12

Projective d-space

Es ist wünschenswert, daß neben der Euklidischen Methode neuere Methoden der Geometrie in den Unterricht auf Gymnasien eingeführt werden.

Felix Klein, 1868
One of the "Theses" of his thesis defense

Mathematics is a game played according to certain simple rules with meaningless marks on paper.

David Hilbert

Different topic! So far, we have dealt almost exclusively with projective geometry of the line and of the plane. We explored the tight and very often elegant relationships between geometric objects and their algebraic representations. Our central issues were:

- Introducing elements at infinity to bypass many special cases of ordinary Euclidean geometry,
- representing geometric objects by homogeneous coordinates,
- performing algebraic operations directly on geometric objects (via homogeneous coordinates),
- performing transformations by matrix multiplication,
- duality,
- expressing geometric relations by bracket expressions.

The close interplay of homogeneous coordinates, finite and infinite elements, and linear algebra made it possible to express geometric relations by very elegant algebraic expressions. This chapter deals with generalizations of these concepts to higher dimensions.

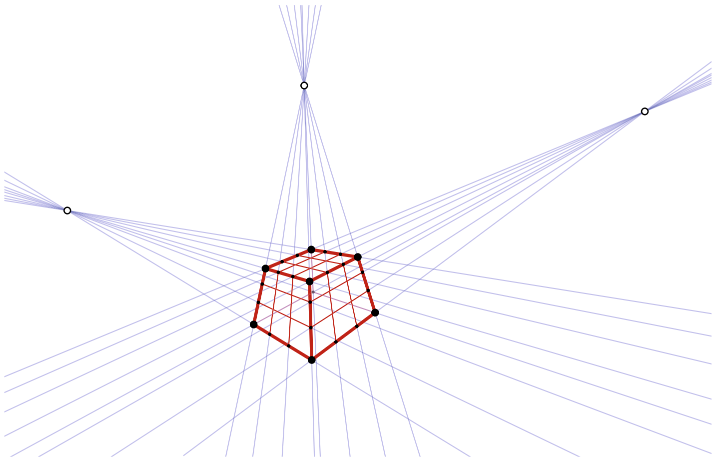

Fig. 12.1 A projectively correct cube.

Here we will just scratch the surface—in the hope that the reader might get a rough impression of the beauty and richness of the entire theory. It would be easy to fill another 600 pages with an in-depth study of projective geometry in higher dimensions. We will restrict ourselves here to representations of basic objects (*points*, *lines*, *planes*, and *transformations*) and elementary operations (*join* and *meet*). Most often we will not give explicit proofs and confine ourselves with the general concepts.

12.1 Elements at Infinity

The three-dimensional projective *space* carries many similarities to the two-dimensional projective plane. Similarly to our treatment of the two-dimensional case we will start with our investigations by considering the usual Euclidean space $\mathbb{R}^3$. Like the Euclidean plane, this space is full of special cases. Planes may, for instance, be parallel or meet in a line. Similarly to the treatment of the projective plane we will extend the usual space $\mathbb{R}^3$ by elements at infinity to get rid of many of these special cases. For every bundle of parallel lines of $\mathbb{R}^3$ we introduce *one point at infinity*. The totality of all points in $\mathbb{R}^3$ together with these infinite points forms the set of points of the three-dimensional projective space.

As usual, certain subsets of these points will be considered (projective) lines, and (since we are in the three-dimensional case) there will also be subsets that are considered projective planes. For every finite line l of $\mathbb{R}^3$ there is a unique point at infinity corresponding to the parallel bundle of this line. The finite part of l together with this point is considered a line in projective space. In addition, there are many lines that lie entirely at an infinite position.

For this consider a usual finite plane h. This plane is extended by all infinite points of lines that are contained in h. The extended plane is nothing but a usual projective plane. All infinite points of this plane form a *line at infinity*. This line consists entirely of infinite points. Every ordinary plane of $\mathbb{R}^3$ is extended by such a unique line at infinity.

There is one object we have not yet covered in our collection of points, lines, and planes. All infinite points taken together again form a projective plane: the *plane at infinity*. The lines of this plane are all the infinite lines. We could say that the real projective three-space may be considered to be $\mathbb{R}^3$ extended by a projective plane at infinity in the same way as we may say that the real projective plane is $\mathbb{R}^2$ together with a line at infinity.

In a way, this distinction of finite and infinite objects is confusing and unnecessary. It is just meant as a dictionary to connect concepts of ordinary Euclidean space to concepts of projective space. As one may expect the projective space is much more homogeneous and symmetric than the usual Euclidean space. The projective space is governed by the following incidence properties (which are again analogues of the corresponding axioms of the projective plane):

- Any *two points* span a unique *line* as long as they do not coincide.
- Any *three points* span a unique *plane* as long as they are not collinear.
- Any *two planes* meet in a unique *line* as long as they do not coincide.
- Any *three planes* meet in a unique *point* as long as they do not meet in a line.
- Any pair of *point and line* span a unique *plane* as long as the point is not on the line.
- Any pair of *plane and line* meet in a unique *point* as long as the line is not contained in the plane.

It will be the task of the next few sections to express these operations of *join* and *meet* in a suitable way that also generalizes to higher dimensions.

12.2 Homogeneous Coordinates and Transformations

How do we represent these elements algebraically? Essentially the process is similar to the setup in the projective plane. Points in projective three-space are represented by nonzero *four-dimensional* vectors. They are the homogeneous coordinates of the points. Nonzero scalar multiples of the vectors are identified. In other words, we may represent the points of the projective three-space as

$$\mathbb{RP}^3 = \frac{\mathbb{R}^4 - \{(0,0,0,0)^T\}}{\mathbb{R} - \{0\}}.$$

In the standard embedding we may “imagine” the Euclidean $\mathbb{R}^3$ embedded in $\mathbb{R}^4$ as an affine space parallel to the $(x, y, z, 0)^T$ space of $\mathbb{R}^4$. If this is too hard to imagine, then one can also proceed purely formally. In the standard embedding a point $(x, y, z)^T \in \mathbb{R}^3$ is represented by a four-dimensional vector $(x, y, z, 1)^T \in \mathbb{R}^4$. Nonzero scalar multiples are identified. The infinite points are exactly those nonzero vectors of the form $(x, y, z, 0)^T$. They do not correspond to Euclidean points. Vectors of the form $(x, y, z, 0)^T$ may also be interpreted as homogeneous coordinates of the usual *projective plane* by ignoring the last entry.

Thus we can literally say that $\mathbb{RP}^3$ consists of $\mathbb{R}^3$ (the homogeneous vectors $(x, y, z, 1)^T$) and a projective plane at infinity (the homogeneous vectors $(x, y, z, 0)^T$). We may also interpret this process inductively and consider the projective plane itself as composed of the Euclidean plane $\mathbb{R}^2$ (the vectors $(x, y, 1, 0)^T$) and a line at infinity (the vectors $(x, y, 0, 0)^T$). The line may be considered a Euclidean line (the vectors $(x, 1, 0, 0)^T$) and finally a point at infinity (represented by $(1, 0, 0, 0)^T$).

The objects dual to points in $\mathbb{RP}^3$ will be *planes*. Similarly to points, planes will also be represented by four-dimensional vectors. In Euclidean terms we may consider the vector $(a, b, c, d)^T$ as representing the parameters that describe the affine plane $\{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz + d = 0\}$. As usual, nonzero scalar multiples of the vector represent the same geometric object. If we interpret the equation in a setup of homogeneous coordinates, a point $(x, y, z, w)^T$ is incident to a plane $(a, b, c, d)^T$ if and only if

$$ax + by + cz + dw = 0.$$

As in the two-dimensional setup, incidence is simply expressed by a scalar product being zero. The plane at infinity has homogeneous coordinates $(0, 0, 0, 1)^T$.

Finding a plane that passes through three given points $p_i = (x_i, y_i, z_i, w_i)$, $i = 1, 2, 3$, thus translates to the task of solving a linear equation:

$$\begin{pmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

In the next section we will deal with methods of performing this calculation explicitly. However, before we do so we will consider transformations in this homogeneous projective setup. The situation here is almost completely analogous to the two-dimensional case. A transformation is represented by a simple matrix multiplication. This time we need a 4×4 matrix. The cases of usual affine transformations in $\mathbb{R}^3$ are again covered by special transformation matrices in which certain entries are zero. The following matrices represent a linear transformation of $\mathbb{R}^3$ with matrix A, a pure translation, a general affine transformation, and a general projective transformation, respectively:

$$\begin{pmatrix} \boxed{\begin{matrix} & & \\ & A & \\ & & \end{matrix}} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ 0\;0\;0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{pmatrix}.$$

A "$\bullet$" stands for an arbitrary number. In every case we must assume that the determinant of the matrix is nonzero.

Such a transformation T maps a point p to the associated image point. The corresponding transformation that applies to the homogeneous coordinates of a plane is (as in the two-dimensional case) given by $(T^{-1})^T$. By this choice incidence of points and planes is preserved by projective transformations:

$$\langle p, h \rangle = 0 \quad \iff \quad \langle Tp, (T^{-1})^T h \rangle = 0.$$

Instead of the transposed inverse $(T^{-1})^T$ it is also possible to use the transposed adjoint $(T^{\triangle})^T$, since they differ by only a scalar factor.

Remark 12.1. A word of caution: One should consider the setup of presenting points and planes by four-dimensional vectors and expressing coincidence by the standard scalar product as a kind of interim solution that will be replaced by something more powerful later on. The problem that we will have to face soon is that lines will be represented by *six-dimensional* vectors and we have to create a notational system that handles points, lines, and planes in a unified way. For this we will have to give up the concept of indexing a vector entry with the position where it is placed in a vector. We will return to this issue in Section 12.4.

12.3 Points and Planes in 3-Space

The task of this section is to give a closed formula for calculating the homogeneous coordinates of a plane spanned by three points. The corresponding two-dimensional situation is governed by the *cross product* operation. The line through two points p, q could be calculated by

$$l = p \times q = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \times \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} +\begin{vmatrix} p_2 & q_2 \\ p_3 & q_3 \end{vmatrix} \\ -\begin{vmatrix} p_1 & q_1 \\ p_3 & q_3 \end{vmatrix} \\ +\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \end{pmatrix}.$$

The entries of the coordinates of the line are the 2×2 subdeterminants of the matrix

$$\begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \\ p_3 & q_3 \end{pmatrix}.$$

One way of explaining this effect is to observe that an arbitrary point $\lambda p + \mu q$ on this line must have a zero scalar product with l. We obtain

$$\langle \lambda p + \mu q, l \rangle = \lambda \langle p, l \rangle + \mu \langle q, l \rangle = \lambda \det(p, p, q) + \mu \det(q, p, q).$$

The last equation holds since if we plug the expression for l into $\langle a, l \rangle$, we obtain the determinant $\det(a, p, q)$, as one can easily see if one develops this determinant by the first column.

In a similar way we can obtain the coordinates of a plane h through three points p, q, r in $\mathbb{RP}^3$. We get

$$h = \mathbf{join}(p, q, r) := \begin{pmatrix} + \begin{vmatrix} p_2 & q_2, & r_2 \\ p_3 & q_3, & r_3 \\ p_4 & q_4, & r_4 \end{vmatrix} \\ - \begin{vmatrix} p_1 & q_1, & r_1 \\ p_3 & q_3, & r_3 \\ p_4 & q_4, & r_4 \end{vmatrix} \\ + \begin{vmatrix} p_1 & q_1, & r_1 \\ p_2 & q_2, & r_2 \\ p_4 & q_4, & r_4 \end{vmatrix} \\ - \begin{vmatrix} p_1 & q_1, & r_1 \\ p_2 & q_2, & r_2 \\ p_3 & q_3, & r_3 \end{vmatrix} \end{pmatrix}.$$

So the coordinates are the 3×3 subdeterminants of the matrix

$$\begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \\ p_4 & q_4 & r_4 \end{pmatrix}.$$

Equipped with alternating signs, they give the plane through these three points. The argument for the correctness of this calculation is the same as in the two-dimensional case. If we consider an arbitrary point a, then the development of the determinant

$$\det \begin{pmatrix} a_1 & p_1 & q_1 & r_1 \\ a_2 & p_2 & q_2 & r_2 \\ a_3 & p_3 & q_3 & r_3 \\ a_4 & p_4 & q_4 & r_4 \end{pmatrix}$$

by the first column is just the scalar product $\langle a, h\rangle$ for the above definition of h. If a is spanned by p, q, and r, then this determinant becomes zero (the point a is on h). If the point is not spanned by p, q, and r, then the columns are linearly independent and the determinant is nonzero. Thus a is not on h. Geometrically speaking, we compute a vector h that is simultaneously orthogonal to all three vectors p, q, and r.

It may happen that the above operation (taking the four 3×3 subdeterminants of the 4×3 matrix) results in a zero vector. However, this could happen only if the three column vectors were linearly dependent. In this case the three points do not span a plane. Geometrically, there are several possibilities how this can happen. Either all points coincide (then the matrix has rank 1) or the points lie on a unique common line (then the matrix has rank 2). In both cases we have a degenerate situation in which the plane through the points is not uniquely determined.

As in the planar case, the procedure described here covers all possible cases of finite and infinite points. For instance, a plane through two finite points p and q and one infinite point r is the unique plane that contains the line $\overline{pq}$ and parallels in direction of r (as long as r is not on $\overline{pq}$, which is a degenerate situation). The plane through three infinite points is the plane at infinity itself.

The same trick can also be used to calculate a point that is simultaneously contained in three planes. Let h, g, f be the homogeneous coordinates of three planes in $\mathbb{RP}^3$. Assume that the three planes do not have a line in common. In this case the planes contain a unique common point. The coordinates of this point can be calculated as the 3×3 subdeterminants of the matrix

$$\begin{pmatrix} h_1 & g_1 & f_1 \\ h_2 & g_2 & f_2 \\ h_3 & g_3 & f_3 \\ h_4 & g_4 & f_4 \end{pmatrix}$$

equipped with alternating signs. As in the dual case, degenerate cases result in a zero vector. All special cases resulting, from finite or infinite points are automatically covered as well. For instance, if h and g are two finite planes and f is the plane at infinity, then the resulting point is the point at infinity on the intersection of h and g.

There is one more or less obvious but remarkable fact that we want to mention for further reference. If p, q, and r span a plane h and if a, b, and c span the same plane, then the join operation that calculates the plane from three points may have different results for p, q, and r and for a, b, and c. However, the two results may differ at most by a non-zero scalar multiple. One may view this result as a consequence of the fact that the plane h is uniquely determined and thus vectors representing it only may differ by a scalar multiple. However, one can obtain it also directly from the fact that the operation $\mathbf{join}(p, q, r)$ is linear in each argument and anticommutative.

Each of the points a, b, c is a linear combination of p, q, and r. So, if we, for instance, replace p by a, we obtain

$$\begin{aligned}\mathbf{join}(a,q,r) &= \mathbf{join}(\lambda p + \mu q + \tau r, q, r)\\ &= \lambda \cdot \mathbf{join}(p,q,r) + \mu \cdot \mathbf{join}(q,q,r) + \tau \cdot \mathbf{join}(r,q,r)\\ &= \lambda \cdot \mathbf{join}(p,q,r).\end{aligned}$$

As a consequence of antisymmetry, terms with repeated letters can be canceled.

12.4 Lines in 3-Space

Now comes the tricky part. What is the good way of representing lines of projective space? One could say that the answer to this question was, after introduction of homogeneous coordinates, one of the major breakthroughs of nineteenth-century geometry. More or less, the answer was independently discovered by at least two people. Perhaps the first was Hermann Günther Grassmann (1809–1877). In his work on *Lineare Ausdehnungslehre* [46] from 1844 he laid at the same time the basis for our modern linear algebra as well as for *multilinear algebra*. One of the essential parts was a formal method that made it possible to directly operate with points, lines, planes, etc. Unfortunately, Grassmann developed a kind of completely new mathematical terminology and notation to deal with these kinds of objects. This caused him to be more or less completely ignored by his contemporaries, and his ideas did not become common mathematical knowledge until he completely rewrote his book and published a second version [47] in 1862[1].

The second person involved was Julius Plücker (whom we have already met frequently in this book). Not aware of Grassmann's work, in 1868 he published the first part of *Neue Geometrie des Raumes, gegründet auf die Betrachtung der geraden Linie als Raumelement* [101] (*New Geometry of space, based on the straight line as space element*) but died before the second part was complete. Felix Klein at this time was his assistant and essentially completed Plücker's thoughts in his doctoral dissertation [64].

From a modern perspective the basic ideas are a straightforward generalization of our considerations in the previous section. Nevertheless, these ideas opened whole new branches of mathematics starting from projective

[1] The first edition of his book was of about 900 copies, from of about 600 were pulped since they simply could not be sold. The remaining 300 books were given away for free to anyone who showed interest. In his second edition, Grassmann expresses his regret that people do not take the time to follow another person's thoughts. It is little known among mathematicians that Grassmann was much more famous in his time for his works on Indo-Germanic linguistics. He published the first German translation of the Rig-Veda (an ancient indian document), and his Indo-German dictionaries are still in use today.

geometry in arbitrary dimensions, via multilinear or exterior algebra up to tensor calculus (which plays an omnipresent role in modern physics).

Let us return to our question: How do we represent a line in space? The correct generalization of our considerations so far is as follows. If we want to calculate the line spanned by two points p and q, then we consider the 4×2 matrix

$$\begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \\ p_3 & q_3 \\ p_4 & q_4 \end{pmatrix}.$$

From this matrix we take all 2×2 subdeterminants and collect them in a six-dimensional vector. For reasons we will investigate later, we will equip the entries of this vector only with positive signs. All in all, the coordinates for the line through the two points are

$$l = p \vee q := \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} \vee \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} +\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \\ +\begin{vmatrix} p_1 & q_1 \\ p_3 & q_3 \end{vmatrix} \\ +\begin{vmatrix} p_1 & q_1 \\ p_4 & q_4 \end{vmatrix} \\ +\begin{vmatrix} p_2 & q_2 \\ p_3 & q_3 \end{vmatrix} \\ +\begin{vmatrix} p_2 & q_2 \\ p_4 & q_4 \end{vmatrix} \\ +\begin{vmatrix} p_3 & q_3 \\ p_4 & q_4 \end{vmatrix} \end{pmatrix}.$$

Let us start to collect a few properties of this new operation $p \vee q$.

Theorem 12.1. *The operation $p \vee q$ is linear in each argument and anticommutative.*

Proof. The result is more or less obvious from the definition of the operation. Each entry of the six-dimensional vector is a 2×2 matrix that is by itself linear in p and q and anticommutative. Thus the whole operation $p \vee q$ must have these properties. □

This implies another immediate result, which is very similar to operations we have had so far.

Theorem 12.2. *Let p, q be homogeneous coordinates of two points that span a line l and let a, b be two other points that span the same line. Then $p \vee q = \lambda \cdot (a \vee b)$ for a suitable factor λ.*

Proof. The result is an immediate consequence of multilinearity and anti-commutativity. If a and b are on l, then we may express them as linear combinations of p and q, say

$$a = \lambda_1 p + \mu_1 q \text{ and } b = \lambda_2 p + \mu_2 q.$$

Since a and b span the line, the determinant of the matrix $\begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{pmatrix}$ is nonzero. Calculating the join, we obtain

$$\begin{aligned} a \vee b &= (\lambda_1 p + \mu_1 q) \vee (\lambda_2 p + \mu_2 q) \\ &= \lambda_1\lambda_2(p \vee p) + \lambda_1\mu_2(p \vee q) + \mu_1\lambda_2(q \vee p) + \mu_1\mu_2(q \vee q) \\ &= (\lambda_1\mu_2 - \mu_1\lambda_2)(p \vee q). \end{aligned}$$

The factor λ turns out to be the determinant $(\lambda_1\mu_2 - \mu_1\lambda_2)$. □

The last result is crucial. It claims that the six-dimensional vector $p \vee q$ is (up to a scalar factor, as usual) a unique representation of the line spanned by p and q. It does not depend on the special choice of the two spanning points. This is similar to our observation in the last section, where we saw that the join of three points depends only on the plane spanned by these points, and not on their particular choice.

If we accept that a good representation of lines in space are six-dimensional vectors, then we have to face another problem. There are only *four* degrees of freedom necessary to parameterize lines in space. Here is a rough but instructive way to see this: Assume you have two parallel planes in $\mathbb{R}^3$. Then almost all lines in $\mathbb{R}^3$ will intersect these planes (except those that are parallel to the planes). The two points of intersection (one on each plane) determine those lines uniquely. This makes four degrees of freedom—two for each plane. So, if we have just four degrees of freedom, how does this relate to the six-dimensional vector that represents the line? One of the degrees of freedom is "eaten up" by the irrelevant scalar factor that we have in any homogeneous approach. So there can be at least five relevant parameters in the six-dimensional vector. The reason that we have five parameters and not four is that the entries in our vector are not independent. They were defined to be the 2×2 subdeterminants of a 4×2 matrix. In Section 6.5 we learned about *Grassmann-Plücker relations* that form dependencies among such systems of sub-determinants. In particular, we have

$$\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \begin{vmatrix} p_3 & q_3 \\ p_4 & q_4 \end{vmatrix} - \begin{vmatrix} p_1 & q_1 \\ p_3 & q_3 \end{vmatrix} \begin{vmatrix} p_1 & q_1 \\ p_4 & q_4 \end{vmatrix} + \begin{vmatrix} p_2 & q_2 \\ p_3 & q_3 \end{vmatrix} \begin{vmatrix} p_2 & q_2 \\ p_4 & q_4 \end{vmatrix} = 0.$$

Thus in general, if we know five entries of a six-dimensional vector that describes a line, then we automatically know the last one (we have only to be a bit careful with degenerate cases).

Conversely, if we have a vector $l = (a, b, c, d, e, f)^T$ that satisfies the equation $af - be + cd = 0$, Theorem 7.1 implies that there are two vectors p and q with $l = p \vee q$. Vectors that satisfy such a condition are called *decomposable*.

12.5 Joins and Meets: A Universal System ...

Before we play around with line coordinates in space we will first clear up our notation and present a universal system that is capable of dealing with points, lines and planes in a unified way. The situation is even better: we will present a system that is capable of dealing with linear objects in arbitrary projective spaces of any dimension. This will be a direct generalization of the three-dimensional case.

The main problem so far is that points and planes are represented by four-dimensional vectors, while lines are represented by six-dimensional vectors. So a priori it is not clear how to define operators that work reasonably on arbitrary collections of such objects. The key observation here is to give up the idea that the coordinate entries belong to certain positions in a vector. It will be much more useful to associate meaningful labels to the different entries of a vector. In the three-dimensional case these labels will be the subsets of the set $\{1, 2, 3, 4\}$. There are exactly *four* one-element subsets; they will label the coordinate entries of a point. There are *six* two-element subsets; they will label the entries of a line. And there are *four* three-element subsets and they will be used to label the coordinate entries of a plane. So points and planes are both represented by four-dimensional vectors, but they have explicitly different meanings.

It is also reasonable to include the empty set $\{\ \}$ and the full set $\{1, 2, 3, 4\}$ in our system. We will learn about their meaning soon. If we consider the numbers of subsets sorted by their cardinality, we obtain the sequence $1, 4, 6, 4, 1$, which is simply a sequence of binomial coefficients—the fifth row of Pascal's triangle.

We will (at least partially) follow Grassmann's footprints in order to see how the whole system of *join* and *meet* operations arises. In modern terms, Grassmann states that if we want to work in projective $(d-1)$-dimensional geometry, we have to fix first of all a system of d units

$$e_1, e_2, e_3, \dots, e_d$$

that are by definition independent. You may think of them as the d-dimensional unit vectors and associate the corresponding projective points to them. These units are in a sense the most fundamental geometric objects, and any other objects will be expressible in terms of units, real numbers, and admissible operations. Furthermore, it is allowed to form *products* of units. These products are assumed to be anticommutative and linear in both factors. Thus we have

$$e_i e_j = -e_j e_i \text{ if } i \neq j \text{ and } e_i e_i = 0.$$

Products of k units are called rank-k units. Thus $e_1, \ldots, e_d$ are rank-1 units, $e_1 e_2, \ldots, e_{d-1} e_d$ are rank-2 units, and so forth. Furthermore it is allowed to form products and sums of objects. A geometric object of rank-k is a linear combination of several rank-k units. One may think of rank-k objects as representing a $(k-1)$-dimensional linear object in the corresponding projective space.

Let us see how these simple rules automatically create a system in which multiplication corresponds to well-known arithmetic operations. Let us start with the first nontrivial case $d = 2$. Rank-1 objects (points) are simply linear combinations

$$\lambda_1 e_1 + \lambda_2 e_2.$$

Up to sign change there is only one nonvanishing rank-2 unit, namely $e_1 e_2 = -e_2 e_1$. The product of two points turns out to be

$$\begin{aligned} &(\lambda_1 e_1 + \lambda_2 e_2)(\mu_1 e_1 + \mu_2 e_2) \\ &= \lambda_1 \mu_1 e_1 e_1 + \lambda_1 \mu_2 e_1 e_2 + \lambda_2 \mu_1 e_2 e_1 + \lambda_2 \mu_2 e_2 e_2 \\ &= (\lambda_1 \mu_2 - \lambda_2 \mu_1) e_1 e_2, \end{aligned}$$

which is up to the factor $e_1 e_2$ just the determinant of the points. Products of more than three points will always vanish completely.

Next is $d = 3$. We have three units e_1, e_2, e_3 and (up to sign) three rank-2 units $e_1 e_2, e_1 e_3, e_2 e_3$. Calculating the product of two points, we get

$$\begin{aligned} &(\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3)(\mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3) \\ &= \ldots \textit{expression with nine summands} \ldots \\ &= (\lambda_1 \mu_2 - \lambda_2 \mu_1) e_1 e_2 + (\lambda_1 \mu_3 - \lambda_3 \mu_1) e_1 e_3 + (\lambda_2 \mu_3 - \lambda_3 \mu_2) e_2 e_3, \end{aligned}$$

which is essentially the cross product of the two points expressed as a linear combination of rank-2 units. Thus the join operation of two points pops out automatically (except for the sign change in the middle entry). If we proceed, we see that the product of three points turns out to be

$$\begin{aligned} &(\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3)(\mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3)(\tau_1 e_1 + \tau_2 e_2 + \tau_3 e_3) \\ &= \ldots \textit{expression with 27 summands} \ldots \\ &= (\lambda_1 \mu_2 \tau_3 + \lambda_2 \mu_3 \tau_1 + \lambda_3 \mu_1 \tau_2 - \lambda_1 \mu_3 \tau_2 - \lambda_3 \mu_2 \tau_1 - \lambda_2 \mu_1 \tau_3) e_1 e_2 e_3, \end{aligned}$$

which is just the determinant of the point coordinates times $e_1 e_2 e_3$.

If we proceed in a similar manner, then we find out that for $d = 4$ the points are represented by four-dimensional objects. The products of two points is a linear combination of the six rank-2 units whose coefficients are exactly the entries of our join operation;

$$
\begin{aligned}
&(\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4)(\mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3 + \mu_4 e_4) \\
&= \ldots \textit{expression with 16 summands} \ldots \\
&= +(\lambda_1\mu_2 - \lambda_2\mu_1)e_1e_2 + (\lambda_1\mu_3 - \lambda_3\mu_1)e_1e_3 + (\lambda_1\mu_4 - \lambda_4\mu_1)e_1e_4 \\
&\quad +(\lambda_2\mu_3 - \lambda_3\mu_2)e_2e_3 + (\lambda_2\mu_4 - \lambda_4\mu_2)e_2e_4 + (\lambda_3\mu_4 - \lambda_4\mu_3)e_3e_4.
\end{aligned}
$$

(Now you see where the positive signs come from.) The product of three points gives the coordinates of the plane expressed in rank-3 units. Finally, the product of four points creates the determinant of the coordinate vectors of the points times the rank-4 unit $e_1e_2e_3e_4$.

We see that all the e_i in our expressions carry essentially no information. The only thing that counts are the indices. Furthermore, the only combinations of indices that really contribute are those with nonrepeating letters, and it suffices to consider one unit for every subset of indices. Thus we can go ahead and say that a rank-k object is a vector with $\binom{d}{k}$ entries labeled by the k-element subsets of $\{1, 2, \ldots, d\}$. We identify these subsets with the sequences of k ordered elements taken from $\{1, 2, \ldots, d\} =: E_d$. Thus the index set of a k-flat (this is a $(k-1)$-dimensional linear object) in $\mathbb{RP}^{d-1}$ is

$$\Lambda(d, k) := \{(i_1, \ldots, i_k) \in E_d^k \mid i_1 < i_2 < \cdots < i_k\}.$$

To further distinguish the Grassmann operation from ordinary multiplication we introduce the symbol "$\vee$" and call this the *join* operation. In general the join operation can be determined by the following rules. If we are working in $\mathbb{RP}^{d-1}$, we can take the join of a k-flat P and an m-flat Q if $k + m \leq d$. If $R = P \vee Q$, then R is indexed by the elements of $\Lambda(d, k+m)$. For an index $\lambda \in \Lambda(d, k+m)$ we can calculate the corresponding entry of R according to the formula

$$R_\lambda = \sum_{\substack{\lambda = \mu \cup \tau \\ \mu \in \Lambda(d,k) \\ \tau \in \Lambda(d,m)}} \operatorname{sign}(\mu, \tau) P_\mu Q_\tau.$$

Here $sign(\mu, \tau)$ is defined to be 1 or -1 depending on the parity of transpositions needed to sort the sequence (μ, τ). We will see in the next section how this formula is used in practice. It allows us to calculate the join of two arbitrary objects as long as the sum of their ranks does not exceed d. The above formula filters exactly those terms that do not cancel in the Grassmann product.

In a similar fashion we can define a *meet* operation. We will not develop here the theory behind the exact definition. It is essentially defined in a way that represents the dual of the join operation. If we work in $\mathbb{RP}^{d-1}$, we are allowed to take the meet of a k-flat P and an m-flat Q whenever $k + m \geq d$. We abbreviate the meet operator by "$\wedge$". If $R = P \wedge Q$, then R is indexed by the elements of $\Lambda(d, k+m-d)$. For an index $\lambda \in \Lambda(d, k+m-d)$ we can calculate the corresponding entry of R according to the formula

$$R_\lambda = \sum_{\substack{\lambda=\mu\cap\tau \\ \mu\in\Lambda(d,k) \\ \tau\in\Lambda(d,m)}} \text{sign}(\mu\backslash\lambda, \tau\backslash\lambda) P_\mu Q_\tau.$$

12.6 ... And How to Use It

In this section we will see what can be done with join and meet operations and how they are calculated in practice. We will restrict our considerations to $d = 4$, the spatial case. Everything carries over in a straightforward way to higher (and lower) dimensions. First of all, we will start with a kind of symbolic table that exemplifies the dimensions of the objects involved under join and meet operations. For the join we obtain

$$\begin{aligned}
\textbf{point} \vee \textbf{point} &= \textbf{line}, \\
\textbf{point} \vee \textbf{line} &= \textbf{plane}, \\
\textbf{point} \vee \textbf{plane} &= \textbf{number}, \\
\textbf{line} \vee \textbf{line} &= \textbf{number}, \\
\textbf{point} \vee \textbf{point} \vee \textbf{point} &= \textbf{plane}, \\
\textbf{point} \vee \textbf{point} \vee \textbf{line} &= \textbf{number}, \\
\textbf{point} \vee \textbf{point} \vee \textbf{point} \vee \textbf{point} &= \textbf{number}.
\end{aligned}$$

Every operation results in either a vector or a number. Whenever a zero vector or the number zero occurs as result, this indicates a degenerate situation in which the objects are dependent. For instance, usually the join of a point and a line results in the plane spanned by the point and the line. However, if the point is on the line, then the join results in the zero vector. The consecutive join of four points results in a number, and this number is just the determinant of the matrix formed by the vectors of the points as column vectors. If the points are coplanar, then this join returns the number zero. The join of two lines is zero if the two lines coincide. The join of a point and a plane is a number. In principle, this number is just the scalar product of the point and the plane. It is zero whenever the point is on the plane.

The meet operator performs the dual operations. Again the occurrence of a zero or a zero vector indicates a degenerate situation. In detail, the meets can be used to perform the following operations:

$$
\begin{aligned}
\textbf{plane} \wedge \textbf{plane} &= \textbf{line},\\
\textbf{plane} \wedge \textbf{line} &= \textbf{point},\\
\textbf{plane} \wedge \textbf{point} &= \textbf{number},\\
\textbf{line} \wedge \textbf{line} &= \textbf{number},\\
\textbf{plane} \wedge \textbf{plane} \wedge \textbf{plane} &= \textbf{point},\\
\textbf{plane} \wedge \textbf{plane} \wedge \textbf{line} &= \textbf{number},\\
\textbf{plane} \wedge \textbf{plane} \wedge \textbf{plane} \wedge \textbf{plane} &= \textbf{number}.
\end{aligned}
$$

Again it should be mentioned that all operations based on join and meet fully support all elements of projective geometry. Thus in the standard embedding of $\mathbb{R}^3$ elements at infinity are also processed correctly. For instance, the meet of two parallel planes results in a line at infinity. To achieve a standard embedding we have to make a choice of which the vectors $e_1, \ldots, e_4$ is chosen for homogenization purposes. If we choose e_4 for this purpose, we may represent a point $(x, y, z) \in \mathbb{R}^3$ as $xe_1 + ye_2 + ze_3 + e_4$. The plane at infinity then corresponds to $e_1e_2e_3$. The notation is much more familiar if we simply represent the objects with associated labels. A Euclidean point (x, y, z) and a Euclidean plane $\{(x, y, z)^t \mid ax + by + cz + d = 0\}$ then correspond to the two vectors

$$
\begin{matrix} 1\\2\\3\\4 \end{matrix}\begin{pmatrix} x\\ y\\ z\\ 1 \end{pmatrix} \quad \text{and} \quad \begin{matrix} 123\\124\\134\\234 \end{matrix}\begin{pmatrix} -d\\ c\\ -b\\ a \end{pmatrix}.
$$

The minus signs in the plane coordinates are used to compensate the sign changes caused by the join operator. The join operation of these two objects results in a single number (labeled by "1234"); this number is simply calculated as $ax + by + cz + d$, as desired. It is zero if the point and the plane coincide.

Let us perform a more elaborate example with concrete coordinates. Let us first calculate the join of two points:

$$
\begin{matrix} 1\\2\\3\\4 \end{matrix}\begin{pmatrix} 3\\ -2\\ 4\\ 1 \end{pmatrix} \vee \begin{matrix} 1\\2\\3\\4 \end{matrix}\begin{pmatrix} -3\\ 3\\ 5\\ 1 \end{pmatrix} = \begin{matrix} 12\\13\\14\\23\\24\\34 \end{matrix}\begin{pmatrix} 3{\cdot}3 - (-2){\cdot}(-3)\\ 3{\cdot}5 - 4{\cdot}(-3)\\ 3{\cdot}1 - 1{\cdot}(-3)\\ (-2){\cdot}5 - 4{\cdot}3\\ (-2){\cdot}1 - 1{\cdot}3\\ 4{\cdot}1 - 1{\cdot}5 \end{pmatrix} = \begin{matrix} 12\\13\\14\\23\\24\\34 \end{matrix}\begin{pmatrix} 3\\ 27\\ 6\\ -22\\ -5\\ -1 \end{pmatrix}.
$$

The result is a six-dimensional vector representing a line. In particular, this vector must satisfy the Grassmann-Plücker relation. The Grassmann-Plücker relation for a line l can be simply expressed as $l \vee l = \mathbf{0}$ (where $\mathbf{0}$ is the zero vector). This has a direct geometric interpretation: a line is incident to itself. In our example, we get

$$3{\cdot}(-1) - 27{\cdot}(-5) + 6{\cdot}(-22) \;=\; -3 + 135 - 132 \;=\; 0,$$

as expected. We proceed by forming the meet of this line with some plane. The result is the point where the plane intersects the line:

$$\begin{matrix}\scriptstyle 12\\ \scriptstyle 13\\ \scriptstyle 14\\ \scriptstyle 23\\ \scriptstyle 24\\ \scriptstyle 34\end{matrix}\begin{pmatrix}3\\27\\6\\-22\\-5\\-1\end{pmatrix} \wedge \begin{matrix}\scriptstyle 123\\ \scriptstyle 124\\ \scriptstyle 134\\ \scriptstyle 234\end{matrix}\begin{pmatrix}4\\2\\1\\5\end{pmatrix} = \begin{matrix}\scriptstyle 1\\ \scriptstyle 2\\ \scriptstyle 3\\ \scriptstyle 4\end{matrix}\begin{pmatrix}3{\cdot}1 - 27{\cdot}2 + 6{\cdot}4\\ 3{\cdot}5 - (-22){\cdot}2 + (-5){\cdot}4\\ 27{\cdot}5 - (-22){\cdot}1 + (-1){\cdot}4\\ 6{\cdot}5 - (-5){\cdot}1 + (-1){\cdot}2\end{pmatrix} = \begin{matrix}\scriptstyle 1\\ \scriptstyle 2\\ \scriptstyle 3\\ \scriptstyle 4\end{matrix}\begin{pmatrix}-27\\39\\153\\33\end{pmatrix}.$$

If everything went right with our calculation, the resulting point must lie on the plane itself. Thus the join of the point and the plane must be simply zero. We obtain

$$\begin{matrix}\scriptstyle 1\\ \scriptstyle 2\\ \scriptstyle 3\\ \scriptstyle 4\end{matrix}\begin{pmatrix}-27\\39\\153\\33\end{pmatrix} \vee \begin{matrix}\scriptstyle 123\\ \scriptstyle 124\\ \scriptstyle 134\\ \scriptstyle 234\end{matrix}\begin{pmatrix}4\\2\\1\\5\end{pmatrix} = {\scriptstyle 1234}\;\big((-27){\cdot}5 - 39{\cdot}1 + 153{\cdot}2 - 33{\cdot}4\big) = 0.$$

We finally will have a closer look at the representation of infinite lines (in the usual embedding of $\mathbb{R}^3$ in $\mathbb{RP}^3$). There are two ways to derive a line at infinity: either we join two infinite points or we meet two parallel planes. Either way leads to the same characterization of infinite lines, though the situations may be interpreted slightly differently from a geometric point of view. Joining two infinite points. we obtain

$$\begin{matrix}\scriptstyle 1\\ \scriptstyle 2\\ \scriptstyle 3\\ \scriptstyle 4\end{matrix}\begin{pmatrix}x_1\\y_1\\z_1\\0\end{pmatrix} \vee \begin{matrix}\scriptstyle 1\\ \scriptstyle 2\\ \scriptstyle 3\\ \scriptstyle 4\end{matrix}\begin{pmatrix}x_2\\y_2\\z_2\\0\end{pmatrix} = \begin{matrix}\scriptstyle 12\\ \scriptstyle 13\\ \scriptstyle 14\\ \scriptstyle 23\\ \scriptstyle 24\\ \scriptstyle 34\end{matrix}\begin{pmatrix}x_1y_2 - x_2y_1\\ x_1y_3 - x_3y_1\\ 0\\ x_2y_3 - x_3y_2\\ 0\\ 0\end{pmatrix}.$$

A line is infinite if the only nonzero entries are those not involving the label 4. We also see that if we restrict the line to the labels 12, 13, and 23, then the join of infinite points behaves like a cross product.

Two planes are parallel if they differ only in the 123entry. The meet of two parallel planes is an infinite line:

$$\begin{matrix}\scriptstyle 123\\ \scriptstyle 124\\ \scriptstyle 134\\ \scriptstyle 234\end{matrix}\begin{pmatrix}d_1\\c\\b\\a\end{pmatrix} \wedge \begin{matrix}\scriptstyle 123\\ \scriptstyle 124\\ \scriptstyle 134\\ \scriptstyle 234\end{matrix}\begin{pmatrix}d_2\\c\\b\\a\end{pmatrix} = \begin{matrix}\scriptstyle 12\\ \scriptstyle 13\\ \scriptstyle 14\\ \scriptstyle 23\\ \scriptstyle 24\\ \scriptstyle 34\end{matrix}\begin{pmatrix}d_1c - d_2c\\ d_1b - d_2b\\ cb - bc\\ d_1a - d_2a\\ ca - ac\\ ba - ab\end{pmatrix} = (d_1 - d_2)\cdot \begin{matrix}\scriptstyle 12\\ \scriptstyle 13\\ \scriptstyle 14\\ \scriptstyle 23\\ \scriptstyle 24\\ \scriptstyle 34\end{matrix}\begin{pmatrix}c\\b\\0\\a\\0\\0\end{pmatrix}.$$

Again, as expected, the 14, 24, and 34 entries are zero. The other three entries encode up to the usual sign changes the common normal vectors of the planes. The factor $d_1 - d_2$ can be interpreted in the following way. If the planes coincide then $d_1 = d_2$ and the meet operation results in a zero vector.