6

Determinants

One person's constant is another person's variable.

Susan Gerhart

While the previous chapters had their focus on the exploration of the logical and structural properties of projective planes, this chapter will focus on the following question: *What is the easiest way to calculate with geometry?* Many textbooks on geometry introduce coordinates (or homogeneous coordinates) and base all analytic computations on calculations on this level. An analytic proof of a geometric theorem is carried out in the parameter space. For a different parameterization of the theorem the proof may look entirely different.

In this chapter we will see that this way of thinking is very often not the most economical one. The reason for this is that the coordinates of a geometric object are in a way a basis-dependent artifact and carry not only information on the geometric object but also on the relation of this object to the basis of the coordinate system. For instance, if a point is represented by its homogeneous coordinates (x, y, z), we have encoded its relative position to a frame of reference. From the perspective of projective geometry the perhaps most important fact that one can say about the point is simply that it *is a point.* All other properties are not projectively invariant. Similarly, if we consider three points $p_1 = (x_1, y_1, z_1)$, $p_2 = (x_2, y_2, z_2)$, $p_3 = (x_3, y_3, z_3)$, the statement that these points are collinear reads

$$x_1y_2z_3 + x_2y_3z_1 + x_3y_1z_2 - x_1y_3z_2 - x_2y_1z_3 - x_3y_2z_1 \ = \ 0,$$

a 3×3 determinant. Again from a structural point of view this expression is far too complicated. It would be much better to encode the collinearity directly into a short algebraic expression and deal with this. The simplest way to do this is to change the role of primary and derived algebraic objects. If we consider the *determinants* themselves as "first-class citizens," the statement of collinearity simply reads $\det(p_1, p_2, p_3) = 0$, where the determinant is considered an unbreakable unit rather than just a shorthand for the above expanded formula. In this chapter we will explore the roles determinants play within projective geometry. For further reading on this fascinating topic we recommend [16, 33, 126, 132].

6.1 A "Determinantal" Point of View

Before we start with the treatment of determinants on a more general level we will review and emphasize the role of determinants in topics we have treated so far.

One of our first encounters of determinants occurred when we expressed the collinearity of points in homogeneous coordinates. Three points p_1, p_2, p_3 are collinear in $\mathbb{RP}^2$ if and only if $\det(p_1, p_2, p_3) = 0$. One can interpret this fact either geometrically (if p_1, p_2, p_3 are collinear, then the corresponding vectors of homogeneous coordinates are coplanar) or algebraically (if p_1, p_2, p_3 are collinear, then the system of linear equations $\langle p_1, l\rangle = \langle p_2, l\rangle = \langle p_3, l\rangle = 0$ has a nontrivial solution $l \neq (0, 0, 0)$). Dually, we can say that the determinant of three lines l_1, l_2, l_3 vanishes if and only if these lines have a point in common (this point may be at infinity).

A second instance in which determinants played a less obvious role occurred when we calculated the join of two points p and q by the cross product $p \times q$. We will give an algebraic interpretation of this. If (x, y, z) is *any* point on the line $p \vee q$, then it satisfies

$$\det \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ x & y & z \end{pmatrix} = 0.$$

If we develop the determinant by the last row, we can rewrite this as

$$\det \begin{pmatrix} p_2 & p_3 \\ q_2 & q_3 \end{pmatrix} \cdot x - \det \begin{pmatrix} p_1 & p_3 \\ q_1 & q_3 \end{pmatrix} \cdot y + \det \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} \cdot z = 0.$$

Or expressed as a scalar product,

$$\left\langle \left(\det \begin{pmatrix} p_2 & p_3 \\ q_2 & q_3 \end{pmatrix}, -\det \begin{pmatrix} p_1 & p_3 \\ q_1 & q_3 \end{pmatrix}, \det \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} \right), (x, y, z) \right\rangle = 0.$$

We can geometrically reinterpret this equation by saying that

$$\left(\det \begin{pmatrix} p_2 & p_3 \\ q_2 & q_3 \end{pmatrix}, -\det \begin{pmatrix} p_1 & p_3 \\ q_1 & q_3 \end{pmatrix}, \det \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} \right)$$

must be the homogeneous coordinates of the line l through p and q, since every vector (x, y, z) on this line satisfies $\langle l, (x, y, z)\rangle = 0$. This vector is nothing, but the cross product $p \times q$.

A third situation in which determinants played a fundamental role was in the definition of cross-ratios. Cross-ratios were defined as the product of two determinants divided by the product of two other determinants.

We will see later on that all three circumstances described here will have nice and interesting generalizations:

- In projective d-space coplanarity will be expressed as the vanishing of a $(d+1) \times (d+1)$ determinant.
- In projective d-space joins and meets will be nicely expressed as vectors of sub-determinants.
- Projective invariants can be expressed as certain rational functions of determinants.

6.2 A Few Useful Formulas

We will now see how we can translate geometric constructions into expressions that involve only determinants and base points of the construction. Since from now on we will have to deal with many determinants at the same time, we first introduce a useful abbreviation. For three points $p, q, r \in \mathbb{RP}^2$ we set

$$[p, q, r] := \det \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{pmatrix}.$$

Similarly, we set for two points in $\mathbb{RP}^1$,

$$[p, q] := \det \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}.$$

We call an expression of the form $[\ldots]$ a *bracket*. Here are a few fundamental and useful properties of 3×3 determinants:

Alternating sign changes:

$$[p, q, r] = [q, r, p] = [r, p, q] = -[p, r, q] = -[r, q, p] = -[q, p, r].$$

Linearity (in every row and column):

$$[\lambda \cdot p_1 + \mu \cdot p_2, q, r] = \lambda \cdot [p_1, q, r] + \mu \cdot [p_2, q, r].$$

Plücker's formula:

$$[p, q, r] = \langle p, q \times r\rangle.$$

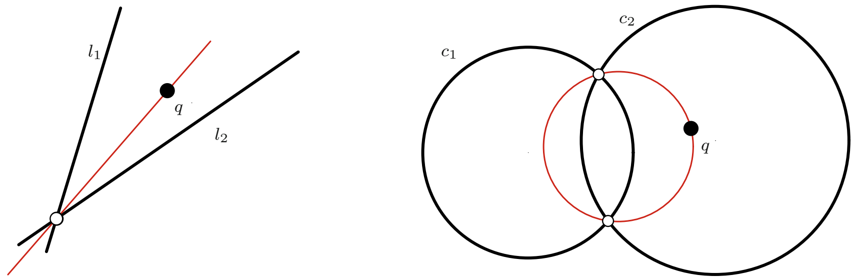


Fig. 6.1 Two applications of Plücker's μ.

The last formula can be considered a shorthand for our developments on cross products, scalar products, and determinants in the previous section.

6.3 Plücker's μ

We now introduce a very useful trick with which one can derive formulas for geometric objects that should simultaneously satisfy several constraints. The trick was frequently used by Plücker and is sometimes called *Plücker's* μ.

Imagine you have an equation $f\colon \mathbb{R}^d \to \mathbb{R}$ whose zero set describes a geometric object. For instance, think of a line equation $(x,y,z) \mapsto a \cdot x + b \cdot y + c \cdot z$ or a circle equation in the plane $(x,y) \mapsto (x-a)^2 + (y-b)^2 - r^2$. Often one is interested in objects that share intersection points with a given reference object and in addition pass through a third object. If the linear combination $\lambda \cdot f(p) + \mu \cdot g(p)$ again describes an object of the same type, then one can apply Plücker's μ. All objects described by

$$p \mapsto \lambda \cdot f(p) + \mu \cdot g(p)$$

will pass through the common zeros of f and g. This is obvious, since whenever $f(p) = 0$ and $g(p) = 0$, any linear combination is also 0. If one in addition wants to have the object pass through a specific point q, then the linear combination

$$p \mapsto g(q) \cdot f(p) - f(q) \cdot g(p)$$

is the desired equation. To see this, simply plug the point q into the equation. Then one gets $g(q) \cdot f(q) - f(q) \cdot g(q) = 0$.

With this trick we can very easily describe the homogeneous coordinates of a line ℓ that passes through the intersection of two other lines l_1 and l_2 and through a third point q by

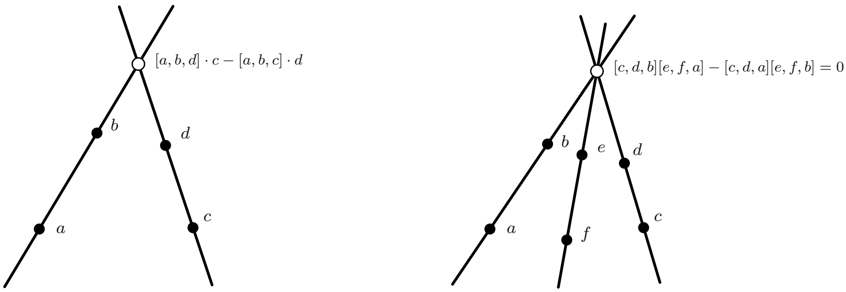


Fig. 6.2 Conditions for lines meeting in a point.

$$\langle l_2, q\rangle l_1 - \langle l_1, q\rangle l_2.$$

Testing whether this line passes through q yields

$$\langle \langle l_2, q\rangle l_1 - \langle l_1, q\rangle l_2, q\rangle = \langle l_2, q\rangle\langle l_1, q\rangle - \langle l_1, q\rangle\langle l_2, q\rangle = 0,$$

which is obviously true. Later on we will make frequent use of this trick whenever we need a fast and elegant way to calculate a specific geometric object. We will now use Plücker's μ to calculate intersections of lines spanned by points.

What is the intersection of the two lines spanned by the point pairs (a, b) and (c, d)? On the one hand, the point has to be on the line $a \vee b$; thus it must be of the form $\lambda \cdot a + \mu \cdot b$. It also has to be on $c \vee d$; hence it must be of the form $\psi \cdot c + \phi \cdot d$. Identifying these two expressions and solving for λ, μ, ψ, and ϕ would be one possibility to solve the problem. But we can directly read off the right parameters using (a dual version of) Plücker's μ. The property that encodes that a point p is on the line $c \vee d$ is simply $[c, d, p] = 0$. Thus we immediately obtain that the point

$$[c, d, b] \cdot a - [c, d, a] \cdot b$$

must be the desired intersection. This point is obviously on $a \vee b$, and it is on $c \vee d$, since we have

$$[c, d, [c, d, b] \cdot a - [c, d, a] \cdot b] \ = \ [c, d, b] \cdot [c, d, a] - [c, d, a] \cdot [c, d, b] \ = \ 0.$$

We could equivalently have applied the calculation with the roles of $a \vee b$ and $c \vee d$ interchanged. Then we can express the same point as

$$[a, b, d] \cdot c - [a, b, c] \cdot d.$$

In fact, it is not a surprise that these two expressions end up at identical points. We will later on, in Section 6.5, see that this is just a reformulation of the well-known Cramer's rule for solving systems of linear equations.

How can we express the condition that three lines $a \vee b$, $c \vee d$, $e \vee f$ meet in a point? For this we simply have to test whether the intersection p of $a \vee b$ and $c \vee d$, is on $e \vee f$. We can do this by testing whether the determinant of these three points is zero. Plugging in the formula for p, we get

$$[e, f, [c, d, b] \cdot a - [c, d, a] \cdot b] = 0.$$

After expansion by multilinearity, we obtain

$$[c, d, b][e, f, a] - [c, d, a][e, f, b] = 0.$$

This is the algebraic condition for the three lines meeting in a point. Taking a look at the above formula, we should pause and make a few observations:

- The first and most important observation is that we could write such a projective condition as a polynomial of determinants evaluating to zero.
- In the formula, each term has the same number of determinants.
- Each letter occurs equally often in each term.

All three observations extend very well to much more general cases. In order to see this, we will have first to introduce the notion of *projectively invariant properties*.

Before we do this we want to use this formula to obtain (another) beautiful proof for Pappos's theorem. Consider the drawing of Pappos's theorem in Figure 6.3 (observe the nice 3-fold symmetry). We can state Pappos's theorem in the following way: If for six points $a, \ldots, f$ in the projective plane the lines $a \vee d$, $c \vee b$, $e \vee f$ meet and the lines $c \vee f$, $e \vee d$, $a \vee b$ meet, then also $e \vee b$, $a \vee f$, $c \vee d$ meet. The two hypotheses can be expressed as

$$[b, c, e][a, d, f] = [b, c, f][a, d, e],$$
$$[c, f, b][d, e, a] = [c, f, a][d, e, b].$$

Using the fact that a cyclic shift of the points of a 3×3 bracket does not change its sign, we observe that the first term of the second equation is identical to the second term of the first equation. So we obtain

$$[f, a, c][e, b, d] = [f, a, d][e, b, c]$$

Which is exactly the desired conclusion of Pappos's theorem.

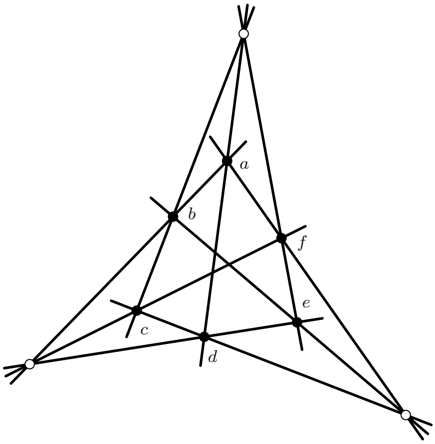


Fig. 6.3 Pappos's theorem, once more.

6.4 Invariant Properties

How can we algebraically characterize that a certain property of a point configuration is invariant under projective transformations? Properties of such type are for instance *three lines being concurrent* or six points $a, \ldots, f$ such that $a \vee b$, $c \vee d$, $e \vee f$ meet.

In general, properties of this type can be expressed as functions in the (homogeneous) coordinates of the points that have to be zero when the property holds. Being invariant means that a property holds for a point configuration P if and only if it also holds for any projective transformation of P. More formally, let us express the point configuration P by a matrix whose columns are the homogeneous coordinates of n points $p_1, \ldots, p_n$:

$$P = \begin{pmatrix} p_{1x} & p_{2x} & \cdots & p_{nx} \\ p_{1y} & p_{2y} & \cdots & p_{ny} \\ p_{1z} & p_{2z} & \cdots & p_{nz} \end{pmatrix}.$$

A projective transformation is then simply represented by left-multiplication by a 3×3 invertible matrix T. A projectively invariant property should also be invariant when we replace a vector p_i by a scalar multiple $\lambda_i \cdot p_i$. We can express the scaling of the points by right multiplication of P by an invertible diagonal matrix D. All matrices obtained from P via

$$T \cdot P \cdot D$$

represent essentially the same projective configuration. A projectively invariant property is any property of P that is invariant under such a transformation.

Very often, our invariant properties will be polynomials being zero, but for now we want to keep things more general and consider any map that associates to P a Boolean value. The matrix P can be considered an element of $\mathbb{R}^{3\cdot n}$. Thus we make the following definition:

Definition 6.1. A *projectively invariant property* of n points in the real projective plane is a map $f\colon \mathbb{R}^{3\cdot n} \to \{\mathbf{true}, \mathbf{false}\}$ such that for all invertible real 3×3 matrices $T \in \mathrm{GL}(\mathbb{R}, 3)$ and $n \times n$ invertible real diagonal matrices $D \in \mathrm{diag}(\mathbb{R}, n)$, we have

$$f(P) = f(T \cdot P \cdot D).$$

In a canonical way we can identify each predicate $P \subseteq X$ on a space X with its characteristic function $f\colon X \to \{\mathbf{true}, \mathbf{false}\}$, where $f(x)$ evaluates to **true** if and only if $x \in P$. Thus we can equivalently speak of projectively invariant predicates.

In this sense, for instance, $[a, b, c] = 0$ defines a projectively invariant property of three points a, b, c in the real projective plane. Also the property that we encountered in the last section,

$$[c, d, b][e, f, a] - [c, d, a][e, f, b] = 0,$$

which encodes the fact that three lines $a \vee b$, $c \vee d$, $e \vee f$ meet in a point, is projectively invariant. Before we state a more general theorem we will analyze why this relation is invariant from an algebraic point of view. Transforming the points by a projective transformation T results in replacing the points $a, \ldots, f$ with $T \cdot a, \ldots, T \cdot f$. Scaling the homogeneous coordinates results in replacing $a, \ldots, f$ by $\lambda_a \cdot a, \ldots, \lambda_f \cdot f$ with nonzero λ's. Thus if P encodes the point configuration, then the overall effect of $T \cdot P \cdot D$ on the expression $[c, d, b][e, f, a] - [c, d, a][e, f, b]$ can be written as

$$\begin{aligned} &[\lambda_c \cdot T \cdot c, \lambda_d \cdot T \cdot d, \lambda_b \cdot T \cdot b][\lambda_e \cdot T \cdot e, \lambda_f \cdot T \cdot f, \lambda_a \cdot T \cdot a] \\ -\, &[\lambda_c \cdot T \cdot c, \lambda_d \cdot T \cdot d, \lambda_a \cdot T \cdot a][\lambda_e \cdot T \cdot e, \lambda_f \cdot T \cdot f, \lambda_b \cdot T \cdot b]. \end{aligned}$$

Since $[T \cdot p, T \cdot q, T \cdot r] = \det(T) \cdot [p, q, r]$, the above expression simplifies to

$$(\det(T)^2 \cdot \lambda_a \cdot \lambda_b \cdot \lambda_c \cdot \lambda_d \cdot \lambda_e \cdot \lambda_f) \cdot ([c, d, b][e, f, a] - [c, d, a][e, f, b]).$$

All λ's were nonzero and T was assumed to be invertible. Hence the expression $[c, d, b][e, f, a] - [c, d, a][e, f, b]$ is zero if and only if the above expression is zero. Observe that it was important that each summand of the bracket polynomial had exactly the same number of brackets. This made it possible to factor out a factor $\det(T)^2$. Furthermore, in each summand each letter occurred equally often. This made it possible to factor out the λ's.

This example is a special case of a much more general fact, namely that all *multihomogeneous bracket polynomials* define projectively invariant properties.

Definition 6.2. Let $P = (p_1, p_2, , \dots, p_n) \in (\mathbb{R}^3)^n$ represent a point configuration of n points. A *bracket monomial* on P is an expression of the form

$$[a_{1,1}, a_{1,2}, a_{1,3}] \cdot [a_{2,1}, a_{2,2}, a_{2,3}] \cdot \dots \cdot [a_{k,1}, a_{k,2}, a_{k,3}],$$

where each $a_{j,k}$ is one of the points p_i. The *degree* $\deg(p_i, M)$ of a point p_i in a monomial is the total number of occurrences of p_i in M. A *bracket polynomial* on P is a sum of bracket monomials on P. A bracket polynomial $Q = M_1 + \dots + M_l$ with monomials $M_1, \dots, M_l$ is *multihomogeneous* if for each point p_i we have

$$\deg(p_i, M_1) = \dots = \deg(p_i, M_l).$$

In other words, a bracket polynomial is multihomogeneous if each point occurs in each summand the same number of times. We can make an analogous definition for points on a projective line. The only difference there is that we have to deal with brackets of length 2 instead of length 3.

As a straightforward generalization of our observations on the multihomogeneous bracket polynomial $[c, d, b][e, f, a] - [c, d, a][e, f, b]$ we obtain the following theorem:

Theorem 6.1. *Let $Q(P)$ be a multihomogeneous bracket polynomial on n points $P = (p_1, p_2, \dots, p_n) \in (\mathbb{R}^3)^n$. Then $Q(P) = 0$ defines a projectively invariant property.*

Proof. Since Q is multihomogeneous, each of the summands contains the same number (say $3k$) of points. Therefore each summand is the product of k brackets. Thus we have for any projective transformation T the relation

$$Q(T \cdot P) = \det(T)^k \cdot Q(P).$$

Furthermore, the degree of the point p_i is the same (say d_i) in each monomial. Scaling the points by scalars $\lambda_1 \cdot \dots \cdot \lambda_d$ can be expressed as multiplication by the diagonal matrix $D = \text{diag}(\lambda_1 \cdot \dots \cdot \lambda_n)$. Since each bracket is linear in each point entry, the scaling induces the following action on Q:

$$Q(P \cdot D) = \lambda_1^{d_1} \cdot \dots \cdot \lambda_n^{d_n} \cdot Q(P).$$

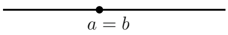
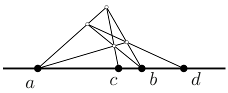
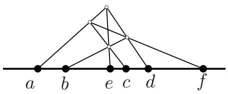
Overall, we obtain

$$Q(T \cdot P \cdot D) = \det(T)^k \cdot \lambda_1^{d_1} \cdot \dots \cdot \lambda_n^{d_n} \cdot Q(P).$$

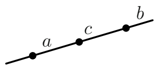
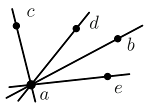
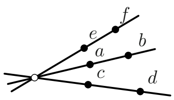
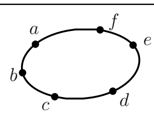
The factors preceding $Q(P)$ are all nonzero, since T is invertible and only nonzero λ_i are allowed. Hence $Q(T \cdot P \cdot D)$ is zero if and only if $Q(P)$ is zero. □

Clearly, a similar statement also holds for points on the projective line (and 2×2 brackets) and also for projective planes over other fields.

We could now begin a comprehensive study of multihomogeneous bracket polynomials and the projective invariants encoded by them. We will encounter several of them later in the book. Here we just give without further proofs a few examples to exemplify the expressive power of multihomogeneous bracket polynomials. We begin with a few examples on the projective line:

$[ab] = 0$	a coincides with b	
$[ac][bd] + [ad][bc] = 0$	$(a, b); (c, d)$ is harmonic	
$[ae][bf][cd] - [af][bd][ce] = 0$	$(a, b); (c, d); (e, f)$ is a quadrilateral set	

Here are some other examples in the projective plane:

$[abc] = 0$	a, b, c are collinear	
$[abd][ace] + [abe][acd] = 0$	The line pairs $(a \vee b, a \vee c); (a \vee d, a \vee e)$ are harmonic	
$[abe][cdf] - [abf][cde] = 0$	$(a \vee b); (c \vee d); (e \vee f)$ meet in a point	
$[abc][aef][dbf][dec]-$ $[def][dbc][aec][abf] = 0$	a, b, c, d, e, f are on a conic	

6.5 Grassmann-Plücker relations

When we studied the example of three lines $a \vee b$, $c \vee d$, $e \vee f$ meeting in a point we ended up with the formula

$$[c, d, b][e, f, a] - [c, d, a][e, f, b] = 0.$$

A closer look at this formula shows that the line $a \vee b$ plays a special role compared to the other two lines. Its points are distributed over the brackets, while the points of the other lines always occur in one bracket. The symmetry of the original property implies that there are two more essentially different ways to encode the property in a bracket polynomial:

$$[a, b, c][e, f, d] - [a, b, d][e, f, c] = 0 \quad \text{and} \quad [a, b, e][c, d, f] - [a, b, f][c, d, e] = 0.$$

The reason for this is that there are multi-homogeneous bracket polynomials that will always evaluate to zero no matter where the points of the configuration are placed. These special polynomials are of fundamental importance whenever one makes calculations in which several determinants are involved. The relations in question are the *Grassmann-Plücker relations.* In principle, such relations exist in any dimension. However, as usual in our exposition we will mainly focus on the case of the projective line and the projective plane, i.e., 2×2 and 3×3 brackets. We start with the 2×2 case. We state the relations on the level of vectors rather than on the level of projective points.

Theorem 6.2. *For any vectors $a, b, c, d \in \mathbb{R}^2$ the following equation holds:*

$$[a, b][c, d] - [a, c][b, d] + [a, d][b, c] = 0.$$

Proof. If one of the vectors is the zero vector, the equation is trivially true. Thus we may assume that each of the vectors represents a point of the projective line. Since $[a, b][c, d] - [a, c][b, d] + [a, d][b, c]$ is a multihomogeneous bracket polynomial, we may assume that all vectors are (if necessary after a suitable projective transformation) finite points and normalized to vectors $\binom{\lambda_a}{1}, \ldots, \binom{\lambda_d}{1}$. The determinants then become simply differences. Rewriting the term gives

$$(\lambda_a - \lambda_b)(\lambda_c - \lambda_d) - (\lambda_a - \lambda_c)(\lambda_b - \lambda_d) + (\lambda_a - \lambda_d)(\lambda_b - \lambda_c) = 0.$$

Expanding all terms, we get equivalently

$$\begin{array}{rcl} (\lambda_a\lambda_c + \lambda_b\lambda_d - \lambda_a\lambda_d - \lambda_b\lambda_c) & & \\ -(\lambda_a\lambda_b + \lambda_c\lambda_d - \lambda_a\lambda_d - \lambda_c\lambda_b) & & \\ +(\lambda_a\lambda_b + \lambda_d\lambda_c - \lambda_a\lambda_c - \lambda_d\lambda_b) & = & 0. \end{array}$$

The last equation can be easily checked. □

Grassmann-Plücker relations can be interpreted in many equivalent ways and, thereby this link several branches of geometry and invariant theory. We will here present three more interpretations (or proofs if you want).

1. Determinant expansion: The Grassmann-Plücker relation $[a, b][c, d] - [a, c][b, d] + [a, d][b, c] = 0$ can be considered a determinant expansion. For this assume without loss of generality that $[a,b] \neq 0$. After a projective transformation we may assume that $a = \binom{1}{0}$ and $b = \binom{0}{1}$. The Grassmann-Plücker relation then reads as

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix} - \begin{vmatrix} 1 & c_1 \\ 0 & c_2 \end{vmatrix} \cdot \begin{vmatrix} 0 & d_1 \\ 1 & d_2 \end{vmatrix} + \begin{vmatrix} 1 & d_1 \\ 0 & d_2 \end{vmatrix} \cdot \begin{vmatrix} 0 & c_1 \\ 1 & c_2 \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix} - c_2 \cdot (-d_1) + d_2 \cdot (-c_1) = 0.$$

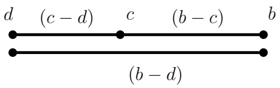
The last expression is easily recognized as the expansion formula for the determinant and obviously evaluates to zero.

2. Area relation: After a projective transformation and rescaling we can also assume that $a = \binom{1}{0}$, $b = \binom{b_1}{1}$, $c = \binom{c_1}{1}$ and $d = \binom{d_1}{1}$. Then the Grassmann-Plücker relation reads

$$\begin{vmatrix} 1 & b_1 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} c_1 & d_1 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & c_1 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} b_1 & d_1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & d_1 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} b_1 & c_1 \\ 1 & 1 \end{vmatrix}$$

$$= 1 \cdot (c_1 - d_1) - 1 \cdot (b_1 - d_1) + 1 \cdot (b_1 - c_1) = 0.$$

This formula can be affinely (!) interpreted as the relation of three directed length segments of three points b, c, d on a line:



3. Cramer's rule: Let us assume that $[a,c] \neq 0$. Cramer's rule gives us an explicit formula to solve the system of equations

$$\begin{pmatrix} a_1 & c_1 \\ a_2 & c_2 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

We get

$$\alpha = \frac{[b,c]}{[a,c]} \quad \text{and} \quad \beta = \frac{[a,b]}{[a,c]}.$$

Inserting this into the original equation and multiplying by $[a,c]$, we get

$$[b,c] \cdot a + [a,b] \cdot c - [a,c] \cdot b = 0.$$

Here "0" means the zero vector. Thus we can find the following expansion of zero:

$$0 = [[b,c] \cdot a + [a,b] \cdot c - [a,c] \cdot b, d] = [b,c][a,d] + [a,b][c,d] - [a,c][b,d].$$

This is exactly the Grassmann-Plücker relation.

What happens in dimension 3 (i.e., the projective plane)? First of all, we obtain a consequence of the Grassmann-Plücker relation on the line when we add the same point to any bracket:

Theorem 6.3. *For any vectors $a, b, c, d, x \in \mathbb{R}^3$ the following equation holds:*

$$[x, a, b][x, c, d] - [x, a, c][x, b, d] + [x, a, d][x, b, c] = 0.$$

Proof. Assuming without loss of generality that $x = (1, 0, 0)$ reduces all determinants of the expression to 2×2 determinants, any of the above proofs translates literally. □

In the projective plane we get another Grassmann-Plücker relation that involves four instead of three summands.

Theorem 6.4. *For any vectors $a, b, c, d, e, f \in \mathbb{R}^3$ the following equation holds:*

$$[a, b, c][d, e, f] - [a, b, d][c, e, f] + [a, b, e][c, d, f] - [a, b, f][c, d, e] = 0.$$

Proof. Applying Cramer's rule to the solution of a 3×3 equation

$$\begin{pmatrix} | & | & | \\ c & d & e \\ | & | & | \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} | \\ f \\ | \end{pmatrix},$$

we can prove the identity

$$[f, d, e] \cdot c + [c, f, e] \cdot d + [c, d, f] \cdot e = [c, d, e] \cdot f.$$

Rearranging the terms yields

$$[d, e, f] \cdot c - [c, e, f] \cdot d + [c, d, f] \cdot e - [c, d, e] \cdot f = 0.$$

Inserting this expansion of the zero vector 0 into $[a, b, 0] = 0$ yields (after expanding the terms by multilinearity) the desired relation. □

Again, we can also interpret this equation in many different ways. Setting (a, b, c) to the unit matrix the Grassmann-Plücker relation encodes the development of the 3×3 determinant (d, e, f) by the first column. We get

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{vmatrix} - \begin{vmatrix} 1 & 0 & d_1 \\ 0 & 1 & d_2 \\ 0 & 0 & d_3 \end{vmatrix} \cdot \begin{vmatrix} 0 & e_1 & f_1 \\ 0 & e_2 & f_2 \\ 1 & e_3 & f_3 \end{vmatrix} + \begin{vmatrix} 1 & 0 & e_1 \\ 0 & 1 & e_2 \\ 0 & 0 & e_3 \end{vmatrix} \cdot \begin{vmatrix} 0 & d_1 & f_1 \\ 0 & d_2 & f_2 \\ 1 & d_3 & f_3 \end{vmatrix} - \begin{vmatrix} 1 & 0 & f_1 \\ 0 & 1 & f_2 \\ 0 & 0 & f_3 \end{vmatrix} \cdot \begin{vmatrix} 0 & d_1 & e_1 \\ 0 & d_2 & e_2 \\ 1 & d_3 & e_3 \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{vmatrix} - d_3 \cdot \begin{vmatrix} e_1 & f_1 \\ e_2 & f_2 \end{vmatrix} + e_3 \cdot \begin{vmatrix} d_1 & f_1 \\ d_2 & f_2 \end{vmatrix} - f_3 \cdot \begin{vmatrix} d_1 & e_1 \\ d_2 & e_2 \end{vmatrix} = 0.$$

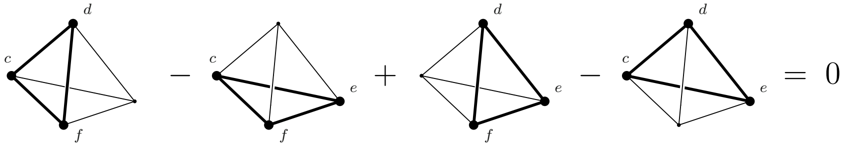


Fig. 6.4 Grassmann-Plücker relation as area formulas.

Observe that we can express each minor of the determinant $[d, e, f]$ as a suitable bracket that involves a, b, c. This point will later be of fundamental importance.

There is also a nice interpretation that generalizes the "area viewpoint." The determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{vmatrix}$$

calculates twice the oriented area $\Delta(a, b, c)$ of the affine triangle a, b, c. After a suitable projective transformation the Grassmann-Plücker relation reads

$$\begin{vmatrix} 1 & 0 & c_1 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 0 & d_1 \\ 0 & 1 & d_2 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} c_1 & e_1 & f_1 \\ c_2 & e_2 & f_2 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & e_1 \\ 0 & 1 & e_2 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} c_1 & d_1 & f_1 \\ c_2 & d_2 & f_2 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 0 & f_1 \\ 0 & 1 & f_2 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} c_1 & d_1 & e_1 \\ c_2 & d_2 & e_2 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

In terms of triangle areas this equation reads as

$$\Delta(d, e, f) - \Delta(c, e, f) + \Delta(c, d, f) - \Delta(c, d, e) = 0.$$

This formula has again a direct geometric interpretation in terms of affine oriented areas of triangles. Assume that c, d, e, f are any four points in the affine plane. The convex hull of these four points can be covered in two ways by triangles spanned by three of the points. These two possibilities must both result in the same total (oriented) area. This is the Grassmann-Plücker relation.

Using Grassmann-Plücker relations we can easily explain why the property that $a \vee b$, $c \vee d$, $e \vee f$ meet can be expressed either by

$$[a, b, e][c, d, f] - [a, b, f][c, d, e] = 0$$

or by

$$[a, b, c][e, f, d] - [a, b, d][e, f, c] = 0.$$

Adding the two expressions yields

$$[a, b, c][e, f, d] - [a, b, d][e, f, c] + [a, b, e][c, d, f] - [a, b, f][c, d, e] = 0,$$

which is exactly a Grassmann-Plücker relation. Hence this equation must be zero, which proves the equivalence of the above two expressions.