2

Projective Planes

> *Möge dieses Büchlein dazu beitragen den Schatz geometrischer Schönheit [...] über unsere Zeit hinwegzuretten.*
>
> *W. Blaschke, Projektive Geometrie 1949*

The basis of all investigations in this book will be projective geometry. Although projective geometry has a tradition of more than 400 years, it gives a fresh look at many problems, even today. One could even say that the essence of this book is to view many well-known geometric effects/setups/statements/environments from a projective viewpoint.

One of the usual approaches to projective geometry is the axiomatic one (see for instance [3, 25, 44, 58]). There, in the spirit of Euclid, a few axioms are set up and a *projective geometry* is defined as any system that satisfies these axioms. We will very briefly meet this approach in this chapter. The main part of this book will, however, be much more concrete and "down to earth." We will predominantly study projective geometries that are defined over a specific coordinate field (most prominently the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$). This gives us the chance to directly investigate the interplay of geometric objects (points, lines, circles, conics, ...) and the algebraic structures (coordinates, polynomials, determinants, ...) that are used to represent them. The largest part of the book will be about surprisingly elegant ways of expressing geometric operations or relations by algebraic formulas (see also [26]). We will in particular focus on understanding the geometry of real and of complex spaces. In the same way as the concept of complex numbers explains many of the seemingly complicated effects for real situations (for instance in calculus, algebra, or function theory), studying the complex projective world will give surprising insights into the geometry over the real numbers (which to a large extent governs our real life).

The usual study of Euclidean geometry leads to a treatment of special cases at a very early stage. Two lines may intersect or not depending on whether they are parallel or not. Two circles may intersect or not depending on their radii and on the position of their midpoints. In fact, these two effects already lead to a variety of special cases in constructions and theorems all over Euclidean geometry. The treatment of these special cases often unnecessarily obscures the beauty of the underlying structures. Our aim in this book is to derive statements and formulas that are elegant and general, and carry as much geometric information as possible. In particular, we will try to reduce the necessity of treating special cases to a minimum. Here we do not strive for complicated formulas but for formulas that carry much structural insight and often simplicity. In a sense, this book is written in the spirit of Julius Plücker (1801–1868), who was, as Felix Klein (1849–1925) expressed it [69], a master of "reading in the equations."

Starting from the usual Euclidean plane we will see that there are two essential extensions needed to bypass the special situations described in the last paragraph. First, one has to introduce *elements at infinity*. These elements at infinity will nicely unify special cases that come from parallel situations. Second (in the third part of this book), we will study the geometry over complex numbers, since they allow us also to treat intersections of circles that are disjoint from each other in real space.

2.1 Drawings and Perspectives

- *In the Garden of Eden, God is giving Adam a geometry lesson: "Two parallel lines intersect at infinity. It can't be proven but I've been there."*
- *If parallel lines meet at infinity, infinity must be a very noisy place with all those lines crashing together!*

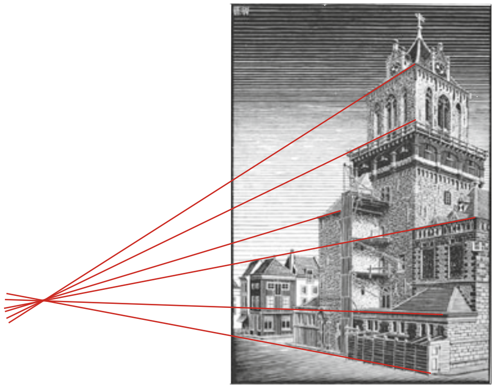
Two math jokes from a website

It was one of the major achievements of the Renaissance period of painting to understand the laws of perspective drawing. If one tries to produce a two-dimensional image of a three-dimensional object (say a cube or a pyramid), the lines of the drawing cannot be in arbitrary position. Lines that are parallel in the original scene must either be parallel or meet in a finite point. Lines that meet in a point in the original scene have either to meet in a point in the drawing or they may become parallel in the picture for very special choices of the viewpoint. The artists of that time (among others Dürer, Leonardo da Vinci and Raphael) used these principles to produce (for the standards of that time) stunningly realistic-looking images of buildings, towns, and other

Fig. 2.1 A page of Dürer's book *Underweysung der Messung, mit dem Zirkel unn Richtscheyt, in Linien, Ebenen unn gantzen corporen.*

scenes. The principles developed at this time still form the basis of most computer-created photorealistic images. The basic idea is simple. To produce a two-dimensional drawing of a three-dimensional scene, fix the position of the canvas and the position of the viewer's eye in space. For each point on the canvas consider a line from the viewer's eye through this point and plot a dot according to the object that your ray meets first (compare Figure 2.1).

By this procedure a line in object space is in general mapped to a line in the picture. One may think of this process in the following way: Any point in object space is connected to the viewpoint by a line. The intersection of this line with the canvas gives the image of the point. For any line in object space we consider the plane spanned by this line and the viewpoint (if the line does not pass through the viewpoint this plane is unique). The intersection of this plane and the canvas plane is the image of the line. This simple construction principle implies that—almost obviously—incidences of points and lines are preserved by the mapping process and that lines are again mapped to lines. Parallelism, orthogonality, distances, and angles, however, are not preserved by this process. So it may happen that lines that were parallel in object space are mapped to concurrent lines in the image space. Two pictures by the Dutch



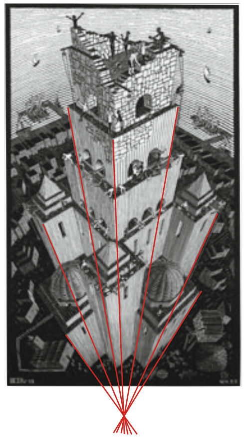


Fig. 2.2 Two copperplates of the dutch graphic artist M.C. Escher with auxiliary lines demonstrating the strong perspectivity.

artist M.C. Escher in which these construction principles are carried out in a very strict sense are reproduced in Figure 2.2[1].

A first systematic treatment of the mathematical laws of perspective drawings was undertaken by the French architect and engineer Girard Desargues (1591–1661) [32] and later by his student Blaise Pascal (1623–1662). They laid foundations of the discipline that we today call *projective geometry*. Unfortunately, many of their geometric investigations had not been anticipated by the mathematicians of their time, since approximately at the same time Réne Descartes (1596–1650) published his groundbreaking work *La géométrie*, which for the first time intimately related the concepts of algebra and geometry by introducing a *coordinate system* (this is why we speak of "Cartesian coordinates"). It was almost 150 years later that large parts of projective geometry were rediscovered by the Frenchman Gaspard Monge (1746–1818), who was, among other occupations, draftsman, lecturer, minister and a strong supporter of Napoleon Bonaparte and his revolution. His mathematical investigations had very practical backgrounds, since they were at least partially directly related to mechanics, architecture, and military applications. In 1799 Monge wrote a book [89] on what we today would call *constructive* or *descriptive geometry*. This discipline deals with the problem of making exact two-dimensional construction sketches of three-dimensional objects. Monge introduced a method (which in essence is still used today by architects and mechanical engineers) of providing different interrelated

[1] M.C. Escher's "Delft: Town Hall" —M.C. Escher's "Tower of Babel" www.mcescher.com.

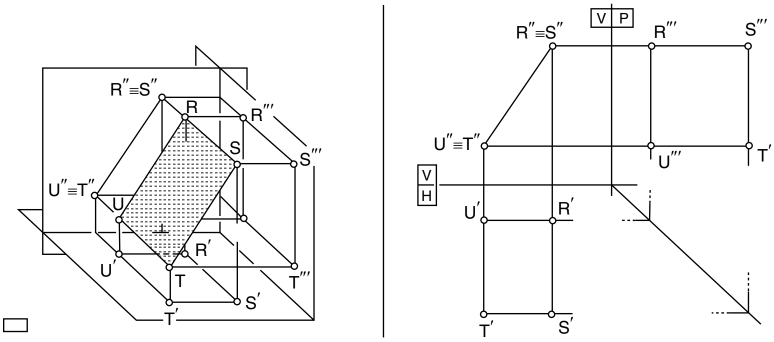


Fig. 2.3 Monge view of a square in space.

perspective drawings of a three-dimensional object in a well-defined way, such that the three-dimensional object is essentially determined by the sketches. Monge's method usually projects an object by parallel rays orthogonally to two or three distinct canvases that are orthogonal to each other. Thus the planar sketch contains, for instance a front view, a side view, and a top view of the same object. The line in which the two canvases intersect is identified and commonly used in both perspective drawings. For an example of this method consider Figure 2.3.

Monge made the exciting observation that relations between geometric objects in space and their perspective drawings may lead to genuinely planar theorems. These planar theorems can be entirely interpreted in the plane and need no further reference to the original spatial object. For instance, consider the triangle in space (see Figure 2.4). Assume that a triangle A, B, C is projected to two different mutually perpendicular projection planes. The vertices of the triangle are mapped to points A', B', C' and A'', B'', C'' in the projection planes. Furthermore, assume that the plane that supports the triangle contains the line ℓ in which the two projection planes meet. Under this condition the images ab' and ab'' of the line supporting the edge AB will also intersect in the line ℓ. The same holds for the images ac' and ac'' and for bc' and bc''. Now let us assume that we are trying to construct such a descriptive geometric drawing without reference to the spatial triangle. The fact that ab' and ab'' meet in ℓ can be interpreted as the fact that the spatial line AB meets ℓ. Similarly, the fact that ac' and ac'' meet in ℓ corresponds to the fact that the spatial line AC meets ℓ. However, this already implies that the plane that supports the triangle contains ℓ. Hence, line BC has to meet ℓ as well and therefore bc' and bc'' also will meet in ℓ. Thus the last coincidence in the theorem will occur automatically. In other words, in the drawing the last coincidence of lines occurs automatically. In fact, this special situation

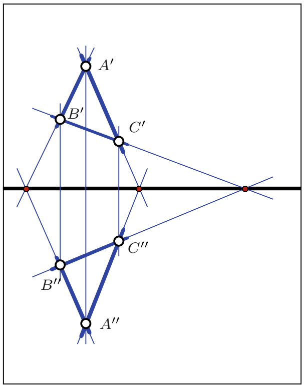


Fig. 2.4 Monge view of a triangle in space.

is nothing other than Desargues's theorem, which was discovered almost 200 years earlier.

Our starting point, and the last person of our little historical review, was Monge's student Jean-Victor Poncelet (1788–1867). He took up Monge's ideas and elaborated on them on a more abstract level. In 1822 he finished his *Traité des propriétés projectives des figures* [103]. In this monumental work (about 1200 big folio pages) he investigated those properties that remain invariant under projection. This two-volume work contains fundamental ideas of projective geometry, such as the cross-ratio, perspective, involution, and the circular points at infinity, that we will meet in many situations throughout the rest of this book. Poncelet was consequently the first to make use of *elements at infinity*, which form the basis of all the elegant treatments that we will encounter later on.

2.2 The Axioms

What happens if we try to untangle planar Euclidean geometry by eliminating special cases arising from parallelism? In planar Euclidean geometry two distinct lines intersect unless they are parallel. Now in the setup of projective geometry one enlarges the geometric setup by claiming that two distinct lines will always intersect. Even if they are parallel, they have an intersection—we just do not see it. In the axiomatic approach a *projective plane* is defined in the following way.

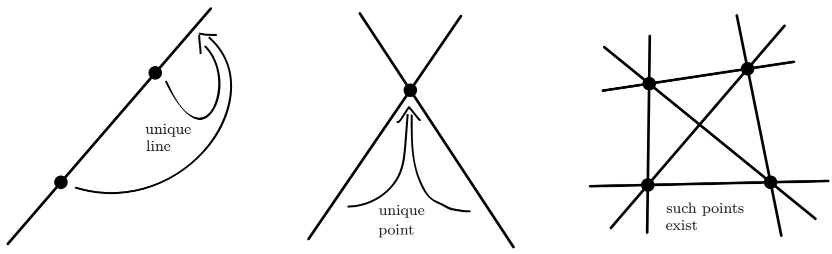


Fig. 2.5 The axioms of projective geometry.

Definition 2.1. A *projective plane* is a triple $(\mathcal{P}, \mathcal{L}, \mathtt{I})$. The set $\mathcal{P}$ consists of the *points*, and the set $\mathcal{L}$ consists of the *lines* of the geometry. The inclusion $\mathtt{I} \subseteq \mathcal{P} \times \mathcal{L}$ is an incidence relation satisfying the following three axioms:

(i) For any two distinct points, there is exactly one line incident with both of them.
(ii) For any two distinct lines, there is exactly one point incident with both of them.
(iii) There are four distinct points such that no line is incident with more than two of them.

Observe that the first two axioms describe a completely symmetric relation of points and lines. The second axiom simply states that (without any exception) two distinct lines will always intersect in a unique point. The first axiom states that (without any exception) two distinct points will always have a line joining them. The third axiom merely ensures that the structure is not a degenerate trivial case in which most of the points are collinear.

It is the aim of this and the following section to give various models for this axiom system. Let us first see how the usual Euclidean plane can be extended to a projective plane in a natural way by including elements at infinity. Let $\mathbb{E} = (\mathcal{P}_\mathbb{E}, \mathcal{L}_\mathbb{E}, \mathtt{I}_\mathbb{E})$ be the usual Euclidean plane with points $\mathcal{P}_\mathbb{E}$, lines $\mathcal{L}_\mathbb{E}$, and the usual incidence relation $\mathtt{I}_\mathbb{E}$ of the Euclidean plane. We can easily identify $\mathcal{P}_\mathbb{E}$ with $\mathbb{R}^2$. Now let us introduce the elements at infinity. For a line l consider the equivalence class $[l]$ of all lines that are parallel to l. For each such equivalence class we define a new point $p_{[l]}$. This point will play the role of the point at infinity in which all the parallels contained in the equivalence class $[l]$ shall meet. This point is supposed to be incident with all lines of $[l]$. Furthermore, we define one *line at infinity* l_∞. All points $p_{[l]}$ are supposed to be incident with this line. More formally, we set

- $\mathcal{P} = \mathcal{P}_\mathbb{E} \cup \{p_{[l]} \mid l \in \mathcal{L}_\mathbb{E}\}$,
- $\mathcal{L} = \mathcal{L}_\mathbb{E} \cup \{l_\infty\}$,
- $\mathtt{I} = \mathtt{I}_\mathbb{E} \cup \{(p_{[l]}, l) \mid l \in \mathcal{L}_\mathbb{E}\} \cup \{(p_{[l]}, l_\infty) \mid l \in \mathcal{L}_\mathbb{E}\}$.

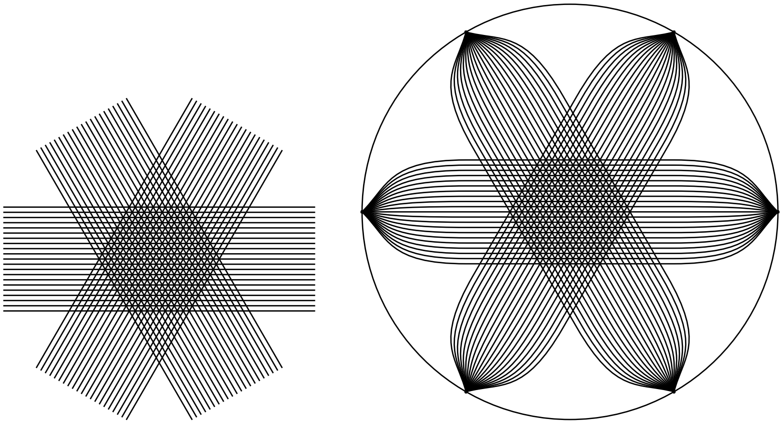


Fig. 2.6 Sketch of some lines in the projective extension of Euclidean geometry.

It is easy to verify that this system $(\mathcal{P}, \mathcal{L}, \mathtt{I})$ satisfies the axioms of a projective plane. Let us start with axiom (ii). Two distinct lines l_1 and l_2 have a point in common: If l_1 and l_2 are nonparallel Euclidean lines, then this intersection is simply their usual Euclidean intersection. If they are parallel, it is the corresponding unique point $p_{[l_1]}$ (which is identical to $p_{[l_2]}$). The intersection of l_∞ with a Euclidean line l is the point at infinity $p_{[l]}$ "on" that line. The first axiom is also easy to check: the unique line incident to two Euclidean points p_1 and p_2 is simply the Euclidean line between them. The line that joins a Euclidean point p and an infinite point p_∞ is the unique line l through p with the property that $p_\infty = p_{[l]}$. Last but not least, the line incident to two distinct infinite points is the line at infinity l_∞ itself. This completes the considerations for axiom (i) and axiom (ii). Axiom (iii) is evidently satisfied. For this one has simply to pick four points of an arbitrary proper rectangle.

Figure 2.6 (left) symbolizes three bundles of parallels in the Euclidean plane. Figure 2.6 (right) indicates how these lines projectively meet in a point and how all these points lie together on the line at infinity (drawn as a large circle for which antipodal points are assumed to be identified). Looking at the process of extending the Euclidean plane to a projective plane, it may seem that the points at infinity and the line at infinity play a special role. We will see later on that this is by far not the case. In a certain sense the projective extension of a Euclidean plane is even more symmetric than the usual Euclidean plane itself, since it allows for even more automorphisms.

2.3 The Smallest Projective Plane

The concept of projective planes as set up by our three axioms is a very general one. The projective extension of the real Euclidean plane is by far not the only model of the axiom system. In fact, even today there is no final classification or enumeration of all possible projective planes. Projective planes do not even have to be infinite objects. There are interesting systems of finitely many points and lines that fully satisfy the axioms of a projective plane. To get a feeling for these structures we will briefly construct and encounter a few small examples.

What is the smallest projective plane? Axiom (iii) tells us that it must contain at least four points, no three of which are collinear. So let us start with four points and search for the smallest system of points and lines that contains these points and at the same time satisfies axioms (i) and (ii). Let the four points be A, B, C, and D. By axiom (ii) any pair of these points has to be connected by a line. This generates exactly $\binom{4}{2} = 6$ lines. Axiom (i) requires that any pair of such lines intersect. There are exactly three missing intersections, namely those of the pairs of lines $(\overline{AB}, \overline{CD})$, $(\overline{AC}, \overline{BD})$, and $(\overline{AD}, \overline{BC})$. This gives an additional three points that must necessarily exist. Now again axiom (i) requires that any pair of points be joined by a line. The only pairs of points that are not joined so far are those formed by the most recently added three points. We can satisfy the axioms by simply adding one line that contains exactly these three points.

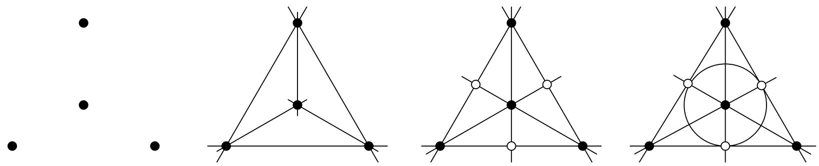


Fig. 2.7 Construction of the smallest projective plane.

The final construction contains seven points and seven lines and is called the *Fano plane*. There are a few interesting observations that can be made in this example:

- There are exactly as many lines as there are points in the drawing.
- On each line there is exactly the same number of points (here 3).
- Through each point passes exactly the same number of lines.

Each of these statements generalizes to general finite projective planes, as the following propositions show. We first fix some notation. Let $(\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a projective plane. For a line $l \in \mathcal{L}$ let $p(l) = \{p \in \mathcal{P} \mid p\mathtt{I}l\}$ be the points on l, and for a point $p \in \mathcal{P}$ let $l(p) = \{l \in \mathcal{L} \mid p\mathtt{I}l\}$ be the lines through p. Furthermore, we agree on a few linguistic conventions. Since in a projective plane the line l that is at the same time incident to two points p and q is by axiom (i) uniquely determined, we will use a language that is more functional than set-theoretic and simply speak of the *join* of the two points. We will express this join operation by $p \vee q$ or by $\mathbf{join}(p, q)$. Similarly, we will call the unique point incident with two lines l and m the *meet* or *intersection* of these lines and denote the corresponding operation by $l \wedge m$ or by $\mathbf{meet}(l, m)$. We also say that a line l *contains* a point p if it is incident with it.

Lemma 2.1. *If for $p, q \in \mathcal{P}$ and $l, m \in \mathcal{L}$ we have $p\mathtt{I}l$, $q\mathtt{I}l$, $p\mathtt{I}m$, and $q\mathtt{I}m$, then either $p = q$ or $l = m$.*

Proof. Assume that $p\mathtt{I}l$, $q\mathtt{I}l$, $p\mathtt{I}m$, and $q\mathtt{I}m$. If $p \neq q$, axiom (i) implies that $l = m$. Conversely, if $l \neq m$, axiom (ii) implies that $p = q$. □

Lemma 2.2. *Every line of a projective plane is incident with at least three points.*

Proof. Let $l \in \mathcal{L}$ be any line of the projective plane and assume to the contrary that l does contain fewer than three points. Let a, b, c, and d be the points of axiom (iii). Assume without loss of generality that a and b are not on l. Consider the lines $a \vee b$, $a \vee c$, $a \vee d$. Since these all pass through a, they must be distinct by axiom (iii) and must by Lemma 2.1 have three distinct intersections with l. □

Lemma 2.3. *For every point p there is at least one line not incident with p.*

Proof. Let p be any point. Let l and m be arbitrary lines. Either one of them does not contain p (then we are done), or we have $p = l \wedge m$. By the last lemma there is a point p_l on l distinct from p, and a point p_m on m distinct from p. The join of these two points cannot contain p, since this would violate axiom (i). □

Theorem 2.1. *Let $(\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a projective plane with finite sets $\mathcal{P}$ and $\mathcal{L}$. Then there exists a number $n \in \mathbb{N}$ such that $|p(l)| = n + 1$ for any $l \in \mathcal{L}$ and $|l(p)| = n + 1$ for any $p \in \mathcal{P}$.*

Proof. Let l and m be two distinct lines. Assume that l contains k points. We will prove that both lines contain the same number of points. Let $p = l \wedge m$ be their intersection and let ℓ be a line through p distinct from l and m. Now consider a point q on ℓ distinct from p, which exists by Lemma 2.2. Let $\{a_1, a_2, \ldots, a_n\} = p(l) - \{p\}$ be the points on l distinct from p and consider

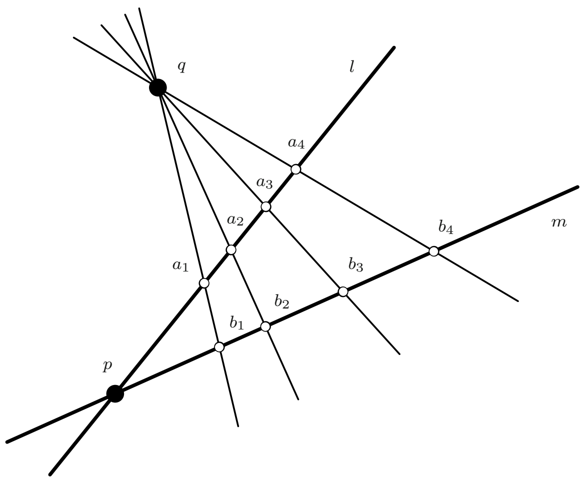


Fig. 2.8 The proof that all lines have the same number of points.

the $n-1$ lines $l_i = p_i \vee q$; $i = 1, \dots, n$. Each of these lines intersects the line m in a point $b_i = l_i \wedge m$. All these points have to be distinct, since otherwise there would be lines l_i, l_j that intersect twice, in contradiction to Lemma 2.1. Thus the number of points on m is at least as big as the number of points on l. Similarly, we can argue that the number of points on l is at least as big as the number of points on m. Hence both numbers have to be equal. Thus the number of points on a line is the same for any line (see Figure 2.8).

Now let p be any point and let l be a line that does not contain p. Let $\{p_1, p_2, \dots, p_{n+1}\}$ be the $n+1$ points on l. Joining these points with p generates $n+1$ lines through p. In fact, these lines must be all lines through p since any line through p, must have an intersection with l by axiom (ii). Hence the number of lines that pass through our (arbitrarily chosen) point p must also be equal to $n+1$. □

The number n of the last proposition (which was the number of points on a line minus one) is usually called the *order* of the projective plane. The following proposition relates the order and the overall number of points and lines in a finite projective plane.

Theorem 2.2. *Let $(\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a projective plane with finite sets $\mathcal{P}$ and $\mathcal{L}$ of order n. Then we have $|\mathcal{P}| = |\mathcal{L}| = n^2 + n + 1$.*

Proof. The last proposition proved that the number of points on each line is $n+1$ and the number of lines through each point is also $n+1$. Let p be any point of the projective plane. Each of the $n+1$ lines through p contains n additional points. They must all be distinct, since otherwise two of these lines intersect twice. We have altogether $(n+1) \cdot n + 1 = n^2 + n + 1$ points. A similar count proves that the number of lines is the same. □

So far we have met two examples of a projective plane. One is the finite Fano plane of order 2; the other (infinite example) was the projective extension of the real Euclidean plane. Our next chapter will show that both can be considered as special examples of a construction that generates a projective plane for every number field.