0, 1, -1)$-matrix obtained from A by replacing each entry by its sign. Notice that in the computations of the entries of the power A^k, an "*ambiguous sign*" may arise when we add a positive sign to a negative sign. So a new symbol "#" has been introduced to denote the ambiguous sign.

For convenience, we call the set $\Gamma = \{0,\ 1,\ -1,\ \#\}$ the generalized sign set and define the addition and multiplication involving the symbol # as follows (the addition and multiplication which do not involve # are obvious):

$$(-1) + 1 = 1 + (-1) = \#,\ a + \# = \# + a = \#\ (for\ all\ a \in \Gamma),$$

$$0 \cdot \# = \# \cdot 0 = 0,\ b \cdot \# = \# \cdot b = \#\ (for\ all\ b \in \Gamma \backslash \{0\}).$$

It is straightforward to check that the addition and multiplication in Γ defined in this way are commutative and associative, and the multiplication is distributive with respect to addition. It is easy to see that a $(0, 1)$-Boolean matrix is a non-negative sign pattern matrix.

⋆ Supported by NSFC(No. 10871166).
⋆⋆ Corresponding author. Supported by the NSFC (No. 10671074, No. 11075057, No. 11071078 and No. 60673048).

W. Wu and O. Daescu (Eds.): COCOA 2010, Part I, LNCS 6508, pp. 114–127, 2010.

Definition 1.1. *Let A be a square sign pattern matrix of order n with powers sequence $A, A^2, \cdots$. Because there are only 4^{n^2} different generalized sign pattern matrices of order n, there must be repetitions in the powers sequence of A. Suppose $A^l = A^{l+p}$ is the first pair of powers that are repeated in the sequence. Then l is called the generalized base (or simply base) of A, and is denoted by $l(A)$. The least positive integer p such that $A^l = A^{l+p}$ holds for $l = l(A)$ is called the generalized period (or simply period) of A, and is denoted by $p(A)$. For a square (0, 1)-Boolean matrix A, $l(A)$ is also known as the convergence index of A, denoted by $k(A)$.*

In 1994, Z. Li, F. Hall and C. Eschenbach [5] extended the concept of the base (or convergence index) and period from nonnegative matrices to sign pattern matrices. They defined powerful and nonpowerful for sign pattern matrices, gave a sufficient and necessary condition that an irreducible sign pattern matrix is powerful and also gave a condition for the nonpowerful case.

Definition 1.2. *A square sign pattern matrix A is powerful if all the powers A^1, A^2, A^3, $\cdots$ are unambiguously defined, namely there is no # in A^k $(k = 1, 2, \cdots)$. Otherwise, A is called nonpowerful.*

If A is a sign pattern matrix, then $|A|$ is the nonnegative matrix obtained from A by replacing a_{ij} with $|a_{ij}|$.

Definition 1.3. *An irreducible $(0, 1)$-Boolean matrix A is primitive if there exists a positive integer k such that all the entries of A^k are non-zero, such least k is called the primitive index of A, denoted by $exp(A) = k$. A square sign pattern matrix A is called primitive if $|A|$ is primitive. The primitive index of A is equal to $exp(|A|)$, denoted by $exp(A)$.*

It is well known that graph theoretical methods are often useful in the study of the powers of square matrices, so we now introduce some graph theoretical concepts.

Definition 1.4. *Let A be a square sign pattern matrix of order n. The associated digraph of A, denoted by $D(A)$, has vertex set $V = \{1,\ 2,\ \cdots,\ n\}$ and arc set $E = \{(i,\ j)|a_{ij} \neq 0\}$. The associated signed digraph of A, denoted by $S(A)$, is obtained from $D(A)$ by assigning sign of a_{ij} to arc (i, j) for all i and j. Let S be a signed digraph of order n and A be a square sign pattern matrix of order n; A is called associated sign pattern matrix of S if $S(A)$=S. The associated sign pattern matrix of a signed digraph S is always denoted by $A(S)$. Note that $D(A) = D(|A|)$, so $D(A)$ is also called the underlying digraph of the associated signed digraph of A or is called the underlying digraph of A simply. We always denote by $D(A(S))$ or $|S|$ simply for the underlying digraph of a signed digraph S. Sometimes, $|A(S)|$ is called the associated or underlying matrix of signed digraph S.*

In this paper, we permit loops but no multiple arcs in a signed digraph. Denote by $V(S)$ the vertex set and denoted by $E(S)$ the arc set for a signed digraph S.

Let $W = v_0e_1v_1e_2 \cdots e_kv_k$ ($e_i = (v_{i-1},\ v_i)$, $1 \leq i \leq k$) be a directed walk of signed digraph S. The sign of W, denoted by $\mathrm{sgn}(W)$, is $\prod_{i=1}^{k} \mathrm{sgn}(e_i)$. Sometimes a directed walk can be denoted simply by $W = v_0v_1 \cdots v_k$, $W = (v_0,\ v_1,\ \cdots,\ v_k)$ or $W = e_1e_2 \cdots e_k$ if there is no ambiguity. The positive integer k is called the length of the directed walk W, denoted by $L(W)$. The definitions of directed cycle and directed path are given in [1]. The length of the shortest directed path from v_i to v_j is called the distance from v_i to v_j in signed digraph S, denoted by $d(v_i, v_j)$. A cycle with length k is always called a k-*cycle*, a cycle with even (odd) length is called an *even cycle* (*odd cycle*). The length of the shortest directed cycle in digraph S is called the *girth* of S usually. When there is no ambiguity, a directed walk, a directed path or a directed cycle will be called a walk, a path or a cycle. A walk is called a *positive walk* if its sign is positive, and a walk is called a *negative walk* if its sign is negative. If p is a positive integer and if C is a cycle, then pC denotes the walk obtained by traversing through C p times. If a cycle C passes through the end vertex of W, $W \bigcup pC$ denotes the the walk obtained by going along W and then going around the cycle C p times; $pC \bigcup W$ is similarly defined. We use the notation $v \xrightarrow{k} u$ ($v \overset{k}{\not\longrightarrow} u$) to denote that there exists (exists no) a directed walk with length k from vertex v to u. For a digraph S, let $R_k(v) = \{u|\ v \xrightarrow{k} u,\ u \in V(S)\}$ and $R_t(v) \xrightarrow{k} u$ mean that there exists a $s \in R_t(v)$ such that $s \xrightarrow{k} u$.

Definition 1.5. *Assume that W_1, W_2 are two directed walks in signed digraph S, they are called a pair of $SSSD$ walks if they have the same initial vertex, the same terminal vertex and the same length, but they have different signs.*

From [5] or [11], we know that a signed digraph S is powerful if and only if there is no pair of $SSSD$ walks in S.

Definition 1.6. *A strongly connected digraph G is primitive if there exists a positive integer k such that for all vertices $v_i, v_j \in V(G)$ (not necessarily distinct), there exists a directed walk of length k from v_i to v_j. The least such k is called the primitive index of G, and is denoted by $\exp(G)$. Let G be a primitive digraph. The least l such that there is a directed walk of length t from v_i to v_j for any integer $t \geq l$ is called the local primitive index from v_i to v_j, denoted by $\exp_G(v_i, v_j) = l$. Similarly, $\exp_G(v_i) = \max_{v_j \in V(G)} \{\exp_G(v_i, v_j)\}$ is called the local primitive index at v_i, so $\exp(G) = \max_{v_i \in V(G)} \{\exp_G(v_i)\}$.*

For a square sign pattern A, let $W_k(i,j)$ denote the set of walks of length k from vertex i to vertex j in $S(A)$; notice that the entry $(A^k)_{ij}$ of A^k satisfies $(A^k)_{ij} = \sum_{W \in W_k(i,j)} \mathrm{sgn}(W)$. Then we have

(1) $(A^k)_{ij} = 0$ if and only if there is no walk of length k from i to j in $S(A)$ (i.e., $W_k(i,j) = \phi$);

(2) $(A^k)_{ij} = 1$ (or -1) if and only if $W_k(i,j) \neq \phi$ and all walks in $W_k(i,j)$ have the same sign 1 (or -1);

(3) $(A^k)_{ij} = \#$ if and only if there is a pair of $SSSD$ walks of length k from i to j.

So the associated signed digraph can be used to study the properties of the powers sequence of a sign pattern matrix, and the signed digraph is taken as the tool in this paper. From the relation between sign pattern matrices and signed digraphs, we know that it is logical to define a sign pattern A to be primitive and to define $\exp(A) = \exp(D(A)) = \exp(|A|)$ if A is primitive.

Definition 1.7. *A signed digraph S is primitive and nonpowerful if there exists a positive integer l such that for any integer $t \geq l$, there is a pair of $SSSD$ walks of length t from any vertex v_i to any vertex v_j $(v_i, v_j \in V(S))$. Such least integer l is called the base of S, denoted by $l(S)$. Let S be a primitive nonpowerful signed digraph of order n. Let $u, v \in V(S)$. The local base from u to v, denoted by $l_S(u,v)$, is defined to be the least integer k such that there is a pair of $SSSD$ walks of length t from u to v for any integer $t \geq k$. The local base at a vertex $u \in V(S)$ is defined to be $l_S(u) = \max_{v \in V(S)} \{l_S(u,v)\}$. So*

$$l(S) = \max_{u \in V(S)} l_S(u) = \max_{u,v \in V(S)} l_S(u,v).$$

Therefore, a sign pattern A is primitive nonpowerful if and only if $S(A)$ is primitive nonpowerful, and the base $l(A) = l(S(A))$ is the least positive integer l such that every entry of A^l is #.

From [5], we know that $l(A) = l(|A|)$ for a powerful sign pattern A. So $l(A) = \exp(A)$ if A is a primitive powerful sign pattern. Moreover, if A is a powerful sign pattern, then A is primitive if and only if every real matrix B in $Q(A)$ ($Q(A) = \{B|$ real matrix B with pattern $A\}$) is primitive. Thus, when A is a primitive, powerful sign pattern, every real matrix B with pattern A is primitive, has $D(B) = D(A)$, and has $\exp(B) = \exp(D(|A|))$. But we say that the result about the base of a powerful sign pattern fails to hold for a nonpowerful sign patterns, see an example as follow:

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Note that A is trivially primitive since $D(A)$ has all possible arcs, that $l(|A|) = 1$, that A^2 contains no 0, but $l(A) = 3$. In particular, a real matrix B with sign pattern A can behave very differently from A:

$$B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

gives $B^4 = -4I$, which means B is not primitive in the usual sense. So the treatments of the bases about the nonpowerful sign patterns require greater care than the treatments for the powerful sign patterns.

Let S be a primitive nonpowerful signed digraph of order n and $V(S) = \{1, 2, \cdots, n\}$; for convenience, the vertices can be ordered so that $l_S(1) \leq l_S(2) \leq \cdots \leq l_S(n)$. We call $l_S(k)$ the kth local base of S. Thus $l(S) = l_S(n)$ and it is easy to see that $l_S(k)$ is the smallest integer l such that there are k all $\sharp$ rows in $[A(S)]^l$. Similarly, $\exp_G(k)$ is defined to be the smallest integer l such that there are l all "1" rows in $|A(G)|^l$ for a primitive digraph G.

Primitivity, base, local base, extremal patterns and other properties of powers sequence of a square sign pattern matrix are of great significance. The bases of sign patterns are closely related to many other problems in various areas of pure and applied mathematics (see [3], [4], [6], [7], [9], [12]). In practice, we consider the *memoryless communication system* [6] in communication field, which is depicted as a digraph D of order n. If D is primitive, the least time t such that each vertex in D receive the n pieces of different information from any vertex is equal to the index of D. If D is a primitive non-powerful signed digraph, the least time t such that each vertex in D receive the n pieces of ambiguous information from any vertex is just equal to the base of D. So studying the bases of the primitive non-powerful signed digraphs is very useful in information communication field, and hence studying the bases of the primitive non-powerful signed digraphs is also very useful for net works and theoretical computer science.

This paper is organized as follows: Section 1 introduces the basic ideas of patterns and their supports. Section 2 introduces series of working lemmas. Section 3 and Section 4 characterize the cycle properties in the associated signed digraphs and some bounds about the bases for the sign pattern matrices with base at least $\frac{3}{2}n^2 - 2n + 4$. Section 5 characterizes some sign pattern matrices with given bases and shows that there are some "*gaps*" in the base set.

2 Preliminaries

Let S be a strongly connecte digraph of order n and $C(S)$ denote the set of all cycle lengths in S.

Definition 2.1. *Let $\{s_1, s_2, \cdots, s_\lambda\}$ be a set of distinct positive integers, gcd$(s_1, s_2, \cdots, s_\lambda) = 1$. The Frobenius number of $s_1, s_2, \cdots, s_\lambda$, denoted by $\phi(s_1, s_2, \cdots, s_\lambda)$, is the smallest positive integer m such that $k = \sum_{i=1}^{\lambda} a_i s_i$ for any positive integer $k \geq m$ where a_i $(i = 1, 2, \cdots, \lambda)$ is non-negative integer.*

Lemma 2.2. **([6])** *If gcd$(s_1, s_2) = 1$, then $\phi(s_1, s_2) = (s_1 - 1)(s_2 - 1)$.*

From Definition 2.1, it is easy to know that $\phi(s_1, s_2, \cdots, s_\lambda) \leq \phi(s_i, s_j)$ if there exist $s_i, s_j \in \{s_1, s_2, \cdots, s_\lambda\}$ such that $\gcd(s_i, s_j) = 1$. So, if $\min\{s_i : 1 \leq i \leq \lambda\} = 1$, then $\phi(s_1, s_2, \cdots, s_\lambda) = 0$.

Lemma 2.3. **([4])** *A Boolean matrix A is primitive if and only if $D(A)$ is strongly connected and* $\gcd(p_1, p_2, \cdots, p_t) = 1$ *where* $C(D(A)) = \{p_1, p_2, \cdots, p_t\}$.

Definition 2.4. *For a primitive digraph S, suppose $C(S) = \{p_1, p_2, \ldots, p_u\}$. Let $d_{C(S)}(v_i, v_j)$ denote the length of the shortest walk from v_i to v_j which meets at least one p_i-cycle for each i $(i = 1, 2, \cdots, u)$. Such shortest directed walk is called a $C(S)$-walk from v_i to v_j. Further, $d_{C(S)}(v_i)$, $d_1(C(S))$ and $d(C(S))$ are defined as follows: $d_{C(S)}(v_i) = \max\{d_{C(S)}(v_i, v_j): v_j \in V(S)\}$, $d(C(S)) = \max\{d_{C(S)}(v_i, v_j): v_i, v_j \in V(S)\}$, $d_i(C(S))$ $(1 \le i \le n)$ is the ith smallest one in $\{d_{C(S)}(v_i)|1 \le i \le n\}$, $d_n(C(S)) = d(C(S))$. In particular, if $C(S) = \{p, q\}$, $d(C(S))$ can be simply denoted by $d\{p, q\}$.*

Lemma 2.5. ([2]) *Let S be a primitive digraph of order n and $C(S) = \{p_1, p_2, \ldots, p_u\}$. Then $exp(v_i, v_j) \le d_{C(S)}(v_i, v_j) + \phi(p_1, p_2, \ldots, p_u)$ for v_i, $v_j \in V(S)$. We have $\exp(S) \le d(C(S)) + \phi(p_1, p_2, \ldots, p_u)$ furthermore.*

Lemma 2.6. ([2]) *Let S be a primitive digraph of order n whose girth is s. Then $exp_S(k) \le s(n-2) + k$ for $1 \le k \le n$.*

Lemma 2.7. ([8]) *Let S be a primitive digraph of order n and $|C(S)| \ge 3$. Then $exp_S(k) \le \lfloor \frac{1}{2}(n-2)^2 \rfloor + k$ for $1 \le k \le n$.*

Lemma 2.8. *Let D be a primitive digraph of order n which has a $s-$cycle C, $v \in V(C)$, and $|R_1(v)| \ge 2$. Then $exp(1) \le exp(v) \le 1 + s(n-2)$.*

Proof. We can take w, $z \in R_1(v)$ such that $(v, w) \in E(C)$ and $(v, z) \notin E(C)$ because of $v \in V(C)$ and $|R_1(v)| \ge 2$. We consider strongly connected digraph D^s (where $A(D^s) = [A(D)]^s$) in which the arc corresponds to the walk of length s in S. In D^s, w has a loop and there is arc (w, z). Thus $R_1(v) \xrightarrow{n-2} u$ for any vertex u in D^s. So there exists a walk of length $1 + s(n-2)$ from vertex v to any vertex u in D. □

Lemma 2.9. ([11]) *Let S be a primitive nonpowerful signed digraph. Then S must contain a p_1-cycle C_1 and a p_2-cycle C_2 satisfying one of the following two conditions:*

(1) p_i is odd, p_j is even and $sgnC_j = -1$ $(i, j = 1, 2; i \neq j)$.
(2) p_1 and p_2 are both odd and $sgnC_1 = -sgnC_2$.

C_1, C_2 satisfying condition (1) or (2) are always called *a distinguished cycle pair.* It is easy to prove that $W_1 = p_2C_1$ and $W_2 = p_1C_2$ have the same length p_1p_2 but different signs if p_1-cycle C_1 and p_2-cycle C_2 are a distinguished cycle pair, namely $(\text{sgn}C_1)^{p_2} = -((\text{sgn}C_2)^{p_1})$.

Lemma 2.10. ([12]) *Let S be a primitive signed digraph. Then S is nonpowerful if and only if S contains a distinguished cycle pair.*

Lemma 2.11. ([12]) *Let S be a primitive nonpowerful signed digraph of order n and $C(S) = \{p_1, p_2, \ldots, p_m\}$. If the cycles in S with the same length have the same sign, p_1-cycle C_1 and p_2-cycle C_2 form a distinguished cycle pair, then*

(i) $l_S(v_i, v_j) \le d_{C(S)}(v_i, v_j) + \phi(p_1, p_2, \ldots, p_m) + p_1p_2,\ v_i, v_j \in V(S)$.
(ii) $l_S(v_i) \le d_{C(S)}(v_i) + \phi(p_1, p_2, \ldots, p_m) + p_1p_2$.
(iii) $l(S) \le d(C(S)) + \phi(p_1, p_2, \ldots, p_m) + p_1p_2$.

Lemma 2.12. **([10])***Let S be a primitive nonpowerful signed digraph of order n and $u \in V(S)$. If there exists a pair of SSSD walks with length r from u to u, then $l_S(u) \leq exp_S(u) + r$.*

Lemma 2.13. **([10])***Let S be a primitive nonpowerful signed digraph of order n. Then we have $l_S(k) \leq l_S(k-1) + 1$ for $2 \leq k \leq n$.*

Let D_1 consist of cycle $(v_n, v_{n-1}, \cdots, v_2, v_1, v_n)$ and arc (v_1, v_{n-1}) and $D_2 = D_1 \bigcup \{(v_2, v_n)\}$. Then we have the following lemmas 2.14, 2.15.

Lemma 2.14. **([11])***Let S_2 be a nonpowerful signed digraph of order $n \geq 3$ with D_2 as its underlying digraph. Then we have*

(1) if the (only) two cycles of length $n-1$ of S_2 have different signs, then $l(S_2) \leq n^2 - n + 2$;

(2) if the two cycles of length $n-1$ of S_2 have the same sign, then $l(S_2) = 2(n-1)^2 + (n-1)$.

Lemma 2.15. **([11])***Let A be an irreducible generalized sign pattern matrix of order $n \geq 3$. Then*

(i) $l(A) \leq 2(n-1)^2 + n$; (1)

(ii) equality holds in (1) if and only if A is a nonpowerful sign pattern matrix and the associated digraph $D(A)$ of A is isomorphic to D_1;

(iii) for each integer k with $2n^2 - 4n + 5 < k < 2n^2 - 3n + 1$, there is no irreducible generalized sign pattern matrix A of order n with $l(A) = k$.

3 Cycle Properties

Theorem 3.1. *Let S be a primitive nonpowerful signed digraph of order $n \geq 6$ whose underlying digraph is $|S|$. If $|C(S)| \geq 3$, then $l(S) \leq \frac{3}{2}n^2 - 2n + 3$.*

Proof. By Lemma 2.9, there exists a distinguished cycle pair p_1-cycle C_1 and p_2-cycle C_2 in S, p_1C_2 and p_2C_1 have different signs.

Case 1. C_1, C_2 have no common vertex.

Then $p_1 + p_2 \leq n$. Suppose $p_1 \leq \frac{n}{2}$ for convenience, Q_1 is one of the shortest walks with length q_1 from C_1 to C_2, $\{v_1\} = V(Q_1) \bigcap V(C_1), \{v_2\} = V(Q_1) \bigcap V(C_2)$, and Q_2 is one of the shortest walks with length q_2 from v_2 to v_1, then $q_1 \leq n - p_1 - p_2 + 1, q_2 \leq n - 1$, $p_2C_1 \bigcup Q_1 \bigcup Q_2$ and $Q_1 \bigcup p_1C_2 \bigcup Q_2$ are a pair of $SSSD$ walks with length $p_1p_2 + q_1 + q_2$ from v_1 to v_1.

Note that

$$p_1p_2 + q_1 + q_2 \leq p_1p_2 + 2n - p_1 - p_2 = (p_1 - 1)(p_2 - 1) + 2n - 1$$

$$\leq [\frac{1}{2}(p_1 + p_2 - 2)]^2 + 2n - 1 \leq [\frac{1}{2}(n-2)]^2 + 2n - 1 = \frac{n^2}{4} + n,$$

and $\exp(v_1) \le p_1(n-2)+1$ by Lemma 2.8, then

$$l_S(1) \le l_S(v_1) \le \exp_S(v_1) + p_1p_2 + q_1 + q_2 \le \frac{n}{2}(n-2) + 1 + \frac{n^2}{4} + n = \frac{3n^2}{4} + 1$$

and $l_S(n) \le l_S(1) + n - 1 \le \dfrac{3n^2}{4} + n$ by Lemmas 2.12, 2.13.

Case 2. C_1, C_2 have common vertices.

Subcase 2.1. $p_1 = p_2$. It is easy to see that p_1 is odd.

1° $p_1 = n$, let $v_1 \in V(S)$, $\exp_S(v_1) = \exp_S(1)$. The underlying digraph $|S|$ is not isomorphic to D_1 or D_2 because $|C(S)| \ge 3$. Thus the girth s of S is at most $n-2$. By Lemma 2.8, then $\exp_S(1) = \exp(v_1) \le s(n-2)+1 \le (n-2)^2+1$. Note that C_1 and C_2 form a pair of $SSSD$ walks from v_1 to itself now, by Lemmas 2.12, 2.13, then

$$l_S(1) \le l_S(v_1) \le n+(n-2)^2+1 = n^2-3n+5,\ l_S(n) \le l_S(1)+n-1 \le n^2-2n+4.$$

2° $p_1 \le n-1$, suppose $v_1 \in V(C_1)\bigcap V(C_2)$ and $|R_1(v_1)| \ge 2$. By Lemmas 2.8, 2.12, 2.13, then

$$\exp_S(v_1) \le p_1(n-2)+1 \le n^2-3n+3,\ l_S(1) \le l_S(v_1) \le p_1+\exp_S(v_1) \le n^2-2n+2$$

and $l_S(n) \le l_S(1) + n - 1 \le n^2 - n + 1$.

Subcase 2.2. $\mathrm{Min}(p_1, p_2) = p_1 \le n-2$.

Suppose $V(C_1)\bigcap V(C_2) = \{v_1,\ v_2,\ \cdots, v_t\}$ and $\exp_D(u) = \exp_D(1)$. Because of $|C(S)| \ge 3$, so $\exp_S(u) \le \lfloor\frac{1}{2}(n-2)^2\rfloor + 1$ by Lemma 2.7. Let $q_i = d(u, v_i)$, $1 \le i \le t$ and suppose $q_1 = \min\limits_{1\le i\le t}\{q_i\}$ for convenience, then $q_1 \le n - (p_1 + p_2 - t) + p_2 - t = n - p_1$. So there exists a pair of $SSSD$ walks with length $q_1 + d(v_1, u) + p_1p_2$ from u to u. Because of

$$d(v_1, u) \le n-1,\ q_1 + d(v_1, u) + p_1p_2 \le 2n - 1 + p_1(p_2 - 1) \le 2n - 1 + (n-2)(n-1) \le n^2 - n + 1,$$

so

$$l_S(1) \le l_S(u) \le q_1+d(v_1,u)+p_1p_2+\exp_S(u) \le \lfloor\frac{1}{2}(n-2)^2\rfloor+1+n^2-n+1 \le \frac{3n^2}{2}-3n+4$$

by Lemma 2.12 and $l_S(n) \le l_S(1) + n - 1 \le \dfrac{3n^2}{2} - 2n + 3$ by Lemma 2.13.

Subcase 2.3. $\{p_1, p_2\} = \{n-1, n\}$.

Let $C_1 = C_{n-1},\ C_2 = C_n$. Suppose $\exp_S(u) = \exp_D(1)$ for convenience, then $\exp_S(u) \le \frac{1}{2}(n-2)^2 + 1$ by Lemma 2.7. Because there exists a pair of $SSSD$ walks with length $n(n-1)$ from u to u if $u \in V(C_{n-1})$, by Lemmas 2.12 and 2.13, so

$$l_S(1) \le l_S(u) \le \frac{(n-2)^2}{2} + 1 + n(n-1) = \frac{3n^2-6n}{2} + 3,\ l_S(n) \le \frac{3n^2}{2} - 2n + 2.$$

If $u \notin V(C_{n-1})$, there exists a vertex $v \in V(S)$ towards u such that (v,u) is an arc in S and $v \in V(C_{n-1})$. Then $\exp_S(v) \le \exp_S(u) + 1 \le \frac{1}{2}(n-2)^2 + 2$. Because there exists a pair of $SSSD$ walks of length $n(n-1)$ from v to v, by Lemmas 2.12 and 2.13, so

$$l_S(1) \le l_S(v) \le \frac{(n-2)^2}{2} + 2 + n(n-1) = \frac{3n^2-6n}{2} + 4,\ l_S(n) \le \frac{3n^2}{2} - 2n + 3. \quad \square$$

Corollary 3.2. *Let S be a primitive nonpowerful signed digraph of order $n \ge 6$. Then $|C(S)| = 2$ if $l(S) \ge \frac{3}{2}n^2 - 2n + 4$.*

Theorem 3.3. *Let S be a primitive nonpowerful signed digraph of order $n \ge 6$. Cycle C_1 with length p_1 and cycle C_2 with length p_2 form a distinguished cycle pair $(p_1 \le p_2)$. If $p_1 + p_2 \le n$, then $l(S) \le \frac{3}{4}n^2 + n$.*

Proof. **Case 1.** C_1 and C_2 have no common vertex.

As proved in case 1 of Theorem 3.1, $l_S(n) \le \frac{3}{4}n^2 + n$ can be obtained.

Case 2. C_1 and C_2 have at least one common vertex.

Subcase 2.1. If $p_1 = p_2$, then $p_1 \le \frac{1}{2}n$. Let $v_1 \in V(C_1) \bigcap V(C_2)$ and $|R_1(v_1)| \ge 2$. By Lemmas 2.8, 2.12, 2.13, then $\exp_S(v_1) \le p_1(n-2) + 1 \le \frac{1}{2}n^2 - n + 1$, $l_S(1) \le l_S(v_1) \le p_1 + \exp_S(v_1) \le \frac{1}{2}n^2 - \frac{1}{2}n + 1$ and $l_S(n) \le l_S(1) + n - 1 \le \frac{1}{2}n^2 + \frac{1}{2}n$.

Subcase 2.2. If $p_1 < p_2$, then $p_1 < \frac{1}{2}n$. Let $v_1 \in V(C_1) \bigcap V(C_2)$ and $|R_1(v_1)| \ge 2$. Similar to Subcase 2.1, note that there is a pair of $SSSD$ walks with length p_1p_2 from v_1 to itself and $p_1p_2 \le (\frac{p_1+p_2}{2})^2$, we get $l_S(v_1) < \frac{3}{4}n^2 - n + 1$ and $l_S(n) < \frac{3}{4}n^2$. $\square$

Corollary 3.4. *Let S be a primitive nonpowerful signed digraph of order $n \ge 6$. Cycle C_1 with length p_1 and cycle C_2 with length p_2 form a distinguished cycle pair $(p_1 \le p_2)$. Then $p_1 + p_2 > n$ if $l(S) \ge \frac{3}{4}n^2 + n + 1$.*

Theorem 3.5. *Let S be a primitive nonpowerful signed digraph with order $n \ge 3$. If there exist two cycles with the same length but different signs, then we have $l(S) \le n^2$.*

Proof. Let C_1 and C_2 be two cycles such that $L(C_1) = p = L(C_2)$ but sgn(C_1)=-sgn(C_2).

Case 1. C_1 and C_2 have at least one common vertex.

Let $v_1 \in V(C_1) \bigcap V(C_2)$ and $|R_1(v_1)| \ge 2$. Note that $p \le n$ and there is a pair of $SSSD$ walks with length p from v_1 to itself, similar to the proof in Subcase 2.1 of Theorem 3.3, we get $l_S(v_1) \le n^2 - n + 1$ and $l_S(n) \le n^2$.

Case 2. C_1 and C_2 have no common vertex. Then $p \leq \frac{n}{2}$.

Let Q_1 be one of the shortest walks with length q_1 from C_1 to C_2, $V(Q_1) \bigcap V(C_1) = \{v_1\}$, $V(Q_1) \bigcap V(C_2) = \{v_2\}$ and Q_2 is one of the shortest walks with length q_2 from v_2 to v_1. Then $q_1 \leq n - 2p + 1, q_2 \leq n - 1$. $C_1 + Q_1 + Q_2$ and $C_2 + Q_1 + Q_2$ are a pair of $SSSD$ walks with length $p + q_1 + q_2$ from v_1 to v_1. Similar to Subcase 2.1 of Theorem 3.3, we get $\exp_S(v_1) \leq p(n-2) + 1, l_S(v_1) \leq \frac{n^2}{2} + \frac{n}{2} + 1$ and $l_S(n) \leq \frac{n^2}{2} + \frac{3n}{2}$.

To sum up, the theorem is proved. □

Corollary 3.6. *Let S be a primitive nonpowerful signed digraph of order $n \geq 6$. Then any two cycles with the same length have the same sign if $l(S) \geq n^2 + 1$.*

Theorem 3.7. *Let A be a primitive nonpowerful square sign pattern matrix with order $n \geq 6$. If $l(A) \geq \frac{3}{2}n^2 - 2n + 4$, then we have the results as follows:*

(i)$|C(S(A))| = 2$. Suppose $C(S(A)) = \{p_1, p_2\}$ $(p_1 < p_2)$, then $\gcd(p_1, p_2) = 1, p_1 + p_2 > n$;

(ii) In $S(A)$, all p_1-cycles have the same sign, all p_2-cycles have the same sign, and every pair of p_1-cycle and p_2-cycle form a distinguished cycle pair.

Proof. The theorem follows from Corollaries 3.2, 3.4, 3.6. □

4 Bounds of the Bases

Lemma 4.1. *Let A be a primitive nonpowerful square sign pattern matrix with order $n \geq 6$. If $C(S(A)) = \{p_1, p_2\}$ $(p_1 < p_2, p_1 + p_2 > n)$, all p_1-cycles have the same sign, all p_2-cycles have the same sign in $S(A)$, then $l(A) \leq 2n - 1 + (2p_1 - 1)(p_2 - 1)$.*

Especially, if $p_1 = n - 1,\ p_2 = n$, then $l(A) \leq 2n^2 - 3n + 2$.

Proof. Let C_1, C_2 form a distinguished cycle pair and $L(C_1) = p_1$, $L(C_2) = p_2$ in $S(A)$. Suppose $V(C_1) \bigcap V(C_2) = \{v_1, v_2, \cdots, v_t\}$. Let $d_0 = \min\limits_{1 \leq i \leq t} \{d(x, v_i)\}$ for $x \in V(S(A))$. Thus $d_0 \leq n - (p_1 + p_2 - t) + p_2 - t = n - p_1$. Let $d_0 = d(x, v_k)$ $(1 \leq k \leq t)$, then $d(v_k, y) \leq n - 1$ for $y \in V(S(A))$. So there exist a pair of $SSSD$ walks of length $d_0 + \phi(p_1, p_2) + p_1p_2 + d(v_k, y)$ $(d_0 + \phi(p_1, p_2) + p_1p_2 + d(v_k, y) \leq 2n - 1 + (2p_1 - 1)(p_2 - 1))$ from x to y. Because x, y are arbitrary, thus $l(S(A)) \leq 2n - 1 + (2p_1 - 1)(p_2 - 1)$. Therefore, the lemma is proved. □

Theorem 4.2. *Let A be a primitive nonpowerful square signed pattern matrix with order $n \geq 10$. If $l(A) \geq \frac{3}{2}n^2 - 2n + 4$ and the girth of $S(A)$ is at most $n - 3$, then $l(A) \leq 2n^2 - 7n + 6$.*

Proof. Note that $\min C(S(A)) \leq n - 3$, then the theorem follows from Theorem 3.7 and Lemma 4.1. □

5 Gaps and Some Digraphs with Given Bases

Theorem 5.1. *Let* $D_{k,i}$ *consist of cycle* $C_n = (v_1, v_n, v_{n-1}, v_{n-2}, \ldots, v_2, v_1)$ *and arcs* $(v_1, v_{n-k}), (v_2, v_{n-k+1}), \ldots, (v_i, v_{n-k+i-1})$ $(1 \le i \le \min\{k+1, n-k-1\})$ *(see Fig. 1) where* $\gcd(n, n-k) = 1$*. Let* $S_{k,i}$ *be a primitive nonpowerful signed digraph with underlying digraph* $D_{k,i}$ $(1 \le i \le \min\{k+1, n-k-1\})$*. If all* $(n-k)$*-cycles have the same sign in* $S_{k,i}$*, then* $l(S_{k,i}) = (2n-2)(n-k)+1-i+n$*.*

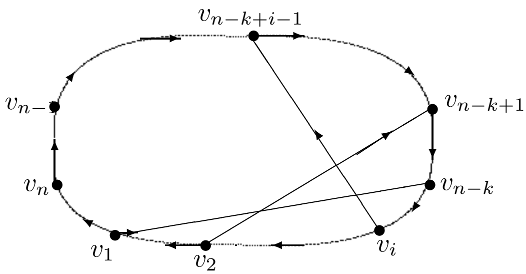

Fig. 1. $D_{k,i}$

Proof. Every pair of $(n-k)$-cycle and n-cycle form a distinguished cycle pair because $S_{k,i}$ is a primitive nonpowerful signed digraph.

Case 1. $i - 1 < k$. Then $n - k + i - 1 < n$.

Now $d(C(S_{k,i})) = d_{C(S_{k,i})}(v_n, v_{n-k+i}) = n + k - i$, by Lemma 2.11, then

$$l(S_{k,i}) \le d(C(S_{k,i})) + \phi(n, n-k) + n(n-k) = (2n-2)(n-k) + 1 - i + n.$$

We assert $l(S_{k,i}) = (2n-2)(n-k) + 1 - i + n$. Now we prove that there is no pair of $SSSD$ walks of length $(2n-2)(n-k) - i + n$ from v_n to v_{n-k+i}.

Otherwise, suppose W_1, W_2 are a pair of $SSSD$ walks with length $(2n-2)(n-k) - i + n$ from v_n to v_{n-k+i}. Let P be the unique path from v_n to v_{n-k+i} on cycle C_n and $W = P \bigcup C_n$. Then W_j $(j = 1, 2)$ must consist of W, some n-cycles and some $(n-k)$-cycles, namely, $|W_j| = (2n-2)(n-k) - i + n = n + k - i + a_i n + b_i(n-k)$ $(a_j, b_j \ge 0,\ j = 1, 2)$. Because of $\gcd(n, n-k) = 1$, so $(a_1 - a_2)n = (b_2 - b_1)(n-k)$, $n|(b_2 - b_1)$, $(n-k)|(a_1 - a_2)$, and then $b_2 - b_1 = nx$, $a_1 - a_2 = (n-k)x$ for some integer x.

We assert $x = 0$. If $x \ge 1$, then $b_2 \ge n$. Thus we have

$$(2n-2)(n-k) - i + n = n + k - i + a_2 n + (b_2 - n)(n-k) + n(n-k)$$

and $\phi(n, n-k) - 1 = a_2 n + (b_2 - n)(n-k)$, which contradicts the definition of $\phi(n, n-k)$. In a same way, we can get analogous contradiction when $x \le -1$. The assertion $x = 0$ is proved. So W_1, W_2 have the same sign because $b_2 = b_1$, $a_1 = a_2$ and all $(n-k)-$cycles have the same sign. This contradicts that W_1, W_2 are a pair of $SSSD$ walks. Thus there is no pair of $SSSD$ walks of length $(2n-2)(n-k) - i + n$ from v_n to v_{n-k+i}, and so

$$l(S_{k,i}) = l_{S_{k,i}}(v_n, v_{n-k+i}) = (2n-2)(n-k) + 1 + n - i.$$

Case 2. $k = i - 1$. Then $n - k + i - 1 = n$.

Now $d(C(S_{k,i})) = d_{C(S_{k,i})}(v_n, v_1) = n - 1$. As the proof of case 1, we can prove

$$l(S_{k,i}) = l_{S_{k,i}}(v_n, v_1) = (2n - 2)(n - k) + 1 + n - i. \qquad \square$$

Suppose that n is odd, let $\mathscr{L}$ consist of cycle $C_n = (v_1, v_n, v_{n-1}, v_{n-2}, v_{n-3}, \ldots, v_2, v_1)$ $(n \geq 6)$ and arcs (v_1, v_{n-2}), (v_3, v_n). Let F consist of cycle $C_{n-1} = (v_1, v_n, v_{n-1}, v_{n-3}, v_{n-4}, \ldots, v_2, v_1)$ $(n \geq 6)$ and arcs (v_1, v_{n-2}), (v_{n-2}, v_{n-3}). Let F_1 consist of cycle $(v_1, v_{n-1}, v_{n-2}, \ldots, v_2, v_1)$ and arcs (v_1, v_{n-2}), (v_2, v_n), (v_n, v_{n-1}). Let F_2 consist of cycle $(v_1, v_n, v_{n-2}, v_{n-3}, v_{n-4}, \ldots, v_2, v_1)$ and arcs (v_1, v_{n-2}), (v_n, v_{n-1}), (v_{n-1}, v_{n-3}). Let F_3 consist of cycle $(v_1, v_{n-2}, v_{n-3}, v_{n-4}, \ldots, v_2, v_1)$ and arcs (v_1, v_{n-1}), (v_{n-1}, v_{n-2}), (v_1, v_n), (v_n, v_{n-2}). Let $F_i^{'}$ $(2 \leq i \leq n-3)$ consist of cycle $(v_1, v_{n-1}, v_{n-2}, \ldots, v_2, v_1)$ and arcs (v_1, v_{n-2}), (v_{i+1}, v_n), (v_n, v_{i-1}). Let F_4 consist of cycle $(v_1, v_{n-1}, v_{n-2}, \ldots, v_2, v_1)$ and arcs (v_1, v_{n-2}), (v_1, v_n), (v_n, v_{n-3}). Let F_5 consist of cycle $(v_1, v_{n-1}, v_{n-2}, \ldots, v_2, v_1)$ and arcs (v_1, v_{n-2}), (v_2, v_n), (v_n, v_{n-2}). Let F_6 consist of cycle $(v_1, v_{n-1}, v_{n-2}, \ldots, v_2, v_1)$ and arcs (v_1, v_n), (v_n, v_{n-3}), (v_2, v_{n-1}). Let F_7 consist of cycle $(v_1, v_{n-1}, v_{n-2}, \ldots, v_2, v_1)$ and arcs (v_1, v_{n-2}), (v_3, v_n), (v_n, v_{n-1}). Let $\mathscr{B}_1$ consist of cycle $C_n = (v_1, v_n, v_{n-1}, \ldots, v_2, v_1)$ and arcs (v_1, v_{n-3}), (v_3, v_{n-1}). Let $\mathscr{B}_2$ consist of cycle $C_n = (v_1, v_n, v_{n-1}, \ldots, v_2, v_1)$ and arcs (v_1, v_{n-3}), (v_4, v_n). Let $\mathscr{B}_3$ consist of cycle $C_n = (v_1, v_n, v_{n-1}, \ldots, v_2, v_1)$ and arcs (v_1, v_{n-3}), (v_2, v_{n-2}), (v_4, v_n). Let $\mathscr{B}_3$ consist of cycle $C_n = (v_1, v_n, v_{n-1}, \ldots, v_2, v_1)$ and arcs (v_1, v_{n-3}), (v_2, v_{n-2}), (v_4, v_n). Let $\mathscr{B}_4$ consist of cycle $C_n = (v_1, v_n, v_{n-1}, \ldots, v_2, v_1)$ and arcs (v_1, v_{n-3}), (v_3, v_{n-1}), (v_4, v_n).

Suppose that n is odd, let $\mathscr{T}$ be a primitive nonpowerful signed digraph with underlying digraph $\mathscr{L}$, all $(n-2)$-cycles have the same sign in $\mathscr{T}$. Let $\mathscr{S}_0$ be a primitive nonpowerful signed digraph with underlying digraph F, all $(n-1)$-cycles have the same sign, all $(n-2)$-cycles have the same sign in $\mathscr{S}_0$. Let $\mathscr{S}_1$ be a primitive nonpowerful signed digraph with underlying digraph F_1, all $(n-1)$-cycles have the same sign, all $(n-2)$-cycles have the same sign in $\mathscr{S}_1$. Let $\mathscr{S}_2$ be a primitive nonpowerful signed digraph with underlying digraph F_2, all $(n-1)$-cycles have the same sign, all $(n-2)$-cycles have the same sign in $\mathscr{S}_2$. Let $\mathscr{S}_3$ be a primitive nonpowerful signed digraph with underlying digraph F_3, all $(n-1)$-cycles have the same sign, all $(n-2)$-cycles have the same sign in $\mathscr{S}_3$. Let $\mathscr{S}_4$ be a primitive nonpowerful signed digraph with underlying digraph F_4, all $(n-1)$-cycles have the same sign, all $(n-2)$-cycles have the same sign in $\mathscr{S}_4$. Let $\mathscr{S}_5$ be a primitive nonpowerful signed digraph with underlying digraph F_5, all $(n-1)$-cycles have the same sign, all $(n-2)$-cycles have the same sign in $\mathscr{S}_5$. Let $\mathscr{S}_6$ be a primitive nonpowerful signed digraph with underlying digraph F_6, all $(n-1)$-cycles have the same sign, all $(n-2)$-cycles have the same sign in $\mathscr{S}_6$. Let $\mathscr{S}_7$ be a primitive nonpowerful signed digraph with underlying digraph F_7, all $(n-1)$-cycles have the same sign, all $(n-2)$-cycles have the same sign in $\mathscr{S}_7$. Let $\mathscr{S}_i$ be a primitive nonpowerful signed digraph with underlying digraph $F_i^{'}$, all $(n-1)$-cycles have the same sign, all $(n-2)$-cycles have the same sign in $\mathscr{S}_i$. Let $\mathscr{Q}_1$ be a primitive nonpowerful signed digraph with underlying digraph $\mathscr{B}_1$, all $(n-3)$-cycles have the same sign in $\mathscr{Q}_1$. Let $\mathscr{Q}_2$ be a primitive nonpowerful signed digraph with underlying digraph $\mathscr{B}_2$, all $(n-3)$-cycles have the same sign in $\mathscr{Q}_2$. Let $\mathscr{Q}_3$ be a primitive nonpowerful signed digraph with underlying digraph $\mathscr{B}_3$, all $(n-3)$-cycles have the same sign in $\mathscr{Q}_3$. Let $\mathscr{Q}_4$ be a primitive nonpowerful signed digraph with underlying digraph $\mathscr{B}_4$, all $(n-3)$-cycles have the same sign in $\mathscr{Q}_4$.

Similar to the proof of Theorem 5.1, we can prove the following Theorem 5.2:

Theorem 5.2. *Let S be a primitive nonpowerful signed digraph with order n. Then*

$$l(S)=\begin{cases}2n^2-5n+2, & S=\mathscr{T},\ n \text{ is odd};\\ 2n^2-7n+8, & S\in\{\mathscr{S}_0,\mathscr{S}_1,\mathscr{S}_2\};\\ 2n^2-7n+7, & S\in\{\mathscr{S}_3,\cdots,\mathscr{S}_7\}\bigcup\{\mathscr{S}_i|2\le i\le n-3\};\\ 2n^2-7n+4, & S=\mathscr{Q}_1;\\ 2n^2-7n+3, & S\in\{\mathscr{Q}_2,\mathscr{Q}_3,\mathscr{Q}_4\}.\end{cases}$$

Theorem 5.3. *Let A be a primitive nonpowerful square sign pattern matrix with order $n(n\ge 10)$. Then we have:*

(1) There is no A such that $l(A)\in[2n^2-7n+9,2n^2-3n]$ if n is a positive even integer.

(2) If n is a positive odd integer, then

(i) There is no A such that $l(A)\in([2n^2-7n+9,2n^2-5n+1]\bigcup[2n^2-5n+5,2n^2-3n])$;

(ii) $l(A)=2n^2-5n+4$ if and only if $D(A)\cong D_{2,1}$;

$l(A)=2n^2-5n+3$ if and only if $D(A)\cong D_{2,2}$, the cycles with the same length have the same sign in $S(A)$;

$l(A)=2n^2-5n+2$ if and only if $D(A)\cong D_{2,3}$ or $D(A)\cong\mathscr{L}$, the cycles with the same length have the same sign in $S(A)$;

(3) For any integer $n\ge 10$, $l(A)=2n^2-7n+8$ if and only if $D(A)$ is isomorphic to one in $\{F, F_1, F_2\}$, the cycles with the same length have the same sign in $S(A)$;

$l(A)=2n^2-7n+7$ if and only if $D(A)$ is isomorphic to one in $\{F_3, F_4, F_5, F_6, F_7\}\bigcup\{F_i^{'}\mid 2\le i\le n-3\}$, the cycles with the same length have the same sign in $S(A)$.

(4) $\{2n^2-7n+m|\ 3\le m\le 6\}\subset E_n$ if $\gcd(n,n-3)=1$ (namely $3\nmid n$), where $E_n=\{l(A)|\ A$ is a primitive nonpowerful square sign pattern matrix with order n $(n\ge 10)\}$.

Proof. Note that $n\ge 10$, then $2n^2-7n+7\ge \dfrac{3}{2}n^2-2n+4$. By Theorem 3.7, then $C(S(A))=\{p_1,p_2\}, p_1<p_2, p_1+p_2>n$, all p_1-cycles have the same sign, all p_2-cycles have the same sign in $S(A)$. By Theorem 4.2, we know that $l(A)\le 2n^2-7n+6$ if $p_1\le n-3$. So, if $l(A)\ge 2n^2-7n+7$, there are just the following possible cases:

(1) $p_2=n, p_1=n-1$;
(2) $p_2=n, p_1=n-2$;
(3) $p_2=n-1, p_1=n-2$.

Note that $l(S_{3,i}) = 2n^2 - 7n + 7 - i$ $(1 \leq i \leq 4)$, then the theorem follows from the Lemmas 2.14, 2.15 and Theorems 5.1, 5.2. □

References

1. Bondy, J.A., Murty, U.S.R.: Graph Theory. Springer, Heidelberg (2008)
2. Brualdi, R.A., Liu, B.: Generalized Exponents of primitive directed graphs. J. of Graph Theory 14, 483–499 (1994)
3. Dulmage, A.L., Mendelsohn, N.S.: Graphs and matrices. In: Harary, F. (ed.) Graph Theory and Theoretical Physics, ch. 6, pp. 167–227
4. Kim, K.H.: Boolean Matrix Theory and Applications. Marcel Dekkez, New York (1982)
5. Li, Z., Hall, F., Eschenbach, C.: On the period and base of a sign pattern matrix. Linear Algebra Appl. 212(213), 101–120 (1994)
6. Liu, B.: Combinatorical matrix theory, second published. Science Press, Beijing (2005)
7. Ljubic, J.I.: Estimates of the number of states that arise in the determinization of a nondeterministic autonomous automaton. Dokl. Akad. Nauk SSSR 155, 41–43 (1964); Soviet Math. Dokl., 5, 345–348 (1964)
8. Miao, Z., Zhang, K.: The local exponent sets of primitive digraphs. Linear Algebra Appl. 307, 15–33 (2000)
9. Schwarz, S.: On the semigroup of binary relations on a finite set. Czech. Math. J. 20(95), 632–679 (1970)
10. Wang, L., Miao, Z., Yan, C.: Local bases of primitive nonpowerful signed digraphs. Discrete Mathematics 309, 748–754 (2009)
11. You, L., Shao, J., Shan, H.: Bounds on the basis of irreducible generalized sign pattern matrices. Linear Algebra Appl. 427, 285–300 (2007)
12. Yu, G.L., Miao, Z.K., Shu, J.L.: The bases of the primitive, nonpowerful sign patterns with exactly d nonzero diagonal entries (revision is submitted)