Chapter 18
Applications in Finance

A portfolio is a linear combination of assets. Each asset contributes with a weight c_j to the portfolio. The performance of such a portfolio is a function of the various returns of the assets and of the weights $c = (c_1, \dots, c_p)^\top$. In this chapter we investigate the "optimal choice" of the portfolio weights c. The optimality criterion is the mean-variance efficiency of the portfolio. Usually investors are risk-averse, therefore, we can define a mean-variance efficient portfolio to be a portfolio that has a minimal variance for a given desired mean return. Equivalently, we could try to optimize the weights for the portfolios with maximal mean return for a given variance (risk structure). We develop this methodology in the situations of (non)existence of riskless assets and discuss relations with the Capital Assets Pricing Model (CAPM).

18.1 Portfolio Choice

Suppose that one has a portfolio of p assets. The price of asset j at time i is denoted as p_{ij}. The return from asset j in a single time period (day, month, year etc.) is:

$$x_{ij} = \frac{p_{ij} - p_{i-1,j}}{p_{i-1,j}}.$$

We observe the vectors $x_i = (x_{i1}, \dots, x_{ip})^\top$ (i.e., the returns of the assets which are contained in the portfolio) over several time periods. We stack these observations into a data matrix $\mathcal{X} = (x_{ij})$ consisting of observations of a random variable

$$X \sim (\mu, \Sigma).$$

The return of the portfolio is the weighted sum of the returns of the p assets:

$$Q = c^\top X, \tag{18.1}$$

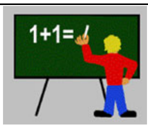
where $c = (c_1, \dots, c_p)^\top$ (with $\sum_{j=1}^p c_j = 1$) denotes the proportions of the assets in the portfolio. The mean return of the portfolio is given by the expected value of

DOI 10.1007/978-3-642-17229-8_18,

Q, which is $c^\top \mu$. The *risk* or *variance* (*squared volatility*) of the portfolio is given by the variance of Q (Theorem 4.6), which is equal to two times

$$\frac{1}{2} c^\top \Sigma c. \tag{18.2}$$

The reason for taking *half of* the variance of Q is merely technical. The optimization of (18.2) with respect to c is of course equivalent to minimizing $c^\top \Sigma c$. Our aim is to maximize the portfolio returns (18.1) given a bound on the volatility (18.2) or vice versa to minimize risk given a (desired) mean return of the portfolio.



Summary

↪ Given a matrix of returns $\mathcal{X}$ from p assets in n time periods, and that the underlying distribution is stationary, i.e., $X \sim (\mu, \Sigma)$, then the (theoretical) return of the portfolio is a weighted sum of the returns of the p assets, namely $Q = c^\top X$.

↪ The expected value of Q is $c^\top \mu$. For technical reasons one considers optimizing $\frac{1}{2} c^\top \Sigma c$. The risk or squared volatility is $c^\top \Sigma c = \mathsf{Var}(c^\top X)$.

↪ The portfolio choice, i.e., the selection of c, is such that the return is maximized for a given risk bound.

18.2 Efficient Portfolio

A variance efficient portfolio is one that keeps the risk (18.2) minimal under the constraint that the weights sum to 1, i.e., $c^\top 1_p = 1$. For a variance efficient portfolio, we therefore try to find the value of c that minimizes the Lagrangian

$$\mathcal{L} = \frac{1}{2} c^\top \Sigma c - \lambda (c^\top 1_p - 1). \tag{18.3}$$

A mean-variance efficient portfolio is defined as one that has minimal variance among all portfolios with the same mean. More formally, we have to find a vector of weights c such that the variance of the portfolio is minimal subject to two constraints:

1. a certain, pre-specified mean return $\overline{\mu}$ has to be achieved,
2. the weights have to sum to one.

Mathematically speaking, we are dealing with an optimization problem under two constraints.

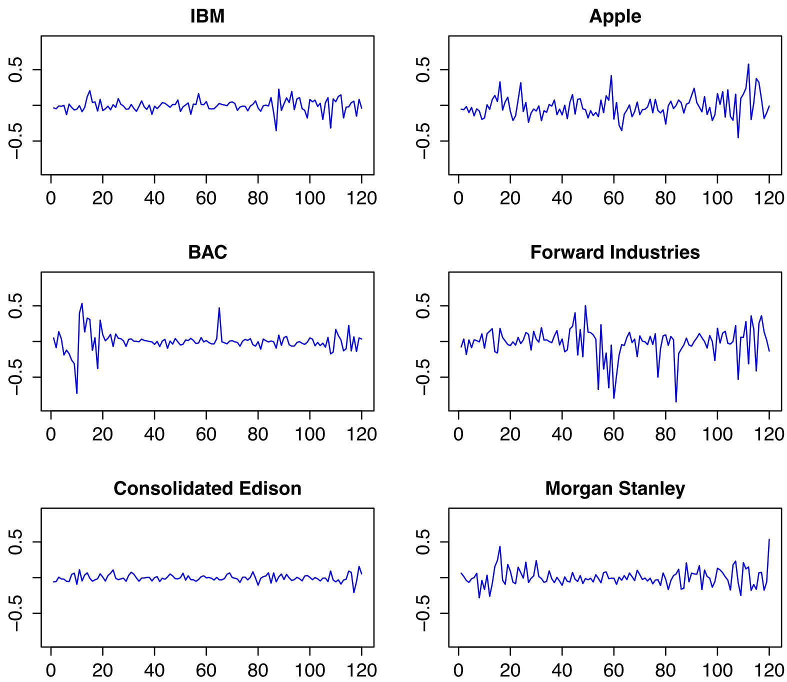


Fig. 18.1 Returns of six firms from January 2000 to December 2009 `MVAreturns`

The Lagrangian function for this problem is given by

$$\mathcal{L} = c^\top \Sigma c + \lambda_1(\overline{\mu} - c^\top \mu) + \lambda_2(1 - c^\top 1_p).$$

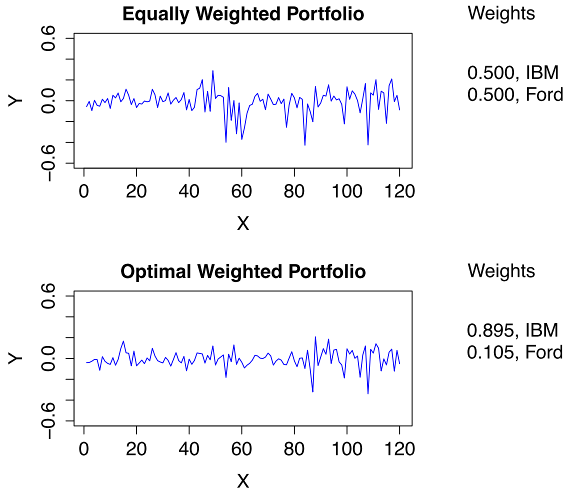
With tools presented in Section 2.4 we can calculate the first order condition for a minimum:

$$\frac{\partial \mathcal{L}}{\partial c} = 2\Sigma c - \lambda_1 \mu - \lambda_2 1_p = 0. \tag{18.4}$$

Example 18.1 Figure 18.1 shows the monthly returns from January 2000 to December 2009 of six stocks. The data is from Yahoo Finance. For each stock we have chosen the same scale on the vertical axis (which gives the return of the stock). Note how the return of some stocks, such as Forward Industries and Apple, are much more volatile than the returns of other stocks, such as IBM or Consolidated Edison (Electric utilities).

As a very simple example consider two differently weighted portfolios containing only two assets, IBM and Forward Industries. Figure 18.2 displays the monthly returns of the two portfolios. The portfolio in the upper panel consists of approximately 10% Forward Industries assets and 90% IBM assets. The portfolio in the lower panel contains an equal proportion of each of the assets. The text windows on the right of Figure 18.2 show the exact weights which were used. We can clearly see

Fig. 18.2 Portfolio of IBM and forward industries assets, equal and efficient weights
MVAportfol



that the returns of the portfolio with a higher share of the IBM assets (which have a low variance) are much less volatile.

For an exact analysis of the optimization problem (18.4) we distinguish between two cases: the existence and nonexistence of a riskless asset. A riskless asset is an asset such as a zero bond, i.e., a financial instrument with a fixed nonrandom return (Franke, Härdle and Hafner, 2011).

Nonexistence of a Riskless Asset

Assume that the covariance matrix Σ is invertible (which implies positive definiteness). This is equivalent to the nonexistence of a portfolio c with variance $c^\top \Sigma c = 0$. If all assets are uncorrelated, Σ is invertible if all of the asset returns have positive variances. A riskless asset (uncorrelated with all other assets) would have zero variance since it has fixed, nonrandom returns. In this case Σ would not be positive definite.

The optimal weights can be derived from the first order condition (18.4) as

$$c = \frac{1}{2}\Sigma^{-1}(\lambda_1 \mu + \lambda_2 1_p). \tag{18.5}$$

Multiplying this by a $(p \times 1)$ vector 1_p of ones, we obtain

$$1 = 1_p^\top c = \frac{1}{2} 1_p^\top \Sigma^{-1}(\lambda_1 \mu + \lambda_2 1_p^\top),$$

which can be solved for λ_2 to get:

$$\lambda_2 = \frac{2 - \lambda_1 1_p^\top \Sigma^{-1} \mu}{1_p^\top \Sigma^{-1} 1_p}.$$

Plugging this expression into (18.5) yields

$$c = \frac{1}{2}\lambda_1 \left(\Sigma^{-1}\mu - \frac{1_p^\top \Sigma^{-1}\mu}{1_p^\top \Sigma^{-1} 1_p} \Sigma^{-1} 1_p \right) + \frac{\Sigma^{-1} 1_p}{1_p^\top \Sigma^{-1} 1_p}. \tag{18.6}$$

For the case of a variance efficient portfolio there is no restriction on the mean of the portfolio ($\lambda_1 = 0$). The optimal weights are therefore:

$$c = \frac{\Sigma^{-1} 1_p}{1_p^\top \Sigma^{-1} 1_p}. \tag{18.7}$$

This formula is identical to the solution of (18.3). Indeed, differentiation with respect to c gives

$$\begin{aligned} \Sigma c &= \lambda 1_p \\ c &= \lambda \Sigma^{-1} 1_p. \end{aligned}$$

If we plug this into (18.3), we obtain

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\lambda^2 1_p \Sigma^{-1} 1_p - \lambda(\lambda 1_p \Sigma^{-1} 1_p - 1) \\ &= \lambda - \frac{1}{2}\lambda^2 1_p \Sigma^{-1} 1_p. \end{aligned}$$

This quantity is a function of λ and is minimal for

$$\lambda = (1_p \Sigma^{-1} 1_p)^{-1}$$

since

$$\frac{\partial^2 \mathcal{L}}{\partial c^\top \partial c} = \Sigma > 0.$$

Theorem 18.1 *The variance efficient portfolio weights for returns $X \sim (\mu, \Sigma)$ are*

$$c_{opt} = \frac{\Sigma^{-1} 1_p}{1_p^\top \Sigma^{-1} 1_p}. \tag{18.8}$$

Existence of a Riskless Asset

If an asset exists with variance equal to zero, then the covariance matrix Σ is not invertible. The notation can be adjusted for this case as follows: denote the return of the riskless asset by r (under the absence of arbitrage this is the interest rate), and partition the vector and the covariance matrix of returns such that the last component is the riskless asset. Thus, the last equation of the system (18.4) becomes

$$2\,\mathsf{Cov}(r, X) - \lambda_1 r - \lambda_2 = 0,$$

and, because the covariance of the riskless asset with any portfolio is zero, we have

$$\lambda_2 = -r\lambda_1. \tag{18.9}$$

Let us for a moment modify the notation in such a way that in each vector and matrix the components corresponding to the riskless asset are excluded. For example, c is the weight vector of the *risky* assets (i.e., assets with positive variance), and c_0 denotes the proportion invested in the riskless asset. Obviously, $c_0 = 1 - 1_p^\top c$, and Σ the covariance matrix of the *risky* assets, is assumed to be invertible. Solving (18.4) using (18.9) gives

$$c = \frac{\lambda_1}{2}\Sigma^{-1}(\mu - r1_p). \tag{18.10}$$

This equation may be solved for λ_1 by plugging it into the condition $\mu^\top c = \overline{\mu}$. This is the mean-variance efficient weight vector of the risky assets if a riskless asset exists. The final solution is:

$$c = \frac{\overline{\mu}\Sigma^{-1}(\mu - r1_p)}{\mu^\top\Sigma^{-1}(\mu - r1_p)}. \tag{18.11}$$

The variance optimal weighting of the assets in the portfolio depends on the structure of the covariance matrix as the following corollaries show.

Corollary 18.1 *A portfolio of uncorrelated assets whose returns have equal variances* ($\Sigma = \sigma^2\mathcal{I}_p$) *needs to be weighted equally*:

$$c_{opt} = p^{-1}1_p.$$

Proof Here we obtain $1_p^\top\Sigma^{-1}1_p = \sigma^{-2}1_p^\top 1_p = \sigma^{-2}p$ and therefore $c = \frac{\sigma^{-2}1_p}{\sigma^{-2}p} = p^{-1}1_p$. □

Corollary 18.2 *A portfolio of correlated assets whose returns have equal variances, i.e.*,

$$\Sigma = \sigma^2\begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{pmatrix}, \quad -\frac{1}{p-1} < \rho < 1$$

needs to be weighted equally:

$$c_{opt} = p^{-1}1_p.$$

Proof Σ can be rewritten as $\Sigma = \sigma^2\{(1-\rho)\mathcal{I}_p + \rho 1_p 1_p^\top\}$. The inverse is

$$\Sigma^{-1} = \frac{\mathcal{I}_p}{\sigma^2(1-\rho)} - \frac{\rho 1_p 1_p^\top}{\sigma^2(1-\rho)\{1+(p-1)\rho\}}$$

since for a $(p \times p)$ matrix $\mathcal{A}$ of the form $\mathcal{A} = (a-b)\mathcal{I}_p + b 1_p 1_p^\top$ the inverse is generally given by

$$\mathcal{A}^{-1} = \frac{\mathcal{I}_p}{(a-b)} - \frac{b\, 1_p 1_p^\top}{(a-b)\{a+(p-1)b\}}.$$

Hence

$$\begin{aligned}\Sigma^{-1} 1_p &= \frac{1_p}{\sigma^2(1-\rho)} - \frac{\rho 1_p 1_p^\top 1_p}{\sigma^2(1-\rho)\{1+(p-1)\rho\}} \\ &= \frac{[\{1+(p-1)\rho\} - \rho p] 1_p}{\sigma^2(1-\rho)\{1+(p-1)\rho\}} = \frac{\{1-\rho\} 1_p}{\sigma^2(1-\rho)\{1+(p-1)\rho\}} \\ &= \frac{1_p}{\sigma^2\{1+(p-1)\rho\}}\end{aligned}$$

which yields

$$1_p^\top \Sigma^{-1} 1_p^\top = \frac{p}{\sigma^2\{1+(p-1)\rho\}}$$

and thus $c = p^{-1} 1_p$. □

Let us now consider assets with different variances. We will see that in this case the weights are adjusted to the risk.

Corollary 18.3 *A portfolio of uncorrelated assets with returns of different variances, i.e.,* $\Sigma = \operatorname{diag}(\sigma_1^2, \dots, \sigma_p^2)$, *has the following optimal weights*

$$c_{j,opt} = \frac{\sigma_j^{-2}}{\sum_{l=1}^p \sigma_l^{-2}}, \quad j = 1, \dots, p.$$

Proof From $\Sigma^{-1} = \operatorname{diag}(\sigma_1^{-2}, \dots, \sigma_p^{-2})$ we have $1_p^\top \Sigma^{-1} 1_p^\top = \sum_{l=1}^p \sigma_l^{-2}$ and therefore the optimal weights are $c_j = \sigma_j^{-2} / \sum_{l=1}^p \sigma_l^{-2}$. □

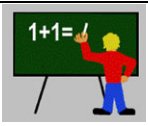
This result can be generalized for covariance matrices with block structures.

Corollary 18.4 *A portfolio of assets with returns* $X \sim (\mu, \Sigma)$, *where the covariance matrix has the form*:

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 & \dots & 0 \\ 0 & \Sigma_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \Sigma_r \end{pmatrix}$$

has optimal weights $c = (c_1, \ldots, c_r)^\top$ *given by*

$$c_{j,opt} = \frac{\Sigma_j^{-1} 1}{1^\top \Sigma_j^{-1} 1}, \quad j = 1, \ldots, r.$$



Summary

↪ An efficient portfolio is one that keeps the risk minimal under the constraint that a given mean return is achieved and that the weights sum to 1, i.e., that minimizes $\mathcal{L} = c^\top \Sigma c + \lambda_1(\overline{\mu} - c^\top \mu) + \lambda_2(1 - c^\top 1_p)$.

↪ If a riskless asset does not exist, the variance efficient portfolio weights are given by

$$c = \frac{\Sigma^{-1} 1_p}{1_p^\top \Sigma^{-1} 1_p}.$$

↪ If a riskless asset exists, the mean-variance efficient portfolio weights are given by

$$c = \frac{\overline{\mu} \Sigma^{-1}(\mu - r 1_p)}{\mu^\top \Sigma^{-1}(\mu - r 1_p)}.$$

↪ The efficient weighting depends on the structure of the covariance matrix Σ. Equal variances of the assets in the portfolio lead to equal weights, different variances lead to weightings proportional to these variances:

$$c_{j,opt} = \frac{\sigma_j^{-2}}{\sum_{l=1}^{p} \sigma_l^{-2}}, \quad j = 1, \ldots, p.$$

18.3 Efficient Portfolios in Practice

We can now demonstrate the usefulness of this technique by applying our method to the monthly market returns computed on the basis of transactions at the New York stock market and the NASDAQ stock market between January 2000 to December 2009 (Berndt, 1990).

Example 18.2 Recall that we had shown the portfolio returns with uniform and optimal weights in Figure 18.2. The covariance matrix of the returns of IBM and

Forward Industries is

$$\mathcal{S} = \begin{pmatrix} 0.0073 & 0.0023 \\ 0.0023 & 0.0454 \end{pmatrix}.$$

Hence by (18.7) the optimal weighting is

$$\widehat{c} = \frac{\mathcal{S}^{-1}1_2}{1_2^\top \mathcal{S}^{-1}1_2} = (0.8952, 0.1048)^\top.$$

The effect of efficient weighting becomes even clearer when we expand the portfolio to six assets. The covariance matrix for the returns of all six firms introduced in Example 18.1 is

$$\mathcal{S} = 10^{-3} \begin{pmatrix} 7.3 & 6.2 & 3.1 & 2.3 & -0.1 & 5.2 \\ 6.2 & 23.9 & 4.3 & 2.1 & 0.4 & 6.4 \\ 3.1 & 4.3 & 19.5 & -0.9 & 1.1 & 3.7 \\ 2.3 & 2.1 & -0.9 & 45.4 & -2.1 & 0.8 \\ -0.1 & 0.4 & 1.1 & -2.1 & 2.4 & -0.1 \\ 5.2 & 6.4 & 3.7 & 0.8 & -0.1 & 14.7 \end{pmatrix}.$$

Hence the optimal weighting is

$$\widehat{c} = \frac{\mathcal{S}^{-1}1_6}{1_6^\top \mathcal{S}^{-1}1_6} = (0.1894, -0.0139, 0.0094, 0.0580, 0.7112, 0.0458)^\top.$$

As we can clearly see, the optimal weights are quite different from the equal weights ($c_j = 1/6$). The weights which were used are shown in text windows on the right hand side of Figure 18.3.

This efficient weighting assumes stable covariances between the assets over time. Changing covariance structure over time implies weights that depend on time as well. This is part of a large body of literature on multivariate volatility models. For a review refer to Franke et al. (2011).

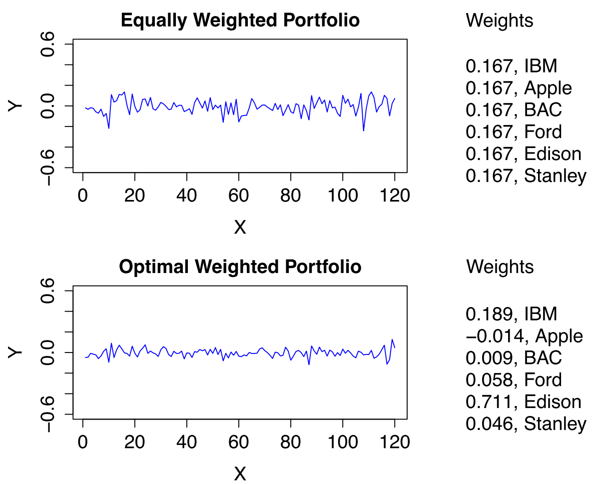
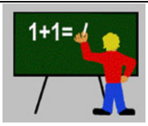


Fig. 18.3 Portfolio of all six assets, equal and efficient weights MVAportfol



Summary

↪ Efficient portfolio weighting in practice consists of estimating the covariances of the assets in the portfolio and then computing efficient weights from this empirical covariance matrix.

↪ Note that this efficient weighting assumes stable covariances between the assets over time.

18.4 The Capital Pricing Model (CAPM)

The CAPM considers the relation between a mean-variance efficient portfolio and an asset uncorrelated with this portfolio. Let us denote this specific asset return by y_0. The riskless asset with constant return $y_0 \equiv r$ may be such an asset. Recall from (18.4) the condition for a mean-variance efficient portfolio:

$$2\Sigma c - \lambda_1 \mu - \lambda_2 1_p = 0.$$

In order to eliminate λ_2, we can multiply (18.4) by $c^\top$ to get:

$$2c^\top \Sigma c - \lambda_1 \bar{\mu} = \lambda_2.$$

Plugging this into (18.4), we obtain:

$$\begin{aligned} 2\Sigma c - \lambda_1 \mu &= 2c^\top \Sigma c 1_p - \lambda_1 \bar{\mu} 1_p \\ \mu &= \bar{\mu} 1_p + \frac{2}{\lambda_1}(\Sigma c - c^\top \Sigma c 1_p). \end{aligned} \tag{18.12}$$

For the asset that is uncorrelated with the portfolio, equation (18.12) can be written as:

$$y_0 = \bar{\mu} - \frac{2}{\lambda_1} c^\top \Sigma c$$

since $y_0 = r$ is the mean return of this asset and is otherwise uncorrelated with the risky assets. This yields:

$$\lambda_1 = 2\frac{c^\top \Sigma c}{\bar{\mu} - y_0} \tag{18.13}$$

and if (18.13) is plugged into (18.12):

$$\begin{aligned} \mu &= \bar{\mu} 1_p + \frac{\bar{\mu} - y_0}{c^\top \Sigma c}(\Sigma c - c^\top \Sigma c 1_p) \\ \mu &= y_0 1_p + \frac{\Sigma c}{c^\top \Sigma c}(\bar{\mu} - y_0) \\ \mu &= y_0 1_p + \beta(\bar{\mu} - y_0) \end{aligned} \tag{18.14}$$

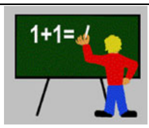
with

$$\beta \stackrel{\text{def}}{=} \frac{\Sigma c}{c^\top \Sigma c}.$$

The relation (18.14) holds if there exists any asset that is uncorrelated with the mean-variance efficient portfolio c. The existence of a riskless asset is not a necessary condition for deriving (18.14). However, for this special case we arrive at the well-known expression

$$\mu = r 1_p + \beta(\bar{\mu} - r), \tag{18.15}$$

which is known as the *Capital Asset Pricing Model* (CAPM), see Franke et al. (2011). The *beta factor* β measures the relative performance with respect to riskless assets or an index. It reflects the sensitivity of an asset with respect to the whole market. The beta factor is close to 1 for most assets. A factor of 1.16, for example, means that the asset reacts in relation to movements of the whole market (expressed through an index like DAX or DOW JONES) 16 percents stronger than the index. This is of course true for both positive and negative fluctuations of the whole market.



Summary

- ↪ The weights of the mean-variance efficient portfolio satisfy $2\Sigma c - \lambda_1 \mu - \lambda_2 1_p = 0$.
- ↪ In the CAPM the mean of X depends on the riskless asset and the pre-specified mean $\overline{\mu}$ as follows $\mu = r 1_p + \beta(\overline{\mu} - r)$.
- ↪ The beta factor β measures the relative performance with respect to riskless assets or an index and reflects the sensitivity of an asset with respect to the whole market.

18.5 Exercises

Exercise 18.1 Prove that the inverse of $\mathcal{A} = (a-b)\mathcal{I}_p + b 1_p 1_p^\top$ is given by

$$\mathcal{A}^{-1} = \frac{\mathcal{I}_p}{(a-b)} - \frac{b\, 1_p 1_p^\top}{(a-b)\{a + (p-1)b\}}.$$

Exercise 18.2 The empirical covariance between the 120 returns of IBM and Forward Industries is 0.0023 (see Example 18.2). Test if the true covariance is zero. Hint: Use Fisher's Z-transform.

Exercise 18.3 Explain why in both Figures 18.2 and 18.3 the portfolios have negative returns just before the end of the series, regardless of whether they are optimally weighted or not! (What happened in in the mid 2007?)

Exercise 18.4 Apply the method used in Example 18.2 on the same data (Table B.5) including also the Digital Equipment company. Obviously one of the weights is negative. Is this an efficient weighting?

Exercise 18.5 In the CAPM the β value tells us about the performance of the portfolio relative to the riskless asset. Calculate the β value for each single stock price series relative to the "riskless" asset IBM.