16

Complex Numbers: A Primer

The Divine Spirit found a sublime outlet in that wonder of analysis, that portent of the ideal world, that amphibian between being and not-being, which we call the imaginary root of negative unity.

Gottfried Wilhelm Leibniz, 1702

So far, almost all our considerations have dealt with *real* projective geometry. The main reason for this is that we wanted to stay with all our considerations as concrete and close to imagination as possible. Nevertheless, almost all considerations we have made so far carry over in a straightforward way to other underlying fields. Only in very rare cases are small twists necessary, and if so, they result from one of the following two facts:

- Depending on the field, certain equations may be solvable or not.
- Other fields may have other field automorphisms.

We now will be interested in particular in projective geometry over the *complex* numbers. Since the complex numbers are algebraically closed (every polynomial equation is solvable over the complex numbers), objects will have intersections in general. For instance, over the complex numbers there is always an intersection of a conic and a line, in contrast to the real case in which we may have situations in which the two objects do not intersect. This is the first benefit we will get from the use of complex numbers. They will help in our efforts to exclude special cases.

Compared to real numbers, the complex numbers have more field automorphisms. Besides the identity (the only automorphism of the real numbers), there is also the automorphism that sends $x + iy$ to $x - iy$.

automorphism!of a field This automorphism corresponds to complex conjugation.[1] Considering this fact under the light of the fundamental theorem of projective geometry (compare Section 5.4), we see that over the complex numbers there will be harmonic maps and collineations that do not correspond to a matrix multiplication but come from complex conjugation. We will have to deal with this difference later.

This section is meant as a very brief introduction to complex numbers. Here we will highlight all ingredients we will need later. Readers who already feel very familiar with complex numbers can skip this chapter with no harm.

16.1 Historical Background

The historical roots of complex numbers are closely related to the task of finding solutions to polynomial equations [137]. It is an interesting fact that complex numbers were first discovered in the context of cubic rather than quadratic equations. Roughly, the story goes as follows:
quadratic equation
equation!quadratic In antiquity it was known how to solve quadratic equations $x^2 + px + q = 0$. The solution is given by the well-known formula

$$x_{1/2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}.$$

It was clear that there are instances in which no (real) solution would be possible due to the fact that the square of a (real) number is always positive. So for instance, the equation $x^2 = -1$ was considered unsolvable.

Since then it remained a major open question how to generalize the solution of the quadratic equation to the next more complicated case, the cubic equation.
cubic equation
equation!cubic In fact, it took several hundred years until a solution was found (in the sixteenth century), and one can say that the developments triggered by the discovery of this formula form the initial point of what one would consider "modern mathematics." The history around the discovery of the solution (we already used a variant of it in Section 11.4 when we intersected two conics) is exciting, full of personal tragedy, amusing—a prototype of a mathematical "crime story" on priority disputes. Again, since this is not a book on the history of mathematics we will only briefly outline the basic plot and refer to history books for details [48, 137]. We also recommend the novel [61]. Briefly, Scipione del Ferro was essentially the first to solve (a special case of) the general cubic equation $x^3+ax^2+bx+c = 0$ around 1515.

[1] These are the only continuous automorphisms of $\mathbb{C}$. Indeed, in the presence of the axiom of choice there are also uncountably many *wild automorphisms* of $\mathbb{C}$.

Unfortunately, he did not publish his result during his lifetime (he died in 1526). However, he told the solution to his scholar and relative Anton Maria Fior (a more or less mediocre mathematician). After Scipione's death, Anton Maria Fior challenged the mathematician Nicolo Tartaglia (1499–1557) to a mathematical tournament (which were quite common at this time). Tartaglia was the best-known mathematician in Italy at this time and practiced in Venice. Such a tournament consisted of 30 mathematical challenges, which were given from either opponent to the other. Somehow, Tartaglia found out that Fior knew the secret of how to solve cubic equations. Tartaglia suspected correctly that all challenges he had to face would involve cubic equations. He figured out how to solve (a special case of) the cubic equation by himself and won the tournament handily—he solved all 30 problems within 2 hours. At that time another Italian mathematician, Girolamo Cardano (1501–1576) was working on a book that covered the mathematical knowledge known at the time (Tartaglia had a similar project). Cardano importuned Tartaglia to reveal his formula so that he could include it in his book. Tartaglia first resisted, but after several attempts, he agreed to tell him the formula under the condition that Cardano would not publish the formula in his own book. Cardano even vowed not to tell Tartaglia's formula in speech or in writing. However, a few years later Cardano found out that Tartaglia was not the first to have solved the cubic equation. Therefore, he felt released from the vow and included Tartaglia's formula in his book *Ars magna* (1545). Cardano even generalized the solution and was able to solve all special cases of the equation. For Tartaglia this was a great shock, and he began a long and public priority fight with Cardano. The solution formula was until recently always called "Cardano's formula." In recent years (and after some historical research) it is more common to call it the "del Ferro/Tartaglia/Cardano formula."

As we will see, despite all the priority disputes, Cardano made a contribution in this context that is perhaps more important than the solution of cubic equations itself. Tartaglia's general formula for solving the cubic equation $x^3+px+q=0$ (every cubic equation can be easily transformed into this form) can in modern terms be formulated as:

$$x = \sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}.$$

For instance, if one wants to find a solution of $x^3 - 24x - 72 = 0$, one gets

$$\begin{aligned} x &= \sqrt[3]{-\frac{-72}{2} + \sqrt{\frac{72^2}{4} + \frac{(-24^3)}{27}}} + \sqrt[3]{-\frac{-72}{2} - \sqrt{\frac{72^2}{4} + \frac{(-24^3)}{27}}} \\ &= \sqrt[3]{36 + \sqrt{1296 - 512}} + \sqrt[3]{36 - \sqrt{1296 - 512}} \\ &= \sqrt[3]{36 + 28} + \sqrt[3]{36 - 28} \\ &= \sqrt[3]{64} + \sqrt[3]{8} \\ &= 4 + 2 \\ &= 6. \end{aligned}$$

And indeed, $x = 6$ is a solution of the cubic equation: $6^3 - 24 \cdot 6 - 72 = 216 - 144 - 72 = 0$.

Now, it is clear that *every* cubic equation of the above form must have at least one solution, since as x runs from $-\infty$ to $+\infty$, the cubic itself runs continuously from $-\infty$ to $+\infty$ and therefore has to pass through zero. Unfortunately, Tartaglia's formula applied in a naive way does not always lead to a solution. Consider, for instance, the problem $x^3 - 15x - 4 = 0$. If we proceed in the same way as before, we get

$$\begin{aligned} x &= \sqrt[3]{-\frac{-4}{2} + \sqrt{\frac{4^2}{4} + \frac{(-15^3)}{27}}} + \sqrt[3]{-\frac{-4}{2} - \sqrt{\frac{4^2}{4} + \frac{(-15^3)}{27}}} \\ &= \sqrt[3]{2 + \sqrt{4 - 125}} + \sqrt[3]{2 - \sqrt{4 - 125}} \\ &= \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} \\ &= \dots \end{aligned}$$

... and we are stuck! What should one do with the term $\sqrt{-121}$? There is no number (at least for del Ferro and Tartaglia) whose square is -121. Nevertheless, there must be a solution of the cubic (in contrast to the quadratic case, where nobody saw a reason for searching for a solution of $x^2 = -1$). This was the place where Cardano made his brilliant innovation. He simply went on calculating, assuming that one could take the term $\sqrt{-1}$ as a purely formal expression with which one could do arithmetic. It behaves almost like a usual number but must be used subject to the rule $\sqrt{-1} \cdot \sqrt{-1} = -1$. If we do so, we can continue the above calculation

$$\begin{aligned} & \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} \\ = & \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}} \\ = & \dots \end{aligned}$$

... and now we need two numbers a and b with the property that $a^3 = 2 + 11\sqrt{-1}$ and $b^3 = 2 - 11\sqrt{-1}$. In fact, $a = 2 + \sqrt{-1}$ and $b = 2 - \sqrt{-1}$ do the job (check it!) and we can proceed:

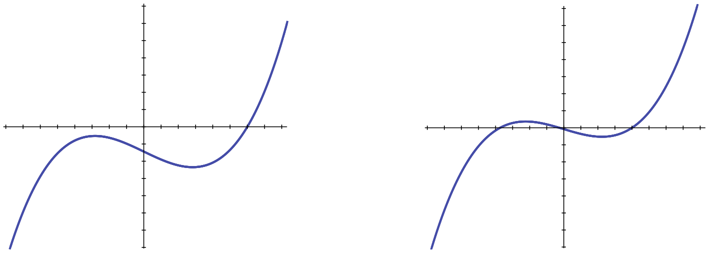

Fig. 16.1 Function plots of the two cubic polynomials $(x^3-24x-72)/50$ and $(x^3-15x-4)/50$.

$$
\begin{aligned}
&\sqrt[3]{2+\sqrt{-1}}+\sqrt[3]{2-\sqrt{-1}}\\
&=(2+11\sqrt{-1})+(2-11\sqrt{-1})\\
&=4.
\end{aligned}
$$

Et voilà, right before the last equation the mysterious $\sqrt{-1}$ disappears again, and we end up with a nice, real (and correct) solution $x=4$. Testing the result we obtain $4^3-15\cdot4-4=64-60-4=0$.

In "modern times" things have been smoothed out. A new symbol "i" was introduced that plays the role of the mysterious $\sqrt{-1}$ and behaves subject to $i^2=-1$. This number i is usually called the *imaginary unit*. We now usually consider $\mathbb{C}=\mathbb{R}[i]$ as a field extension of the real numbers, so that we can consider complex numbers as numbers of the form $x+i\cdot y$. Complex numbers play a great unifying role in modern mathematics. With their help, seemingly unrelated effects and topics may be interpreted as different sides of the same coin. We will have a brief look at some of them.

16.2 The Fundamental Theorem

The fundamental theorem of algebra is a great example for the unifying power of complex numbers. Complex numbers were originally introduced to perform the calculations to solve cubic equations. However, they also generalize the structure of the solution set. A view to the solution set of cubic equation from the "real perspective" tells us that they will have one, two, or three solutions, depending on the values of their parameters. From a "complex perspective" one can prove that a cubic x^3+ax^2+bx+c can always be written in the form $(x-x_1)(x-x_2)(x-x_3)$, where x_1,x_2,x_3 are three (possibly complex) numbers. Since this expression is zero if and only if x itself equals one of these numbers, x_1,x_2,x_3 must be solutions of the cubic equation $x^3+ax^2+bx+c=0$. Thus in a certain sense we could say that a cubic equation always has three solutions.

They may occur with a *multiplicity* if the same linear expression is used more than once in the expression $(x - x_1)(x - x_2)(x - x_3)$.

This is a special case of a much more general theorem: the fundamental theorem of algebra. This theorem states that every polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

of degree n may be written as a product of n linear factors

$$(x - x_1) \cdot (x - x_2) \cdot \cdots \cdot (x - x_n).$$

The numbers $x_1, \ldots, x_n$ are all solutions of the equation $f(x) = 0$. These numbers may be real or complex, and they may occur with a certain multiplicity in the product expression.

Thus not only do complex numbers help to solve quadratic and cubic equations, they also allow one to find all solutions of arbitrary polynomials. The fundamental theorem is by far not easy to prove, and both proofs of this theorem first known (one by Gauss and one by d'Alembert) turned out to have some minor flaws.[2] It is important to mention that the fundamental theorem of algebra does not tell how to find the solutions of a polynomial equation. It only states their existence.

Applied to geometric problems, the fundamental theorem has many important consequences concerning the intersection multiplicity of geometric objects. It gives us the right to speak of intersections of objects even if we do not see them. Thus a line and a nondegenerate conic will always have two intersections, either real or complex. These two intersections coincide if the line is tangent to the conic. Similarly, two conics will in general have four points of intersection. Again these intersections may coincide.

16.3 Geometry of Complex Numbers

One of the most important aspects in relation to our investigations will be a geometric interpretation of complex numbers. A complex number $a + i \cdot b$ may be associated to the point (a, b) in the real plane $\mathbb{R}^2$. Thus we may identify the field of complex numbers $\mathbb{C}$ with the real Euclidean plane $\mathbb{R}^2$. Every statement about complex numbers immediately possesses a geometric counterpart. It is amazing that this (from a modern perspective almost obvious) interpretation was made quite a while after the invention of complex numbers. The geometric interpretation was first published by Caspar Wessel in 1799 (this is about 250 years after complex numbers were introduced!). Later,

[2] In fact, in 1799 Gauss published a proof in response to d'Alembert's proof, since he thought that this proof was not rigorous. However Gauss's proof (based on a topological argument) also contained some flaws. Perhaps the first complete and correct proof was given in 1816 by Gauss.

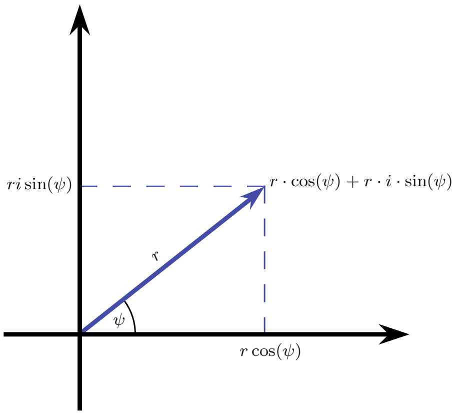

Fig. 16.2 Expressing complex numbers by trigonometric functions.

the geometric interpretation was rediscovered independently by Argand and by Gauss. The geometric interpretation of complex numbers as points in the plane will be crucial for all our further investigations.

Let us see how elementary arithmetic operations translate into geometric terms. If we add two complex numbers $z_1 = a_1 + i \cdot b_1$ and $z_2 = a_2 + i \cdot b_2$ we simply have to add the real and imaginary parts, and we obtain

$$z_1 + z_2 = a_1 + i \cdot b_1 + a_2 + i \cdot b_2 = (a_1 + a_2) + i \cdot (b_1 + b_2).$$

This is nothing but usual addition of vectors. Thus we can say that adding a complex number $z = a + i \cdot b$ causes a translation by the vector (a, b).

Multiplication is a bit more intricate. Using our rules for calculations with i we get:

$$\begin{aligned} z_1 \cdot z_2 &= (a_1 + i \cdot b_1) \cdot (a_2 + i \cdot b_2) \\ &= a_1 a_2 + i \cdot b_1 a_2 + i \cdot a_1 b_2 + i^2 \cdot b_1 b_2 \\ &= (a_1 a_2 - b_1 b_2) + i \cdot (b_1 a_2 + a_1 b_2). \end{aligned}$$

At first sight, this formula does not reveal an obvious geometric interpretation. Nevertheless such an interpretation will turn out to be the major key to all our applications of complex numbers to geometry.

To analyze what is going on, we introduce two magnitudes: the *length* $|a + i \cdot b|$ of a complex number, which is defined by

$$|a + i \cdot b| = a^2 + b^2,$$

and the *angle* of $z = a + i \cdot b$, which is the angle between the vector (a, b) and the vector $(1, 0)$ on the x-axis. The length is also sometimes called *absolute value* or *modulus* of z. The angle is also called *argument* or *phase* of z. We will

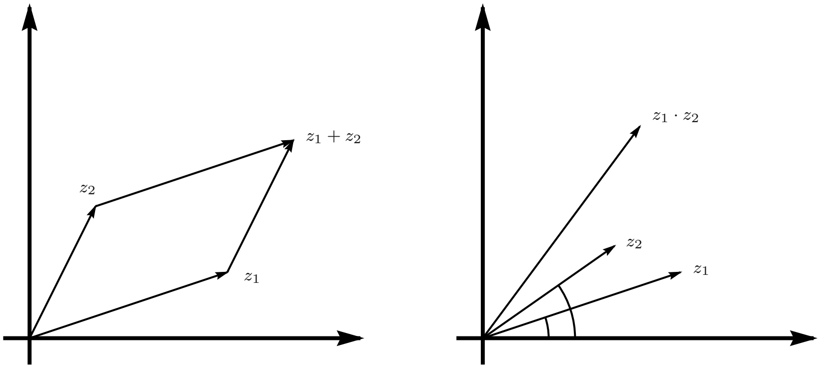

Fig. 16.3 Geometric interpretation of addition and multiplication.

prefer the geometric terms "length" and "angle." It is clear that a complex number is completely determined if we know its length r and its angle ψ. The corresponding complex number then calculates (as can be seen by simple trigonometry) to

$$z = r \cdot \cos(\psi) + r \cdot i \cdot \sin(\psi).$$

Let us see what happens if we multiply a complex number $z_1 = a + i \cdot b$ by another number z_2 that is defined by its length r and its angle ψ. We get

$$\begin{aligned} z_1 \cdot z_2 &= (a + i \cdot b) \cdot (r \cdot \cos(\psi) + r \cdot i \cdot \sin(\psi)) \\ &= (ar \cdot \cos(\psi) - br \sin(\psi)) + i \cdot (br \cdot \cos(\psi) + ar \sin(\psi)). \end{aligned}$$

If we abbreviate $z_1 \cdot z_2 = a' + i \cdot b'$, we can express this formula by a matrix multiplication:

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = r \cdot \begin{pmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}.$$

This gives an immediate interpretation in geometric terms: multiplying by a complex number with length r and angle ψ results in a rotation around the origin by an angle of ψ combined with a stretch (or dilatation) by a factor r. Figure 16.3 illustrates the geometric interpretation of addition and multiplication of two complex numbers.

16.4 Euler's Formula

Expressing complex numbers by trigonometric functions is a nice feature, but it is not the most compact form in which we can write a complex number given by angle and length. There is a beautiful formula known as *Euler's formula* that closely relates trigonometric functions, complex numbers, and the exponential function. One way to express this formula is

$$e^{ix} = \cos(x) + i \cdot \sin(x).$$

If x is a real number, this means that the exponential of the purely imaginary number ix can be expressed as combination of $\cos(x)$, which forms the real part, and $i \cdot \sin(x)$, which forms the imaginary part. In this form the formula was published by Euler in 1748 (although it was discovered earlier by Roger Cotes in 1714). Since the geometric interpretation of complex numbers was not known at this time, both Cotes and Euler considered this result a purely analytical statement.

When trying to interpret this formula, we encounter an important difficulty: What is e^{ix}? How should we define the result of an analytic function applied to a complex argument? The short answer to this goes as follows: *Whenever a function is expressible as a formal power series, one can use this power series to evaluate the function also for complex arguments, as long as it converges.* In particular, the functions e^x, $\sin(x)$, and $\cos(x)$ have formal power series that converge for all complex numbers. These power series are

$$\begin{aligned} e^x &= 1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \tfrac{x^4}{4!} + \tfrac{x^5}{5!} + \tfrac{x^6}{6!} + \tfrac{x^7}{7!} + \cdots, \\ \sin(x) &= x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} - \tfrac{x^7}{7!} + \cdots, \\ \cos(x) &= 1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} - \tfrac{x^6}{6!} + \cdots. \end{aligned}$$

Comparing the entries of these power series, we observe a striking similarity of the summands of the three power series. Up to sign changes, all summands of e^x occur in $\sin(x)$ or $\cos(x)$. So, how do we get the signs right? This is where the number i enters the game. If we expand the function e^{ix}, we see that depending on the power of the summand, the summand occurs either with a factor i (for even powers) or not (for odd powers). By the rule $i^2 = -1$ also the signs are altered according to a very regular pattern. In detail, we get

$$\begin{aligned} e^{ix} &= 1 + ix + \tfrac{(ix)^2}{2!} + \tfrac{(ix)^3}{3!} + \tfrac{(ix)^4}{4!} + \tfrac{(ix)^5}{5!} + \tfrac{(ix)^6}{6!} + \tfrac{(ix)^7}{7!} + \dots \\ &= 1 + ix - \tfrac{x^2}{2!} - i\tfrac{x^3}{3!} + \tfrac{x^4}{4!} + i\tfrac{x^5}{5!} - \tfrac{x^6}{6!} - i\tfrac{x^7}{7!} + \cdots, \\ i\sin(x) &= ix - i\tfrac{x^3}{3!} + i\tfrac{x^5}{5!} - i\tfrac{x^7}{7!} + \cdots, \\ \cos(x) &= 1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} - \tfrac{x^6}{6!} + \cdots. \end{aligned}$$

We have printed the power series for $\cos(x)$ and $i\sin(x)$ for reference purposes. We see that e^{ix} splits into these two functions. So we have

$$e^{ix} = \cos(x) + i \cdot \sin(x).$$

This formula has many consequences. First of all, we see that if x is real, the length of e^{ix} is one, since $|e^{ix}| = |\cos(x) + i \cdot \sin(x)| = \cos(x)^2 + \sin(x)^2 = 1$. Thus these numbers all lie on the unit circle in the complex plane. We may think of x as an angle, and as x increases, the number e^{ix} rotates counter-clockwise along the unit circle. In particular, we get $e^{i\pi} = -1$ and $e^{2i\pi} = 1$.[3]

If we consider $\ln(x)$ as the inverse function of e^x, we must consider $\ln(x)$ to be a *many-valued* function. If we search, for instance, for a number x with $e^x = 1$, any number in the set

$$\{\ldots, -4i\pi, -2i\pi, 0, 2i\pi, 4i\pi, \ldots\}$$

will do. So we could say that $\ln(1)$ could be any of these numbers. In general, the value of $\ln(x)$ is defined only up to additive constants of the form $2ki\pi$, where k is an integer. Still, usually one prefers to define $\ln(x)$ as a single-valued function and defines the *principal value* of $\ln(x)$ to be the value $a + ib$, where b is in the half-open interval $(-\pi, \pi]$. We will have to deal with the many-valuedness of ln later on. It simply reflects the geometric fact that a rotation by 360° is indistinguishable from the identity.

There is another important application of Euler's formula. It allows us to express a complex number directly by its length and its angle. If r is the length and ψ is the angle, we get

$$z = r \cdot e^{i\psi}.$$

This is the so-called polar representation of a complex number. It allows us also immediately to understand how one can calculate the product of two complex numbers using the rule $e^x \cdot e^y = e^{x+y}$. We get in polar coordinates

$$z_1 \cdot z_2 = r_1 \cdot e^{i\psi_1} \cdot r_2 \cdot e^{i\psi_2} = r_1 r_2 \cdot e^{i(\psi_1+\psi_2)}.$$

The angles add and the lengths multiply. Figure 16.4 compares the vector and the polar representations of a complex number $z = a + i \cdot b = r \cdot e^{i\psi}$.

Polar representations allow us also to easily describe division by a complex number. If a number $z \neq 0$ is given by $z = r \cdot e^{i\psi}$, then its inverse z^{-1} is given by $z = \frac{1}{r} \cdot e^{-i\psi}$, since we have

$$re^{i\psi} \cdot \tfrac{1}{r}e^{-i\psi} = e^{i\psi - i\psi} = e^0 = 1.$$

In particular, numbers of the form $e^{i\psi}$ can be considered numbers with *pure angle* ψ. Thus these numbers can be considered synonymously to angles. Adding angles corresponds to multiplication of these numbers. The behavior of these numbers nicely reflects the fact that angles are usually defined only

[3] Many consider $e^{i\pi} + 1 = 0$ to be one of the most beautiful formulas in mathematics, since it connects five very important mathematical constants: 0, 1, π, e, and i and nothing else. Furthermore, this formula involves just four different basic operators: equality, addition, multiplication, and exponentiation.

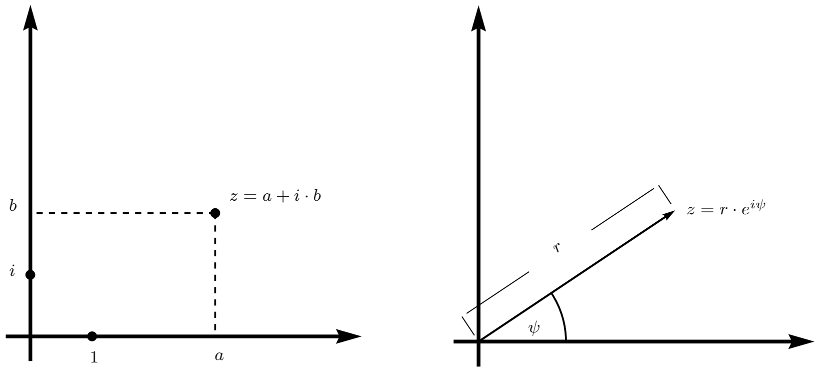

Fig. 16.4 Vector and polar representation of a complex number.

modulo 2π. Adding two angles of 270° and 180° results in an angle of 450°. For most applications, this angle is equivalent to the angle $450° - 360° = 90°$. This is reflected by calculations with numbers of the form $e^{i\psi}$. We have

$$e^{\frac{3}{2}i\pi} \cdot e^{i\pi} = e^{\frac{5}{2}i\pi} = e^{(2+\frac{1}{2})i\pi} = e^{2i\pi} \cdot e^{\frac{1}{2}i\pi} = 1 \cdot e^{\frac{1}{2}i\pi} = e^{\frac{1}{2}i\pi}.$$

16.5 Complex Conjugation

We have seen that complex addition and complex multiplication can be used nicely to express geometric transformations. Addition corresponds to translation, and multiplication can be used to express scaling and rotation (or a combination of both). There is one Euclidean transformation that is still missing: reflection.

Reflections are closely related to another operation on complex numbers: *conjugation*. Conjugation has no counterpart in the field of real numbers. Conjugation expresses a mirror reflection in the real axis of the complex number plane. The conjugate of $z = a + i \cdot b$ is denoted by $\overline{z}$ and is defined by

$$\overline{z} = \overline{a + i \cdot b} = a - i \cdot b.$$

If $z = r \cdot e^{i\psi}$ is given in polar coordinates, then the conjugate calculates as

$$\overline{z} = \overline{r \cdot e^{i\psi}} = r \cdot e^{-i\psi}.$$

Thus the conjugate of a real number is the number itself. The conjugate of a purely imaginary number is its negative. Complex conjugation is a nontrivial field automorphism. We have

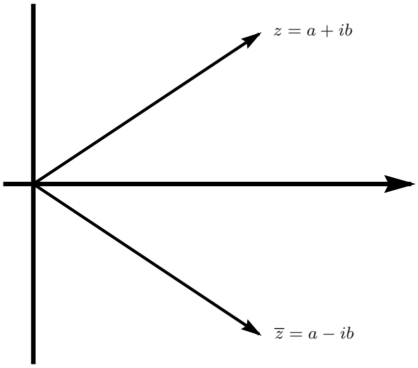

Fig. 16.5 Complex conjugation.

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \quad \text{and} \quad \overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}.$$

The first equation follows immediately from the vector representation; the second follows immediately from the polar representation. In respect of the fundamental theorem of projective geometry this means that in complex projective spaces we will have harmonic maps or collinearities that do not come from projective transformations. If we consider only continuous automorphisms, complex conjugation is the only nontrivial field automorphism of $\mathbb{C}$, so that we do not get too many types of continuous harmonic maps or collineations.

There are a few facts on complex conjugation that should be mentioned, since they will turn out to be useful later. The first couple of facts concern the relation of complex conjugates to the elementary arithmetic operations:

- Adding a number z to its own conjugate, we obtain twice the real part of the number:
$$z + \overline{z} = (a + ib) + (a - ib) = 2a.$$
- Subtracting $\overline{z}$ from z, we obtain twice the imaginary part of the number:
$$z - \overline{z} = (a + ib) - (a - ib) = 2ib.$$
- Multiplying z by $\overline{z}$, we obtain the square of the absolute value of z:
$$z \cdot \overline{z} = re^{i\psi} \cdot re^{-i\psi} = r^2 e^{i\psi - i\psi} = r^2 e^0 = r^2.$$
- Dividing z by $\overline{z}$, we obtain the number that has length 1 and twice the angle of z:
$$z/\overline{z} = re^{i\psi}/re^{-i\psi} = e^{i\psi + i\psi} = e^{2i\psi}.$$

It is a remarkable fact that complex conjugates and the four arithmetic operations are so closely related to the parameters of a complex number. We will make use of this later on. In particular, the third equation implies that one can define the absolute value of a complex number by $|z| = \sqrt{z \cdot \overline{z}}$.

Another important fact is that if we have a polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ with real parameters a_i, then as mentioned before, not all zeros of this polynomial may be real. However, if we have a complex root z of this polynomial, then $\overline{z}$ will also be a root. This can be seen easily by evaluating $f(\overline{z})$:

$$f(\overline{z}) = \sum_{i=0}^{n} a_i \overline{z}^i = \sum_{i=0}^{n} \overline{a_i}\, \overline{z}^i = \sum_{i=0}^{n} \overline{a_i z^i} = \overline{\sum_{i=0}^{n} a_i z^i} = \overline{0} = 0.$$

The second equality holds since the a_i were assumed to be real and hence we have $\overline{a_i} = a_i$.

The proof strategy we have used here may be viewed as a special case of a more general concept. If we have any complex function $f(z_1, z_2, \ldots, z_n)$ that is composed only of the four arithmetic operations and complex conjugation, then we will automatically have

$$f(\overline{z_1}, \overline{z_2}, \ldots, \overline{z_n}) = \overline{f(z_1, z_2, \ldots, z_n)}.$$