Chapter 5
Fuzzy Rough Multiple Objective Decision Making

The concept of a rough set was first raised by Pawlak [340]. Then Liu [226] proposed the fuzzy rough (Fu-Ro) variable by combining the fuzzy variable and rough variable. Xu and Zhao [343] discussed the properties of Fu-Ro variable, and introduced the Fu-Ro multi-objective decision making models and the ways to deal with them, some crisp equivalent models are given and relative algorithms are proposed to solve the problems.

In this chapter, we first introduce the Fu-Ro variable, then the arithmetic and the properties of the Fu-Ro variable. Based on the expected value operator and chance operator of the Fu-Ro variable, three parts are presented respectively:

(1) Fuzzy rough expected value decision-making model(Fu-Ro EVM).

(2) Fuzzy rough chance constraint decision-making model(Fu-Ro CCM).

(3) Fuzzy rough dependent chance decision-making model(Fu-Ro DCM).

Finally, an application to reuse an integrated logistics network design problem under fuzzy rough environment is presented to show the effectiveness of the above three models.

5.1 Integrated Logistics Problem under Fuzzy Rough Environment

In recent years, the logistics system has been gaining importance due to the increasing market globalization competitiveness. At first, most of the scholars just researched forward logistics, and there are many papers on this subject. Then scholars realized a growth of interest in transporting items that could be recycled or reused, and the possible commercial returns and saw that reverse logistics is an important problem requiring careful consideration.

Reverse logistics was first mentioned in the early 1990s. Two papers about reverse logistics from the American GLM (Council of Logistics Management) mark the start of research into reverse logistics [81, 82]. The first paper represents the research results of Stock (1992), who proposed the relevance among the fields of reverse

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logistics, business and social. A year later, Kopicki and other scholars researched the actual operation and rules of reverse logistics, including the factors of re-use and recycling. Kostecki (1998) discussed reverse logistics as the way to extend the life cycle of products. The same year, Stock reported in detail how to set up a reverse logistics and implementation plan. Roger (1999) and Tibber-Lembke collected extensive data on reverse logistics business operation examples, especially in the United States, where the two researchers wrote a lot of logistics optimization management articles. After that, many problems regarding reverse logistics were discussed. Some papers under assumed a crisp environment, like [83, 84, 85, 86, 87], but many uncertainties were found to exist in a reverse logistics system, so the research of uncertain reverse logistics began, as in [88, 89].

In more recent years, forward and reverse logistics were integrated to build the integrated logistics system. In 1997, Fleischmann and Jacqueline first did some research on integrated logistics[90], wherein they integrated forward logistics and reverse logistics to construct a close-loop integrated logistics system. Hyun Jeung Ko[91] presented a mixed integer nonlinear programming model from the perspective of the third party for the design of a dynamic integrated distribution network to account for the integrated goals of simultaneously optimizing the forward and return network. There was very little literature about integrated logistics. The integrated logistics system is often applied in practice, but the theoretical study of integrated logistics has lagged behind. It is therefore necessary for scholars to research and develop this field.

The integrated logistics system is composed of the forward logistics system and the reverse logistics system. Forward logistics systems are usually the same, they all deliver the new product from the first producers to the last customers. However, there are different reverse logistics networks structures according to different kinds of reverse goods, such as reuse, remanufacturing, recycling and commercial return. If the purpose of the reverse logistics system is for reusing items, then the integrated logistics can be called a reuse integrated logistics system.

In every day life, re-usable packages and containers such as glass bottles, plastic bottles, cans, boxes and pallets are widely used in the food and chemical industries. As more and more people realize the importance of environmental protection, more and more producers want to reuse the recycled items to reduce resource waste. So in this paper, we concentrate on the reuse reverse logistics system.

The process of the reuse integrated logistics network is that the re-usable packages are gathered by collectors and processed by recyclers to then be sent to the producers to reuse. New products which use the recycled packages are produced and again get into the forward logistics and are finally consumed by the customers. After that, the same process recurs. Thus, the whole integrated logistics system operates in cycles like this and forms a closed loop system.

There are three main establishments in an integrated logistics network:

(1) Collectors: have the responsibility of collecting reusable packages that are scattered around.

(2) Recyclers or expanded distributors: receive the items from collectors, and their work concentrates on detecting, cleaning and processing the used items to such a state that they are undifferentiated from new items, and then these items will be delivered to enterprises again.
(3) Disposal places: process the waste items that can no longer be used. The integrated logistics system also can be described as in the following Figure. 5.1.

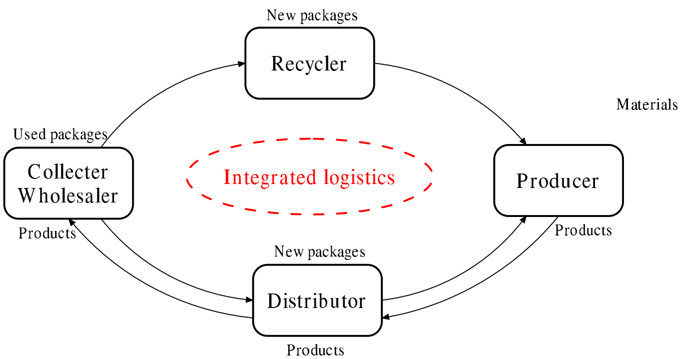

Fig. 5.1 Integrated logistics system

The integrated logistics network also includes the forward logistics network and the reverse logistics network. In forward logistics, there are producers, distributors and main wholesalers, and in reuse reverse logistics network there are collectors, recyclers/ expanded distributors, final disposal places and producers, and the associated transport routes.

Unfortunately, the integrated logistics network design problem is subject to many sources of uncertainty. In a practical decision-making process, we often face a hybrid uncertain environment. To deal with this twofold uncertainty, the concept of the fuzzy rough variable was proposed to depict the phenomena in which fuzziness and roughness appear simultaneously.

In this next section, we consider the reuse integrated logistics system, items which can be re-used after simple treatment, mainly package containers and auxiliary materials such as trays. In the integrated logistics network problem, it is hard to describe these problem parameters as known variables. For instance, since people usually drink more beer in summer and autumn, and less beer in winter and spring, that is, the demand of beer is seasonal. When we forecast the demand in a period, we may use the fuzzy variable to estimate, for example, we give a middle value μ, two spread α and β. Further more, the middle value μ is usually not a certain number, because when we design the network of the network of a reuse integrated logistics network, the period we consider will definitely cover the whole season, so the it is appropriate to use a rough variable to describe the middle value μ. So until now, in this situation, we can use fuzzy rough variables to describe the demand of the beer. Because the amount of the used packages is relevant to the consumption of the product, so it is natural to consider that the quantity of used packages is also a fuzzy rough variable, just as is that of the demand for the products.

The following Figure. 5.2 describes this kind of reuse integrated logistics network.

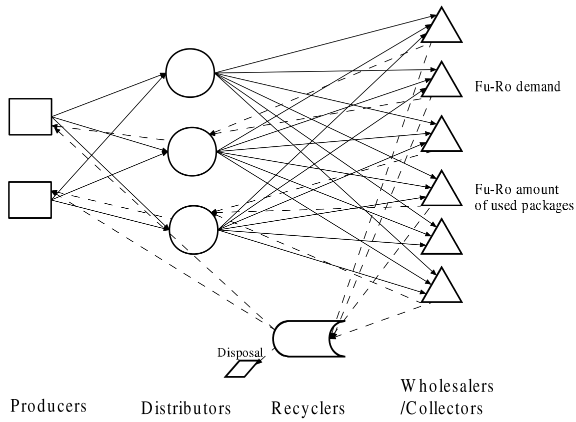

Fig. 5.2 Conceptual model of reuse integrated logistics network

However, there is no attempt to research another mixed environment, where fuzziness and roughness both appear simultaneously. For some seasonal items (Ice cream, Christmas trees, woolen materials), the demand may vary year to year. According to the historical data or abundance of information or the experiences of experts, we can know the demand in one year is a certain fuzzy variable. However, the middle value of the fuzzy variable is vague and varies year to year. The result is that decision makers are unable to achieve a better decision. Hence, we have to consider it as an uncertain variable. A rough variable can be applied to depict it well if the average sold amount is clear according to the statistical data of every year. Thus, the demand of some seasonal items can be described as a fuzzy rough variable to help decision makers develop better strategies.

5.2 Fu-Ro Variable

Let's introduce the basic knowledge of Fu-Ro variables, which include the definition, the chance measure, the expected value, and the optimistic and pessimistic value.

5.2.1 Definition of Fu-Ro Variable

Before the introduction of the concept of fuzzy rough variables, let's recall some definitions and properties of rough sets.

The rough sets theory introduced by Pawlak [340, 377] has often proved to be an excellent mathematical tool for the analysis of a vague description of objects (called actions in decision problems). The adjective vague, referring to the quality of information, means inconsistency or ambiguity which follows from information granulation. The rough sets philosophy is based on the assumption that with every object of the universe there is associated a certain amount of information (data, knowledge), expressed by means of some attributes used for object description. Objects having the same description are indiscernible (similar) with respect to the available information. The indiscernibility relation thus generated constitutes a mathematical basis for the rough sets theory; it induces a partition of the universe into blocks of indiscernible objects, called elementary sets, that can be used to build knowledge about a real or abstract world. The use of the indiscernibility relation results in information granulation. The rough sets theory, dealing with the representation and processing of vague information, presents a series of intersections and complements with respect to many other theories and mathematical techniques handling imperfect information, like probability theory, evidence theory of DempsterShafer, fuzzy sets theory, discriminant analysis and mereology [373, 374, 375, 376, 341, 378, 379, 380].

For algorithmic reasons, the information regarding the objects is supplied in the form of a data table, whose separate rows refer to distinct objects (actions), and whose columns refer to di.erent attributes considered. Each cell of this table indicates an evaluation (quantitative or qualitative) of the object placed in that row by means of the attribute in the corresponding column.

Formally, a data table is the 4-tuple $\mathbf{S} = (U,Q,V,f)$, where U is a finite set of objects (universe), $Q = q_1, q_2, \cdots, q_n$is a finite set of attributes, V_q is the domain of the attribute, $V = \bigcup_{q \in Q} V_q$ and $f : U \times Q \to V$ is a total function such that $f(x,q) \in V_q$ for each $x \in U, q \in Q$, called *information function*.

Therefore, each object x of U is described by a vector (string) $Des_Q(x) = (f(x,q_1), f(x,q_2), \cdots, f(x,q_m))$, called description of x in terms of the evaluations of the attributes from Q; it represents the available information about x.

To every (non-empty) subset of attributes P is associated an indiscernibility relation on U, denoted by I_P:

$$I_p = \{(x,y)| \in U \times U : f(x,q) = f(y,q) \forall q \P\}.$$

If $(x,y) \in I_p$, it is said that the objects x and y are P-indiscernible. Clearly, the indiscernibility relation thus de.ned is an equivalence relation (reflexive, symmetric and transitive). The family of all the equivalence classes of the relation I_P is denoted by $U|I_P$ and the equivalence class containing an element $x \in U$ by $I_p(x)$. The equivalence classes of the relation I_P are called *P-elementary sets*. If $P = Q$, the Q-elementary sets are called *atoms*.

Let $\mathbf{S}$ be a data table, X a non-empty subset of U and $\Phi \neq P \subseteq Q$. The *P-lower* approximation and the *P-upper* approximation of X in $\mathbf{S}$ are defined, respectively, by:

$$\underline{P}(X) = \{x \in U : I_p(x) \subseteq X\},$$

$$\bar{P}(X) = \bigcup_{x \in X} I_P(X).$$

The elements of $\underline{P}(X)$ are all and only those objects $x \in U$ which belong to the equivalence classes generated by the indiscernibility relation I_P, contained in X; the elements of $\bar{P}(X)$ are all and only those objects $x \in U$ which belong to the equivalence classes generated by the indiscernibility relation I_P, containing at least one object x belonging to X. In other words, $\underline{P}X$ is the largest union of the P-elementary sets included in X, while $\bar{P}(X)$ is the smallest union of the P-elementary sets containing X.

(1) The *P-boundary* of X in S, denoted by $Bn_P(X)$, is $Bn_P(X) = \bar{P}(X) - \underline{P}(X)$.

(2) The following relation holds: $\underline{P}(X) \subseteq X \subseteq \bar{P}(X)$.

Therefore, if an object x belongs to $\underline{P}(X)$, it is certainly also an element of X, while if x belongs to $\bar{P}(X)$, it may belong to the set X. $Bn_P(X)$ constitutes the "doubtful region" of X: nothing can be said with certainty about the belonging of its elements to the set X.

The following relation, called *complementarity property*, is satisfied: $\underline{P}(X) = U - \bar{P}(U - X)$.

If the P-boundary of X is empty, $Bn_P(X) = \Phi$, then the set X is an ordinary (exact) set with respect to P, that is, it may be expressed as the union of a certain number of P-elementary sets; otherwise, if $Bn_P(X) \neq \Phi$, the set X is an approximate (rough) set with respect to P and may be characterized by means of the approximations $\underline{P}(X)$ and $\bar{P}(X)$. The family of all the sets $X \subseteq U$ having the same P-lower and P-upper approximations is called a *rough set*.

The following ratio defines an accuracy of the approximation of X, $X \neq \Phi$ by means of the attributes from P:

$$\alpha_P(X) = \frac{|\underline{P}(X)|}{|\bar{P}(X)|},$$

where $|Y|$ indicates the cardinality of a (finite) set Y. Obviously, $0 \leq \alpha_P(X) \leq 1$; if $\alpha_P(X) = 1$, X is an ordinary (exact) set with respect to P; if $\alpha_P(X) = 1$, X is a rough (vague) set with respect to P.

Another ratio defines a quality of the approximation of X by means of the attributes from P:

$$\gamma_P(X) = \frac{|\underline{P}(X)|}{|X|}.$$

The quality $\gamma_P(X)$ represents the relative frequency of the objects correctly classified by means of the attributes from P. Moreover, $0 \leq \alpha_P(X) \leq \gamma_P(X) \leq 1$, and $\gamma_P(X) = 0$ iff $\alpha_P(X) = 0$, while $\gamma_P(X) = 1$ iff $\alpha_P(X) = 1$.

The definition of approximations of a subset $X \subseteq U$ can be extended to a classi.cation, i.e. a partition $Y = \{Y_1, Y_2, \cdots, Y_n\}$ of U. Subsets $Y_i, i = 1, 2, \cdots, n$ are disjunctive classes of Y. By P-lower (P-upper) approximation of Y in S, we mean sets $\underline{P}(Y) = \{\underline{P}(Y_1), \underline{P}(Y_2), \cdots, \underline{P}(Y_n)\}$ and $\bar{P}(Y) = \{\bar{P}(Y_1), \bar{P}(Y_2), \cdots, \bar{P}(Y_n)\}$, respectively. The coefficient

$$\gamma_P(X) = \frac{|\sum_{1=1}^{n} \underline{P}(X)|}{|U|}$$

is called quality of the approximation of classication Y by set of attributes P, or in short, quality of classification. It expresses the ratio of all P-correctly classified objects to all objects in the system.

The main preoccupation of the rough sets theory is approximation of subsets or partitions of U, representing a knowledge about U, with other sets or partitions built up using available information about U. From the viewpoint of a particular object $x \in U$, it may be interesting, however, to use the available information to assess the degree of its membership to a subset X of U. The subset X can be identified with a concept of knowledge to be approximated. Using the rough set approach one can calculate the membership function $\mu_X^P(x)$ (rough membership function) as

$$\mu_X^P(x) = \frac{X \cap I_p(x)}{I_p(x)}.$$

The value of $\mu_X^P(x)$ may be interpreted analogously to conditional probability and may be understood as the degree of certainty (credibility) to which x belongs to X. Observe that the value of the membership function is calculated from the available data, and not subjectively assumed, as it is the case of membership functions of fuzzy sets.

Between the rough membership function and the approximations of X the following relationships hold (Pawlak [340]):

$$\underline{P}(X) = \{x \in U : \mu_X^P(x) = 1\}, \bar{P}(X) = \{x \in U : \mu_X^P(x) > 0\},$$

$$Bn_P(X) = \{x \in U : 0 < \mu_X^P(x) < 1\}, \underline{P}(U - X) = \{x \in U : \mu_X^P(x) = 0\}.$$

In the rough sets theory there is, therefore, a close link between vagueness (granularity) connected with rough approximation of sets and uncertainty connected with rough membership of objects to sets.

Trust theory [145] is the branch of mathematics that studies the behavior of rough events. It is the foundation for rough programming as the probability theory for stochastic programming as well as the possibility theory for fuzzy programming. Liu [145] also combined trust measure with probability measure and possibility measure to describe the two-fold uncertain events, such as random rough variable, fuzzy rough variable, rough random variable and rough fuzzy variable. In this section, we will define the fuzzy rough variable from another perspective, i.e. the rough approximation.

After the rough set was initialized by Pawlak [340], it has been applied to many fields to deal with the vague description of objectives. He asserted that any vague information can be approximated by other crisp information. In this section, we will recall these fundamental concepts and introduce its application to the statistical field and programming problem.

Definition 5.1. (Slowinski and Vanderpooten [381]) Let U be a universe, and X a set representing a concept. Then its lower approximation is defined by

$$\underline{X} = \{x \in U | R^{-1}(x) \subset X\}, \tag{5.1}$$

and the upper approximation is defined by

$$\overline{X} = \bigcup_{x \in X} R(x), \tag{5.2}$$

where R is the similarity relationship on U. Obviously, we have $\underline{X} \subseteq X \subseteq \overline{X}$.

Definition 5.2. (Pawlak [340]) The collection of all sets having the same lower and upper approximations is called a rough set, denoted by $(\underline{X}, \overline{X})$. Its boundary is defined as follows,

$$Bn_R(X) = \overline{X} - \underline{X}. \tag{5.3}$$

In order to know the degree of the upper and lower approximation describing the set X, the concept of the *accuracy* of approximation is proposed by Greco et al. [382],

$$\alpha_R(X) = \frac{|\underline{X}|}{|\overline{X}|}, \tag{5.4}$$

where $X \neq \Phi$, $|X|$ expresses the cardinal number of the set X when X is a finite set, otherwise it expresses the Lebesgue measure.

Another ratio defines a *quality* of the approximation of X by means of the attributes from R according to Greco et al. [382],

$$\gamma_R(X) = \frac{|\underline{X}|}{|X|}. \tag{5.5}$$

The quality $\gamma_R(X)$ represents the relative frequency of the objects correctly classified by means of the attributes from R.

Remark 5.1. For any set A we can represents its frequency of the objects correctly approximated by $(\underline{X}, \overline{X})$ as follows,

$$\beta_R(A) = \frac{|\underline{X} \cap A|}{|\overline{X} \cap A|}.$$

If $\underline{X} \subseteq A \subseteq \overline{X}$, namely, A has the upper approximation $\overline{X}$ and the lower approximation $\underline{X}$, we have that $\beta_R(A)$ degenerates to the *quality* $\gamma_R(A)$ of the approximation.

As we know, the *quality* $\gamma_R(A)$ of the approximation describes the frequency of A, and when $\gamma_R(A) = 1$, we only have $|A| = |\underline{X}|$, namely, the set A is well approximated by the lower approximation. If we we want to make A be a definable set, there must be $\gamma_R(A) = 1$ and $\alpha_R(X) = 1$ both holds. Then we could make use the following definition to combine them into together.

Definition 5.3. Let $(\underline{X}, \overline{X})$ be a rough set under the similarity relationship R and A be any set satisfying $\underline{X} \subseteq A \subseteq \overline{X}$. Then we define the approximation function as follows expressing the relative frequency of the objects of A correctly classified into $(\underline{X}, \overline{X})$,

$$Appr_R(A) = 1 - \eta(1 - \frac{|A|}{|\overline{X}|}), \tag{5.6}$$

where η is a predetermined by the decision maker's preference.

From Definition 5.3, we know that $\frac{|A|}{|\overline{X}|}$ which keeps accord with $\gamma_R(A)$ describes the relative frequency of the objects correctly classified by R from the view of the upper approximation $\overline{X}$. Obviously, $\mathrm{Appr}_R(A)$ is a number between 0 and 1, and is increasing along with the increase of $|A|$. The extreme case $\mathrm{Appr}_R(A) = 1$ means that $|A| = |\overline{X}|$, namely, A is completely described by $\overline{X}$.

Lemma 5.1. *Let $(\underline{X}, \overline{X})$ be a rough set under the similarity relationship R and A be any set satisfying $\underline{X} \subseteq A \subseteq \overline{X}$. Then we have*

$$Appr_R(A) = \frac{\eta \alpha_R(A) + (1-\eta)\gamma_R(A)}{\gamma_R(A)}.$$

Proof. Since $\underline{X} \subseteq A \subseteq \overline{X}$, it means that A has the lower approximation $\underline{X}$ and the upper approximation $\overline{X}$, and it follows from Greco et al. [382] that

$$\alpha_R(A) = \frac{|\underline{X}|}{|\overline{X}|}, \quad \gamma_R(A) = \frac{|\underline{X}|}{|A|}.$$

Thus,

$$\frac{|A|}{|\overline{X}|} = \frac{\alpha_R(A)}{\gamma_R(A)}.$$

It follows that

$$\begin{aligned} \mathrm{Appr}_R(A) &= 1 - \eta(1 - \tfrac{|A|}{|\overline{X}|}) \\ &= 1 - \eta(1 - \tfrac{\alpha_R(A)}{\gamma_R(A)}) \\ &= \tfrac{\eta \alpha_R(A) + (1-\eta)\gamma_R(A)}{\gamma_R(A)}. \end{aligned}$$

This completes the proof. □

Lemma 5.2. *Let $(\underline{X}, \overline{X})$ be a rough set on the finite universe under the equivalence relationship R, A be any set satisfying $\underline{X} \subseteq A \subseteq \overline{X}$ and $\eta \in (0,1)$. Then $Appr_R(A) = 1$ holds if and only if $\underline{X} = A = \overline{X}$.*

Proof. If $\underline{X} = A = \overline{X}$ holds, it is obvious that $\mathrm{Appr}_R(A) = 1$ according to Definition 5.4. Let's proved the necessity of the condition.

If $\mathrm{Appr}_R(A) = 1$ holds for any A satisfying $\underline{X} \subseteq A \subseteq \overline{X}$, it follows from Lemma 5.1 that, for $0 < \eta \leq 1$,

$$\frac{\eta \alpha_R(A) + (1-\eta)\gamma_R(A)}{\gamma_R(A)} = 1 \Rightarrow \alpha_R(A) = \gamma_R(A) \Rightarrow |\overline{X}| = |A|.$$

Since $A \subseteq \overline{X}$ and the universe is finite, we have that $A = \overline{X}$. Because A is any set satisfying $\underline{X} \subseteq A \subseteq \overline{X}$, let $A = X$, then we have $X = \overline{X}$. It follows from the property proposed by Pawlak [340] that $\underline{X} = X = \overline{X}$. Thus, we have $\underline{X} = A = \overline{X}$. □

Lemma 5.1 shows that the approximation function Appr inherits the accuracy and quality of the approximation, and extends it to the relationship between any set A and the rough set $(\underline{X}, \overline{X})$. Lemma 5.2 shows that the approximation function is complete and well describes the property in traditional rough set theory, and describe the property only by one index.

Lemma 5.3. *Let $(\underline{X}, \overline{X})$ be a rough set on the infinite universe under the similarity relationship R, A be any set satisfying $\underline{X} \subseteq A \subseteq \overline{X}$ and $\eta \in (0,1)$. If $Appr_R(A) = 1$ holds, then there exist the similarity relationship R^* such that $|\underline{X}| = |A| = |\overline{X}|$, where $|\cdot|$ expresses the Lebesgue measure.*

Proof. According to Lemma 5.2, we know that $|A| = |\overline{X}|$ must hold. Let $\underline{X} = \overline{X}/\partial\overline{X}$ under the similarity relationship R^*, where $\partial\overline{X}$ is composed by all the elements such that $|\partial\overline{X}| = 0$, namely, the measure of $\partial\overline{X}$ is 0. Next, we will prove that $\overline{X}/\partial\overline{X} \subseteq A$.

(1) If $|\overline{X}| = 0$, then $\overline{X}/\partial\overline{X} = \Phi$. Thus, $|\underline{X}| = |A| = |\overline{X}| = 0$.

(2) If $|\overline{X}| \neq 0$, we only need to prove that for any $x^0 \in \overline{X}/\partial\overline{X}$, $x^0 \in A$. In fact, when $x^0 \in \overline{X}/\partial\overline{X}$, then $x^0 \in int(\overline{X})$ holds, where $int(\overline{X})$ is the internal part of $\overline{X}$. It follows that there exists $r > 0$ such that $N(x^0, r) \subset int(\overline{X})$ and $|N(x^0, r)| > 0$. There exist four cases describing the relationship between A and $N(x^0, r)$.

Case 1. $A \cap N(x^0, r) = \Phi$ (see Figure 5.3) . Since $N(x^0, r) \subset int(\overline{X}) \subset \overline{X}$ and $A \subseteq \overline{X}$, we have that

$$|\overline{X}| \geq |N(x^0, r) \cup A| = |N(x^0, r)| + |A|.$$

This conflicts with $|A| = |\overline{X}|$.

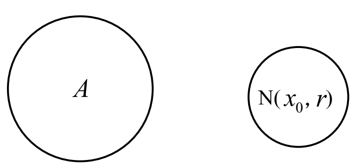

Fig. 5.3 Apartment

Case 2. $A \cap N(x^0, r) = P$, where the set P includes countable points (see Figure 5.4). Obviously, we have $|P| = 0$, thus $|N(x_0, r)/P| = |N(x_0, r)| > 0$. Then we have

$$|\overline{X}| \geq |N(x^0, r) \cup A| = |N(x^0, r)/P| + |A|.$$

This also conflicts with $|A| = |\overline{X}|$.

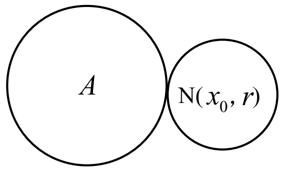

Fig. 5.4 Tangent

Case 3. $A \cap N(x^0, r) = P'$, where $P' \subset N(x^0, r)/\{x_0\}$. As Figure 5.5 shows, we can divide it into three parts, namely, $(N(x^0, r)/P) = P' \cup \{x_0\} \cup T$, where P', T and $\{x_0\}$ don't have the same element with each other. Then $|T| > 0$, it follows that

$$|\overline{X}| \geq |N(x^0, r) \cup A| = |T| + |A|.$$

This also conflicts with $|A| = |\overline{X}|$.

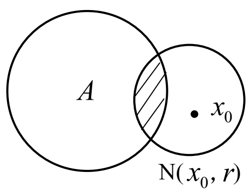

Fig. 5.5 Intersection

Case 4. $A \supset (N(x^0, r)/x_0)$ (see Figure 5.6). This means that for any $x_0 \in int(A)$, $x_0 \notin A$. It follows that $A \cap int(A) = \Phi$, then we have

$$|\overline{X}| \geq |int(A) \cup A| = |int(A)| + |A|.$$

This also conflicts with $|A| = |\overline{X}|$. In above, we can get $\overline{X}/\partial\overline{X} \subseteq A$. Thus, there exists the lower approximation $\underline{X} = \overline{X}/\partial\overline{X}$ such that $\underline{X} \subseteq A \subseteq \overline{X}$ under the similarity relationship $R*$. $\square$

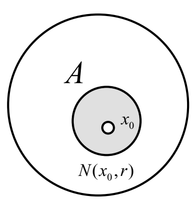

Fig. 5.6 Inclusion

Remark 5.2. In fact, we can extend Definition 5.3 to more general set. when $\underline{X} \subseteq A \subseteq \overline{X}$, we have the following equivalent formula,

$$\begin{aligned}\mathrm{Appr}_R(A) &= 1-\eta(1-\frac{|A|}{|\overline{X}|})\\ &= \frac{|A\cap\underline{X}|}{|\underline{X}|}\left(1-\eta\left(1-\frac{|A\cap\overline{X}|}{|\overline{X}|}\right)\right)\\ &= \frac{|A\cap\underline{X}|}{|\underline{X}|}+\eta\left(\frac{|A\cap\overline{X}|}{|\overline{X}|}-\frac{|A\cap\underline{X}|}{|\underline{X}|}\right).\end{aligned}$$

Furthermore, we get the definition of the approximation function for any set A.

Definition 5.4. Let $(\underline{X},\overline{X})$ be the rough set generated by X under the similarity relationship R, for any set A, the approximation function of event A by $(\underline{X},\overline{X})$ is defined as follows

$$Appr_R(A) = \frac{|A\cap\underline{X}|}{|\underline{X}|}+\eta\left(\frac{|A\cap\overline{X}|}{|\overline{X}|}-\frac{|A\cap\underline{X}|}{|\underline{X}|}\right),$$

where η is a given parameter predetermined by the decision maker's preference.

From Definition 5.4, we know that $\mathrm{Appr}_R(A)$ expresses the relationship between the set A and the set $(\underline{X},\overline{X})$ generated by X, that is, the frequency of A correctly classified into $(\underline{X},\overline{X})$ according to the similarity relationship R. It has the internal link with the *accuracy* α_R of the approximation and the *quality* γ_R of the approximation in some extent. α_R expresses the degree of the upper and lower approximation describing the set X. $\gamma_R(X)$ represents the relative frequency of the objects correctly classified by means of the attributes from R. Then Appr_R combines both of them together and considers the level which A has the attributes correctly classified by $(\underline{X},\overline{X})$ for any A.

Lemma 5.4. *Let $(\underline{X},\overline{X})$ be a rough set, for any set A, we have the following conclusion,*

$$Appr_R = \begin{cases} 1, & \text{if } A \supseteq \overline{X}\\ 1-\eta(1-\frac{\alpha_R(A)}{\gamma_R(A)}), & \text{if } \underline{X}\subset A\subset\overline{X}\\ \frac{1-\eta(1-\alpha_R(A))}{\gamma_R(A)}, & \text{if } A\subseteq\underline{X}\\ 0, & \text{if } A\cap\overline{X}=\Phi\\ \frac{|A\cap\overline{X}|}{|\overline{X}|}\left(\frac{\beta_R(A)}{\alpha_R(A)}+\eta(1-\frac{\beta_R(A)}{\alpha_R(A)})\right), & \text{otherwise.}\end{cases} \quad (5.7)$$

Proof. (1) If $A \supseteq \overline{X}$, we have that $A\cap\underline{X}=\underline{X}$ and $A\cap\overline{X}=\overline{X}$. Then $\mathrm{Appr}_R = 1$.

(2) If $\underline{X}\subseteq A\subseteq\overline{X}$, we have that $A\cap\underline{X}=\underline{X}$ and $A\cap\overline{X}=A$. It follows that $\mathrm{Appr}_R = 1-\eta(1-\frac{|A|}{|\overline{X}|})$.

(3) If $A\subset\underline{X}$, we have that $A\cap\underline{X}=A$ and $A\cap\overline{X}=A$. It follows that $\mathrm{Appr}_R = \frac{1-\eta(1-\alpha_R(A))}{\gamma_R(A)}$.

(4) If $A\cap\overline{X}=\Phi$, we have that $A\cap\underline{X}=\Phi$ and $A\cap\overline{X}=\Phi$. It follows that $Appr_R = 0$.

(5) For the others, we have

$$
\begin{aligned}
Appr_R(A) &= \frac{|A\cap \underline{X}|}{|\underline{X}|} + \eta\left(\frac{|A\cap \overline{X}|}{|\overline{X}|} - \frac{|A\cap \underline{X}|}{|\underline{X}|}\right)\\
&= \frac{|A\cap \overline{X}|}{|\overline{X}|}\left(\frac{|A\cap \underline{X}|}{|\underline{X}|}\cdot\frac{|\overline{X}|}{|A\cap \overline{X}|} + \eta\left(1 - \frac{|A\cap \underline{X}|}{|\underline{X}|}\cdot\frac{|\overline{X}|}{|A\cap \overline{X}|}\right)\right)\\
&= \frac{|A\cap \overline{X}|}{|\overline{X}|}\left(\frac{\beta_R(A)}{\alpha_R(A)} + \eta(1 - \frac{\beta_R(A)}{\alpha_R(A)})\right).
\end{aligned}
$$

This completes the proof. □

For the different purposes, we can respectively discuss the extreme case as follows.

Remark 5.3. When $\eta = 1$, we have $Appr_R(A) = \frac{|A\bigcap \underline{X}|}{|\underline{X}|}$. It means that the decision maker only consider the level that A includes the frequency of A correctly classified into $\underline{X}$ according to the similarity relationship R.

Remark 5.4. When $\eta = 0$, we have $Appr_R(A) = \frac{|A\bigcap \overline{X}|}{|\overline{X}|}$. It means that the decision maker only consider the level that A includes the frequency of A correctly classified into $\overline{X}$ according to the similarity relationship R.

In fact, the rough set theory is increasingly developed by many scholars and applied to many fields, for example, data mining, decision reduction, system analysis and so on. Figure 5.7 shows that the rough approximation. The curves including the internal points is X. The two thick curves including their internal points are the upper and lower approximation.

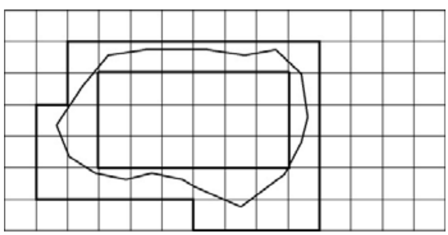

Fig. 5.7 Rough approximation

Let's focus on the continuous set in the one dimension real space **R**. There are still some vague sets which cannot be directly fixed and need to be described by the rough approximation. For example, set **R** be the universe, a similarity relation $\simeq$ is defined as $a \simeq b$ if and only if $|a-b| \le 10$. We have that for the set $[20,50]$, its lower approximation $\underline{[20,50]} = [30,40]$ and its upper approximation $\overline{[20,50]} = [10,60]$. Then the upper and lower approximation of the set [20,50] make up a rough set ([30,40],[10,60]) which is the collection of all sets having the same lower approximation [30,40] and upper approximation [10,60].

Definition 5.5. A fuzzy rough variable ξ is a fuzzy variable with uncertain parameter $\rho \in X$, where X is approximated by $(\underline{X}, \overline{X})$ according to the similarity relation R, namely, $\underline{X} \subseteq X \subseteq \overline{X}$.

For convenience, we usually denote $\rho \vdash (\underline{X}, \overline{X})_R$ expressing that ρ is in some set A which is approximated by $(\underline{X}, \overline{X})$ according to the similarity relation R, namely, $\underline{X} \subseteq A \subseteq \overline{X}$.

Example 5.1. Let's consider the *LR* fuzzy variable ξ with the following membership function,

$$\mu_\xi(x) = \begin{cases} L\left(\frac{\rho - x}{\alpha}\right), & \text{if } \rho - \alpha < x < \rho \\ 1, & \text{if } x = \rho \\ R\left(\frac{x-\rho}{\beta}\right), & \text{if } \rho < x_j < \rho + \beta, \end{cases} \tag{5.8}$$

where $\rho \vdash ([1,2],[0,3])_R$. Then ξ is a fuzzy rough variable.

5.2.2 Expected Value Operator of Fu-Ro Variables

Definition 5.6. Let ξ be a Fu-Ro variable with the uncertain parameter λ, where $\lambda \vdash (\underline{X}, \overline{X})_R$, then its expected value is defined by

$$E[\xi] = \int_0^{\infty} \text{Appr}\{E[\xi(\lambda)] \geq r\}dr - \int_{-\infty}^{0} \text{Appr}\{E[\xi(\lambda)] \leq r\}dr \tag{5.9}$$

Lemma 5.5 ([220]). *Assume that ξ and η are the introduction of variables with finite expected values. Then for any real numbers a and b, we have*

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]. \tag{5.10}$$

Proposition 5.1. *Let ξ be a Fu-Ro variable with the membership function*

$$\mu_\xi(x) = \begin{cases} 1, & \text{if} \quad x \in [\bar{a}, \bar{b}] \\ 0, & \text{otherwise}, \end{cases}$$

where $\bar{a}, \bar{b}$ are rough variables defined on $(\Lambda, \Theta, \mathscr{A}, \pi)$, and $\bar{a} = ([m_2, m_3], [m_1, m_4])$, $0 < m_1 \leq m_2 < m_3 \leq m_4$, $\bar{b} = ([n_2, n_3], [n_1, n_4])$, $0 < n_1 \leq n_2 < n_3 \leq n_4$.

Then the expected value of ξ is

$$E[\xi] = \frac{1}{8}\sum_{i=1}^{4}(m_i + n_i).$$

Proof. Since $\xi(\lambda) = [\bar{a}, \bar{b}]$, and according to proposition 2.1, we have

$$E^{Cr}[\xi(\lambda)] = \frac{1}{2}(\bar{a} + \bar{b}).$$

By the rough arithmetic operators, it follows that,

$$\frac{\bar{a}+\bar{b}}{2} = \frac{1}{2}\{([m_2,m_3],[m_1,m_4]) + ([n_2,n_3],[n_1,n_4])\}$$
$$= \left([\frac{m_2+n_2}{2},\frac{m_3+n_3}{2}],[\frac{m_1+n_1}{2},\frac{m_4+n_4}{2}]\right). \tag{5.11}$$

From the definition of trust measure, we have

$$Appr\{\xi \geq r\} = \begin{cases} 0, & \text{if } \frac{m_4+n_4}{2} \leq r \\ \frac{\frac{m_4+n_4}{2}-r}{m_4+n_4-m_1-n_1}, & \text{if } \frac{m_3+n_3}{2} \leq r \leq \frac{m_4+n_4}{2} \\ \frac{1}{2}(\frac{\frac{m_4+n_4}{2}-r}{m_4+n_4-m_1-n_1} + \frac{\frac{n_3+m_3}{2}-r}{m_3+n_3-m_2-n_2}), & \text{if } \frac{m_2+n_2}{2} \leq r \leq \frac{m_3+n_3}{2} \\ \frac{\frac{m_4+n_4}{2}-r}{m_4+n_4-m_1-n_1} + \frac{1}{2}, & \text{if } \frac{m_1+n_1}{2} \leq r \leq \frac{m_2+n_2}{2} \\ 1, & \text{if } r \leq \frac{m_1+n_1}{2}. \end{cases}$$

It follows that,

$$\begin{aligned} E[\xi] &= \int_0^{+\infty} Appr\{\tfrac{\bar{a}+\bar{b}}{2} \geq r\}dr - \int_{-\infty}^0 Appr\{\tfrac{\bar{a}+\bar{b}}{2} \leq r\}dr \\ &= \int_0^{\frac{m_1+n_1}{2}} 1dr + \int_{\frac{m_1+n_1}{2}}^{\frac{m_2+n_2}{2}} (\frac{\frac{m_4+n_4}{2}-r}{m_4+n_4-m_1-n_1} + \tfrac{1}{2})dr \\ &+ \int_{\frac{m_2+n_2}{2}}^{\frac{m_3+n_3}{2}} \tfrac{1}{2}(\frac{\frac{m_4+n_4}{2}-r}{m_4+t_4-m_1-n_1} + \frac{\frac{m_3+n_3}{2}-r}{m_3+n_3-m_2-n_2})dr + \int_{\frac{m_3+n_3}{2}}^{\frac{m_4+n_4}{2}} \frac{\frac{m_4+n_4}{2}-r}{m_4+n_4-m_1-n_1}dr \\ &= \tfrac{1}{8}\sum_{i=1}^{4}(m_i+n_i). \end{aligned}$$

The proof is complete. □

Proposition 5.2. *Let ξ be a trapezoidal Fu-Ro variable $\xi = (\bar{r}_1,\bar{r}_2,\bar{r}_3,\bar{r}_4)$, where $\bar{r}_1,\bar{r}_2,\bar{r}_3,\bar{r}_4$ are rough variables defined on $(\Lambda,\Theta,\mathscr{A},\pi)$, and*

$$\begin{aligned} \bar{r}_1 &= ([m_2,m_3],[m_1,m_4]), 0 < m_1 \leq m_2 < m_3 \leq m_4, \\ \bar{r}_2 &= ([n_2,n_3],[n_1,n_4]), 0 < n_1 \leq n_2 < n_3 \leq n_4, \\ \bar{r}_3 &= ([s_2,s_3],[s_1,s_4]), 0 < s_1 \leq s_2 < s_3 \leq s_4, \\ \bar{r}_4 &= ([t_2,t_3],[t_1,t_4]), 0 < t_1 \leq t_2 < t_3 \leq t_4. \end{aligned}$$

Then the expected value of ξ is

$$E[\xi] = \frac{1}{16}\sum_{i=1}^{4}(m_i+n_i+s_i+t_i).$$

Proof. Since $\xi(\lambda) = [\bar{r}_1,\bar{r}_2,\bar{r}_3,\bar{r}_4]$, and according to proposition 2.2, we have

$$E^{Cr}[\xi(\lambda)] = \frac{1}{4}(\bar{r}_1+\bar{r}_2+\bar{r}_3+\bar{r}_4).$$

By the rough arithmetic operators, it follows that,

$$\begin{aligned}&\frac{\bar{r}_1+\bar{r}_2+\bar{r}_3+\bar{r}_4}{4}\\&=\frac{1}{4}\{([m_2,m_3],[m_1,m_4])+([n_2,n_3],[n_1,n_4])+([s_2,s_3],[s_1,s_4])+([t_2,t_3],[t_1,t_4])\}\\&=\left([\frac{m_2+n_2+s_2+t_2}{4},\frac{m_3+n_3+s_3+t_3}{4}],[\frac{m_1+n_1+s_1+t_1}{4},\frac{m_4+n_4+s_4+t_4}{4}]\right).\end{aligned}$$

From the definition of trust measure, we have

$$Appr\{\xi \geq r\} =$$

$$\begin{cases} 0, & \text{if } \frac{m_4+n_4+s_4+t_4}{4} \leq r \\ \frac{\frac{m_4+n_4+s_4+t_4}{2}-2r}{m_4+n_4+s_4+t_4-(m_1+n_1+s_1+t_1)}, & \text{if } \frac{m_3+n_3+s_3+t_3}{4} \leq r \leq \frac{m_4+n_4+s_4+t_4}{4} \\ \frac{1}{2}\Big(\frac{\frac{m_4+n_4+s_4+t_4}{2}-2r}{m_4+n_4+s_4+t_4-(m_1+n_1+s_1+t_1)} \\ \qquad +\frac{\frac{m_3+n_3+s_3+t_3}{2}-2r}{m_3+n_3+s_3+t_3-(m_2+n_2+s_2+t_2)}\Big), & \\ & \text{if } \frac{m_2+n_2+s_2+t_2}{4} \leq r \leq \frac{m_3+n_3+s_3+t_3}{4} \\ \frac{\frac{m_4+n_4+s_4+t_4}{2}-2r}{m_4+n_4+s_4+t_4-(m_1+n_1+s_1+t_1)}+\frac{1}{2}, & \text{if } \frac{m_1+n_1+s_1+t_1}{4} \leq r \leq \frac{m_2+n_2+s_2+t_2}{4} \\ 1, & \text{if } r \leq \frac{m_1+n_1+s_1+t_1}{4}. \end{cases}$$

It follows that,

$$\begin{aligned}E[\xi] &= \int_0^{+\infty} Appr\{\frac{\bar{r}_1+\bar{r}_2+\bar{r}_3+\bar{r}_4}{4} \geq r\}dr - \int_{-\infty}^0 Appr\{\frac{\bar{r}_1+\bar{r}_2+\bar{r}_3+\bar{r}_4}{4} \leq r\}dr\\ &= \int_0^{\frac{m_1+n_1+s_1+t_1}{4}} 1dr + \int_{\frac{m_1+n_1+s_1+t_1}{2}}^{\frac{m_2+n_2+s_2+t_2}{4}} \left(\frac{\frac{m_4+n_4+s_4+t_4}{2}-2r}{m_4+n_4+s_4+t_4-(m_1+n_1+s_1+t_1)}+\frac{1}{2}\right)dr\\ &+ \int_{\frac{m_2+n_2+s_2+t_2}{2}}^{\frac{m_3+n_3+s_3+t_3}{2}} \frac{1}{2}\left(\frac{\frac{m_4+n_4+s_4+t_4}{2}-2r}{m_4+t_4+s_4+t_4-(m_1+n_1+s_1+t_1)}+\frac{\frac{m_3+n_3+s_3+t_3}{2}-2r}{m_3+n_3+s_3+t_3-(m_2+n_2+s_2+t_2)}\right)dr\\ &+ \int_{\frac{m_3+n_3+s_3+t_3}{2}}^{\frac{m_4+n_4+s_4+t_4}{2}} \frac{\frac{m_4+n_4+s_4+t_4}{2}-2r}{m_4+n_4+s_4+t_4-(m_1+n_1+s_1+t_1)}dr\\ &= \frac{1}{16}\sum_{i=1}^{4}(m_i+n_i+s_i+t_i).\end{aligned}$$

The proof is complete. □

Proposition 5.3. *Let ξ be a LR Fu-Ro variable with the membership function of fuzzy variable ξ has the following form*

$$\mu_{\xi}(x) = \begin{cases} L(\frac{\bar{z}-x}{\alpha}), & \bar{z}-\alpha < x \leq \bar{z} \\ 1, & x = \bar{z} \\ R(\frac{x-\bar{z}}{\beta}), & \bar{z} < x < \bar{z}+\beta, \end{cases} \tag{5.12}$$

where $\bar{z}$ is a rough variable and $\bar{z} = ([z_2,z_3],[z_1,z_4]), \alpha < z_1 < z_2 < z_3 < z_4$. And here we just consider the situation when the reference function $L(x) = R(x) = 1-x$, then this LR fuzzy rough variable is triangular type, and the left and right spread $\alpha, \beta > 0$.

Then the expected value of ξ *is*

$$E[\xi] = \frac{1}{4}(z_1 + z_2 + z_3 + z_4 + \alpha + \beta).$$

Proof. Since $\xi(\lambda) = (\bar{z}, \alpha, \beta)_{\text{LR}}$ is triangular LR Fu-Ro variable, which means the reference functions $L(x) = R(x) = 1 - x$, according to remark 2.11, we have

$$E^{Cr}[\xi(\lambda)] = \bar{z} + \frac{1}{4}(\alpha + \beta).$$

Then we have

$$\begin{aligned} E[\xi] &= E[\bar{z} + \tfrac{1}{4}(\alpha + \beta)] \\ &= E[\bar{z}] + E[\tfrac{1}{4}(\alpha + \beta)]. \end{aligned}$$

It follows that

$$\begin{aligned} E[\xi] &= \tfrac{1}{4}(z_1 + z_2 + z_3 + z_4) + \tfrac{1}{4}(\alpha + \beta) \\ &= \tfrac{z_1 + z_2 + z_3 + z_4 + \alpha + \beta}{4}. \end{aligned}$$

The proof is complete. □

5.2.3 Chance Operator of Fu-Ro Variables

To begin with, we give the three types of primitive chance of Fu-Ro event as follows.

Definition 5.7. Let $\xi = (\xi_1, \xi_2, \cdots \xi_n)$ be a Fu-Ro vector defined on $(\Lambda, \Delta, \mathscr{A}, \pi)$, and $f : \mathbf{R}^n \to \mathbf{R}$ is Borel measurable function. Then the primitive chance of a Fu-Ro event characterized by $f(\xi) \le 0$ is a function from (0, 1] to [0, 1], defined as

(1). $Appr - Pos$ chance,

$$Ch\{f(\xi) \le 0\}(\alpha) = \sup_{\alpha \in [0,1]} \{\beta | Appr\{\lambda \in \Lambda | Pos\{\, f(\xi(\lambda)) \le 0 \,\} \ge \beta\} \ge \alpha\}. \tag{5.13}$$

(2). $Appr - Nec$ Chance,

$$Ch\{f(\xi) \le 0\}(\alpha) = \sup_{\alpha \in [0,1]} \{\beta | Appr\{\lambda \in \Lambda | Nec\{\, f(\xi(\lambda)) \le 0 \,\} \ge \beta\} \ge \alpha\}. \tag{5.14}$$

(3). $Appr - Cr$ Chance,

$$Ch\{f(\xi) \le 0\}(\alpha) = \sup_{\alpha \in [0,1]} \{\beta | Appr\{\lambda \in \Lambda | Cr\{\, f(\xi(\lambda)) \le 0 \,\} \ge \beta\} \ge \alpha\}. \tag{5.15}$$

Remark 5.5. The primitive chance of a Fu-Ro event characterized by $f(\xi) \le 0$ defined as (5.13), (5.14), (5.15) have the equivalent forms respectively.

$$Ch\{f(\xi) \le 0\}(\alpha) = \sup_{Appr\{A\} \ge \alpha} \inf_{\lambda \in A} Pos\{f(\xi(\lambda)) \le 0\}, \tag{5.16}$$

$$Ch\{f(\xi) \le 0\}(\alpha) = \sup_{Appr\{A\} \ge \alpha} \inf_{\lambda \in A} Nec\{f(\xi(\lambda)) \le 0\}, \tag{5.17}$$

$$Ch\{f(\xi)\leq 0\}(\alpha)=\sup_{Appr\{A\}\geq\alpha}\inf_{\lambda\in A}Cr\{f(\xi(\lambda))\leq 0\}. \tag{5.18}$$

Lemma 5.6. *For any confidence levels α,β.*
(1). Appr-Pos Chance $Ch\{f(\xi)\leq 0\}(\alpha)\geq\beta$ holds if and only if

$$Appr\{\lambda\in\Lambda|Pos\{f(\xi(\lambda))\leq 0\}\geq\beta\}\geq\alpha.$$

(2). Appr-Nec Chance $Ch\{f(\xi)\leq 0\}(\alpha)\geq\beta$ holds if and only if

$$Appr\{\lambda\in\Lambda|Nec\{f(\xi(\lambda))\leq 0\}\geq\beta\}\geq\alpha.$$

(3). Appr-Cr Chance $Ch\{f(\xi)\leq 0\}(\alpha)\geq\beta$ holds if and only if

$$Appr\{\lambda\in\Lambda|Cr\{f(\xi(\lambda))\leq 0\}\geq\beta\}\geq\alpha.$$

5.3 Fu-Ro EVM

For the multi-objective model (5.19) with Fu-Ro parameters, we cannot deal with it directly, we should use some tools to make it have mathematical meaning, we then can solve it. In this section, we employ the expected value operator to transform the fuzzy rough model into Fu-Ro EVM. Consider the following multi-objective decision making model (5.19) with fuzzy rough coefficients:

$$\begin{cases}\max\ \{f_1(x,\xi),f_2(x,\xi),\cdots,f_m(x,\xi)\}\\ \text{s.t.}\begin{cases}g_r(x,\xi)\leq 0,\ r=1,2,\cdots,p\\ x\in X,\end{cases}\end{cases} \tag{5.19}$$

where x is a n-dimensional decision vector, $\xi=(\xi_1,\xi_2,\cdots,\xi_n)$ is a Fu-Ro vector, $f_i(x,\xi)$ are objective functions, $i=1,2,\cdots,m$. Because of the existence of Fu-Ro vector ξ, problem (5.19) is not well-defined. That is, the meaning of maximizing $f_i(x,\xi),i=1,2,\cdots,m$ is not clear and constraints $g_r(x,\xi)\leq 0,r=1,2,\cdots,p$ do not define a deterministic feasible set. In the following, we use Fu-Ro EVM to deal with the meaningless model.

5.3.1 *General Model for Fu-Ro EVM*

Based on the definition of the expected value of fuzzy rough events f_i and g_r, the general model for Fu-Ro EVM is proposed as follows,

$$\begin{cases}\max\ E[f_1(x,\xi),f_2(x,\xi),\cdots,f_m(x,\xi)]\\ \text{s.t.}\begin{cases}E[g_r(x,\xi)]\leq 0,\ r=1,2,\cdots,p\\ x\in X,\end{cases}\end{cases} \tag{5.20}$$

where x is n-dimensional decision vector and ξ is n-dimensional fuzzy rough variable.

Definition 5.8. If x^* is an efficient solution of problem (5.20), we call it as a fuzzy rough expected efficient solution.

Clearly, the problem (5.20) is a multi-objective with crisp parameters. Then we can convert it into a single-objective programming by traditional method of weight sum.

$$\begin{cases} \max \sum_{i=1}^{m} w_i E[f_i(x,\xi)] \\ \text{s.t.} \begin{cases} E[g_r(x,\xi)] \le 0,\ r=1,2,\cdots,p \\ x \in X \\ w_1 + w_2 + \cdots + w_m = 1. \end{cases} \end{cases} \tag{5.21}$$

Theorem 5.1. *Problem (5.21) is equivalent to problem (5.20), i.e., the efficient solution of problem (5.20) is the optimal solution of problem (5.21) and the optimal solution of problem (5.21) is the efficient solution of problem (5.20).*

Proof. The proof is the same as the proof of Theorem 3.1.

Theorem 5.2. *Let $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ be a fuzzy rough vector on the rough space $(\Lambda, \Delta, \mathscr{A}, \pi)$, and f_i and $g_r : \mathscr{A}^n \to \mathscr{A}$ be convex continuous functions with respect to x, $i = 1,2,\cdots,m; r = 1,2,\cdots,p$. Then the expected value programming problem (5.21) is a convex programming.*

Proof. It is similar to the proof of Theorem 3.2, and thus omit. □

We can also formulate a fuzzy rough decision system as an expected value goal programming (EVGP) model according to the priority structure and target levels set by the decision-maker:

$$\begin{cases} \min \sum_{j=1}^{l} P_j \sum_{i=1}^{m} (u_{ij} d_i^+ + v_{ij} d_i^-) \\ \text{s.t.} \begin{cases} E[f_i(x,\xi)] + d_i^- - d_i^+ = b_i,\ i=1,2,\cdots,m \\ E[g_r(x,\xi)] \le 0,\ r=1,2,\cdots,p \\ d_i^-, d_i^+ \ge 0,\ i=1,2,\cdots,m \\ x \in X, \end{cases} \end{cases} \tag{5.22}$$

where P_j is the preemptive priority factor which expresses the relative importance of various goals, $P_j >> P_{j+1}$, for all j, u_{ij} is the weighting factor corresponding to positive deviation for goal i with priority j assigned, v_{ij} is the weighting factor corresponding to negative deviation for goal i with priority j assigned, d_i^+ is the positive deviation from the target of goal i, defined as

$$d_i^+ = [E[f_i(x,\xi)] - b_i] \vee 0,$$

d_i^- is the negative deviation from the target of goal i, defined as

$$d_i^- = [b_i - E[f_i(x,\xi)]] \vee 0,$$

f_i is a function in goal constraints, g_j is a function in real constraints, b_i is the target value according to goal i, l is the number of priorities, m is the number of goal constraints, and p is the number of real constraints.

5.3.2 Linear Fu-Ro EVM and Minimax Point Method

For the regular fuzzy rough linear programming problem, we can use the expected value operator to handle it,

$$\begin{cases} \max\ E[\sum_{j=1}^{n} \tilde{\bar{c}}_{ij}x_j,\ i=1,2,\cdots,m] \\ \text{s.t.} \begin{cases} E[\tilde{\bar{a}}_{rj}x_j] \geq E[\tilde{\bar{b}}_r],\ r=1,2,\ldots,p \\ x_j \geq 0,\ j=1,2,\ldots,n, \end{cases} \end{cases} \tag{5.23}$$

where $\tilde{\bar{c}}, \tilde{\bar{a}}, \tilde{\bar{b}}$ are fuzzy rough variables.

5.3.2.1 Crisp Equivalent Model

In order to solve the model (5.23), we must compute the crisp expected value of ξ. However, as we know, this process is usually a hard work at most of time. In this section, we will consider a special cases and present their results.

$$\begin{cases} \max\ [E[\tilde{\bar{c}}_1^T x], E[\tilde{\bar{c}}_2^T x], \cdots, E[\tilde{\bar{c}}_m^T x]] \\ \text{s.t.} \begin{cases} E[\tilde{\bar{a}}_r^T x] \leq E[\tilde{\bar{b}}_r], r=1,2,\cdots,p \\ x \geq 0, \end{cases} \end{cases} \tag{5.24}$$

where $\tilde{\bar{c}}_i = (\tilde{\bar{c}}_{i1}, \tilde{\bar{c}}_{i1}, \cdots, \tilde{\bar{c}}_{in})^T, \tilde{\bar{a}}_r = (\tilde{\bar{a}}_{r1}, \tilde{\bar{a}}_{r1}, \cdots, \tilde{\bar{a}}_{rn})^T$ are fuzzy rough vectors, $\tilde{\bar{b}}_r$ are fuzzy rough variables, $i=1,2,\cdots,m, r=1,2,\cdots,p$. If these fuzzy vectors, as well as rough variables have special forms, we have the following theorem.

Theorem 5.3. *If fuzzy rough variables $\tilde{\bar{c}}_{ij}$ are defined as*

$$\tilde{\bar{c}}_{ij}(\lambda) = (\bar{c}_{ij1}, \bar{c}_{ij2}, \bar{c}_{ij3}, \bar{c}_{ij4}), \quad \textit{with} \quad \bar{c}_{ijt} \vdash ([c_{ijt1}, c_{ijt2}], [c_{ijt3}, c_{ijt4}])$$

for $i=1,2,\cdots,m, j=1,2,\cdots,n, t=1,2,3,4$, then

$$E[\tilde{\bar{c}}_1^T x], E[\tilde{\bar{c}}_2^T x], \cdots, E[\tilde{\bar{c}}_m^T x]$$

is equivalent to

$$\frac{1}{16}\sum_{j=1}^{n}\sum_{t=1}^{4}\sum_{k=1}^{4} c_{1jtk}x_j, \frac{1}{16}\sum_{j=1}^{n}\sum_{t=1}^{4}\sum_{k=1}^{4} c_{2jtk}x_j, \cdots, \frac{1}{16}\sum_{j=1}^{n}\sum_{t=1}^{4}\sum_{k=1}^{4} c_{mjtk}x_j.$$

Proof. First, we verify that $E[\tilde{\bar{c}}_{ij}] = \frac{1}{16}\sum_{t=1}^{4}\sum_{k=1}^{4} c_{ijtk}, i=1,2,\cdots,m$. In fact $\forall\, \lambda \in \Lambda$,

$$\begin{aligned} E[\tilde{\bar{c}}_{ij}(\lambda)] &= \tfrac{1}{4}(\bar{c}_{ij1} + \bar{c}_{ij2} + \bar{c}_{ij3} + \bar{c}_{ij4}) \\ &= ([\tfrac{1}{4}\sum_{t=1}^{4} c_{ijt1}, \tfrac{1}{4}\sum_{t=1}^{4} c_{ijt2}], [\tfrac{1}{4}\sum_{t=1}^{4} c_{ijt3}, \tfrac{1}{4}\sum_{t=1}^{4} c_{ijt4}]). \end{aligned}$$

Suppose $A=\frac{1}{4}\sum_{t=1}^{4}c_{ijt1}, B=\frac{1}{4}\sum_{t=1}^{4}c_{ijt2}, C=\frac{1}{4}\sum_{t=1}^{4}c_{ijt3}, D=\sum_{t=1}^{4}c_{ijt4}$, then we have

$$Appr\{E[\tilde{\bar{c}}_{ij}(\lambda)]\geq r\}=\begin{cases}0, & if \quad D\leq r\\ \frac{D-r}{2(D-C)}, & if \quad B\leq r\leq D\\ \frac{1}{2}(\frac{D-r}{D-C}+\frac{B-r}{B-A}), & if \quad A\leq r\leq B\\ \frac{1}{2}(\frac{D-r}{D-C}+1), & if \quad C\leq r\leq A\\ 1, & if \quad r\leq C\end{cases}$$

and

$$Appr\{E[\tilde{\bar{c}}_{ij}(\lambda)]\leq r\}=\begin{cases}0, & if \quad r\leq C\\ \frac{r-C}{2(D-C)}, & if \quad C\leq r\leq A\\ \frac{1}{2}(\frac{r-C}{D-C}+\frac{r-A}{B-A}), & if \quad A\leq r\leq B\\ \frac{1}{2}(\frac{r-C}{D-C}+1), & if \quad B\leq r\leq D\\ 1, & if \quad D\leq r.\end{cases}$$

There are five cases when we compute the expected value of ξ. Let's discuss every case in turn.

Case 1: $0\leq C\leq A\leq B\leq D$.

$$\begin{aligned}E[\tilde{\bar{c}}_{ij}]&=\int_0^{+\infty}Appr\{\lambda\in\Lambda|E[\tilde{\bar{c}}_{ij}(\lambda)]\geq r\}dr-\int_{-\infty}^{0}Appr\{\lambda\in\Lambda|E[\tilde{\bar{c}}_{ij}]\leq r\}dr\\&=\int_0^C 1dr+\int_C^A\frac{1}{2}(\frac{D-r}{D-C}+1)dr\\&\quad+\int_A^B\frac{1}{2}(\frac{D-r}{D-C}+\frac{B-r}{B-A})dr+\int_B^D\frac{D-r}{2(D-C)}dr\\&=\frac{1}{4}(A+B+C+D).\end{aligned}$$

Case 2: $C\leq 0\leq A\leq B\leq D$.

$$\begin{aligned}E[\tilde{\bar{c}}_{ij}]&=\int_0^{+\infty}Appr\{\lambda\in\Lambda|E[\tilde{\bar{c}}_{ij}(\lambda)]\geq r\}dr-\int_{-\infty}^{0}Appr\{\lambda\in\Lambda|E[\tilde{\bar{c}}_{ij}]\leq r\}dr\\&=\int_0^A\frac{1}{2}(\frac{D-r}{D-C}+1)dr+\int_A^B\frac{1}{2}(\frac{D-r}{D-C}+\frac{B-r}{B-A})dr\\&\quad+\int_B^D\frac{D-r}{2(D-C)}dr-\int_C^0\frac{r-C}{2(D-C)}dr\\&=\frac{1}{4}(A+B+C+D).\end{aligned}$$

Case 3: $C\leq A\leq 0\leq B\leq D$.

$$\begin{aligned}E[\tilde{\bar{c}}_{ij}]&=\int_0^{+\infty}Appr\{\lambda\in\Lambda|E[\tilde{\bar{c}}_{ij}(\lambda)]\geq r\}dr-\int_{-\infty}^{0}Appr\{\lambda\in\Lambda|E[\tilde{\bar{c}}_{ij}(\lambda)]\leq r\}dr\\&=\int_0^B\frac{1}{2}(\frac{D-r}{D-C}+\frac{B-r}{B-A})dr+\int_B^D\frac{D-r}{2(D-C)}dr\\&\quad-\int_C^A\frac{r-C}{2(D-C)}dr-\int_A^0\frac{1}{2}(\frac{r-C}{D-C}+\frac{r-A}{B-A})dr\\&=\frac{1}{4}(A+B+C+D).\end{aligned}$$

Case 4: $C\leq A\leq B\leq 0\leq D$.

$$\begin{aligned}E[\tilde{\bar{c}}_{ij}]&=\int_0^{+\infty}Appr\{\lambda\in\Lambda|E[\tilde{\bar{c}}_{ij}(\lambda)]\geq r\}dr-\int_{-\infty}^{0}Appr\{\lambda\in\Lambda|E[\tilde{\bar{c}}_{ij}(\lambda)]\leq r\}dr\\&=\int_0^D\frac{D-r}{2(D-C)}dr-\int_C^A\frac{r-C}{2(D-C)}dr\\&\quad-\int_A^B\frac{1}{2}(\frac{r-C}{D-C}+\frac{r-A}{B-A})dr-\int_B^0\frac{1}{2}(\frac{r-C}{D-C}+1)dr\\&=\frac{1}{4}(A+B+C+D).\end{aligned}$$

Case 5: $C \le A \le B \le D \le 0$.

$$\begin{aligned} E[\tilde{\bar{c}}_{ij}] &= \int_0^{+\infty} Appr\{\lambda \in \Lambda | E[\tilde{\bar{c}}_{ij}(\lambda)] \ge r\} dr - \int_{-\infty}^{0} Appr\{\lambda \in \Lambda | E[\tilde{\bar{c}}_{ij}(\lambda)] \le r\} dr \\ &= -\int_C^A \frac{r-C}{2(D-C)} dr - \int_A^B \frac{1}{2}\left(\frac{r-C}{D-C} + \frac{r-A}{B-A}\right) dr \\ &\quad - \int_B^D \frac{1}{2}\left(\frac{r-C}{D-C} + 1\right) - dr \int_D^0 1 dr \\ &= \frac{1}{4}(A+B+C+D). \end{aligned}$$

So we always have $E[\tilde{\bar{c}}_{ij}] = \frac{1}{16} \sum\limits_{t=1}^{4} \sum\limits_{k=1}^{4} c_{ijtk}, i = 1, 2, \cdots, m.$

It follows from the nonnegativity of $x_j (j = 1, 2, \cdots, n)$ and linearity of expected value operator that

$$\begin{aligned} E[\tilde{\bar{c}}_i^T x] &= E[\sum_{j=1}^{n} \tilde{\bar{c}}_{ij} x_j] \\ &= \sum_{j=1}^{n} E[\tilde{\bar{c}}_{ij}] x_j \\ &= \frac{1}{16} \sum_{j=1}^{n} \sum_{t=1}^{4} \sum_{k=1}^{4} c_{rjtk} x_j. \end{aligned}$$

Thus the theorem is proved. □

Theorem 5.4. *If fuzzy rough variables* $\tilde{\bar{a}}_{rj}, \tilde{\bar{b}}_r$ *are defined as follows,*

$$\begin{aligned} &\tilde{\bar{a}}_{rj}(\lambda) = (\bar{a}_{rj1}, \bar{c}_{rj2}, \bar{a}_{rj3}, \bar{a}_{rj4}), \quad \textit{with} \quad \bar{a}_{rjt} \vdash ([a_{rjt1}, a_{rjt2}], [a_{rjt3}, a_{rjt4}]), \\ &\tilde{\bar{b}}_r(\lambda) = (\bar{b}_{r1}, \bar{b}_{r2}, \bar{b}_{r3}, \bar{c}_{r4}), \quad \textit{with} \quad \bar{b}_{rt} \vdash ([b_{rt1}, b_{rt2}], [b_{rt3}, b_{rt4}]), \end{aligned}$$

for $r = 1, 2, \cdots, p, j = 1, 2, \cdots, n, t = 1, 2, 3, 4$, *then*

$$E[\tilde{\bar{a}}_r^T x] \le E[\tilde{\bar{b}}_r], r = 1, 2, \cdots, p$$

is equivalent to

$$\sum_{j=1}^{n} \sum_{t=1}^{4} \sum_{k=1}^{4} a_{rjtk} x_j \le \sum_{t=1}^{4} \sum_{k=1}^{4} b_{rtk}, r = 1, 2, \cdots, p.$$

Proof. Similar to Theorem 5.4, we have

$$E[\tilde{\bar{a}}_i^T x] = \frac{1}{16} \sum_{j=1}^{n} \sum_{t=1}^{4} \sum_{k=1}^{4} a_{rjtk} x_j$$

and

$$E[\tilde{\bar{b}}_r] = \frac{1}{16} \sum_{t=1}^{4} \sum_{k=1}^{4} b_{rtk},$$

for $i = 1, 2, \cdots, m, r = 1, 2, \cdots, p$.

Thus the theorem holds. □

According to Theorems 5.3-5.4, Model (5.24) with the Fu-Ro coefficients described as Theorem 5.3 and Theorem 5.4 is equivalent to the conventional multi-objective linear programming

$$\begin{cases} \max \left[\frac{1}{16}\sum_{j=1}^{n}\sum_{t=1}^{4}\sum_{k=1}^{4} c_{1jtk}x_j, \frac{1}{16}\sum_{j=1}^{n}\sum_{t=1}^{4}\sum_{k=1}^{4} c_{2jtk}x_j, \cdots, \frac{1}{16}\sum_{j=1}^{n}\sum_{t=1}^{4}\sum_{k=1}^{4} c_{mjtk}x_j \right] \\ \text{s.t.} \begin{cases} \sum_{j=1}^{n}\sum_{t=1}^{4}\sum_{k=1}^{4} a_{rjtk}x_j \le \sum_{t=1}^{4}\sum_{k=1}^{4} b_{rtk}, r = 1,2,\cdots,p \\ x_j \ge 0, j = 1,2,\cdots,n. \end{cases} \end{cases} \tag{5.25}$$

5.3.2.2 Minimax Point Method

In this section, we use the minimax point method proposed in [92] to deal with the crisp multiobjective problem (5.26).

$$\begin{cases} \max\ [H_1(x), H_2(x), \cdots, H_m(x)] \\ \text{s.t. } x \in X. \end{cases} \tag{5.26}$$

To maximize the objectives, the minimax point method firstly constructing an evaluation function by seeking the minimal objective value after respectively computing all objective functions, that is, $u(\mathbf{H}(x)) = \min_{1\le i\le m} H_i(x)$, where $\mathbf{H}(x) = (H_1(x), H_2(x), \cdots, H_m(x))^T$. Then the objective function of problem (5.26) is came down to solve the maximization problem as follows,

$$\max_{x\in X'} u(\mathbf{H}(x)) = \max_{x\in X'} \min_{1\le i\le m} H_i(x). \tag{5.27}$$

Sometimes, decision makers need considering the relative importance of various goals, then the weight can be combined into the evaluation function as follows,

$$\max_{x\in X'} u(\mathbf{H}(x)) = \max_{x\in X'} \min_{1\le i\le m} \{\omega_i H_i(x)\}, \tag{5.28}$$

where the weight $\sum_{i=1}^{m} \omega_i = 1(\omega_i > 0)$ and is predetermined by decision makers.

Theorem 5.5. *The optimal solution x^* of problem (5.28) is the weak efficient solution of problem (5.26).*

Proof. Assume that $x^* \in X'$ is the optimal solution of the problem (5.28). If there exists an x such that $H_i(x) \ge H_i(x^*)(i = 1,2,\cdots,m)$, we have

$$\min_{1\le i\le m} \{\omega_i H_i(x^*)\} \le \omega_i H_i(x^*) \le \omega_i H_i(x),\ 0 < \omega_i < 1.$$

Denote $\delta = \min_{1\le i\le m}\{\omega_i H_i(x)\}$, then $\delta \ge \min_{1\le i\le m}\{\omega_i H_i(x^*)\}$. This means that x^* isn't the optimal solution of the problem (5.28). This conflict with the condition.

Thus, there doesn't exist $x \in X'$ such that $H_i(x) \geq H_i(x^*)$, namely, x^* is a weak efficient solution of the problem (5.26). □

By introducing an auxiliary variable, the minimax problem (5.28) can be converted into a single objective problem. Let

$$\lambda = \min_{1 \leq i \leq m} \{\omega_i H_i(x)\},$$

then the problem (5.28) is converted into

$$\begin{cases} \max \lambda \\ \text{s.t.} \begin{cases} \omega_i H_i(x) \geq \lambda, i = 1, 2, \cdots, m \\ x \in X'. \end{cases} \end{cases} \tag{5.29}$$

Theorem 5.6. *The problem (5.28) is equivalent to the problem (5.29).*

Proof. Assume that $x^* \in X'$ is the optimal solution of the problem (5.28) and let $\lambda^* = \min_{1 \leq i \leq m}\{\omega_i H_i(x^*)\}$, then it is apparent that $H_i(x^*) \geq \lambda^*$. This means that (x^*, λ^*) is a feasible solution of the problem (5.29). Assume that (x, λ) is any feasible solution of the problem (5.29). Since x^* is the optimal solution of the problem (5.28), wa have

$$\lambda^* = \min_{1 \leq i \leq m} \{\omega_i H_i(x^*)\} \geq \min_{1 \leq i \leq m} \{\omega_i H_i(x)\} \geq \lambda,$$

namely, (x^*, λ^*) is the optimal solution of the problem (5.29).

On the contrary, assume that (x^*, λ^*) is an optimal solution of the problem (5.29). Then $\omega_i H_i(x^*) \geq \lambda^*$ holds for any i, this means $\min_{1 \leq i \leq m}\{\omega_i H_i(x^*)\} \geq \lambda^*$. It follows that for any any feasible $x \in X'$,

$$\min_{1 \leq i \leq m} \{\omega_i H_i(x)\} = \lambda \leq \lambda^* \leq \min_{1 \leq i \leq m} \{\omega_i H_i(x^*)\}$$

holds, namely, x^* is the optimal solution of the problem (5.28). □

In a word, the minimax point method can be summarized as follows:
Step 1. Compute the weight for each objective function by solving the two problems, $\max_{x \in X'} H_i(x)$ and $\omega_i = H_i(x^*)/\sum_{i=1}^{m} H_i(x^*)$.
Step 2. Construct the auxiliary problem as follows,

$$\begin{cases} \max \lambda \\ \text{s.t.} \begin{cases} \omega_i H_i(x) \geq \lambda, i = 1, 2, \cdots, m \\ x \in X'. \end{cases} \end{cases}$$

Step 3. Solve the above problem to obtain the optimal solution.

5.3.2.3 Numerical Example

Example 5.2. Let us consider the following problem.

$$\begin{cases} \max\ 0.375x_1 + 0.625x_2 + 0.875x_3 \\ \max\ E[c_1\xi_1 x_1 + c_2\xi_2 x_2 + c_3\xi_3 x_3] \\ \text{s.t.} \begin{cases} x_1 + x_2 + x_3 \le 250 \\ x_1 + x_2 + x_3 \ge 200 \\ \xi_4 x_1 + \xi_5 x_2 + \xi_6 x_3 \le 600 \\ x_1 \ge 20, x_2 \ge 20, x_3 \ge 20. \end{cases} \end{cases} \tag{5.30}$$

The following is the relevant data, ξ is fuzzy rough variable,

$$\begin{array}{l} (c_1, c_2, c_3) = (1.2, 0.8, 1.5), \\ \xi_1 = (\rho_1 - 2, \rho_1 - 1, \rho_1 + 1, \rho_1 + 2), \text{ with } \rho_1 \vdash ([0,1],[0,3]), \\ \xi_2 = (\rho_2 - 2, \rho_2 - 1, \rho_2 + 1, \rho_2 + 2), \text{ with } \rho_2 \vdash ([1,2],[0,3]), \\ \xi_3 = (\rho_3 - 2, \rho_3 - 1, \rho_3 + 1, \rho_3 + 2), \text{ with } \rho_3 \vdash ([2,3],[0,3]). \\ \xi_4 = (\rho_4 - 2, \rho_4 - 1, \rho_4 + 1, \rho_4 + 2), \text{ with } \rho_1 \vdash ([2,5],[0,9]), \\ \xi_5 = (\rho_5 - 2, \rho_5 - 1, \rho_5 + 1, \rho_5 + 2), \text{ with } \rho_2 \vdash ([10,20],[4,30]), \\ \xi_6 = (\rho_6 - 2, \rho_6 - 1, \rho_6 + 1, \rho_6 + 2), \text{ with } \rho_3 \vdash ([6,10],[4,12]). \end{array}$$

It follows from Proposition 5.4 that problem (5.30) is equivalent to

$$\begin{cases} \max\ F_1(x) = 0.375x_1 + 0.625x_2 + 0.875x_3 \\ \max\ F_2(x) = 0.3x_1 + 0.3x_2 + 0.75x_3 \\ \text{s.t.} \begin{cases} x_1 + x_2 + x_3 \le 250 \\ x_1 + x_2 + x_3 \ge 200 \\ x_1 + 2x_2 + 4x_3 \le 600 \\ x_1 \ge 20, x_2 \ge 20, x_3 \ge 20. \end{cases} \end{cases} \tag{5.31}$$

According to the minimax point method, first we compute the weight by solving the two single objective models,

$$\begin{array}{l} w_1 = F_1^*/(F_1^* + F_2^*) = 166.25/(166.25 + 124.5) = 0.572, \\ w_2 = F_2^*/(F_1^* + F_2^*) = 0.428. \end{array}$$

Then according to Equation (5.29) we construct the following mode (5.32),

$$\begin{cases} \max\ \lambda \\ \text{s.t.} \begin{cases} w_1 * (0.375x_1 + 0.625x_2 + 0.875x_3) + w_2 * (0.3x_1 + 0.3x_2 + 0.75x_3) \ge \lambda \\ x_1 + x_2 + x_3 \le 250 \\ x_1 + x_2 + x_3 \ge 200 \\ x_1 + 2x_2 + 4x_3 \le 600 \\ x_1 \ge 20, x_2 \ge 20, x_3 \ge 20. \end{cases} \end{cases} \tag{5.32}$$

After solving the model (5.32), we can get a efficient solution as follows,

$$(x_1, x_2, x_3) = (120, 20, 110).$$

5.3.3 Non-linear Fu-Ro EVM and Fu-Ro Simulation-Based TS

For the non-linear Fu-Ro EVM, we use the Fu-Ro simulation 1 based TS to solve.

5.3.3.1 Fu-Ro Simulation 1 for Expected Valie

First, we introduce the procedure to simulate the expected value of a Fu-Ro variable.

Assume that ξ is an n-dimensional Fu-Ro vector defined on the rough space $(\Lambda,\Delta,\mathscr{A},\pi)$, and $f:\mathbf{R}^n\to\mathbf{R}^m$ is a measurable function. In order to calculate the expected value $E[f(\xi)]$, we sample $\underline{\lambda}_1,\underline{\lambda}_2,\cdots,\underline{\lambda}_N$ from Δ and $\overline{\lambda}_1,\overline{\lambda}_2,\cdots,\overline{\lambda}_N$ from Λ. For each $\underline{\lambda}_n$ and $\overline{\lambda}_n$, $n=1,2,\cdots,N$, $\xi(\underline{\lambda}_n)$ and $\xi(\overline{\lambda}_n)$ are both fuzzy variables, and $f(\xi(\underline{\lambda}_n))$ and $F(\xi(\overline{\lambda}_n))$ are both fuzzy variables. Then we can apply the fuzzy simulation 1 to get their expected values $E[f(\xi(\underline{\lambda}_n))]$ and $E[f(\xi(\overline{\lambda}_n))]$.

Since $E[f(\xi)]$ is essentially the expected value of rough variable $E[f(\xi(\lambda))]$, and the following (5.33) will be used to get the expected value of the rough variables.

$$E[f(\xi)]=\frac{\sum_{n=1}^{N}(\eta E[f(\xi(\overline{\lambda}_n))]+(1-\eta)E[f(\xi(\underline{\lambda}_n))])}{2N}. \tag{5.33}$$

So we may combine rough simulation and fuzzy simulation to produce a fuzzy rough simulation as follows.

Step 1. Set $L=0$.
Step 2. Generate $\underline{\lambda}$ from Δ according to the measure.
Step 3. Generate $\overline{\lambda}$ from Λ according to the measure π.
Step 4. $L\leftarrow L+E[f(\xi(\underline{\lambda}))]+E[f(\xi(\overline{\lambda}))]$.
Step 5. Repeat the second to fourth steps N times.
Step 6. Return $L/(2N)$.

Example 5.3. We employ the Fu-Ro simulation 1 to calculate the expected value of $\xi_1\xi_2$, where ξ_1 and ξ_2 are Fu-Ro variables defined as

$$\xi_1=(\rho_1,\rho_1+1,\rho_1+2),\text{ with }\rho_1=([1,2],[0,3]),$$
$$\xi_2=(\rho_2,\rho_2+1,\rho_2+2),\text{ with }\rho_1=([2,3],[1,4]).$$

After a run of Fu-Ro simulation 1 with 5000 cycles, and we get $E[\xi_1\xi_2]=8.93$.

5.3.3.2 TS

In the following, let's introduce the Tabu search algorithm (TS) algorithm.

Local search employs the idea that a given solution x may be improved by making small changes. Those solutions obtained by modifying solution x are called neighbors of x. The local search algorithm starts with some initial solution and moves from neighbor to neighbor as long as possible while decreasing the objective function value. The main problem with this strategy is to escape from local minima where the search cannot find any further neighborhood solution that decreases the objective function value. Different strategies have been proposed to solve this problem. One of the most efficient strategies is tabu search. Tabu search allows the search to explore solutions that do not decrease the objective function value only in those cases where these solutions are not forbidden. This is usually obtained by keeping track of the last solutions in term of the action used to transform one solution to the next. When an action is performed it is considered tabu for the next T iterations, where T is the tabu status length. A solution is forbidden if it is obtained by applying a tabu action to the current solution. The Tabu Search metaheuristic has been defined by Fred Glover [385]. The basic ideas of TS have also been sketched by P. Hansen [386]. After that, TS has achieved widespread success in solving practical optimization problems in different domains(such as resource management, process design, logistics and telecommunications).

A *tabu list* is a set of solutions determined by historical information from the last t iterations of the algorithm, where t is fixed or is a variable that depends on the state of the search, or a particular problem. At each iteration, given the current solution x and its corresponding neighborhood $N(x)$, the procedure moves to the solution in the neighborhood $N(x)$ that most improves the objective function. However, moves that lead to solutions on the tabu list are forbidden, or are tabu . If there are no improving moves, TS chooses the move which least changes the objective function value. The tabu list avoids returning to the local optimum from which the procedure has recently escaped. A basic element of tabu search is the *aspiration criterion*, which determines when a move is admissible despite being on the tabu list. One *termination* criterion for the tabu procedure is a limit in the number of consecutive moves for which no improvement occurs. Given an objective function $f(x)$ over a feasible domain D, a generic tabu search for finding an approximation of the global minimum of $f(x)$ is given in Figure 5.8.

We introduce the detailed steps on how to apply a special TS algorithm–Enhanced Continuous Tabu Search(ECTS) proposed by R. Chelouah and P. Siarry [323] based on fuzzy rough simulation to solve a multi-objective expected value model with fuzzy rough parameters.

Setting of parameters. Two of the parameters must be set before any execution of ECTS:

(1) initialization,

(2) control parameters.

For each of these categories, some parameter values must be chosen by the user and some parameter values must be calculated. These four subsets of parameters are listed in Table 5.1.

Initialization. In this stage, we will list the representation of the solution. We have resumed and adapted the method described in detail in [324]. Randomly generate a solution x and check its feasibility by the fuzzy rough simulation such that $E[g_r(x,\xi)] \leq 0 (r = 1,2,\cdots,p)$. Then generate its neighborhood by the concept of 'ball' defined in [324]. A ball $B(x,r)$ is centered on x with radius r, which contains all points x' such that $||x'-x|| \leq 4$(the symbol $||\cdot||$ denotes the Euclidean norm). To obtain a homogeneous exploration of the space, we consider a set of balls centered on the current solution x, with $h_0, h_1, \cdots, h_\eta$. Hence the space is partitioned into concentric 'crowns' $C_i(x, h_{i-1}, h_i)$, such that

Input: *A problem instance*

Output: *A (sub-optimal) solution*

1. Intialization:
   (a) Generate an intial solution x and set $x^* = x$
   (b) Intialize the tabu list $T = \Phi$
   (c) Set iteration counters $k = 0$ and $l = 0$
2. while ($N(x)\setminus T \neq \Phi$) do
   (a) $k = k+1; l = l+1$
   (b) Select x as the best solution from the set $N(x)\setminus T$
   (c) If $f(x) < f(x^*)$ then update $x^* = x$ and set $l = 0$
   (d) If $k = \bar{k}$ or if $l = \bar{l}$ go to step 3
3. Output the best solution found x^*

Fig. 5.8 Layout of tabu search

$$C_i(x, h_{i-1}, h_i) = \{x' | h_{i-1} \leq ||x' - x|| \leq h_i\}.$$

The η neighbors of s are obtained by random selection of one point inside each crown C_i, for i varying from 1 to η. Finally, we select the best neighbor of x among these η neighbors, even if it is worse than x. In ECTS, we replace the balls by hyperrectangles for the partition of the current solution neighborhood (see Figure 5.9), and we gener-

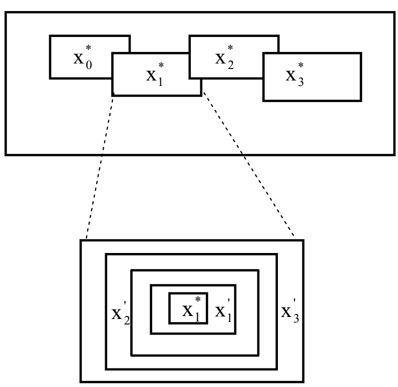

Fig. 5.9 Partition of current solution neighborhood

Table 5.1 Listing of the ECTS parameters

A. *Initialization parameters chosen by the user*
Search domain of each function variable
Starting point
Content of the tabu list
Content of the promising list
B. *Initialization parameters calculated*
Length δ of the smallest edge of the initial hyperrectangular search domain
Initial threshold for the acceptance of a promising area
Initial best point
Number η of neighbors of the current solution investigated at each iteration
Maximum number of successive iterations without any detection of a promising area
Maximum number of successive iterations without any improvement of the objective function value
Maximum number of successive reductions of the hyperrectangular neighborhood and of the radius of tabu balls with out any improvement
Maximum number of iterations
C. *Control parameters chosen by the user*
Length N_t of the tabu list
Length N_p of the promising list
Parameter ρ_t allowing to calculate the initial radius of tabu balls
Parameter ρ_{neigh} allowing to calculate the initial size of the hyperrectangular neighborhood
D. *Control parameters calculated*
Initial radius ε_t of tabu balls
Initial radius ε_p of promising balls
Initial size of the hyperrectangular neighborhood

ate neighbors in the same way. The reason for using a hyperrectangular neighborhood instead of crown 'balls' is the following: it is mathematically much easier to select a point inside a specified hyperrectangular zone than to select a point inside a specified crown ball. Therefore in the first case, we only have to compare the coordinates of the randomly selected points with the bounds that define the hyperrectangular zone at hand.

Next, we will describe the initialization of some parameters and the tuning of the control parameters. In other words, we give the 'definition' of all the parameters of ECTS. The parameters in part A of Table 5.1 are automatically built by using the parameters fixed at the beginning. The parameters in part B of Table 5.1 are valued in the following way:

(1) the search domain of analytical test functions is set as prescribed in the literature, the initial solution x^* is randomly chosen and checked if it is feasible by the fuzzy rough simulation,

(2) the tabu list is initially empty,

(3) to complete the promising list, the algorithm randomly draws a point. This point is accepted as the center of an initial promising ball, if it does not belong to an already generated ball. In this way the algorithm generates N_p sample points which are uniformly dispersed in the whole space solution S,

(4) the initial threshold for the acceptance of a promising area is taken equal to the average of the objective function values over the previous N_p sample points,

(5) the best point found is taken equal to the best point among the previous N_p,

(6) the number η of neighbors of the current solution investigated at each iteration is set to twice the number of variables, if this number is equal or smaller than five, otherwise η is set to 10;

(7) the maximum number of successive iterations without any detection of a new promising area is equal to twice the number of variables,

(8) the maximum number of successive iterations without any improvement of the objective function value is equal to five times the number of variables,

(9) the maximum number of successive reductions of the hyperrectangular neighborhood and of the radius of tabu balls without any improvement of the objective function value is set to twice the number of variables,

(10) the maximum number of iterations is equal to 50 times the number of variables.

There exist two types of control parameters. Some parameters are chosen by the user. Other ones are deduced from the chosen parameters. The fixed parameters are the length of the tabu list (set to 7, which is the usual tuning advocated by Glover), the length of the promising list (set to 10, like in [325]) and the parameters ρ_t, ρ_p and $\rho_n eigh$ (set to 100, 50, and 5, respectively). The expressions of ε_t and ε_p are δ/ρ_t and δ/ρ_p respectively, and the initial size of the hyperrectangular neighborhood of the current solution (the more external hyperrectangle) is obtained by dividing δ by the factor ρ_{neigh}.

Diversification. At this stage, the process starts with the initial solution, used as the current one. ECTS generates a specified number of neighbors: one point is selected inside each hyperrectangular zone around the current solution. Each neighbor is accepted only if it does not belong to the tabu list. The best of these neighbors becomes the new current solution, even if it is worse than the previous one. A new promising solution is detected and generated according to the procedure described above. This promising solution defines a new promising area if it does not already belong to a promising ball. If a new promising area is accepted, the worst area of the promising list is replaced by the newly accepted promising area. The use of the promising and tabu lists stimulates the search for solutions far from the starting one and the identified promising areas. The diversification process stops after a given number of successive iterations without any detection of a new promising area. Then the algorithm determines the most promising area among those present in the promising list.

Search for the most promising area. In order to determine the most promising area, we proceed in three steps. First, we calculate the average value of the

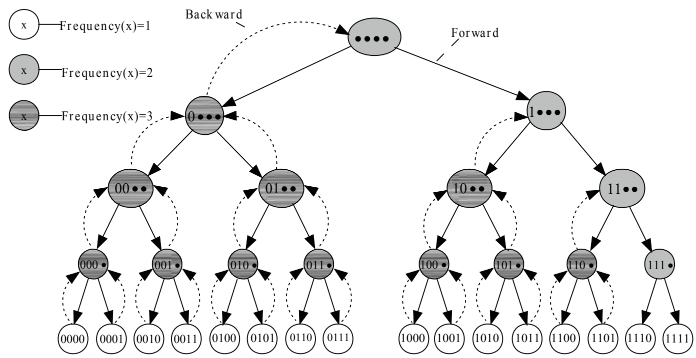

Fig. 5.10 A standard "backtracking"(depth first) branch-and-bound approach

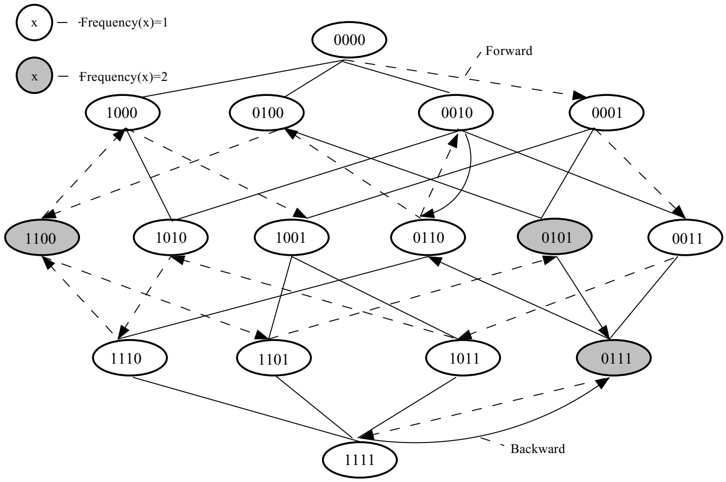

Fig. 5.11 TTS approach

objective function over all the solutions present in the promising list. Secondly, we eliminate all the solutions for which the function value is higher than this average value. Thirdly, we deal with the thus reduced list in the following way. We halve the radius of the tabu balls and the size of the hyperrectangular neighborhood. For each remaining promising solution, we perform the generation of the neighbors and selection of the best. We replace the promising solution by the best neighbor located, yet only if this neighbor is better than that solution. After having scanned the whole promising list, the algorithm removes the least promising solution. This process is reiterated after halving again the above two parameters. It stops when just one promising area remains.

5.3.3.3 Numerical Example

Example 5.4. Let us consider a multi-objective programming with Fu-Ro coefficients.

$$\begin{cases} \max F_1(x,\xi) = 2\xi_1 x_1^2 + 3\xi_2 x_2 - \xi_3 x_3 + \sqrt{(1-\xi_7)^2+(3-\xi_8)^2+(2-\xi_9)^2} \\ \max F_2(x,\xi) = 5\xi_4 x_2 - 2\xi_5 x_1 + 2\xi_6 x_3 + \sqrt{(5-\xi_{10})^2+(2-\xi_{11})^2+(1-\xi_{12})^2} \\ \text{s.t.} \begin{cases} 5x_1 - 3x_2^2 + 6\sqrt{x_3} \le 50 \\ 4\sqrt{x_1} + 6x_2 - 4.5x_3 \le 20 \\ x_1 + x_2 + x_3 \le 15 \\ x_1, x_2, x_3 \ge 0, \end{cases} \end{cases} \tag{5.34}$$

where $\xi_i (i = 1,2,\cdots,12)$ are Fu-Ro variables subject to follows,

$$\begin{array}{l} \xi_1 = (1,\lambda_1,1)_{LR}, \text{with } \lambda_1 \vdash ([0.2,0.5],[0,1]), \\ \xi_2 = (1,\lambda_2,1)_{LR}, \text{with } \lambda_2 \vdash ([0.6,0.8],[0,1]), \\ \xi_3 = (1,\lambda_3,1)_{LR}, \text{with } \lambda_3 \vdash ([0.45,0.95],[0,1]), \\ \xi_4 = (1,\lambda_4,1)_{LR}, \text{with } \lambda_4 \vdash ([0.4,0.5],[0,1]), \\ \xi_5 = (1,\lambda_5,1)_{LR}, \text{with } \lambda_5 \vdash ([0.36,0.64],[0,1]), \\ \xi_6 = (1,\lambda_6,1)_{LR}, \text{with } \lambda_6 \vdash ([0.55,0.65],[0,1]), \\ \xi_7 = (1,\lambda_7,1)_{LR}, \text{with } \lambda_7 \vdash ([1,2],[0,4]), \\ \xi_8 = (1,\lambda_8,1)_{LR}, \text{with } \lambda_8 \vdash ([3,4],[2,8]), \\ \xi_9 = (1,\lambda_9,1)_{LR}, \text{with } \lambda_9 \vdash ([1,2],[0,2]), \\ \xi_{10} = (1,\lambda_{10},1)_{LR}, \text{with } \lambda_{10} \vdash ([2,3],[1,5]), \\ \xi_{11} = (1,\lambda_{11},1)_{LR}, \text{with } \lambda_{11} \vdash ([1,3],[0,4]), \\ \xi_{12} = (1,\lambda_{12},1)_{LR}, \text{with } \lambda_{12} \vdash ([2,4],[2,8]). \end{array}$$

By Fu-Ro simulation, after 3000 cycles, we firstly have

$$E[\sqrt{(1-\xi_7)^2+(3-\xi_8)^2+(2-\xi_9)^2}] = 16.3514,$$

$$E[\sqrt{(5-\xi_{10})^2+(2-\xi_{11})^2+(1-\xi_{12})^2}] = 7.0568.$$

Next, we apply the tabu search algorithm based on the Fu-Ro simulation to solve the nonlinear programming problem (5.34) with the Fu-Ro parameters.

Step 1. Set the move step $h = 0.5$ and the h neighbor $N(x,h)$ for the present point x is defined as follows,

$$N(x,h) = \left\{ \mathbf{y} | \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2} \le h \right\}.$$

The random move of point x to point $\mathbf{y}$ in its h neighbor along direction s is given by

$$\mathbf{y}_s = x_s + rh,$$

where r is a random number that belongs to [0,1], $s = 1,2,3$.

Step 2. Denote $X = \{(x_1,x_2,x_3)|5x_1 - 3x_2^2 + 6\sqrt{x_3} \le 50; x_1+x_2+x_3 \le 15; x_i \ge 0, i = 1,2,3\}$. Give the step set $H = \{h_1,h_2,\cdots,h_r\}$ and randomly generate a feasible point $x_0 \in X$. One should empty the Tabu list T (the list of inactive steps) at the beginning.

Step 3. For each active neighbor $N(x,h)$ of the present point x, where $h \in H - T$, a feasible random move that satisfies all the constraints in problem (5.34) is to be generated.

Step 4. Construct the single objective function as follows,

$$f(x,\xi) = w_1\left(2\xi_1x_1^2 + 3\xi_2x_2 - \xi_3x_3 + \sqrt{(1-\xi_7)^2 + (3-\xi_8)^2 + (2-\xi_9)^2}\right) \\ +w_2\left(5\xi_4x_2 - 2\xi_5x_1 + 2\xi_6x_3 + \sqrt{(5-\xi_{10})^2 + (2-\xi_{11})^2 + (1-\xi_{12})^2}\right),$$

where $w_1 + w_2 = 1$. Compare the $f(x,\xi)$ of the feasible moves with that of the current solution by the fuzzy rough simulation. If an augmenter in new objective function of the feasible moves exists, one should save this feasible move as the updated current one by adding the corresponding step to the Tabu list T and go to the next step; otherwise, go to the next step directly.

Step 5. Stop if the termination criteria are satisfied; other wise, empty T if it is full; then go to Step 3. Here, we set the computation is determined if the better solution doesn't change again.

Table 5.2 Another taxonomy dimension for parallel TS algorithms

w_1	w_2	x_1	x_2	x_3	$f_1(x)$	$f_2(x)$	$f(x)$	Gen
0.1	0.9	0.217	7.861	6.721	219.496	523.976	493.528	1568
0.2	0.8	0.336	7.648	6.712	213.055	511.281	451.638	1470
0.3	0.7	11.356	2.173	1.471	2269.226	-84.484	621.629	1760
0.4	0.6	11.157	2.258	1.573	2194.900	-74.222	833.427	1633
0.5	0.5	11.257	2.151	1.444	2230.700	-84.1427	1073.279	2010
0.6	0.4	11.151	2.244	1.519	2192.769	-75.955	1285.279	1807
0.7	0.3	11.268	2.155	1.532	2234.000	-82.237	1539.189	1834
0.8	0.2	11.205	2.148	1.444	2210.736	-83.245	1752.061	2762
0.9	0.1	11.075	2.288	1.579	2164.917	-71.025	1941.315	1792

5.4 Fu-Ro CCM

Another way to tackle the multi-objective model with Fu-Ro parameters is that we use the chance to measure the uncertainty of fuzzy rough event. In order to compare the degree of occurrence of fuzzy rough events, several kinds of chance measure are introduced.

5.4.1 General Model for Fu-Ro CCM

It has been increasingly recognized that many real-world decision-making problems involve multiple and conflicting objectives which should be considered simultaneously. Fuzzy programming of the multi-objective has been well developed, and as an extension of the fuzzy multi-objective decision-making case, the Fu-Ro multi-objective linear decision-making model is defined as a means for optimizing multiple different objective functions subject to a number of constrains.

Let's introduce the general Fu-Ro CCM to deal with the uncertain model (5.40) as follows.

$$\begin{cases} \max\ [f_1, f_2, \cdots, f_m] \\ \text{s.t.} \begin{cases} Ch\{f_i(x,\xi) \geq f_i\}(\gamma_i) \geq \delta_i,\ i = 1,2,\cdots,m \\ Ch\{g_r(x,\xi) \leq 0\}(\eta_r) \geq \theta_r,\ r = 1,2,\cdots,p, \\ x \in X, \end{cases} \end{cases}$$

where Ch is the chace measure of the fuzzy rough events, $\gamma_i, \delta_i, \eta_r, \theta_r$ are the predetermined confidence level, f_i and x_i are the decision variables, $i = 1,2,\cdots,m$.

According to the definition 5.7 of the primitive chance measure:

$$Ch\{\tilde{\bar{e}}_r^{\mathrm{T}} x \leq \tilde{\bar{b}}_r\}(\eta_r) \geq \theta_r \Leftrightarrow Appr\{\lambda | Pos\{\tilde{\bar{e}}_r(\lambda)^{\mathrm{T}} x \leq \tilde{\bar{b}}_r(\lambda)\} \geq \theta_r\} \geq \eta_r, \tag{5.35}$$
$$Ch\{\tilde{\bar{c}}_i^{\mathrm{T}} x \geq f_i\}(\gamma_i) \geq \delta_i \Leftrightarrow Appr\{\lambda | Pos\{\tilde{\bar{c}}_i(\lambda)^{\mathrm{T}} x \geq f_i\} \geq \delta_i\} \geq \gamma_i. \tag{5.36}$$

So we can get the general Fu-Ro CCM,

$$\begin{cases} \max\ [f_1, f_2, \cdots, f_m] \\ \text{s.t.} \begin{cases} Appr\{\lambda | Pos\{\tilde{\bar{c}}_i(\lambda)^{\mathrm{T}} x \geq f_i\} \geq \delta_i\} \geq \gamma_i, \quad i = 1,2,\cdots,m \\ Appr\{\lambda | Pos\{\tilde{\bar{e}}_r(\lambda)^{\mathrm{T}} x \leq \tilde{\bar{b}}_r(\lambda)\} \geq \theta_r\} \geq \eta_r, \quad r = 1,2,\cdots,p \\ x \geq 0, \end{cases} \end{cases} \tag{5.37}$$

where $\delta_i, \gamma_i, \theta_r, \eta_r \in [0,1]$ are the predetermined confidence level, $Pos\{\cdot\}$ denotes the possibility of the fuzzy events in $\{\cdot\}$, and $Appr\{\cdot\}$ denotes the approximation degree of the rough events in $\{\cdot\}$.

Definition 5.9. Suppose a feasible solution x^* of the problem (5.43) satisfies

$$Appr\{\lambda | Pos\{\tilde{\bar{c}}_i(\lambda)^{\mathrm{T}} x^* \geq f_i(x^*)\} \geq \delta_i\} \geq \gamma_i, \quad i = 1,2,\cdots,m,$$

where confidence levels $\delta_i, \gamma_i \in [0,1]$. x^* is a fuzzy rough efficient solution at $\delta_i - Appr\ \gamma_i - Pos$ levels to the problem (5.40) if and only if there exists no other feasible solution x such that

$$Appr\{\lambda | Pos\{\tilde{\bar{c}}_i(\lambda)^{\mathrm{T}} x \geq f_i(x)\} \geq \delta_i\} \geq \gamma_i, \quad i = 1,2,\cdots,m,$$

$f_i(x) \geq f_i(x^*)$ for all i and $f_{i_0}(x) \geq f_{i_0}(x^*)$ for at least one $i_0 \in \{1,2,\cdots,m\}$.

From Definition (5.9), we know that x^* is a fuzzy rough efficient solution at $\delta_i - Appr$ $\gamma_i - Pos$ levels to the problem (5.40) if x^* is a Pareto optimal solution of the problem (5.43).

Remark 5.6. If the fuzzy rough vector $\tilde{\bar{c}}_i$ delegates to rough vector $\tilde{c}_i$, then $\tilde{c}_i^{\mathrm{T}} x \geq f_i$ is a rough event. For $\lambda \in \Lambda$, $Pos\{\tilde{c}_i(\lambda)^{\mathrm{T}} x \geq f_i\} \geq \delta_i$ means $\tilde{c}_i(\lambda)^{\mathrm{T}} x \geq f_i$. So,

$$Appr\{\lambda | Pos\{\tilde{c}_i(\lambda)^{\mathrm{T}} x \geq f_i\} \geq \delta_i\} \geq \gamma_i$$

is equivalent to $Appr\{\lambda | \tilde{c}_i(\lambda)^{\mathrm{T}} x \geq f_i\} \geq \gamma_i, i = 1, 2, \cdots, m$.

If the fuzzy rough vectors $\tilde{\bar{e}}_r$ and $\tilde{\bar{b}}_r$ delegate to rough vectors $\tilde{e}_r$ and $\tilde{b}_r$ respectively, then the constraint

$$Appr\{\lambda | Pos\{\tilde{e}_r(\lambda)^{\mathrm{T}} x \leq \tilde{b}_r(\lambda)\} \geq \theta_r\} \geq \eta_r$$

is equivalent to $Appr\{w | \tilde{e}_r(\lambda)^{\mathrm{T}} x \leq \tilde{b}_r(\lambda)\} \geq \eta_r, r = 1, 2, \cdots, p$. So, the model (5.43) can be rewritten as

$$\begin{cases} \max\ [f_1, f_2, \cdots, f_m] \\ \text{s.t.} \begin{cases} Appr\{\lambda | \tilde{c}_i(\lambda)^{\mathrm{T}} x \geq f_i\} \geq \gamma_i, & i = 1, 2, \cdots, m \\ Appr\{\lambda | \tilde{e}_r(\lambda)^{\mathrm{T}} x \leq \tilde{b}_r(\lambda)\} \geq \eta_r, & r = 1, 2, \cdots, p \\ x \geq 0. \end{cases} \end{cases}$$

Remark 5.7. If the fuzzy rough vector $\tilde{\bar{c}}_i$ delegates to fuzzy vector $\bar{c}_i$, then $Pos\{\bar{c}_i^{\mathrm{T}} x \geq f_i\} \geq \delta_i$ is a crisp event. In order to satisfy $t_i := Appr\{w | Pos\{\bar{c}_i^{\mathrm{T}} x \geq f_i\} \geq \delta_i\} \geq \gamma_i$, the trust t_i should be 1.

So the constraint

$$Appr\{w | Pos\{\bar{c}_i^{\mathrm{T}} x \geq f_i\} \geq \delta_i\} = 1 \geq \gamma_i,$$

is equivalent to $Pos\{\bar{c}_i^{\mathrm{T}} x \geq f_i\} \geq \delta_i$, $i = 1, 2, \cdots, m$.

And similarly, When the fuzzy rough vectors $\tilde{\bar{e}}_r$ and $\tilde{\bar{b}}_r$ delegate to the fuzzy vector $\bar{e}_r$ and $\bar{b}_r$ respectively, the constraint

$$Appr\{w | Pos\{\bar{e}_r^{\mathrm{T}} x \leq \bar{b}_j\} \geq \theta_r\} \geq \eta_r$$

is equivalent to $\mathrm{Pos}\{\bar{e}_r^{\mathrm{T}} x \leq \bar{b}_r\} \geq \theta_r, r = 1, 2, \cdots, p$. So the model (5.43) is equivalent to

$$\begin{cases} \max\ [f_1, f_2, \cdots, f_m] \\ \text{s.t.} \begin{cases} Pos\{\bar{c}_i^{\mathrm{T}} x \geq f_i\} \geq \delta_i, & i = 1, 2, \cdots, m \\ Pos\{\bar{e}_r^{\mathrm{T}} x \leq \bar{b}_r\} \geq \theta_r, & r = 1, 2, \cdots, p \\ x \geq 0. \end{cases} \end{cases}$$

This is coincident to the fuzzy programming introduced in section 2.

Also by the definition 5.7, there are two other kinds of fuzzy rough multi-objective chance-constrained linear decision making models (5.38, 5.38),

$$\begin{cases} \max\ [f_1, f_2, \cdots, f_m] \\ \text{s.t.} \begin{cases} Appr\{\lambda | Nec\{\tilde{\bar{c}}_i(\lambda)^{\mathrm{T}} x \geq f_i\} \geq \delta_i\} \geq \gamma_i, \quad i = 1, 2, \cdots, m \\ Appr\{\lambda | Nec\{\tilde{\bar{e}}_r(\lambda)^{\mathrm{T}} x \leq \tilde{\bar{b}}_r(\lambda)\} \geq \theta_r\} \geq \eta_r, \quad r = 1, 2, \cdots, p \\ x \geq 0, \end{cases} \end{cases} \tag{5.38}$$

$$\begin{cases} \max\ [f_1, f_2, \cdots, f_m] \\ \text{s.t.} \begin{cases} Appr\{\lambda | Cr\{\tilde{\bar{c}}_i(\lambda)^{\mathrm{T}} x \geq f_i\} \geq \delta_i\} \geq \gamma_i, \quad i = 1, 2, \cdots, m \\ Appr\{\lambda | Cr\{\tilde{\bar{e}}_r(\lambda)^{\mathrm{T}} x \leq \tilde{\bar{b}}_r(\lambda)\} \geq \theta_r\} \geq \eta_r, \quad r = 1, 2, \cdots, p \\ x \geq 0, \end{cases} \end{cases} \tag{5.39}$$

where $\delta_i, \gamma_i, \theta_r, \eta_r \in [0, 1]$ are the predetermined confidence levels, Nec$\{\cdot\}$ and Cr$\{\cdot\}$ denote the necessary and the credibility of the fuzzy events in $\{\cdot\}$ respectively, and $Appr\{\cdot\}$ denotes the approximations of the rough events in $\{\cdot\}$.

For simpleness, the parameters $\delta, \gamma, \theta, \eta$ can be the same confidence level, i.e. $\delta_i = \delta, \gamma_i = \gamma, \theta_r = \theta, \eta_r = \eta, i = 1, 2, \cdots, m, r = 1, 2, \cdots, p.$

5.4.2 *Linear Fu-Ro CCM and Fuzzy Goal Method*

So let's consider the multi-objective linear programming problem with Fu-Ro coefficients:

$$\begin{cases} \max\ [\tilde{\bar{c}}_1^{\mathrm{T}} x, \tilde{\bar{c}}_2^{\mathrm{T}} x, \cdots, \tilde{\bar{c}}_m^{\mathrm{T}} x] \\ \text{s.t.} \begin{cases} \tilde{\bar{e}}_r^{\mathrm{T}} x \leq \tilde{\bar{b}}_r, \quad r = 1, 2, \cdots, p \\ x \geq 0. \end{cases} \end{cases} \tag{5.40}$$

According to the definition 5.7 of the primitive chance measure:

$$Ch\{\tilde{\bar{e}}_r^{\mathrm{T}} x \leq \tilde{\bar{b}}_r\}(\eta_r) \geq \theta_r \Leftrightarrow Appr\{\lambda | Pos\{\tilde{\bar{e}}_r(\lambda)^{\mathrm{T}} x \leq \tilde{\bar{b}}_r(\lambda)\} \geq \theta_r\} \geq \eta_r, \tag{5.41}$$

$$Ch\{\tilde{\bar{c}}_i^{\mathrm{T}} x \geq f_i\}(\gamma_i) \geq \delta_i \Leftrightarrow Appr\{\lambda | Pos\{\tilde{\bar{c}}_i(\lambda)^{\mathrm{T}} x \geq f_i\} \geq \delta_i\} \geq \gamma_i. \tag{5.42}$$

So we can get linear Fu-Ro CCM,

$$\begin{cases} \max\ [f_1, f_2, \cdots, f_m] \\ \text{s.t.} \begin{cases} Appr\{\lambda | Pos\{\tilde{\bar{c}}_i(\lambda)^{\mathrm{T}} x \geq f_i\} \geq \delta_i\} \geq \gamma_i, \quad i = 1, 2, \cdots, m \\ Appr\{\lambda | Pos\{\tilde{\bar{e}}_r(\lambda)^{\mathrm{T}} x \leq \tilde{\bar{b}}_r(\lambda)\} \geq \theta_r\} \geq \eta_r, \quad r = 1, 2, \cdots, p \\ x \geq 0, \end{cases} \end{cases} \tag{5.43}$$

where $\delta_i, \gamma_i, \theta_r, \eta_r \in [0, 1]$ are the predetermined confidence levels, $Pos\{\cdot\}$ denotes the possibility of the fuzzy events in $\{\cdot\}$, and $Appr\{\cdot\}$ denotes the approximation degree of the rough events in $\{\cdot\}$.

5.4.2.1 Crisp Equivalent Model

We introduce the $Appr - Pos$ and $Appr - Nec$ crisp equivalent models for the chance-constrained model, respectively.

Appr-Pos constrained multi-objective linearity model

One way of solving the problem (5.43) is to convert it into its crisp equivalent.

Theorem 5.7. *Assume that $\tilde{\bar{c}}_{ij}$ is a Fu-Ro variable, for any $\lambda \in \Lambda$, the fuzzy variable $\tilde{\bar{c}}_{ij}(\lambda)$ is characterized by the following membership function*

$$\mu_{\tilde{\bar{c}}_{ij}(\lambda)}(t) = \begin{cases} L\left(\frac{c_{ij}(\lambda)-t}{\alpha^c_{ij}}\right), \ t \le c_{ij}(\lambda), \alpha^c_{ij} > 0 \\ R\left(\frac{t-c_{ij}(\lambda)}{\beta^c_{ij}}\right), \ t \ge c_{ij}(\lambda), \beta^c_{ij} > 0 \end{cases} \quad \lambda \in \Lambda, \tag{5.44}$$

where $\alpha^c_{ij}, \beta^c_{ij}$ are positive numbers expressing the left and right spread of $\tilde{\bar{c}}_{ij}(\lambda)$, reference function $L, R : [0,1] \to [0,1]$ with $L(1) = R(1) = 0$, and $L(0) = R(0) = 1$ are non-increasing, continuous function. And $(c_{ij}(\lambda))_{n\times 1} = (c_{i1}(\lambda), c_{i2}(\lambda), \cdots, c_{in}(\lambda))^T$ is a rough vector. It follows that $c_i(\lambda)^T x = ([a,b],[c,d])$ (where $c \le a < b \le d$) is a rough variable and characterized by the following trust measure function,

$$Appr\{c_i(\lambda)^T x \ge t\} = \begin{cases} 0, & \text{if } d \le t \\ \frac{d-t}{2(d-c)}, & \text{if } b \le t \le d \\ \frac{1}{2}(\frac{d-t}{d-c} + \frac{b-t}{b-a}), & \text{if } a \le t < b \\ \frac{1}{2}(\frac{d-t}{d-c} + 1), & \text{if } c \le t \le a \\ 1, & \text{if } t \le c. \end{cases} \tag{5.45}$$

Then we have $Appr\{\lambda | \mathrm{Pos}\{\tilde{\bar{c}}_i(\lambda)^{\mathrm{T}} x \ge f_i\} \ge \delta_i\} \ge \gamma_i$ if and only if

$$\begin{cases} f_i \le d - 2\gamma_i(d-c) + R^{-1}(\delta_i)\beta_i^{cT} x, & \text{if } b \le f_i - R^{-1}(\delta_i)\beta_i^{cT} x \le d \\ f_i \le \frac{d(b-a)+b(d-c)-2\gamma_i(d-c)(b-a)}{d-c+b-a} + R^{-1}(\delta_i)\beta_i^{cT} x, & \text{if } a \le f_i - R^{-1}(\delta_i)\beta_i^{cT} x < b \\ f_i \le d - (d-c)(2\gamma_i - 1) + R^{-1}(\delta_i)\beta_i^{cT} x, & \text{if } c \le f_i - R^{-1}(\delta_i)\beta_i^{cT} x \le a \\ f_i \le c + R^{-1}(\delta_i)\beta_i^{cT} x, & \text{if } f_i - R^{-1}(\delta_i)\beta_i^{cT} x \le c, \end{cases} \tag{5.46}$$

where $\gamma_i, \delta_i \in [0,1]$ are predetermined confidence levels.

Proof. From the assumption we know that $c_i(\lambda) = (c_{i1}(\lambda), c_{i2}(\lambda), \cdots, c_{in}(\lambda))^T$ and $c_{ij}(\lambda)$ is a rough variable . Let $c_{ij}(\lambda) = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}])$ and $x = (x_1, x_2, \cdots, x_n)^T$ then

$$x_j c_{ij}(\lambda) = ([x_j a_{ij}, x_j b_{ij}], [x_j c_{ij}, x_j d_{ij}]),$$

$$\begin{aligned} c_i(\lambda)^T x &= \textstyle\sum_{j=1}^n c_{ij}(\lambda) x_j = \sum_{j=1}^n ([x_j a_{ij}, x_j b_{ij}], [x_j c_{ij}, x_j d_{ij}]) \\ &= ([\textstyle\sum_{j=1}^n a_{ij} x_j, \sum_{j=1}^n a_{ij} x_j], [\sum_{j=1}^n c_{ij} x_j, \sum_{j=1}^n d_{ij} x_j]). \end{aligned}$$

Therefore, $c_i(\lambda)^T x$ is also a rough variable. Now we can assume that

$$\begin{aligned} &a = \textstyle\sum_{j=1}^n a_{ij} x_j, \ b = \sum_{j=1}^n a_{ij} x_j, \\ &c = \textstyle\sum_{j=1}^n c_{ij} x_j, \ d = \sum_{j=1}^n d_{ij} x_j. \end{aligned}$$

then $c_i(\lambda)^T x = ([a,b],[c,d])$.

Moreover, we know that $\tilde{\bar{c}}_{ij}(\lambda)$ is a fuzzy number with the membership function $\mu_{\tilde{\bar{c}}_{ij}(\lambda)}(t)$ for given $\lambda \in \Lambda$. It follows from the extension principle [22] that the fuzzy number $\widehat{c}_i(\lambda)^T x$ is characterized by the membership function in the following

$$\mu_{\tilde{\bar{c}}_i(\lambda)^T x}(r) = \begin{cases} L(\frac{c_i(\lambda)^T - r}{\alpha_i^{cT} x}), & r \leq c_i(\lambda)^T x \\ R(\frac{r - c_i(\lambda)^T x}{\beta_i^{cT} x}), & r \geq c_i(\lambda)^T x \end{cases} \qquad i = 1, 2, \ldots, m.$$

By Lemma 2.2, we have that

$$Pos\{\tilde{\bar{c}}_i(\lambda)^T x \geq f_i\} \geq \delta_i \Leftrightarrow c_i(\lambda)^T x + R^{-1}(\delta_i)\beta_i^{cT} x \geq f_i, \quad i = 1, 2, \ldots, m.$$

For the given confidence level $\delta_i \in [0,1]$, we have

$$\begin{aligned} & Appr\{\lambda | Pos\{\tilde{\bar{c}}_i(\lambda)^T x \geq f_i\} \geq \delta_i\} \geq \gamma_i \\ \Leftrightarrow\ & Appr\{\lambda | c_i(\lambda)^T x \geq f_i - R^{-1}(\delta_i)\beta_i^{cT} x\} \geq \gamma_i \\ \Leftrightarrow\ & \gamma_i \leq \begin{cases} \frac{d - f_i + R^{-1}(\delta_i)\beta_i^{cT} x}{2(d-c)}, & \text{if } b \leq f_i - R^{-1}(\delta_i)\beta_i^{cT} x \leq d \\ \frac{1}{2}(\frac{d - f_i + R^{-1}(\delta_i)\beta_i^{cT} x}{d-c} + \frac{b - f_i + R^{-1}(\delta_i)\beta_i^{cT} x}{b-a}), & \text{if } a \leq f_i - R^{-1}(\delta_i)\beta_i^{cT} x < b \\ \frac{1}{2}(\frac{d - f_i + R^{-1}(\delta_i)\beta_i^{cT} x}{d-c} + 1), & \text{if } c \leq f_i - R^{-1}(\delta_i)\beta_i^{cT} x \leq a \\ 1, & \text{if } f_i - R^{-1}(\delta_i)\beta_i^{cT} x \leq c \end{cases} \end{aligned}$$

$$\Leftrightarrow \begin{cases} f_i \leq d - 2\gamma_i(d-c) + R^{-1}(\delta_i)\beta_i^{cT} x, & \text{if } b \leq f_i - R^{-1}(\delta_i)\beta_i^{cT} x \leq d \\ f_i \leq \frac{d(b-a) + b(d-c) - 2\gamma_i(d-c)(b-a)}{d-c+b-a} + R^{-1}(\delta_i)\beta_i^{cT} x, & \text{if } a \leq f_i - R^{-1}(\delta_i)\beta_i^{cT} x < b \\ f_i \leq d - (d-c)(2\gamma_i - 1) + R^{-1}(\delta_i)\beta_i^{cT} x, & \text{if } c \leq f_i - R^{-1}(\delta_i)\beta_i^{cT} x \leq a \\ f_i \leq c + R^{-1}(\delta_i)\beta_i^{cT} x, & \text{if } f_i - R^{-1}(\delta_i)\beta_i^{cT} x \leq c. \end{cases}$$

This completes the proof. □

Theorem 5.8. *Suppose that $\tilde{\bar{e}}_{rj}, \tilde{\bar{b}}_r$ are fuzzy rough variables, for any $\lambda \in \Lambda$, fuzzy variables $\tilde{\bar{e}}_{rj}(\lambda), \tilde{\bar{b}}_r(\lambda)$ are characterized by the membership function in the following*

$$\mu_{\tilde{\bar{e}}_{rj}(\lambda)}(t) = \begin{cases} L(\frac{e_{rj}(\lambda) - t}{\alpha_{rj}^e}), & t \leq e_{rj}(\lambda), \alpha_{rj}^e > 0 \\ R(\frac{t - e_{rj}(\lambda)}{\beta_{rj}^m}), & t \geq e_{rj}(\lambda), \beta_{rj}^e > 0 \end{cases} \qquad \lambda \in \Lambda \tag{5.47}$$

and

$$\mu_{\tilde{\bar{b}}_r(\lambda)}(t) = \begin{cases} L(\frac{b_r(\lambda) - t}{\alpha_r^b}), & t \leq b_r(\lambda),\ \alpha_r^b > 0 \\ R(\frac{t - b_r(\lambda)}{\beta_r^b}), & t \geq b_r(\lambda),\ \beta_r^b > 0 \end{cases} \qquad \lambda \in \Lambda, \tag{5.48}$$

where $\alpha_{rj}^e, \beta_{rj}^e$ are positive numbers expressing the left and right spread of $\tilde{\bar{e}}_{rj}(\lambda)$, α_r^b, β_r^b are the left and right spread of $\tilde{\bar{b}}_r(\lambda)$, and reference functions $L, R : [0,1] \to [0,1]$ with $L(1) = R(1) = 0$, and $L(0) = R(0) = 1$ are non-increasing, continuous functions. And $(e_{rj}(\lambda))_{n \times 1} = (e_{r1}(\lambda), e_{r2}(\lambda), \cdots, e_{rn}(\lambda))^T$ is a rough *vector, $e_{rj}(\lambda), b_r(\lambda)$ are rough variables, $r = 1,2,\cdots,p$, $j = 1,2,\cdots,n$. By Proposition 5.7, we have $e_r(\lambda)^T x, b_r(\lambda)$ are rough variables, then $e_r(\lambda)^T x - b_r(\lambda) = [(a,b),(c,d)]$($c \le a < b \le d$) is also a rough variable. We assume that it is characterized by the following trust measure function*

$$Appr\{e_r(\lambda)^T x - b_r(\lambda) \le t\} = \begin{cases} 0, & if\, t \le c \\ \frac{t-c}{2(d-c)}, & if\, c \le t \le a \\ \frac{1}{2}(\frac{t-c}{d-c} + \frac{t-a}{b-a}), & if\, a \le t < b \\ \frac{1}{2}(\frac{t-c}{d-c} + 1), & if\, b \le t \le d \\ 1, & if \quad d \le t. \end{cases} \tag{5.49}$$

Then, we have that $Appr\{\lambda | Pos\{\tilde{\bar{e}}_r(\lambda)^T x \le \tilde{\bar{b}}_r(\lambda)\} \ge \theta_r\} \ge \eta_r$ if and only if

$$\begin{cases} W \ge c + 2(d-c)\eta_r, & if\, c \le W \le a \\ W \ge \frac{2\eta_r(d-c)(b-a)+c(b-a)+a(d-c)}{b-a+d-c}, & if\, a \le W < b \\ W \ge (2\eta_r - 1)(d-c) + c, & if\, b \le W \le d \\ W \ge d, & if\, d \le W, \end{cases} \tag{5.50}$$

where $W = R^{-1}(\theta_r)\beta_r^b + L^{-1}(\theta_k)\alpha_r^{eT} x$.

Proof. From the assumption, we know

$$Pos\{\tilde{\bar{e}}_r(\lambda)^T x \le \tilde{\bar{b}}_r(\lambda)\} \ge \theta_r \Leftrightarrow b_r(\lambda) + R^{-1}(\theta_r)\beta_r^b \ge e_r(\lambda)^T x - L^{-1}(\theta_r)\alpha_r^{eT} x.$$

Since $e_r(\lambda)^T x - b_r(\lambda) = [(a,b),(c,d)]$, for given confidence levels $\theta_r, \eta_r \in [0,1]$, we have that,

$$\begin{aligned} & Appr\{\lambda | Pos\{\tilde{\bar{e}}_r(\lambda)^T x \le \tilde{\bar{b}}_r(\lambda)\} \ge \theta_r\} \ge \eta_r \\ \Leftrightarrow\ & Appr\{\lambda | e_r(\lambda)^T x - b_r(\lambda) \le R^{-1}(\theta_r)\beta_r^b + L^{-1}(\theta_r)\alpha_r^{eT} x\} \ge \eta_r \\ \Leftrightarrow\ & \eta_r \le \begin{cases} \frac{W-c}{2(d-c)}, & \text{if } c \le W \le a. \\ \frac{1}{2}(\frac{W-c}{d-c} + \frac{W-a}{b-a}), & \text{if } a \le W < b \\ \frac{1}{2}(\frac{W-c}{d-c} + 1), & \text{if } b \le W \le d \\ 1, & \text{if } W \ge d \end{cases} \\ \Leftrightarrow\ & \begin{cases} W \ge c + 2(d-c)\eta_r, & \text{if } c \le W \le a \\ W \ge \frac{2\eta_r(d-c)(b-a)+c(b-a)+a(d-c)}{b-a+d-c}, & \text{if } a \le W < b \\ W \ge (2\eta_r - 1)(d-c) + c, & \text{if } b \le W \le d \\ W \ge d, & \text{if } d \le W, \end{cases} \end{aligned}$$

where $W = R^{-1}(\theta_r)\beta_r^b + L^{-1}(\theta_r)\alpha_r^{eT} x$.

This completes the proof. □

From Propositions 5.7 and 5.8, we know that the problem (5.8) is equivalent to the following multi-objective programming problems,

$$\begin{cases} \max\ [f_1, f_2, \cdots, f_m] \\ \text{s.t.} \begin{cases} f_i \le d - 2\gamma_i(d-c) + R^{-1}(\delta_i)\beta_i^{cT}x, \\ \qquad\qquad\qquad\qquad \text{if } b \le f_i - R^{-1}(\delta_i)\beta_i^{cT}x \le d \\ f_i \le \frac{d(b-a)+b(d-c)-2\alpha_i(d-c)(b-a)}{d-c+b-a} + R^{-1}(\delta_i)\beta_i^{cT}x, \\ \qquad\qquad\qquad\qquad \text{if } a \le f_i - R^{-1}(\delta_i)\beta_i^{cT}x < b \\ f_i \le d - (d-c)(2\gamma_i - 1) + R^{-1}(\delta_i)\beta_i^{cT}x, \\ \qquad\qquad\qquad\qquad \text{if } c \le f_i - R^{-1}(\delta_i)\beta_i^{cT}x \le a \\ f_i \le c + R^{-1}(\delta_i)\beta_i^{cT}x, \\ \qquad\qquad\qquad\qquad \text{if } f_i - R^{-1}(\delta_i)\beta_i^{cT}x \le c \\ W \ge c + 2(d-c)\eta_r, \text{ if } c \le W \le a \\ W \ge \frac{2\eta_r(d-c)(b-a)+c(b-a)+a(d-c)}{b-a+d-c}, \text{ if } a \le W < b \\ W \ge (2\eta_r - 1)(d-c) + c, \text{ if } b \le W \le d \\ W \ge d, \text{ if } d \le W \\ x \ge 0, \end{cases} \end{cases} \tag{5.51}$$

where $W = R^{-1}(\theta_r)\beta_r^b + L^{-1}(\theta_r)\alpha_r^{eT}x$.

Appr-Nec constrained multi-objective linearity model

Similar to the $Appr - Pos$ constrained multi-objective linearity model, we assume that $\tilde{\bar{c}}_{ij}$, $\tilde{\bar{e}}_{rj}$ and $\tilde{\bar{b}}_r$ are fuzzy rough variables, we give the following two theorems to transform the chance-constrained model (5.43) into its crisp model based on $Appr - Nec$ if the decision maker is comparatively pessimistic.

Theorem 5.9. *Assume that $\tilde{\bar{c}}_{ij}$ is a fuzzy rough variable, for any $\lambda \in \Lambda$, the fuzzy variable $\tilde{\bar{c}}_{ij}(\lambda)$ is characterized by the following membership function*

$$\mu_{\tilde{\bar{c}}_{ij}(\lambda)}(t) = \begin{cases} L\left(\frac{c_{ij}(\lambda)-t}{\alpha_{ij}^c}\right), \ t \le c_{ij}(\lambda), \alpha_{ij}^c > 0 \\ R\left(\frac{t-c_{ij}(\lambda)}{\beta_{ij}^c}\right), \ t \ge c_{ij}(\lambda), \beta_{ij}^c > 0 \end{cases} \quad \lambda \in \Lambda, \tag{5.52}$$

where $\alpha_{ij}^c, \beta_{ij}^c$ are positive numbers expressing the left and right spread of $\tilde{\bar{c}}_{ij}(\lambda)$, reference function $L,R : [0,1] \to [0,1]$ with $L(1) = R(1) = 0$, and $L(0) = R(0) = 1$ are non-increasing, continuous function. And $(c_{ij}(\lambda))_{n\times 1} = (c_{i1}(\lambda), c_{i2}(\lambda), \cdots, c_{in}(\lambda))^T$ is a rough vector. It follows that $c_i(\lambda)^T x = ([a,b],[c,d])$ (where $c \le a < b \le d$) is a rough variable and characterized by the following trust measure function,

$$Appr\{c_i(\lambda)^T x \ge t\} = \begin{cases} 0, & \text{if } d \le t \\ \frac{d-t}{2(d-c)}, & \text{if } b \le t \le d \\ \frac{1}{2}\left(\frac{d-t}{d-c} + \frac{b-t}{b-a}\right), & \text{if } a \le t < b \\ \frac{1}{2}\left(\frac{d-t}{d-c} + 1\right), & \text{if } c \le t \le a \\ 1, & \text{if } t \le c. \end{cases} \tag{5.53}$$

Then we have $Appr\{\lambda|Nec\{\tilde{\bar{c}}_i(\lambda)^{\mathrm{T}}x \geq f_i\} \geq \delta_i\} \geq \gamma_i$ *if and only if*

$$\begin{cases} f_i \leq d - 2\gamma_i(d-c) - L^{-1}(1-\delta_i)\alpha_i^{cT}x, \\ \qquad\qquad\qquad\qquad \text{if } b \leq f_i + L^{-1}(1-\delta_i)\alpha_i^{cT}x \leq d \\ f_i \leq \frac{d(b-a)+b(d-c)-2\gamma_i(d-c)(b-a)}{d-c+b-a} - L^{-1}(1-\delta_i)\alpha_i^{cT}x, \\ \qquad\qquad\qquad\qquad \text{if } a \leq f_i + L^{-1}(1-\delta_i)\alpha_i^{cT}x < b \\ f_i \leq d - (d-c)(2\gamma_i - 1) - L^{-1}(1-\delta_i)\alpha_i^{cT}x, \\ \qquad\qquad\qquad\qquad \text{if } c \leq f_i + L^{-1}(1-\delta_i)\alpha_i^{cT}x \leq a \\ f_i \leq c - L^{-1}(1-\delta_i)\alpha_i^{cT}x, \\ \qquad\qquad\qquad\qquad \text{if } f_i + L^{-1}(1-\delta_i)\alpha_i^{cT}x \leq c, \end{cases} \tag{5.54}$$

where $\gamma_i, \delta_i \in [0,1]$ *are predetermined confidence levels.*

Proof. From the assumption we know that $c_i(\lambda) = (c_{i1}(\lambda), c_{i2}(\lambda), \cdots, c_{in}(\lambda))^T$ and $c_{ij}(\lambda)$ is a rough variable . Let $c_{ij}(\lambda) = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}])$ and $x = (x_1, x_2, \cdots, x_n)^T$ then

$$x_j c_{ij}(\lambda) = ([x_j a_{ij}, x_j b_{ij}], [x_j c_{ij}, x_j d_{ij}]),$$

$$\begin{aligned} c_i(\lambda)^T x &= \textstyle\sum_{j=1}^n c_{ij}(\lambda)x_j = \sum_{j=1}^n([x_j a_{ij}, x_j b_{ij}], [x_j c_{ij}, x_j d_{ij}]) \\ &= ([\textstyle\sum_{j=1}^n a_{ij}x_j, \sum_{j=1}^n a_{ij}x_j], [\sum_{j=1}^n c_{ij}x_j, \sum_{j=1}^n d_{ij}x_j]). \end{aligned}$$

Therefore, $c_i(\lambda)^T x$ is also a rough variable. Now we can assume that

$$\begin{aligned} a = \textstyle\sum_{j=1}^n a_{ij}x_j,\ b = \sum_{j=1}^n a_{ij}x_j, \\ c = \textstyle\sum_{j=1}^n c_{ij}x_j,\ d = \sum_{j=1}^n d_{ij}x_j. \end{aligned}$$

then $c_i(\lambda)^T x = ([a,b],[c,d])$.

Moreover, we know that $\tilde{\bar{c}}_{ij}(\lambda)$ is a fuzzy number with the membership function $\mu_{\tilde{\bar{c}}_{ij}(\lambda)}(t)$ for given $\lambda \in \Lambda$. It follows from the extension principle that the fuzzy number $\widehat{c}_i(\lambda)^T x$ is characterized by the membership function in the following

$$\mu_{\tilde{\bar{c}}_i(\lambda)^T x}(r) = \begin{cases} L(\frac{c_i(\lambda)^T - r}{\alpha_i^{cT}x}), & r \leq c_i(\lambda)^T x \\ R(\frac{r - c_i(\lambda)^T x}{\beta_i^{cT}x}), & r \geq c_i(\lambda)^T x \end{cases} \quad i = 1, 2, \ldots, m.$$

By Lemma 2.2, we have that

$$Nec\{\tilde{\bar{c}}_i(\lambda)^T x \geq f_i\} \geq \delta_i \Leftrightarrow c_i(\lambda)^T x - L^{-1}(1-\delta_i)\alpha_i^{cT}x \geq f_i, \quad i = 1, 2, \ldots, m.$$

For the given confidence level $\delta_i \in [0,1]$, we have

$$
\begin{array}{l}
Appr\{\lambda|Nec\{\tilde{\bar{c}}_i(\lambda)^T x \geq f_i\} \geq \delta_i\} \geq \gamma_i \\
\Leftrightarrow Appr\{\lambda|c_i(\lambda)^T x \geq f_i + L^{-1}(1-\delta_i)\alpha_i^{cT} x\} \geq \gamma_i \\
\Leftrightarrow \gamma_i \leq \begin{cases}
\frac{d-f_i-L^{-1}(1-\delta_i)\alpha_i^{cT}x}{2(d-c)}, \\
\qquad\qquad\qquad \text{if } b \leq f_i + L^{-1}(1-\delta_i)\alpha_i^{cT} x \leq d \\
\frac{1}{2}(\frac{d-f_i-L^{-1}(1-\delta_i)\alpha_i^{cT}x}{d-c} + \frac{b-f_i-L^{-1}(1-\delta_i)\alpha_i^{cT}x}{b-a}), \\
\qquad\qquad\qquad \text{if } a \leq f_i + L^{-1}(1-\delta_i)\alpha_i^{cT} x < b \\
\frac{1}{2}(\frac{d-f_i-L^{-1}(1-\delta_i)\alpha_i^{cT}x}{d-c} + 1), \\
\qquad\qquad\qquad \text{if } c \leq f_i + L^{-1}(1-\delta_i)\alpha_i^{cT} x \leq a \\
1, \\
\qquad\qquad\qquad \text{if } f_i + L^{-1}(1-\delta_i)\alpha_i^{cT} x \leq c
\end{cases}
\end{array}
$$

$$
\Leftrightarrow \begin{cases}
f_i \leq d - 2\gamma_i(d-c) - L^{-1}(1-\delta_i)\alpha_i^{cT} x, \\
\qquad\qquad\qquad \text{if } b \leq f_i + L^{-1}(1-\delta_i)\alpha_i^{cT} x \leq d \\
f_i \leq \frac{d(b-a)+b(d-c)-2\gamma_i(d-c)(b-a)}{d-c+b-a} - L^{-1}(1-\delta_i)\alpha_i^{cT} x, \\
\qquad\qquad\qquad \text{if } a \leq f_i + L^{-1}(1-\delta_i)\alpha_i^{cT} x < b \\
f_i \leq d - (d-c)(2\gamma_i - 1) - L^{-1}(1-\delta_i)\alpha_i^{cT} x, \\
\qquad\qquad\qquad \text{if } c \leq f_i + L^{-1}(1-\delta_i)\alpha_i^{cT} x \leq a \\
f_i \leq c - L^{-1}(1-\delta_i)\alpha_i^{cT} x, \\
\qquad\qquad\qquad \text{if } f_i + L^{-1}(1-\delta_i)\alpha_i^{cT} x \leq c.
\end{cases}
$$

This completes the proof. □

Theorem 5.10. *Suppose that $\tilde{\bar{e}}_{rj}, \tilde{\bar{b}}_r$ are fuzzy rough variables, for any $\lambda \in \Lambda$, fuzzy variables $\tilde{\bar{e}}_{rj}(\lambda), \tilde{\bar{b}}_r(\lambda)$ are characterized by the membership function in the following*

$$
\mu_{\tilde{\bar{e}}_{rj}(\lambda)}(t) = \begin{cases}
L(\frac{e_{rj}(\lambda)-t}{\alpha_{rj}^e}),\ t \leq e_{rj}(\lambda), \alpha_{rj}^e > 0 \\
R(\frac{t-e_{rj}(\lambda)}{\beta_{rj}^m}),\ t \geq e_{rj}(\lambda), \beta_{rj}^e > 0
\end{cases} \quad \lambda \in \Lambda \tag{5.55}
$$

and

$$
\mu_{\tilde{\bar{b}}_r(\lambda)}(t) = \begin{cases}
L(\frac{b_r(\lambda)-t}{\alpha_r^b}),\ t \leq b_r(\lambda), \alpha_r^b > 0 \\
R(\frac{t-b_r(\lambda)}{\beta_r^b}),\ t \geq b_r(\lambda), \beta_r^b > 0
\end{cases} \quad \lambda \in \Lambda, \tag{5.56}
$$

where $\alpha_{rj}^e, \beta_{rj}^e$ are positive numbers expressing the left and right spread of $\tilde{\bar{e}}_{rj}(\lambda)$, α_r^b, β_r^b are the left and right spread of $\tilde{\bar{b}}_r(\lambda)$, and reference functions $L, R : [0,1] \to [0,1]$ with $L(1) = R(1) = 0$, and $L(0) = R(0) = 1$ are non-increasing, continuous functions. And $(e_{rj}(\lambda))_{n\times 1} = (e_{r1}(\lambda), e_{r2}(\lambda), \cdots, e_{rn}(\lambda))^T$ is a rough vector, $e_{rj}(\lambda), b_r(\lambda)$ are rough variables, $r = 1,2,\cdots,p$, $j = 1,2,\cdots,n$.

By Theorem 5.7, we have $e_r(\lambda)^T x, b_r(\lambda)$ *are rough variables, then* $e_r(\lambda)^T x - b_r(\lambda) = [(a,b),(c,d)](c \le a < b \le d)$ *is also a rough variable. We assume that it is characterized by the following trust measure function*

$$Appr\{e_r(\lambda)^T x - b_r(\lambda) \le t\} = \begin{cases} 0, & if\, t \le c \\ \frac{t-c}{2(d-c)}, & if\, c \le t \le a \\ \frac{1}{2}(\frac{t-c}{d-c} + \frac{t-a}{b-a}), & if\, a \le t < b \\ \frac{1}{2}(\frac{t-c}{d-c} + 1), & if\, b \le t \le d \\ 1, & if \quad d \le t. \end{cases} \tag{5.57}$$

Then, we have that $Appr\{\lambda | Nec\{\tilde{\bar{e}}_r(\lambda)^T x \le \tilde{\bar{b}}_r(\lambda)\} \ge \theta_r\} \ge \eta_r$ *if and only if*

$$\begin{cases} W \ge c + 2(d-c)\eta_r, & if\, c \le W \le a \\ W \ge \frac{2\eta_r(d-c)(b-a)+c(b-a)+a(d-c)}{b-a+d-c}, & if\, a \le W < b \\ W \ge (2\eta_r - 1)(d-c) + c, & if\, b \le W \le d \\ W \ge d, & if\, d \le W, \end{cases} \tag{5.58}$$

where $W = -R^{-1}(\theta_r)\beta_r^{eT} x - L^{-1}(1-\theta_r)\alpha_r^b$.

Proof. From the assumption, we know

$$Nec\{\tilde{\bar{e}}_r(\lambda)^T x \le \tilde{\bar{b}}_r(\lambda)\} \ge \theta_r \Leftrightarrow b_r(\lambda) - L^{-1}(1-\theta_r)\alpha_r^b \ge e_r(\lambda)^T x + R^{-1}(\theta_r)\beta_r^{eT} x.$$

Since $e_r(\lambda)^T x - b_r(\lambda) = [(a,b),(c,d)]$, for given confidence levels $\theta_r, \eta_r \in [0,1]$, we have that,

$$\begin{aligned} & Appr\{\lambda | Nec\{\tilde{\bar{e}}_r(\lambda)^T x \le \tilde{\bar{b}}_r(\lambda)\} \ge \theta_r\} \ge \eta_r \\ \Leftrightarrow\ & Appr\{\lambda | e_r(\lambda)^T x - b_r(\lambda) \le -R^{-1}(\theta_r)\beta_r^{eT} x - L^{-1}(1-\theta_r)\alpha_r^b\} \ge \eta_r \\ \Leftrightarrow\ & \eta_r \le \begin{cases} \frac{W-c}{2(d-c)}, & \text{if } c \le W \le a \\ \frac{1}{2}(\frac{W-c}{d-c} + \frac{W-a}{b-a}), & \text{if } a \le W < b \\ \frac{1}{2}(\frac{W-c}{d-c} + 1), & \text{if } b \le W \le d \\ 1, & \text{if } W \ge d \end{cases} \\ \Leftrightarrow\ & \begin{cases} W \ge c + 2(d-c)\eta_r, & \text{if } c \le W \le a \\ W \ge \frac{2\eta_r(d-c)(b-a)+c(b-a)+a(d-c)}{b-a+d-c}, & \text{if } a \le W < b \\ W \ge (2\eta_r - 1)(d-c) + c, & \text{if } b \le W \le d \\ W \ge d, & \text{if } d \le W, \end{cases} \end{aligned}$$

where $W = -R^{-1}(\theta_r)\beta_r^{eT} x - L^{-1}(1-\theta_r)\alpha_r^b$.

This proof is completed. □

From Propositions 5.7 and 5.8, we know that the problem (5.8) is equivalent to the following multi-objective programming problems,

$$\begin{cases} \max\ [f_1, f_2, \cdots, f_m] \\ \text{s.t.} \begin{cases} f_i \le d - 2\gamma_i(d-c) - L^{-1}(1-\delta_i)\alpha_i^{cT} x, \\ \qquad\qquad\qquad\qquad \text{if } b \le f_i + L^{-1}(1-\delta_i)\alpha_i^{cT} x \le d \\ f_i \le \frac{d(b-a)+b(d-c)-2\gamma_i(d-c)(b-a)}{d-c+b-a} - L^{-1}(1-\delta_i)\alpha_i^{cT} x, \\ \qquad\qquad\qquad\qquad \text{if } a \le f_i + L^{-1}(1-\delta_i)\alpha_i^{cT} x < b \\ f_i \le d - (d-c)(2\gamma_i - 1) - L^{-1}(1-\delta_i)\alpha_i^{cT} x, \\ \qquad\qquad\qquad\qquad \text{if } c \le f_i + L^{-1}(1-\delta_i)\alpha_i^{cT} x \le a \\ f_i \le c - L^{-1}(1-\delta_i)\alpha_i^{cT} x, \\ \qquad\qquad\qquad\qquad \text{if } f_i + L^{-1}(1-\delta_i)\alpha_i^{cT} x \le c \\ W \ge c + 2(d-c)\eta_r, \text{ if } c \le W \le a \\ W \ge \frac{2\eta_r(d-c)(b-a)+c(b-a)+a(d-c)}{b-a+d-c}, \text{ if } a \le W < b \\ W \ge (2\eta_r - 1)(d-c) + c, \text{ if } b \le W \le d \\ W \ge d, \text{ if } d \le W \\ x \ge 0, \end{cases} \end{cases} \tag{5.59}$$

where $W = -R^{-1}(\theta_r)\beta_r^{eT} x - L^{-1}(1-\theta_r)\alpha_r^b$.

5.4.2.2 Fuzzy Goal Method

In this section, we introduce how to use the fuzzy goal method to solve multi-objective programming problems. As we know, the standard distribution function $\Phi(x)$ is a nonlinear function, so it is difficult to solve using the usual technique. Here we introduce the fuzzy goal method proposed by Sakawa [128] to solve this kind of nonlinear multi-objective programming problems (5.60),

$$\begin{cases} \max[H_1(x), H_2(x), \cdots, H_m(x)] \\ \text{s.t. } x \in X. \end{cases} \tag{5.60}$$

Assume that decision makers have fixed the membership function $\mu_k(H_k(x))$ and given the goal membership function value $\bar{\mu}_k$ $(k = 1, 2, \cdots, m)$. Let's consider the following programming problem,

$$\begin{cases} \max \sum_{k=1}^{m} d_k^- \\ \text{s.t.} \begin{cases} \mu_k(H_k(x)) + d_k^+ - d_k^- = \bar{\mu}_k, k = 1, 2, \cdots, m \\ x \in X \\ d_k^+ d_k^- = 0, d_k^+, d_k^- \ge 0, k = 1, 2, \cdots, m, \end{cases} \end{cases} \tag{5.61}$$

where d_k^+, d_k^- is the positive and negative deviation. Then we have the following result between the optimal solution of the problem (5.61) and the efficient solution of the problem (5.60).

Theorem 5.11. *(Sakawa [128]) (1) If x^* is the optimal solution of the problem (5.61), and $0 < \mu_k(H_k(\mathbf{x}^*)) < 1, d_k^+ = 0 (k = 1, 2, \cdots, m)$ holds, then x^* is an efficient solution of the problem (5.60).*

(2) If x^* is an efficient solution of the problem (5.60), and $0 < \mu_k(H_k(\mathbf{x}^*)) < 1 (k = 1,2,\cdots,m)$*, then x^* is an efficient solution of the problem (5.61) and* $d_k^+ = 0 (k = 1,2,\cdots,m)$ *holds.*

5.4.2.3 Numerical Example

Example 5.5. An industry will produce three kinds of products which are seasonal. Because the demand amount is seasonal, the profits are fuzzy rough variables, i.e., the profits are fuzzy variables, but the excepted values of these fuzzy variables are rough variables. When producing every product, the efficiency of the machinery is also a fuzzy rough variable, but the coefficient is different. Each product is no less than 20, and the gross amount is no less than 200 but no more than 250. The other coefficients can be seen in Table 5.3. The problem is how many products to produce in order to get the predetermined levels.

Table 5.3 The resource demand in producing process

product	1	2	3	possible using amount
workman amount	1	1	1	250
storage capacity	1	4	2	600
using efficiency	$c_1\tilde{\tilde{\xi}}_4$	$c_2\tilde{\tilde{\xi}}_5$	$c_3\tilde{\tilde{\xi}}_6$	
profit	$\tilde{\tilde{\xi}}_1$	$\tilde{\tilde{\xi}}_2$	$\tilde{\tilde{\xi}}_3$	

Then we can get the following Appr-pos constrained multi-objective programming problem

$$
\begin{array}{l}
\max\{f_1, f_2\} \\
s.t. \begin{cases}
Appr\{\lambda | Pos\{\tilde{\tilde{\xi}}_1 x_1 + \tilde{\tilde{\xi}}_2 x_2 + \tilde{\tilde{\xi}}_3 x_3 \geq f_1\} \geq \delta_1\} \geq \gamma_1 \\
Appr\{\lambda | Pos\{c_1\tilde{\tilde{\xi}}_4 x_1 + c_2\tilde{\tilde{\xi}}_5 x_2 + c_3\tilde{\tilde{\xi}}_6 x_3 \geq f_2\} \geq \delta_2\} \geq \gamma_2 \\
x_1 + x_2 + x_3 \leq 250 \\
x_1 + x_2 + x_3 \geq 200 \\
x_1 + 4x_2 + 2x_3 \leq 600 \\
x_1, x_2, x_3 \geq 20,
\end{cases}
\end{array}
$$

where $c = (c_1, c_2, c_3) = (1.2, 0.8, 1.5)$,

$$
\begin{array}{l}
\tilde{\tilde{\xi}}_1 = (\rho_1, 0.5, 0.5)_{LR}, \text{ with } \rho_1 \vdash ([1,2],[0,3]), \\
\tilde{\tilde{\xi}}_2 = (\rho_2, 2, 2)_{LR}, \text{ with } \rho_2 \vdash ([2,3],[1,4]), \\
\tilde{\tilde{\xi}}_3 = (\rho_3, 1, 1)_{LR}, \text{ with } \rho_3 \vdash ([3,4],[2,5]), \\
\tilde{\tilde{\xi}}_4 = (\rho_4, 1, 1)_{LR}, \text{ with } \rho_4 \vdash ([0,1],[0,3]), \\
\tilde{\tilde{\xi}}_5 = (\rho_5, 0.5, 0.5)_{LR}, \text{ with } \rho_5 \vdash ([1,2],[0,3]), \\
\tilde{\tilde{\xi}}_6 = (\rho_6, 0.5, 0.5)_{LR}, \text{ with } \rho_6 \vdash ([2,3],[0,3]).
\end{array}
$$

According to the knowledge of fuzzy variable and rough variable, we have that

$$\begin{aligned}
&\tilde{\bar{\xi}}_1 x_1 + \tilde{\bar{\xi}}_2 x_2 + \tilde{\bar{\xi}}_3 x_3 \\
&= (\rho_1 x_1 + \rho_2 x_2 + \rho_3 x_3, 0.5x_1 + 2x_2 + x_3, 0.5x_1 + 2x_2 + x_3)_{LR}, \\
&c_1 \tilde{\bar{\xi}}_4 x_1 + c_2 \tilde{\bar{\xi}}_5 x_2 + c_3 \tilde{\bar{\xi}}_6 x_3 \\
&= (\rho_4 c_1 x_1 + \rho_5 c_2 x_2 + \rho_6 c_3 x_3, 1.2x_1 + 0.4x_2 + 0.75x_3, \\
&\quad 1.2x_1 + 0.4x_2 + 0.75x_3)_{LR}.
\end{aligned} \tag{5.62}$$

and

$$\begin{aligned}
\rho_1 x_1 &= ([x_1, 2x_1], [0, 3x_1]) \\
\rho_2 x_2 &= ([2x_2, 3x_2], [x_2, 4x_2]) \\
\rho_3 x_3 &= ([3x_3, 4x_3], [2x_3, 5x_3]) \\
\rho_4 c_1 x_1 &= ([0, c_1 x_1], [0, 3c_1 x_1]) \\
\rho_5 c_2 x_2 &= ([c_2 x_2, 2c_2 x_2], [0, 3c_2 x_2]) \\
\rho_6 c_3 x_3 &= ([2c_3 x_3, 3c_3 x_3], [0, 3c_3 x_3]),
\end{aligned}$$

and

$$\begin{aligned}
&\rho_1 x_1 + \rho_2 x_2 + \rho_3 x_3 \\
&= ([x_1 + 2x_2 + 3x_3, 2x_1 + 3x_2 + 4x_3], [x_2 + 2x_3, 3x_1 + 4x_2 + 5x_3]) \\
&\rho_4 c_1 x_1 + \rho_5 c_2 x_2 + \rho_6 c_3 x_3 \\
&= ([0.8x_2 + 3x_3, 1.2x_1 + 1.6x_2 + 4.5x_3], [0, 3.6x_1 + 2.4x_2 + 4.5x_3]).
\end{aligned}$$

Here we consider the case when $b \le f_i - R^{-1}(\delta_i)\beta_i^{cT} x \le d$, and readers can try another three cases through the following method.

According to Propositions 5.7 and 5.8, the problem (5.63) is equivalent to the following multi-objective programming problem,

$$\begin{cases}
\max\ [f_1, f_2] \\
\text{s.t.} \begin{cases}
f_1 \le 3x_1 + 4x_2 + 5x_3 - 2\gamma_1(3x_1 + 3x_2 + 3x_3) \\
\qquad\qquad + R^{-1}(\delta_1)(0.5x_1 + 2x_2 + x_3) \\
f_2 \le 3.6x_1 + 2.4x_2 + 4.5x_3 - 2\gamma_2(3.6x_1 + 2.4x_2 + 4.5x_3) \\
\qquad\qquad + R^{-1}(\delta_2)(1.2x_1 + 0.4x_2 + 0.75x_3) \\
x_1 + x_2 + x_3 \le 250 \\
x_1 + x_2 + x_3 \ge 200 \\
x_1 + 4x_2 + 2x_3 \le 600 \\
x_1, x_2, x_3 \ge 20
\end{cases}
\end{cases} \tag{5.63}$$

or equivalently

$$\begin{cases}
\max\ [H_1(x), H_2(x)] \\
\text{s.t.} \begin{cases}
x_1 + x_2 + x_3 \le 250 \\
x_1 + x_2 + x_3 \ge 200 \\
x_1 + 4x_2 + 2x_3 \le 600 \\
x_1, x_2, x_3 \ge 20,
\end{cases}
\end{cases} \tag{5.64}$$

where

$$
\begin{aligned}
H_1(x) &:= 3x_1+4x_2+5x_3-2\gamma_1(3x_1+3x_2+3x_3)+R^{-1}(\delta_1)(0.5x_1+2x_2+x_3),\\
H_2(x) &:= 3.6x_1+2.4x_2+4.5x_3-2\gamma_2(3.6x_1+2.4x_2+4.5x_3)\\
&\qquad +R^{-1}(\delta_2)(1.2x_1+0.4x_2+0.75x_3).
\end{aligned}
$$

When $\gamma_1=\gamma_2=\delta_1=\delta_2=0.9$, H_i^0 and $H_i^1(i=1,2)$ are calculated by solving the two single objective model as follows:

$$
\begin{aligned}
H_1^1 &= -119,(x_1,x_2,x_3)=(20,20,160),H_2^0=-656.9,\\
H_2^1 &= -455.83,(x_1,x_2,x_3)=(53.33,126.67,20),H_1^0=-283.33.
\end{aligned}
$$

We give the membership functions as follows,

$$
\begin{aligned}
\mu_1(H_1(x)) &= \frac{H_1(x)-H_1^0}{H_1^1-H_1^0}=\frac{H_1(x)+283.33}{164.33},\\
\mu_2(H_2(x)) &= \frac{H_2(x)-H_2^0}{H_2^1-H_2^0}=\frac{H_2(x)+656.9}{200.97}.
\end{aligned}
$$

According to the fuzzy goal method, we construct the fuzzy goal programming model (5.65) as follows,

$$
\left\{
\begin{array}{l}
\max\ d_1^-+d_2^- \\
\text{s.t.}\left\{
\begin{array}{l}
\frac{H_1(x)+283.33}{164.33}+d_1^++d_1^-=\bar{\mu}_1\\
\frac{H_2(x)+656.9}{200.97}+d_2^++d_2^-=\bar{\mu}_2\\
x_1+x_2+x_3\le 250\\
x_1+x_2+x_3\ge 200\\
x_1+4x_2+2x_3\le 600\\
x_1,x_2,x_3\ge 20\\
d_1^+d_1^-=0,d_1^+,d_1^-\ge 0\\
d_2^+d_2^-=0,d_2^+,d_2^-\ge 0.
\end{array}
\right.
\end{array}
\right. \tag{5.65}
$$

Set $\bar{\mu}_1=\bar{\mu}_2=0.9$, and we obtain the best solution of model (5.65), this solution is also the efficient solution of model (5.63),

$$(x_1,x_2,x_3)=(210,20,20).$$

5.4.3 Non-linear Fu-Ro CCM and Fu-Ro Simulation-Based Parametric TS

For the Fu-Ro CCM, we use the Fu-Ro simulation 2 based parametric TS algorithm to solve.

5.4.3.1 Fu-Ro Simulation 2 for Critical Value

First, we use the Fu-Ro simulation 2 to obtain the critical value which is important in CCM.

Assume that ξ is an n-dimensional fuzzy rough vector defined on the rough space $(\Lambda,\Delta,\mathscr{A},\pi)$, and $f:\mathbf{R}^n \to \mathbf{R}^m$ is a measurable function. For any real number $\alpha \in (0,1]$, we find the maximal value $\overline{f}$ such that

$$Ch\{f(\xi)\geq \overline{f}\}(\alpha)\geq \beta \tag{5.66}$$

holds. That is we should compute the maximal value $\overline{f}$ such that

$$Appr\{\lambda \in \Lambda | Cr\{f(\xi(\lambda))\geq \overline{f}\}\geq \beta\}\geq \alpha. \tag{5.67}$$

We sample $\underline{\lambda}_1,\underline{\lambda}_2,\cdots,\underline{\lambda}_N$ from $\triangle$ and $\overline{\lambda}_1,\overline{\lambda}_2,\cdots,\overline{\lambda}_N$ from Λ according to the measure π. For any number v, let $\underline{N}(v)$ denote the number of $\underline{\lambda}_k$ satisfying $Cr\{f(\xi(\underline{\lambda}_k))\leq v\}\geq \beta$ for $k=1,2,\cdots,N$, and $\overline{N}(v)$ denote the number of $\overline{\lambda}_k$ satisfying

$$Cr\{f(\xi(\overline{\lambda}_k))\leq v\}\geq \beta, \tag{5.68}$$

for $k=1,2,\cdots,N$, where $Cr\{\cdot\}$ may be estimated by fuzzy simulation. Then we may find the maximal value v such that

$$\frac{\underline{N}(v)+\overline{N}(v)}{2N}\geq \alpha. \tag{5.69}$$

This value is an estimation of $\overline{f}$. The procedure is as follows:
Step 1. Generate $\underline{\lambda}_1,\underline{\lambda}_2,\cdots,\underline{\lambda}_N$ from $\triangle$ according to the measure π.
Step 2. Generate $\overline{\lambda}_1,\overline{\lambda}_2,\cdots,\overline{\lambda}_N$ from Λ according to the measure π.
Step 3. Find the maximal value v such that (5.69) holds.
Step 4. Return v.

Example 5.6. We employ Fu-Ro simulation 2 to find the maximal value $\bar{f}$ such that $Ch\{\xi_1^2+\xi_2^2\geq \bar{f}\}(0.9)\geq 0.9$, where xi_1 and ξ_2 are Fu-Ro variables defined as

$$\xi_1=(\rho_1,\rho_1+1,\rho_1+2),\ \text{with}\ \rho_1=([1,2],[0,3]),$$
$$\xi_2=(\rho_2,\rho_2+1,\rho_2+2),\ \text{with}\ \rho_2=([2,3],[1,4]).$$

A run of Fu-Ro simulation with 5000 cycles shows that $\bar{f}=6.39$.

5.4.3.2 Parametric TS

Let's recall the detail of the parametric TS introduced by F. Glover [335]. The solution approach consists of a parametric form of tabu-search utilizing moves based on the approach of parametric branch and bound [336]. Various levels of tabu-search can be used to guide the foregoing processes. We begin by sketching the elements of a basic approach and illustrate its application.

Tabu conditions. At an initial rudimentary level, we attach a tabu restriction to an (R-DN) or (R-UP) response for a particular variable x_j , thereby forbidding the response from being executed, if the opposing response ((R-UP) or (R-DN), respectively) was executed for x_j within the most recent TabuTenure iterations. (That is, we forbid a move in a direction that is contrary to the direction of a move made within the selected span of TabuTenure iterations.) To simplify the discussion, we allow (R-DN) and (R-UP) to refer also to the responses (R-DN^o) and (R-UP^o). The value of TabuTenure varies according to the variable x_j concerned and the history of the search. We represent this value as TabuTenure$_j$(UP) and TabuTenure$_j$(DN) according to whether the tabu condition was launched by an (R-UP) or an (R-DN) response. When such a response is made we use TabuTenure$_j$(UP) or TabuTenure$_j$(DN) and knowledge of the current iteration, which we denote by Iter, to identify the iteration TabuTenure$_j$(UP) or TabuTenure$_j$(DN) that marks the end of x_js tabu tenure. Specifically, when an (R-UP) response occurs, we set

$$\text{TabuEnd}_j(\text{DN}) = \text{Iter} + \text{TabuTenure}_j(\text{DN})$$

to forbid the opposing (R-DN) response from being made for the period of TabuTenure$_j$(DN) iterations in the future. Similarly, when an (R-DN) response occurs, we set

$$\text{TabuEnd}_j(\text{UP}) = \text{Iter} + \text{TabuTenure}_j(\text{UP})$$

to forbid the opposing (R-UP) response from being made for the period of TabuTenure$_j$ (UP) iterations in the future.

By this means, an (R-DN) response is tabu for x_j as long as the (updated) current iteration satisfies

$$\text{Iter} \leq \text{TabuEnd}_j(\text{DN})$$

and an (R-UP) response is tabu for x_j as long as the current iteration satisfies

$$\text{Iter} \leq \text{TabuEnd}_j(\text{UP})$$

Initially, before any responses have been made and before associated tabu conditions have been created, TabuEnd$_j$(UP) and TabuEnd$_j$(DN) are set equal to -1, causing this value to be smaller than every value of Iter and hence assuring that no tabu restrictions will be in effect.

We refer to the values TabuEnd$_j$(UP)-Iter and TabuEnd$_j$(DN)-Iter as *residual tabu tenures*. Hence, a response will be tabu as long as its residual tabu tenure is nonnegative. (A negative residual tabu tenure accordingly indicates the response is free from a tabu restriction.) By convention, we refer to the residual tabu tenure of a variable x_j by taking it to be the residual tabu tenure of the response that is selected for this variable. We refer to the variable itself as being tabu when its associated response is tabu. (This reference is unambiguous since each goal infeasible and potentially goal infeasible variable has a single associated response.) Rules for generating the TabuTenure$_j$(DN) and TabuTenure$_j$(UP) values used to determine TabuEnd$_j$(UP) and TabuEnd$_j$(DN) are given in the next section.

In the application of the tabu tenures, a simple form of probabilistic tabu search can be used that replaces TabuTenure$_j$(DN) and TabuTenure$_j$(UP) in the formulas TabuEnd$_j$(DN)=Iter
+TabuTenure$_j$(DN) and TabuEnd$_j$(DN)=Iter +TabuTenure$_j$(UP) by values that are randomly selected from an interval around the respective tabu tenure values. A fuller use of this type of randomizing effect occurs by making such a replacement each time the inequalities Iter-TabuEnd$_j$(DN) and Iter-TabuEnd$_j$(UP) are checked.

By design, tabu restrictions are prohibitions against returning to a state previously occupied. We only create these restrictions for states that seek to enforce a goal condition, hence that involve the responses (R-UP) and (R-DN) (understanding these to include reference to the responses (R-UP^o) and (R-DN^o)). Moreover, we only check tabu conditions when at least one variable is goal infeasible. In the case where no explicit goal infeasibility exists, and hence the only responses to consider are those applicable to unrestricted free variables, then no attention is paid to tabu restrictions. The situation where all goal conditions are satisfied (no goal infeasibility exists) may be viewed as meeting the requirements of a special type of aspiration criterion, which overrules all tabu conditions. We now examine the use of criteria that operate when goal infeasibility is present.

Aspiration criteria. As is customary in tabu-search, we allow a tabu response to be released from a tabu restriction if the response satisfies an auxiliary *aspiration criterion* that indicates the response has special merit or novelty (i.e., exhibits a feature not often encountered). A common instance of such a criterion, called *aspiration by objective*, permits the response to be made if it yields a better objective function evaluation than any response previously executed. In the present setting, we find it convenient to additionally consider an *aspiration by resistance*, based on the greatest resistance a particular response has generated in the past.

Specifically, let Aspire$_j$(DN) and Aspire$_j$(UP) denote the largest goal resistance values GR$_j$(DN) and GR$_j$(UP) that have occurred for x_j on any iteration, where x_j was selected to execute an (R-DN) or (R-UP) response, respectively. Then we disregard the tabu restriction for an (R-DN) response (identified by Iter$\leq$TabuEnd$_j$(DN)) if

$$\mathrm{GR}_j(\mathrm{DN}) > \mathrm{Aspire}_j(UP)$$

and disregard tabu restriction for an (R-UP) response (identified by
Iter$\leq$TabuEnd$_j$(UP)) if

$$\mathrm{GR}_j(\mathrm{UP}) > \mathrm{Aspire}_j(DN)$$

The rationale for these aspiration criteria is that a move can be allowed if its current resistance value, measured by GR$_j$(DN) or GR$_j$(UP), exceeds the greatest resistance value previously identified for moving in the opposite direction (Aspire$_j$(UP) or Aspire$_j$(DN), respectively). We initially set Aspire$_j$(UP) and Aspire$_j$(DN) to a large negative number, so that the first time a variable x_j is evaluated for a potential response (R-UP) or (R-DN), the response will automatically be allowed, and it will continue to be allowed until the opposing response is made, which establishes a resistance to be exceeded.

We call a response admissible if it is either not tabu or else satisfies the aspiration criterion, and call it *inadmissible* otherwise. If the unique available response for a goal infeasible variable is inadmissible, then the variable is not permitted to enter the sets G_P and G_S, even if this makes it impossible for one or both of these sets to attain its targeted size g_P or g_S. The only exception to this rule is that G_P is not permitted to be empty in the case of goal infeasibility. Hence in the extreme case where no variables would enter G_P the typical aspiration by default rule is invoked that allows G_P to contain a variable with a smallest residual tabu tenure. (Probabilistic variations of the aspiration by default rule can also be applied, by assigning larger probabilities to selecting variables with smaller residual tabu tenures.)

As observed earlier, $GR_j = GR_j(DN)$ or GR_j(UP) may be treated as a 2-element vector, with a dominant component for an overt goal infeasibility and a secondary GR_j^o component for potential goal infeasibility. The Aspire$_j$ values are treated in the same way, as 2-element vectors that include a secondary component Aspire$_j^o$ for potential goal infeasibility. Since overt and potential goal infeasibility for a given variable x_j never occur simultaneously, and since overt goal infeasibility is the dominant component, only a single component of the vector is relevant to considerłthe overt component if it exists, and the potential component otherwise.

It is to be emphasized that Aspire$_j$(UP) and Aspire$_j$(DN) do not record the greatest values of GR_j(UP) and GR_j(DN) encountered over the history of the search, but only the greatest values that occurred in the instances where x_j was selected as a variable to be assigned a goal condition, and only in response to overt or potential goal infeasibility (i.e., not in response to integer infeasibility, which occurs only when x_j is an unrestricted fractional variable).

5.4.3.3 Numerical Example

Example 5.7. Let us consider the following problem,

$$\begin{cases} \max\ [f_1, f_2] \\ \text{s.t.} \begin{cases} Appr\{\lambda|Pos\{\tilde{\bar{\xi}}_1 x_1 + \tilde{\bar{\xi}}_2 x_2 + \tilde{\bar{\xi}}_3 x_3 \geq f_1\} \geq \delta_1\} \geq \gamma_1 \\ Appr\{\lambda|Pos\{c_1\tilde{\bar{\xi}}_4 x_1 + c_2\tilde{\bar{\xi}}_5 x_2 + c_3\tilde{\bar{\xi}}_6 x_3 \geq f_2\} \geq \delta_2\} \geq \gamma_2 \\ x_1 + x_2 + x_3 \leq 250 \\ x_1 + x_2 + x_3 \geq 200 \\ x_1 + 4x_2 + 2x_3 \leq 600 \\ x_1 \geq 20, x_2 \geq 20, x_3 \geq 20, \end{cases} \end{cases} \tag{5.70}$$

where $c = (c_1, c_2, c_3) = (1.2, 0.8, 1.5)$,

$$\begin{array}{l}
\tilde{\tilde{\xi}}_1 = (1, \rho_1, 1)_{LR}, \text{ with } \rho_1 \vdash ([1,2],[0,3]), \\
\tilde{\tilde{\xi}}_2 = (1, \rho_2, 1)_{LR}, \text{ with } \rho_2 \vdash ([2,3],[1,4]), \\
\tilde{\tilde{\xi}}_3 = (1, \rho_3, 1)_{LR}, \text{ with } \rho_3 \vdash ([3,4],[2,5]), \\
\tilde{\tilde{\xi}}_4 = (1, \rho_4, 1)_{LR}, \text{ with } \rho_4 \vdash ([0,1],[0,3]), \\
\tilde{\tilde{\xi}}_5 = (1, \rho_5, 1)_{LR}, \text{ with } \rho_5 \vdash ([1,2],[0,3]), \\
\tilde{\tilde{\xi}}_6 = (1, \rho_6, 1)_{LR}, \text{ with } \rho_6 \vdash ([2,3],[0,3]),
\end{array}$$

and $\rho_i (i = 1, 2, \cdots, 6)$ are rough variables. We set $\delta_i = \gamma_i = 0.9$, then $\Phi^{-1}(1 - \delta_i) = -1.28, i = 1, 2$.

Next, we apply the parallel tabu search algorithm based on the Fu-Ro simulation to solve the nonlinear programming problem (5.70) with the fuzzy rough parameters.

Step 1. Set the move step $h = 0.5$ and the h neighbor $N(x,h)$ for the present point x is defined as follows,

$$N(x,h) = \left\{ \mathbf{y} | \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2} \le h \right\}.$$

The random move of point x to point $\mathbf{y}$ in its h neighbor along direction s is given by

$$\mathbf{y}_s = x_s + rh,$$

where r is a random number that belongs to [0,1], $s = 1, 2, 3$.

Step 2. Give the step set $H = \{h_1, h_2, \cdots, h_r\}$ and randomly generate a feasible point x_0 checked by the fuzzy rough simulation. One should empty the Tabu list T (the list of inactive steps) at the beginning.

Step 3. For each active neighbor $N(x,h)$ of the present point x, where $h \in H - T$, a feasible random move that satisfies all the constraints in problem (5.70) is to be generated.

Step 4. Construct the single objective function as follows,

$$f(x) = w_1 f_1 + w_2 f_2$$

where $w_1 + w_2 = 1$ and $w_i (i = 1, 2)$ is predetermined by the decision maker. Compare the $f(x)$ of the feasible moves with that of the current solution by the fuzzy rough simulation. If an augmenter in new objective function of the feasible moves exists, one should save this feasible move as the updated current one by adding the corresponding step to the Tabu list T and go to the next step; otherwise, go to the next step directly.

Step 5. Stop if the termination criteria are satisfied; other wise, empty T if it is full; then go to Step 3. Here, we set the computation is determined if the better solution doesn't change again.

We apply compute the programming problem (5.70) by the parallel tabu search algorithm. The table 5.4 shows the results.

Table 5.4 The result computed by parallel TS algorithm at different weights

w_1	w_2	x_1	x_2	x_3	H	Gen
0.1	0.9	90.68	25.19	84.13	-2304.55	270
0.2	0.8	90.25	25.08	84.66	-2287.08	240
0.3	0.7	89.82	24.99	85.19	-2269.57	256
0.4	0.6	89.39	24.89	85.72	-2252.01	269
0.5	0.5	88.99	24.79	86.25	-2234.40	294
0.6	0.4	88.53	24.70	86.78	-2216.74	291
0.7	0.3	88.10	24.60	87.30	-2199.03	268
0.8	0.2	87.67	24.50	87.83	-2181.29	281
0.9	0.1	87.24	24.40	88.36	-2163.48	276

5.5 Fu-Ro DCM

This section provides Fu-Ro DCM in which the underling philosophy is based on selecting the decision with the maximum chance to meet the event.

5.5.1 General Model for Fu-Ro DCM

A generally uncertain dependent chance model has the following form,

$$\begin{cases} \max\ [Ch\{f_i(x,\xi) \le f_i\}(\alpha_i), i = 1,2,\cdots,m] \\ \text{s.t.} \begin{cases} g_r(x,\xi) \le 0, r = 1,2,\cdots,p \\ x \in X, \end{cases} \end{cases} \tag{5.71}$$

where x is an n-dimensional decision vector, ξ is a fuzzy rough vector, the event ξ is characterized by $h_k(x,\xi) \le 0, k = 1,2,\ldots q$, and the fuzzy rough environment is described by the fuzzy rough constraints $g_r(x,\xi) \le 0, r = 1,2,\ldots p$. Here, the constraints are all certain. For uncertain constraints, we can deal with them by the technique of chance-constrained programming.

When the fuzzy rough variable degenerates to the single uncertain variable, we obtain the following results.

Remark 5.8. If the fuzzy rough variable ξ degenerates to a fuzzy variable, for any given α_i,

$$Ch\{f_i(x,\xi) \le f_i\}(\alpha_i) = Cr\{f_i(x,\xi) \le f_i\}(\alpha_i), i = 1,2,\cdots,m.$$

Thus, the problem (5.71) is equivalent to

$$\begin{cases} \max\ [Cr\{f_i(x,\xi) \le f_i\}(\alpha_i), i = 1,2,\cdots,m] \\ \text{s.t.} \begin{cases} g_r(x,\xi) \le 0, r = 1,2,\cdots,p \\ x \in X, \end{cases} \end{cases} \tag{5.72}$$

where ξ is a fuzzy variable, and this model is a standard fuzzy DCM.

Remark 5.9. If the fuzzy rough variable ξ degenerates to a rough variable, for any given α_i. This means

$$Ch\{f_i(x,\xi) \le f_i\}(\alpha_i) = Appr\{f_i(x,\xi) \le f_i\}(\alpha_i), i = 1,2,\cdots,m.$$

Thus, the problem (5.71) is converted into

$$\begin{cases} \max\ [Appr\{f_i(x,\xi) \le f_i\}(\alpha_i), i = 1,2,\cdots,m] \\ \text{s.t.} \begin{cases} g_r(x,\xi) \le 0, r = 1,2,\cdots,p \\ x \in X, \end{cases} \end{cases} \tag{5.73}$$

where ξ is a rough variable, and this model is a standard rough DCM.

If there are multiple events in the fuzzy rough environment, a typical formulation of Fu-Ro DCM is given as follows,

$$\begin{cases} \max \begin{bmatrix} Ch\{h_{1k}(x,\xi) \le 0,\ k = 1,2,\cdots,q_1\} \\ Ch\{h_{2k}(x,\xi) \le 0,\ k = 1,2,\cdots,q_2\} \\ \cdots \\ Ch\{h_{mk}(x,\xi) \le 0,\ k = 1,2,\cdots,q_m\} \end{bmatrix} \\ \text{s.t.} \begin{cases} g_r(x,\xi) \le 0,\ r = 1,2,\cdots,p \\ x \in X, \end{cases} \end{cases} \tag{5.74}$$

where $h_{ik}(x,\xi) \le 0,\ k = 1,2,\cdots,q_i$ represent events ε_i for $i = 1,2,\cdots,m$, respectively.

Fuzzy rough dependent-chance goal programming is employed to formulate fuzzy rough decision systems according to the priority structure and target levels set by the decision-maker,

$$\begin{cases} \max \sum\limits_{j=1}^{l} P_j \sum\limits_{i=1}^{m} (u_{ij} d_i^+ \vee 0 + v_{ij} d_i^- \vee 0) \\ \text{s.t.} \begin{cases} Ch\{h_{1k}(x,\xi) \le 0,\ k = 1,2,\cdots,q_i\} - b_i = d_i^+, i = 1,2,\cdots,m \\ b_i - Ch\{h_{1k}(x,\xi) \le 0,\ k = 1,2,\cdots,q_i\} = d_i^-, i = 1,2,\cdots,m \\ g_r(x,\xi) \le 0,\ r = 1,2,\cdots,p \\ x \in X, \end{cases} \end{cases} \tag{5.75}$$

where P_j is the preemptive priority factor which expresses the relative importance of various goals, $P_j \gg P_{j+1}$, for all j, u_{ij} is the weighting factor corresponding to positive deviation for goal i with priority j assigned, v_{ij} is the weighting factor corresponding to negative deviation for goal i with priority j assigned, $d_i^+ \vee 0$ is the positive deviation from the target of goal i, $d_i^- \vee 0$ is the negative deviation from the target of goal i, g_j is a function in system constraints, b_i is the target value according to goal i, l is the number of priorities, m is the number of goal constraints, and p is the number of system constraints.

5.5.2 Linear Fu-Ro DCM and ε-Constraint Method

Let's still consider the linear model with Fu-Ro coefficients as follows,

$$\begin{cases} \max\ [\tilde{\bar{c}}_1^T x, \tilde{\bar{c}}_2^T x, \cdots, \tilde{\bar{c}}_m^T x] \\ \text{s.t.} \begin{cases} e_r^T x \le b_r, \quad r = 1,2,\cdots,p \\ x \in X, \end{cases} \end{cases} \tag{5.76}$$

where $\tilde{\bar{c}}_i$ is Fu-Ro vector, $i = 1,2,\cdots,m$.

5.5.2.1 Crisp Equivalent Model

$Appr - Pos$ constrained multi-objective linearity model

Because there are Fu-Ro variables in the model (5.76), so the model doesn't have mathematical meaning. We can use its DCM on $Appr - Pos$ to deal with it as follows

$$\begin{cases} \max\ [\delta_1, \delta_2, \cdots, \delta_m] \\ \text{s.t.} \begin{cases} Appr\{\lambda | Pos\{\tilde{\bar{c}}_i^T(\lambda) x \le f_i\} \ge \delta_i\} \ge \gamma_i, \quad i = 1,2,\cdots,m \\ e_r^T x \le b_r, \quad r = 1,2,\cdots,p \\ x \in X, \end{cases} \end{cases} \tag{5.77}$$

where $\xi = (\xi_1, \xi_2, \cdots, \xi_n)^T$ is a fuzzy rough vector, γ_i is the given confidence level and f_i is the predetermined value.

We can use the following theorem to obtain the equivalent form of the crisp dependent-chance model (5.77).

Theorem 5.12. *Assume that $\tilde{\bar{c}}_{ij}$ is a Fu-Ro variable, for any $\lambda \in \Lambda$, the fuzzy variable $\tilde{\bar{c}}_{ij}(\lambda)$ is characterized by the following membership function*

$$\mu_{\tilde{\bar{c}}_{ij}(\lambda)}(t) = \begin{cases} L\left(\frac{c_{ij}(\lambda) - t}{\alpha_{ij}^c}\right), \ t \le c_{ij}(\lambda), \alpha_{ij}^c > 0 \\ R\left(\frac{t - c_{ij}(\lambda)}{\beta_{ij}^c}\right), \ t \ge c_{ij}(\lambda), \beta_{ij}^c > 0 \end{cases} \quad \lambda \in \Lambda \tag{5.78}$$

where $\alpha_{ij}^c, \beta_{ij}^c$ are positive numbers expressing the left and right spread of $\tilde{\bar{c}}_{ij}(\lambda)$, reference function $L, R : [0,1] \to [0,1]$ with $L(1) = R(1) = 0$, and $L(0) = R(0) = 1$ are non-increasing, continuous function. And $(c_{ij}(\lambda))_{n\times 1} = (c_{i1}(\lambda), c_{i2}(\lambda), \cdots, c_{in}(\lambda))^T$ is a rough vector. It follows that $c_i(\lambda)^T x = ([a,b],[c,d])$

(where $c \le a < b \le d$) is a rough variable and characterized by the following trust measure function,

$$Appr\{c_i(\lambda)^T x \ge t\} = \begin{cases} 0, & if\ d \le t \\ \frac{d-t}{2(d-c)}, & if\ b \le t \le d \\ \frac{1}{2}(\frac{d-t}{d-c} + \frac{b-t}{b-a}), & if\ a \le t < b \\ \frac{1}{2}(\frac{d-t}{d-c} + 1), & if\ c \le t \le a \\ 1, & if\ t \le c. \end{cases} \tag{5.79}$$

Then we have $Appr\{\lambda | \mathrm{Pos}\{\tilde{\bar{c}}_i(\lambda)^{\mathrm{T}} x \ge f_i\} \ge \delta_i\} \ge \gamma_i$ if and only if

$$\begin{cases} R^{-1}(\delta_i) \ge \frac{f_i - d + 2(d-c)\gamma_i}{\beta_i^{cT} x}, & if\ b \le f_i - R^{-1}(\delta_i)\beta_i^{cT} x \le d \\ R^{-1}(\delta_i) \ge \frac{(d-c+b-a)f_i - d(b-a) - b(d-c) + 2(d-c)(b-a)\gamma_i}{\beta_i^{cT} x}, & if\ a \le f_i - R^{-1}(\delta_i)\beta_i^{cT} x < b \\ R^{-1}(\delta_i) \ge \frac{f_i - d + 2(d-c)(2\gamma_i - 1)}{\beta_i^{cT} x}, & if\ c \le f_i - R^{-1}(\delta_i)\beta_i^{cT} x \le a \\ R^{-1}(\delta_i) \ge \frac{f_i - c}{\beta_i^{cT} x}, & if\ f_i - R^{-1}(\delta_i)\beta_i^{cT} x \le c, \end{cases} \tag{5.80}$$

where $\gamma_i, \delta_i \in [0,1]$ are predetermined confidence levels.

And accordingly we can get the crisp equivalent models in every cases.

Proof. By theorem 5.7, we have

$$\begin{cases} f_i \le d - 2\gamma_i(d-c) + R^{-1}(\delta_i)\beta_i^{cT} x, & \text{if } b \le f_i - R^{-1}(\delta_i)\beta_i^{cT} x \le d \\ f_i \le \frac{d(b-a) + b(d-c) - 2\gamma_i(d-c)(b-a)}{d-c+b-a} + R^{-1}(\delta_i)\beta_i^{cT} x, & \text{if } a \le f_i - R^{-1}(\delta_i)\beta_i^{cT} x < b \\ f_i \le d - (d-c)(2\gamma_i - 1) + R^{-1}(\delta_i)\beta_i^{cT} x, & \text{if } c \le f_i - R^{-1}(\delta_i)\beta_i^{cT} x \le a \\ f_i \le c + R^{-1}(\delta_i)\beta_i^{cT} x, & \text{if } f_i - R^{-1}(\delta_i)\beta_i^{cT} x \le c. \end{cases}$$

Because γ_i is a given confidence level between 0 and 1, this is no optimal solution for $L \ge d$. We can discuss the following four cases.

Case 1: $b \le f_i - R^{-1}(\delta_i)\beta_i^{cT} x \le d$.

From the assumption we know that $(d-c) > 0$, so we have

$$\begin{aligned} & f_i \le d - 2\gamma_i(d-c) + R^{-1}(\delta_i)\beta_i^{cT} x \\ \Leftrightarrow\ & R^{-1}(\delta_i) \ge \frac{f_i - d + 2(d-c)\gamma_i}{\beta_i^{cT} x}. \end{aligned}$$

From the assumption, the reference function $R(\cdot)$ is non-increasing continuous function, so max δ_i is equivalent to min $R^{-1}(\delta_i)$, so the problem (5.77) can be transformed into

$$\begin{cases} \max\ [\frac{f_i - d + 2(d-c)\gamma_i}{\beta_i^{cT} x}] \\ \text{s.t.} \begin{cases} e_r^T x \le b_r, \quad r = 1, 2, \cdots, p \\ x \in X \\ x \ge 0. \end{cases} \end{cases}$$

Case 2: $a \le f_i - R^{-1}(\delta_i)\beta_i^{cT}x < b$. We have

$$\begin{array}{c} f_i \le \frac{d(b-a)+b(d-c)-2\gamma_i(d-c)(b-a)}{d-c+b-a} + R^{-1}(\delta_i)\beta_i^{cT}x \\ \Leftrightarrow R^{-1}(\delta_i) \ge \frac{(d-c+b-a)f_i-d(b-a)-b(d-c)+2(d-c)(b-a)\gamma_i}{\beta_i^{cT}x}, \end{array}$$

so the problem (5.77) can be transformed into

$$\begin{cases} \max\ [\frac{(d-c+b-a)f_i-d(b-a)-b(d-c)+2(d-c)(b-a)\gamma_i}{\beta_i^{cT}x}] \\ \text{s.t.} \begin{cases} e_r^T x \le b_r, \quad r = 1,2,\cdots,p \\ x \in X \\ x \ge 0. \end{cases} \end{cases}$$

Case 3: $c \le f_i - R^{-1}(\delta_i)\beta_i^{cT}x \le a$. We have

$$\begin{array}{c} f_i \le d-(d-c)(2\gamma_i-1)+R^{-1}(\delta_i)\beta_i^{cT}x \\ \Leftrightarrow R^{-1}(\delta_i) \ge \frac{f_i-d+2(d-c)(2\gamma_i-1)}{\beta_i^{cT}x}, \end{array}$$

so the problem (5.77) can be transformed into

$$\begin{cases} \max\ [\frac{f_i-d+2(d-c)(2\gamma_i-1)}{\beta_i^{cT}x}] \\ \text{s.t.} \begin{cases} e_r^T x \le b_r, \quad r = 1,2,\cdots,p \\ x \in X \\ x \ge 0. \end{cases} \end{cases}$$

Case 4: $f_i - R^{-1}(\delta_i)\beta_i^{cT}x \le c$. We have

$$f_i \le c + R^{-1}(\delta_i)\beta_i^{cT}x \Leftrightarrow R^{-1}(\delta_i) \ge \frac{f_i - c}{\beta_i^{cT}x},$$

so the problem (5.77) can be transformed into

$$\begin{cases} \max\ [\frac{f_i-c}{\beta_i^{cT}x}] \\ \text{s.t.} \begin{cases} e_r^T x \le b_r, \quad r = 1,2,\cdots,p \\ x \in X \\ x \ge 0. \end{cases} \end{cases}$$

This completes the proof. □

Appr-Nec constrained multi-objective linearity model

Also we can use the DCM based on $Appr-Nec$ to deal with model (5.76) as follows

$$\begin{cases} \max\ [\delta_1, \delta_2, \cdots, \delta_m] \\ \text{s.t.} \begin{cases} Appr\{\lambda | Nec\{\tilde{\bar{c}}_i^T(\lambda)x \leq f_i\} \geq \delta_i\} \geq \gamma_i, \quad i = 1, 2, \cdots, m \\ e_r^T x \leq b_r, \quad r = 1, 2, \cdots, p \\ x \in X, \end{cases} \end{cases} \tag{5.81}$$

where $\xi = (\xi_1, \xi_2, \cdots, \xi_n)^T$ is a fuzzy rough vector, γ_i is the given confidence level and f_i is the predetermined value.

We can use the following theorem to obtain the equivalent form of the crisp DCM (5.81).

Theorem 5.13. *Assume that $\tilde{\bar{c}}_{ij}$ is a fuzzy rough variable, for any $\lambda \in \Lambda$, the fuzzy variable $\tilde{\bar{c}}_{ij}(\lambda)$ is characterized by the following membership function*

$$\mu_{\tilde{\bar{c}}_{ij}(\lambda)}(t) = \begin{cases} L\left(\frac{c_{ij}(\lambda)-t}{\alpha_{ij}^c}\right), \ t \leq c_{ij}(\lambda), \alpha_{ij}^c > 0 \\ R\left(\frac{t-c_{ij}(\lambda)}{\beta_{ij}^c}\right), \ t \geq c_{ij}(\lambda), \beta_{ij}^c > 0 \end{cases} \quad \lambda \in \Lambda, \tag{5.82}$$

where $\alpha_{ij}^c, \beta_{ij}^c$ are positive numbers expressing the left and right spread of $\tilde{\bar{c}}_{ij}(\lambda)$, reference function $L, R : [0,1] \to [0,1]$ with $L(1) = R(1) = 0$, and $L(0) = R(0) = 1$ are non-increasing, continuous function. And $(c_{ij}(\lambda))_{n\times 1} = (c_{i1}(\lambda), c_{i2}(\lambda), \cdots, c_{in}(\lambda))^T$ is a rough vector. It follows that $c_i(\lambda)^T x = ([a,b],[c,d])$ (where $c \leq a < b \leq d$) is a rough variable and characterized by the following trust measure function,

$$Appr\{c_i(\lambda)^T x \geq t\} = \begin{cases} 0, & \text{if } d \leq t \\ \frac{d-t}{2(d-c)}, & \text{if } b \leq t \leq d \\ \frac{1}{2}\left(\frac{d-t}{d-c} + \frac{b-t}{b-a}\right), & \text{if } a \leq t < b \\ \frac{1}{2}\left(\frac{d-t}{d-c} + 1\right), & \text{if } c \leq t \leq a \\ 1, & \text{if } t \leq c. \end{cases} \tag{5.83}$$

Then we have $Appr\{\lambda | Nec\{\tilde{\bar{c}}_i(\lambda)^{\mathrm{T}} x \geq f_i\} \geq \delta_i\} \geq \gamma_i$ if and only if

$$\begin{cases} L^{-1}(1-\delta_i) \leq \frac{-f_i + d - 2(d-c)\gamma_i}{\alpha_i^{cT} x}, \\ \qquad \text{if } b \leq f_i + L^{-1}(1-\delta_i)\alpha_i^{cT} x \leq d \\ L^{-1}(1-\delta_i) \leq \frac{-(d-c+b-a)f_i + d(b-a) - b(d-c) - 2(d-c)(b-a)\gamma_i}{\alpha_i^{cT} x}, \\ \qquad \text{if } a \leq f_i + L^{-1}(1-\delta_i)\alpha_i^{cT} x < b \\ L^{-1}(1-\delta_i) \leq \frac{-f_i + d - 2(d-c)(2\gamma_i - 1)}{\alpha_i^{cT} x}, \\ \qquad \text{if } c \leq f_i + L^{-1}(1-\delta_i)\alpha_i^{cT} x \leq a \\ L^{-1}(1-\delta_i) \geq \frac{-f_i + c}{\alpha_i^{cT} x}, \\ \qquad \text{if } f_i + L^{-1}(1-\delta_i)\alpha_i^{cT} x \leq c, \end{cases} \tag{5.84}$$

where $\gamma_i, \delta_i \in [0,1]$ are predetermined confidence levels.

And accordingly we can get the crisp equivalent models in every cases.

Proof. By theorem 5.9, we have

$$\begin{cases} f_i \le d - 2\gamma_i(d-c) - L^{-1}(1-\delta_i)\alpha_i^{cT}x, \\ \qquad\qquad\qquad \text{if } b \le f_i + L^{-1}(1-\delta_i)\alpha_i^{cT}x \le d \\ f_i \le \frac{d(b-a)+b(d-c)-2\gamma_i(d-c)(b-a)}{d-c+b-a} - L^{-1}(1-\delta_i)\alpha_i^{cT}x, \\ \qquad\qquad\qquad \text{if } a \le f_i + L^{-1}(1-\delta_i)\alpha_i^{cT}x < b \\ f_i \le d-(d-c)(2\gamma_i-1) - L^{-1}(1-\delta_i)\alpha_i^{cT}x, \\ \qquad\qquad\qquad \text{if } c \le f_i + L^{-1}(1-\delta_i)\alpha_i^{cT}x \le a \\ f_i \le c - L^{-1}(1-\delta_i)\alpha_i^{cT}x, \\ \qquad\qquad\qquad \text{if } f_i + L^{-1}(1-\delta_i)\alpha_i^{cT}x \le c. \end{cases}$$

Because γ_i is a given confidence level between 0 and 1, this is no optimal solution for $L \ge d$. We can discuss the following four cases.
Case 1: $b \le f_i + L^{-1}(1-\delta_i)\alpha_i^{cT}x \le d$.
From the assumption we know that $\alpha_i^{cT}x > 0$, so we have

$$\begin{array}{c} f_i \le d - 2\gamma_i(d-c) - L^{-1}(1-\delta_i)\alpha_i^{cT}x \\ \Leftrightarrow L^{-1}(1-\delta_i) \le \frac{-f_i+d-2(d-c)\gamma_i}{\alpha_i^{cT}x}. \end{array}$$

From the assumption, the reference function $L(\cdot)$ is non-increasing continuous function, so max δ_i is equivalent to max $L^{-1}(1-\delta_i)$, so the problem (5.77) can be transformed into

$$\begin{cases} \max \{\frac{f_i-d+2(d-c)\gamma_i}{\beta_i^{cT}x}\} \\ \text{s.t.} \begin{cases} e_r^T x \le b_r, \quad r = 1,2,\cdots,p \\ x \in X \\ x \ge 0. \end{cases} \end{cases}$$

Case 2: $a \le f_i + L^{-1}(1-\delta_i)\alpha_i^{cT}x < b$. We have

$$\begin{array}{c} f_i \le \frac{d(b-a)+b(d-c)-2\gamma_i(d-c)(b-a)}{d-c+b-a} - L^{-1}(1-\delta_i)\alpha_i^{cT}x \\ \Leftrightarrow L^{-1}(1-\delta_i) \le \frac{-(d-c+b-a)f_i+d(b-a)-b(d-c)-2(d-c)(b-a)\gamma_i}{\alpha_i^{cT}x}, \end{array}$$

so the problem (5.81) can be transformed into

$$\begin{cases} \max [\frac{-(d-c+b-a)f_i+d(b-a)-b(d-c)-2(d-c)(b-a)\gamma_i}{\alpha_i^{cT}x}] \\ \text{s.t.} \begin{cases} e_r^T x \le b_r, \quad r = 1,2,\cdots,p \\ x \in X \\ x \ge 0. \end{cases} \end{cases}$$

Case 3: $c \le f_i + L^{-1}(1-\delta_i)\alpha_i^{cT}x \le a$. We have

$$\begin{array}{c} f_i \le d-(d-c)(2\gamma_i-1) - L^{-1}(1-\delta_i)\alpha_i^{cT}x \\ \Leftrightarrow L^{-1}(1-\delta_i) \le \frac{-f_i+d-2(d-c)(2\gamma_i-1)}{\alpha_i^{cT}x}, \end{array}$$

so the problem (5.77) can be transformed into

$$\begin{cases} \max\ [\frac{-f_i+d-2(d-c)(2\gamma_i-1)}{\alpha_i^{cT}x}] \\ \text{s.t.} \begin{cases} e_r^T x \leq b_r, \quad r=1,2,\cdots,p \\ x \in X \\ x \geq 0. \end{cases} \end{cases}$$

Case 4: $f_i + L^{-1}(1-\delta_i)\alpha_i^{cT}x \leq c$. We have

$$f_i \leq c - L^{-1}(1-\delta_i)\alpha_i^{cT}x \Leftrightarrow L^{-1}(1-\delta_i) \geq \frac{-f_i+c}{\alpha_i^{cT}x},$$

so the problem (5.77) can be transformed into

$$\begin{cases} \max\ [\frac{-f_i+c}{\alpha_i^{cT}x}] \\ \text{s.t.} \begin{cases} e_r^T x \leq b_r, \quad r=1,2,\cdots,p \\ x \in X \\ x \geq 0. \end{cases} \end{cases}$$

This completes the proof. □

5.5.2.2 ε-Constraint Method

ε-constraint method was proposed by Haimes[5, 6] in 1971. The idea of this method is that we choose a main referenced objective f_{i0}, put the other objective functions into the constraints.

Let's consider the following multi-objective model:

$$\begin{cases} \min\ [f_i(x), i=1,2,\cdots,m] \\ \text{s.t. } x \in X. \end{cases} \tag{5.85}$$

So we use the ε-constraint method, we can get the single objective model (5.86):

$$\begin{cases} \min\ f_{i_0}(x) \\ \text{s.t.} \begin{cases} f_i(x) \leq \varepsilon_i,\ i=1,2,\cdots,m, i \neq i_0 \\ c \in X, \end{cases} \end{cases} \tag{5.86}$$

where the parameter ε_i is predetermined by the decision maker, it denote the threshold value that the decision maker will accept, we denote the feasible domain of model (5.86) as X_1.

Theorem 5.14. *If $\bar{x}$ is the optimal solution of model (5.86), then $\bar{x}$ is a weak efficient solution of model (5.85).*

Proof. Let $\bar{x}$ be the optimal solution of model (5.86), but it is not a weak efficient solution of model (5.85), then there exists $x' \in X$, such that for $\forall\ i \in \{1,2,\cdots,m\}$, $f_i(x') < f_i(\bar{x})$ holds. Since $\bar{x} \in X_1$, $f_i(\bar{x}) \le \varepsilon_i$ $(i = 1,2,\cdots,m, i \neq i_0)$, So we have

$$f_i(x') < f_i(\bar{x}) \le \varepsilon_i,\ i = 1,2,\cdots,m, i \neq i_0. \tag{5.87}$$

We can obtain from (5.87) that $x' \in X_1$, and $f_{i_0}(x') < f_{i_0}(\bar{x})$. This conflicts with that $\bar{x}$ is the optimal solution. □

Theorem 5.15. *Let $\bar{x}$ be a efficient solution of model (5.85), then there exists a parameter $\varepsilon_i (i = 1,2,\cdots,m, i \neq i_0)$, such that $\bar{x}$ is the optimal solution of model (5.86).*

Proof. Take $\varepsilon_i = f_i(\bar{x})$ $(i = 1,2,\cdots,m, i \neq i_0)$, by the definition of efficient solution, $\bar{x}$ is a optimal solution of model (5.86).

So the advantage of the ε-constraint method is that:
(1). Every efficient solution of model (5.85) can be get by properly choosing parameter $\varepsilon_i (i = 1,2,\cdots,m, i \neq i_0)$.
(2). The i_0th objective are mainly guaranteed, and the other objectives are considered meanwhile.

It is worth for us noticing that the parameter ε_i is important, we should carefully choose it. If the value of every ε_i is too small, then it is possible that the model (5.86) will have no solutions; otherwise, is the value of ε_i is too large, then besides the main objective, the other objective will lose more with higher possibility. Commonly, we can offer the decision maker $f_i^0 = \min_{x\in X} f_i(x)$ $(i = 1,2,\cdots,m)$ and the objective value $(f_1(x), f_2(x),\cdots,f_m(x))^T$ of a certain feasible solution x. And then the decision maker can decide ε_i. For more details, the readers can refer to Chankong [4].

5.5.2.3 Numerical Example

Example 5.8. Let's still consider the Example 5.5. This time we use fuzzy rough dependent chance constrained model to deal with.

$$\begin{cases} \max \left[Ch\{f_1(x,\tilde{\bar{\xi}}) \ge \bar{f}_1\}(\gamma_1), Ch\{f_2(x,\tilde{\bar{\xi}}) \ge \bar{f}_2\}(\gamma_2) \right] \\ \text{s.t.} \begin{cases} x_1 + x_2 + x_3 \le 250 \\ x_1 + x_2 + x_3 \ge 200 \\ x_1 + 4x_2 + 2x_3 \le 600 \\ x_1 \ge 20, x_2 \ge 20, x_3 \ge 20, \end{cases} \end{cases}$$

$$f_1 = \tilde{\bar{\xi}}_1 x_1 + \tilde{\bar{\xi}}_2 x_2 + \tilde{\bar{\xi}}_3 x_3 \text{ and } f_2 = c_1\tilde{\bar{\xi}}_4 x_1 + c_2\tilde{\bar{\xi}}_5 x_2 + c_3\tilde{\bar{\xi}}_6 x_3,$$

where $\tilde{\bar{\xi}}$ are Fu-Ro vectors, $Ch\{\cdot\}$ is the chance of the Fu-Ro event, γ_i, $i = 1,2$ is the predetermined confidence level, $\bar{f}_i$, $i = 1,2$ are the ideal levels of every objectives.

Model (5.88) can be written as the following equivalent form (5.88) by introducing δ_i, $i = 1,2$.

$$
\begin{array}{l}
\max\ [\delta_1, \delta_2] \\
s.t. \begin{cases}
Ch\{\tilde{\bar{\xi}}_1 x_1 + \tilde{\bar{\xi}}_2 x_2 + \tilde{\bar{\xi}}_3 x_3 \geq \bar{f}_1\}(\gamma_1) \geq \delta_1 \\
Ch\{c_1 \tilde{\bar{\xi}}_4 x_1 + c_2 \tilde{\bar{\xi}}_5 x_2 + c_3 \tilde{\bar{\xi}}_6 x_3 \geq \bar{f}_2\}(\gamma_2) \geq \delta_2 \\
x_1 + x_2 + x_3 \leq 250 \\
x_1 + x_2 + x_3 \geq 200 \\
x_1 + 4x_2 + 2x_3 \leq 600 \\
x_1 \geq 20, x_2 \geq 20, x_3 \geq 20.
\end{cases}
\end{array}
$$

When we use the $Appr - pos$ chance measure, model (5.88) and (5.88) can also be written as (5.88).

$$
\begin{cases}
\max\ [\delta_1, \delta_2] \\
s.t. \begin{cases}
App\{\lambda | Pos\{\tilde{\bar{\xi}}_1 x_1 + \tilde{\bar{\xi}}_2 x_2 + \tilde{\bar{\xi}}_3 x_3 \geq \bar{f}_1\} \geq \delta_1\} \geq \gamma_1 \\
Appr\{\lambda | Pos\{c_1 \tilde{\bar{\xi}}_4 x_1 + c_2 \tilde{\bar{\xi}}_5 x_2 + c_3 \tilde{\bar{\xi}}_6 x_3 \geq \bar{f}_2\} \geq \delta_2\} \geq \gamma_2 \\
x_1 + x_2 + x_3 \leq 250 \\
x_1 + x_2 + x_3 \geq 200 \\
x_1 + 4x_2 + 2x_3 \leq 600 \\
x_1 \geq 20, x_2 \geq 20, x_3 \geq 20.
\end{cases}
\end{cases}
$$

The following is the relevant data:

$$
\begin{array}{l}
c = (c_1, c_2, c_3) = (1.2, 0.8, 1.5), \\
\tilde{\bar{\xi}}_1 = (\rho_1, 0.5, 0.5)_{LR}, \text{ with } \rho_1 \vdash ([1,2],[0,3]), \\
\tilde{\bar{\xi}}_2 = (\rho_2, 2, 2)_{LR}, \text{ with } \rho_2 \vdash ([2,3],[1,4]), \\
\tilde{\bar{\xi}}_3 = (\rho_3, 1, 1)_{LR}, \text{ with } \rho_3 \vdash ([3,4],[2,5]), \\
\tilde{\bar{\xi}}_4 = (\rho_4, 1, 1)_{LR}, \text{ with } \rho_4 \vdash ([0,1],[0,3]), \\
\tilde{\bar{\xi}}_5 = (\rho_5, 0.5, 0.5)_{LR}, \text{ with } \rho_5 \vdash ([1,2],[0,3]), \\
\tilde{\bar{\xi}}_6 = (\rho_6, 0.5, 0.5)_{LR}, \text{ with } \rho_6 \vdash ([2,3],[0,3]).
\end{array}
$$

According to the knowledge of fuzzy variable and rough variable, we have that

$$
\begin{array}{rl}
& \tilde{\bar{\xi}}_1 x_1 + \tilde{\bar{\xi}}_2 x_2 + \tilde{\bar{\xi}}_3 x_3 \\
= & (\rho_1 x_1 + \rho_2 x_2 + \rho_3 x_3, 0.5x_1 + 2x_2 + x_3, 0.5x_1 + 2x_2 + x_3)_{LR}, \\
& c_1 \tilde{\bar{\xi}}_4 x_1 + c_2 \tilde{\bar{\xi}}_5 x_2 + c_3 \tilde{\bar{\xi}}_6 x_3 \\
= & (\rho_4 c_1 x_1 + \rho_5 c_2 x_2 + \rho_6 c_3 x_3, 1.2x_1 + 0.4x_2 + 0.75x_3, \\
& 1.2x_1 + 0.4x_2 + 0.75x_3)_{LR}.
\end{array}
\tag{5.88}
$$

and

$$\begin{aligned}
\rho_1 x_1 &= ([x_1, 2x_1], [0, 3x_1]),\\
\rho_2 x_2 &= ([2x_2, 3x_2], [x_2, 4x_2]),\\
\rho_3 x_3 &= ([3x_3, 4x_3], [2x_3, 5x_3]),\\
\rho_4 c_1 x_1 &= ([0, c_1 x_1], [0, 3c_1 x_1]),\\
\rho_5 c_2 x_2 &= ([c_2 x_2, 2c_2 x_2], [0, 3c_2 x_2]),\\
\rho_6 c_3 x_3 &= ([2c_3 x_3, 3c_3 x_3], [0, 3c_3 x_3]),
\end{aligned}$$

and

$$\begin{aligned}
&\rho_1 x_1 + \rho_2 x_2 + \rho_3 x_3\\
= &([x_1 + 2x_2 + 3x_3, 2x_1 + 3x_2 + 4x_3], [x_2 + 2x_3, 3x_1 + 4x_2 + 5x_3])\\
&\rho_4 c_1 x_1 + \rho_5 c_2 x_2 + \rho_6 c_3 x_3\\
= &([0.8x_2 + 3x_3, 1.2x_1 + 1.6x_2 + 4.5x_3], [0, 3.6x_1 + 2.4x_2 + 4.5x_3]).
\end{aligned}$$

Here we consider the case when $b \le f_i - R^{-1}(\delta_i)\beta_i^{cT} x \le d$, and readers can try another three cases through the following method.

According to Propositions 5.7 and 5.8, the problem (5.63) is equivalent to the following multi-objective programming problem,

$$\begin{cases}
\max\ [\delta_1, \delta_2]\\
\text{s.t.} \begin{cases}
R^{-1}(\delta_1) \le \frac{-\bar{f}_1 + 3x_1 + 4x_2 + 5x_3 - 2\gamma_1(3x_1 + 3x_2 + 3x_3)}{0.5x_1 + 2x_2 + x_3}\\
R^{-1}(\delta_2) \le \frac{-\bar{f}_2 + 3.6x_1 + 2.4x_2 + 4.5x_3 - 2\gamma_2(3.6x_1 + 2.4x_2 + 4.5x_3)}{1.2x_1 + 0.4x_2 + 0.75x_3}\\
x_1 + x_2 + x_3 \le 250\\
x_1 + x_2 + x_3 \ge 200\\
x_1 + 4x_2 + 2x_3 \le 600\\
x_1, x_2, x_3 \ge 0.
\end{cases}
\end{cases} \tag{5.89}$$

Since reference function $R(\cdot)$ is non-increasing continuous function, so $\max \delta_i$ is equal to $\min R^{-1}(\delta_i)$, and it is equal to $\min -R^{-1}(\delta_i)$ or model (5.90),

$$\begin{cases}
\max\ [F_1(x), F_2(x)]\\
\text{s.t.} \begin{cases}
x_1 + x_2 + x_3 \le 250\\
x_1 + x_2 + x_3 \ge 200\\
x_1 + 4x_2 + 2x_3 \le 600\\
x_1, x_2, x_3 \ge 0,
\end{cases}
\end{cases} \tag{5.90}$$

where

$$\begin{aligned}
F_1(x) &:= \frac{-\bar{f}_1 + 3x_1 + 4x_2 + 5x_3 - 2\gamma_1(3x_1 + 3x_2 + 3x_3)}{0.5x_1 + 2x_2 + x_3},\\
F_2(x) &:= \frac{-\bar{f}_2 + 3.6x_1 + 2.4x_2 + 4.5x_3 - 2\gamma_2(3.6x_1 + 2.4x_2 + 4.5x_3)}{1.2x_1 + 0.4x_2 + 0.75x_3}.
\end{aligned}$$

H_i^0 and $H_i^1 (i = 1, 2)$ are calculated as follows:

$$H_1^1 = -599.91, \quad H_1^0 = -850.83, \quad H_2^1 = -751.47, \quad H_2^0 = -1126.83.$$

Then we can use the ε-constraint method to solve it. We suppose that the first objective F_1 is the main objective, and we set $\varepsilon_2 = 4$, $\bar{f}_1 = -800$, $\bar{f}_2 = -1000$, and $\gamma_1 = \gamma_2 = 0.9$. We can get the following model (5.91),

$$\begin{cases} \max\ F_1(x) \\ \text{s.t.} \begin{cases} F_2(x) \geq \varepsilon_2 \\ x_1 + x_2 + x_3 \leq 250 \\ x_1 + x_2 + x_3 \geq 200 \\ x_1 + 4x_2 + 2x_3 \leq 600 \\ x_1, x_2, x_3 \geq 0. \end{cases} \end{cases} \tag{5.91}$$

After calculating the model (5.91), we obtain the follow efficient solution

$$(x_1, x_2, x_3) = (26.09, 113.04, 60.87).$$

5.5.3 Non-linear Fu-Ro DCM and Fu-Ro Simulation Based Parallel TS

For the Fu-Ro DCM, we adopted the Fu-Ro simulation 3 based parallel TS to solve.

5.5.3.1 Fu-Ro Simulation 3 for Chance

First we introduce the simulation for α-chance of Fu-Ro variables which is very important in Fu-Ro DCM.

Suppose that ξ is an n-dimensional fuzzy rough vector defined on the rough space $(\Lambda, \Delta, \mathscr{A}, \pi)$, and $f : \mathbf{R}^n \to \mathbf{R}^m$ is a measurable function. For any real number $\alpha \in (0,1]$, we design a fuzzy rough simulation to compute the α-chance $Ch\{f(\xi) \leq 0\}(\alpha)$. That is, we should find the supremum $\bar{\beta}$ such that

$$Appr\{\lambda \in \Lambda | Cr\{f(\xi(\lambda)) \leq 0\} \geq \bar{\beta}\} \geq \alpha. \tag{5.92}$$

We sample $\underline{\lambda}_1, \underline{\lambda}_2, \cdots, \underline{\lambda}_N$ from $\triangle$ and $\overline{\lambda}_1, \overline{\lambda}_2, \cdots, \overline{\lambda}_N$ from Λ according to the measure π. For any number v, let $\underline{N}(v)$ denote the number of $\underline{\lambda}_k$ satisfying $Cr\{f(\xi(\underline{\lambda}_k)) \leq 0\} \geq v$ for $k = 1, 2, \cdots, N$, and $\overline{N}(v)$ denote the number of $\overline{\lambda}_k$ satisfying

$$Cr\{f(\xi(\overline{\lambda}_k)) \leq 0\} \geq v, \tag{5.93}$$

for $k = 1, 2, \cdots, N$, where $Cr\{\cdot\}$ may be estimated by fuzzy simulation. Then we may find the maximal value v such that

$$\frac{\underline{N}(v) + \overline{N}(v)}{2N} \geq \alpha. \tag{5.94}$$

This value is an estimation of $\bar{\beta}$.

The procedure is as follows:
Step 1. Generate $\underline{\underline{\lambda}}_1, \underline{\underline{\lambda}}_2, \cdots, \underline{\underline{\lambda}}_N$ from $\triangle$ according to the measure π.
Step 2. Generate $\overline{\overline{\lambda}}_1, \overline{\overline{\lambda}}_2, \cdots, \overline{\overline{\lambda}}_N$ from Λ according to the measure π.
Step 3. Find the maximal value v such that (5.94) holds.
Step 4. Return v.

Example 5.9. Suppose the Fu-Ro variables ξ_1 and ξ_2 are defined as follows:

$$\xi_1 = (\rho_1, \rho_1 + 1, \rho_1 + 2), \text{ with } \rho_1 = ([1,2],[0,3]),$$
$$\xi_1 = (\rho_2, \rho_2 + 1, \rho_2 + 2), \text{ with } \rho_1 = ([2,3],[1,4]).$$

After a run of Fu-Ro simulation 3 with 5000 cycles, we get that

$$Ch\{\xi_1 + \xi_2\}(0.9) = 0.72.$$

5.5.3.2 Parallel TS

We introduce the parallel tabu search algorithm to solve the multi-objective problem.

TS is an efficient tool to solve the multi-objective problems. However, as the problem size gets larger, TS has some drawbacks:

(a) TS needs to compute the objective function for solution candidates in the neighborhood around a solution at each iteration. The calculation is very time consuming in large-scale problems. The large size problem often gives a large neighborhood even though the neighborhood is defined as a set of solution candidates with the Hamming distance equal to 1.

(b) The complicated non-linear optimal problem has many local minima in large scale problems. That implies that one-point search does not give satisfactory solutions due to the huge search space. Complicated optimal problems require solution diversity.

In this section, the decomposition of the neighborhood accommodates drawback. The neighborhood is decomposed into several sub-neighborhoods. A processor may be assigned to each sub-neighborhood so that the best solution candidate is selected independently in each sub-neighborhood. After selecting the best solution in each sub-neighborhood, the best solution is eventually selected from the best solutions in the sub-neighborhood. Also, the multiple Tabu lengths is proposed to deal with the multi-objective problem with fuzzy rough parameters. TS itself has only one Tabu length. Moreover, it is important to find out better solutions from different directions rather than from only one direction for a longer period. Namely it is effective to make the solution search process more diverse.

Many classifications of parallel TS algorithms have been proposed [326, 327]. They are based on many criteria: number of initial solutions, identical or different parameter settings, control and communications strategies. We have identified two main categories (Figure 5.12).

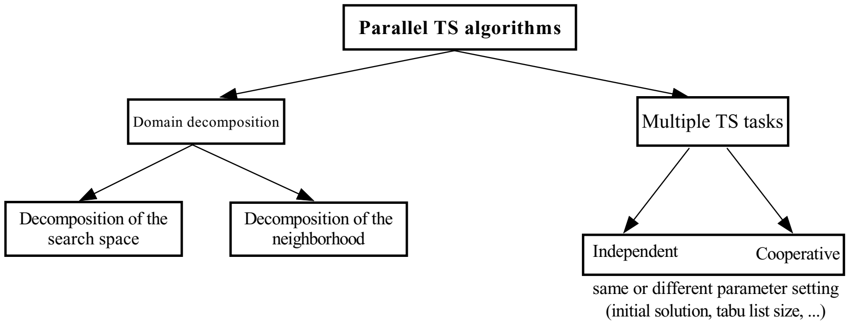

Fig. 5.12 Hierarchical classification of parallel TS strategies.

Domain decomposition: Parallelism in this class of algorithms relies exclusively on:

(1) **The decomposition of the search space:** the main problem is decomposed into a number of smaller subproblems, each subproblem being solved by a different TS algorithm [328].

(2)**The decomposition of the neighborhood:** the search for the best neighbor at each iteration is performed in parallel, and each task evaluates a different subset of the partitioned neighborhood [329, 330].

A high degree of synchronisation is required to implement this class of algorithms.

Multiple tabu search tasks: This class of algorithms consists in executing multiple TS algorithms in parallel. The di.erent TS tasks start with the same or di.erent parameter values (initial solution, tabu list size, maximum number of iterations, etc.). Tabu tasks may be independent (without communication)[331, 332] or cooperative. A cooperative algorithm has been proposed in [327], where each task performs a given number of iterations, then broadcasts the best solution. The best of all solutions becomes the initial solution for the next phase.

Parallelizing the exploration of the search space or the neighborhood is problem-dependent. This assumption is strong and is met only for few problems. The second class of algorithms is less restrictive and then more general. A parallel algorithm that combines the two approaches (two-level parallel organization) has been proposed in [333].

We can extend this classification by introducing a new taxonomy dimension: the way scheduling of tasks over processors is done. Parallel TS algorithms fall into three categories depending on whether the number and/or the location of work (tasks, data) depend or not on the load state of the parallel machine (Table 5.5):

Non-adaptive: This category represents parallel TS in which both the number of tasks of the application and the location of work (tasks or data) are generated at compile time (static scheduling). The allocation of processors to tasks (or data)

Table 5.5 Another taxonomy dimension for parallel TS algorithms

	Tasks or Data	
	Number	Location
Non-adaptive	Static	Static
Semi-adaptive	Static	Dynamic
Adaptive	Dynamic	Dynamic

remains unchanged during the execution of the application regardless of the current state of the parallel machine. Most of the proposed algorithms belong to this class.

An example of such an approach is presented in [334]. The neighborhood is partitioned in equal size partitions depending on the number of workers, which is equal to the number of processors of the parallel machine. In [330], the number of tasks generated depends on the size of the problem and is equal to n^2, where n is the problem size.

When there are noticeable load or power differences between processors, the search time of the non-adaptive approach presented is derived by the maximum execution time over all processors (highly loaded processor or the least powerful processor). A significant number of tasks are often idle waiting for other tasks to complete their work.

Semi-adaptive: To improve the performance of the parallel non adaptive TS algorithms, dynamic load balancing must be introduced [333, 334]. This class represents applications for which the number of tasks is fixed at compile-time, but the locations of work (tasks, data) are determined and/or changed at run-time (as seen in Table 5.5). Load balancing requirements are met in [334] by a dynamic redistribution of work between processors. During the search, each time a task finishes its work, it proceeds to a work-demand. Dynamic load balancing through partition of the neighborhood is done by migrating data.

However, the parallelism degree in this class of algorithms is not related to load variation in the parallel system: when the number of tasks exceeds the number of idle nodes, multiple tasks are assigned to the same node. Moreover, when there are more idle nodes than tasks, some of them will not be used.

Adaptive: A parallel adaptive program refers to a parallel computation with a dynamically changing set of tasks. Tasks may be created or killed function of the load state of the parallel machine. Di.erent types of load state dessimination schemes may be used [337]. A task is created automatically when a processor becomes idle. When a processor becomes busy, the task is killed. Next, let's introduce the design about the parallel adaptive TS introduced by Talbi [338].

The programming style used is the master/workers paradigm. The master task generates work to be processed by the workers. Each worker task receives a work from the master, computes a result and sends it back to the master. The master/ workers paradigm works well in adaptive dynamic environments because:

(1) when a new node becomes available, a worker task can be started there,
(2) when a node becomes busy, the master task gets back the pending work which was being computed on this node, to be computed on the next available node.

The master implements a central memory through which passes all communication, and that captures the global knowledge acquired during the search. The number of workers created initially by the master is equal to the number of idle nodes in the parallel platform. Each worker implements a sequential TS task. The initial solution is generated randomly and the tabu list is empty. The parallel adaptive TS algorithm reacts to two events (Figure 5.13):

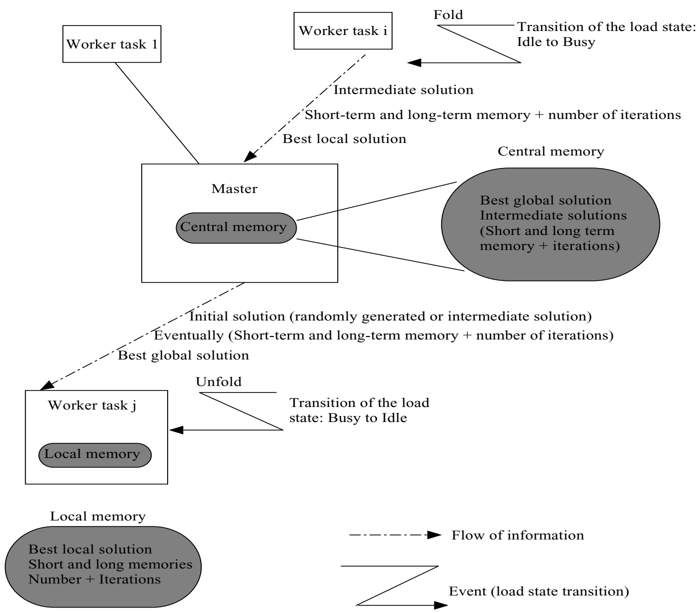

Fig. 5.13 Architecture of the parallel adaptive TS.

Appransition of the load state of a node from idle to busy: If a node hosting a worker becomes loaded, the master folds up the application by withdrawing the worker. The concerned worker puts back all pending work to the master and dies. The pending work is composed of the current solution, the best local solution found, the short-term memory, the long-term memory and the number of iterations done without improving the best solution. The master updates the best global solution if it's worst than the best local solution received.

Appransition of the load state of a node from busy to idle: When a node becomes available, the master unfolds the application by starting a new worker on it. Before starting a sequential TS, the worker task gets the values of the different parameters from the master: the best global solution and an initial solution which may be an intermediate solution found by a folded TS task, which constitutes a "good" initial solution. In this case, the worker receives also the state of the short-term memory, the long-term memory and the number of iterations done without improving the best solution.

The local memory of each TS task which defines the pending work is composed of (Figure 5.13): the best solution found by the task, the number of iterations applied, the intermediate solution and the adaptive memory of the search (short-term and long-term memories). The central memory in the master is then composed of (Figure 5.13): the best global solution found by all TS tasks, the dierent intermediate solutions with the associated number of iterations and adaptive memory.

5.5.3.3 Numerical Example

Example 5.10. Consider the following problem,

$$\begin{cases} \max f_1(x) = Ch\{\tilde{\tilde{\xi}}_1 x_1 + \tilde{\tilde{\xi}}_2 x_2 + \tilde{\tilde{\xi}}_3 x_3 \geq f_1\}(\alpha) \\ \max f_2(x) = Ch\{\sqrt{c_1 \tilde{\tilde{\xi}}_4 x_1 + c_2 \tilde{\tilde{\xi}}_5 x_2 + c_3 \tilde{\tilde{\xi}}_6 x_3 \geq f_2}\}(\beta) \\ \text{s.t.} \begin{cases} x_1 + x_2 + x_3 \leq 250 \\ x_1 + x_2 + x_3 \geq 200 \\ x_1 + 4x_2 + 2x_3 \leq 600 \\ x_1 \geq 20, x_2 \geq 20, x_3 \geq 20, \end{cases} \end{cases}$$

where $c = (c_1, c_2, c_3) = (1.2, 0.8, 1.5)$,

$$\begin{aligned} \tilde{\tilde{\xi}}_1 &= (\rho_1, 1, 1), \text{ with } \rho_1 \vdash ([1,2],[0,3]), \\ \tilde{\tilde{\xi}}_2 &= (\rho_2, 1, 1), \text{ with } \rho_2 \vdash ([2,3],[1,4]), \\ \tilde{\tilde{\xi}}_3 &= (\rho_3, 1, 1), \text{ with } \rho_3 \vdash ([3,4],[2,5]), \\ \tilde{\tilde{\xi}}_4 &= (\rho_4, 1, 1), \text{ with } \rho_4 \vdash ([0,1],[0,3]), \\ \tilde{\tilde{\xi}}_5 &= (\rho_5, 1, 1), \text{ with } \rho_5 \vdash ([1,2],[0,3]), \\ \tilde{\tilde{\xi}}_6 &= (\rho_6, 1, 1), \text{ with } \rho_6 \vdash ([2,3],[0,3]), \end{aligned}$$

and $\tilde{\tilde{\xi}}_i (i = 1, 2, \cdots, 6)$ are Fu-Ro variables. We set $\alpha = \beta = 0.9$, $f_1 = 1500$, and $f_2 = 1300$.

Next, we apply the tabu search algorithm based on the fuzzy rough simulation to solve the nonlinear programming problem (5.10) with the fuzzy rough parameters.

Step 1. Set the move step $h = 0.5$ and the h neighbor $N(x,h)$ for the present point x is defined as follows,

$$N(x,h) = \left\{ \mathbf{y} | \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2} \leq h \right\}.$$

The random move of point x to point $\mathbf{y}$ in its h neighbor along direction s is given by

$$\mathbf{y}_s = x_s + rh,$$

where r is a random number that belongs to [0,1], $s = 1, 2, 3$.

Step 2. Give the step set $H = \{h_1, h_2, \cdots, h_r\}$ and randomly generate a feasible point $x_0 \in X$. One should empty the Tabu list T (the list of inactive steps) at the beginning.

Step 3. For each active neighbor $N(x,h)$ of the present point x, where $h \in H - T$, a feasible random move that satisfies all the constraints in problem (5.10) is to be generated.

Step 4. Construct the single objective function as follows,

$$f(x,\xi) = w_1 Ch\{\tilde{\bar{\xi}}_1 x_1 + \tilde{\bar{\xi}}_2 x_2 + \tilde{\bar{\xi}}_3 x_3 \geq f_1\}(\alpha)$$
$$+w_2 Ch\{c_1 \tilde{\bar{\xi}}_4 x_1 + c_2 \tilde{\bar{\xi}}_5 x_2 + c_3 \tilde{\bar{\xi}}_6 x_3 \geq f_2\}(\beta),$$

where $w_1 + w_2 = 1$. Compare the $f(x,\xi)$ of the feasible moves with that of the current solution by the fuzzy rough simulation. If an augmenter in new objective function of the feasible moves exists, one should save this feasible move as the updated current one by adding the corresponding step to the Tabu list T and go to the next step; otherwise, go to the next step directly.

Step 5. Stop if the termination criteria are satisfied; other wise, empty T if it is full; then go to Step 3. Here, we set the computation is determined if the better solution doesn't change again.

Table 5.6 The result computed by parametric TS algorithm

ω_1	ω_2	ω_3	ω_4	ω_5	x_1	x_2	x_3	x_4	x_5
0.40	0.15	0.15	0.15	0.15	50.48	59.14	80.17	50.12	60.00
0.15	0.40	0.15	0.15	0.15	50.47	59.15	80.17	50.12	60.00
0.15	0.15	0.40	0.15	0.15	50.47	59.14	80.18	50.12	60.00
0.15	0.15	0.15	0.40	0.15	50.47	59.14	80.17	50.13	60.00
0.15	0.15	0.15	0.15	0.40	50.48	59.14	80.17	50.12	60.00

5.6 Application to Integrated Logistics Network Design Problem

Here we consider the problem proposed in section 5.1, and we consider the demand and the amount of the collected recycling packages as triangular fuzzy variables $(\xi - l, \xi, \xi + r)$ from the view point of credibility theory, in which the value of ξ is a rough variable $([a,b],[c,d])$, and l, r are the left spread and the right spread of the triangular fuzzy variable. Therefore, a logistics network design problem with fuzzy rough parameters appears. In this case, a fuzzy rough variable can be used to deal with this kind of combined uncertainty of fuzziness and roughness. Building the model and solving the problem of logistics network design in a fuzzy rough environment is a new area of research interest.

5.6.1 Modelling for Integrated Reuse Logistics Network under Fuzzy Rough Environment

In the following text of this section, we present the details of modelling for the reuse integrated logistics network.

5.6.1.1 Notation

The symbols of the proposed model are defined as follows:

(1) Indices:

i: the location of producers ($i = 1,2,\cdots,I$),
j: the location of distributors ($i = 1,2,\cdots,I$),
k: the location of collectors/wholesalers ($i = 1,2,\cdots,I$),
t: the alternative place of recyclers ($i = 1,2,\cdots,I$).

(2) Variables:

x_{ij}^{PD}: the quantity of products from producer i to distributor j,
x_{jk}^{DC}: the quantity of products from distributor j to wholesaler k,
x_{kt}^{CR}: the quantity of packages from collector k to recycler t,
x_{kj}^{CD}: the quantity of packages from collector k to distributor j,
x_{ti}^{RP}: the quantity of packages from recycler t to producer i,
x_{ji}^{DP}: the quantity of packages from distributor j to producer i,
x_i: the quantity of new packages bought by producer i,
y_t^R: 0-1 variable, whether the alternative recycler t will be chosen or not, 0 denotes we don't choose, 1 denotes we choose it,
y_j^D: 0-1 variable, whether the distributor j will be expanded or not, 0 denotes we don't expand, 1 denotes we expand it,
y_{jk}^{DC}: 0-1 variable, whether the distributor j will send products to wholesaler k, 0 denotes will not send, 1 denotes will send.

(3) Fu-Ro parameters:

$\tilde{\bar{D}}_k$ denote the demand of wholesaler K,
$\tilde{\bar{R}}_k$ denote the quantity of the recycling packages collected by collector k,
$\tilde{\bar{T}}_k^{Lim}$: the time limit of wholesaler k.

(4) Certain parameters:

C_{ab}: the unit transport cost from a to b (a and b could denote producer, distributor, recycling center or wholesaler),
T_{ab}: the transport time from a to b,
V_t^R: the variable cost of recycler t processing unit package,
V_j^D: the variable cost of expanded distributor j processing unit package,
T_t^R: the time of recycler t processing unit package,
T_j^D: the time of expanded distributor j processing unit package. F_t^R: the fixed cost of building a recycler t,
F_j^D: the fixed cost of expanding a distributor j,
Q_t^R: the capacity of recycler t processing packages,
Q_j^D: the capacity of expanded distributor j processing packages,

α_t^R: the discard proportion after recycler t processing packages,
α_j^D: the discard proportion after expanded distributor j processing packages,
N^R: the ceiling number of recyclers,
N^D: the ceiling number of expanded distributors,
U_k^l: the unit default cost when the demand of wholesaler k are not met,
U_k^e: the processing cost when the supply to wholesaler k are excessive,
P_i: the variable cost of producer i buying unit package,
U_k^D: the disposal cost when the time limits are not satisfied.

5.6.1.2 Modelling

We built the following mathematical model according to the conceptual model.

The first objective is minimizing total costs. After analysis, we conclude that there are six parts which should be included in this objective, as follows: The first part of the objective is the total transportation cost,

$$\begin{aligned}&[\sum_{i\in I}\sum_{j\in J} x_{ij}^{PD}\tilde{\bar{C}}_{ij} + \sum_{j\in J}\sum_{k\in K} x_{jk}^{DC}\tilde{\bar{C}}_{jk} + \sum_{k\in K}\sum_{t\in T} x_{kt}^{CR}\tilde{\bar{C}}_{kt}\\ &+ \sum_{t\in T}\sum_{i\in I} x_{ti}^{RP}\tilde{\bar{C}}_{ti} + \sum_{k\in K}\sum_{j\in J} x_{kj}^{CD}\tilde{\bar{C}}_{kj} + \sum_{j\in J}\sum_{i\in I} x_{ji}^{DP}\tilde{\bar{C}}_{ji}]_1.\end{aligned}$$

The second part is the total fixed cost of building recycling centers and expanding the distribution centers,

$$[\sum_{t\in T} y_t^R F_t^R + \sum_{j\in J} y_j^D F_j^D]_2.$$

The third part is the total variable cost of processing packages,

$$[\sum_{k\in K}\sum_{t\in T} x_{kt}^{CR} V_t^R + \sum_{k\in K}\sum_{j\in J} x_{kj}^{CD} V_j^D]_3.$$

The fourth part is the cost of buying new packages,

$$[\sum_{j\in J} x_i P_i]_4.$$

The fifth part is the cost when there exists an imbalance between supply and demand, when the supply is less than the demand, there will occur default costs, or when the supply is more than the demand, the redundant products will be processed at a cost,

$$[\sum_{k\in K} U_k^l \max(\tilde{\bar{D}}_k - \sum_{j\in J} x_{jk}^{DC}, 0) + \sum_{k\in K} U_k^e \max(\sum_{j\in J} x_{jk}^{DC} - \tilde{\bar{D}}_k, 0)]_5.$$

The sixth part is the cost of disposing of the un-useable packages,

$$[\sum_{k\in K}\sum_{t\in T} x_{kt}^{CR}\alpha_t^R U^P + \sum_{k\in K}\sum_{j\in J} x_{kj}^{CD}\alpha_j^D U^P]_6.$$

However, minimizing total costs is not the only objective of a logistics company. Shortening the time taken in the distribution and recycling is also required. Hence the second objective is to minimize total time.

$$TotalTime = \sum_{t\in T}\sum_{k\in K}\bar{\tilde{T}}_{kt}y_t^R + \sum_{t\in T}(T_t^R\sum_{k\in K}x_{kt}^{CR}) + \sum_{i\in I}\sum_{t\in T}\bar{\tilde{T}}_{ti}y_t^R$$
$$\sum_{j\in J}\sum_{k\in K}\bar{\tilde{T}}_{kj}y_j^D + \sum_{j\in J}(T_j^D\sum_{k\in K}x_{kj}^{CD}) + \sum_{i\in I}\sum_{j\in J}\bar{\tilde{T}}_{ji}y_j^D.$$

Now we can obtain the objectives function as shown in (5.95):

$$\begin{aligned}
&\min C = \\
&\sum_{i\in I}\sum_{j\in J}x_{ij}^{PD}\bar{\tilde{C}}_{ij} + \sum_{j\in J}\sum_{k\in K}x_{jk}^{DC}\bar{\tilde{C}}_{jk} + \sum_{k\in K}\sum_{t\in T}x_{kt}^{CR}\bar{\tilde{C}}_{kt} + \sum_{t\in T}\sum_{i\in I}x_{ti}^{RP}\bar{\tilde{C}}_{ti} \\
&+ \sum_{k\in K}\sum_{j\in J}x_{kj}^{CD}\bar{\tilde{C}}_{kj} + \sum_{j\in J}\sum_{i\in I}x_{ji}^{DP}\bar{\tilde{C}}_{ji} + \sum_{t\in T}y_t^R F_t^R + \sum_{j\in J}y_j^D F_j^D + \sum_{k\in K}\sum_{t\in T}x_{kt}^{CR}V_t^R \\
&+ \sum_{k\in K}\sum_{j\in J}x_{kj}^{CD}V_j^D + \sum_{j\in J}x_iP_i + \sum_{k\in K}U_k^l\max(\bar{\tilde{D}}_k - \sum_{j\in J}x_{jk}^{DC}, 0) \\
&+ \sum_{k\in K}U_k^e\max(\sum_{j\in J}x_{jk}^{DC} - \bar{\tilde{D}}_k, 0) + \sum_{k\in K}\sum_{t\in T}x_{kt}^{CR}\alpha_t^R U^P + \sum_{k\in K}\sum_{j\in J}x_{kj}^{CD}\alpha_j^D U^P, \\
&\min T = \\
&\sum_{t\in T}\sum_{k\in K}\bar{\tilde{T}}_{kt}y_t^R + \sum_{t\in T}(T_t^R\sum_{k\in K}x_{kt}^{CR}) + \sum_{i\in I}\sum_{t\in T}\bar{\tilde{T}}_{ti}y_t^R \\
&\sum_{j\in J}\sum_{k\in K}\bar{\tilde{T}}_{kj}y_j^D + \sum_{j\in J}(T_j^D\sum_{k\in K}x_{kj}^{CD}) + \sum_{i\in I}\sum_{j\in J}\bar{\tilde{T}}_{ji}y_j^D.
\end{aligned} \tag{5.95}$$

These two objectives are subject to the following constraints.

(1) Balance constraints:

For every node in Figure 2, the inflow and the outflow must be balanced, such that the total recycling quantity of the recycling centers and expanded distribution centers should be less than or equal to the quantity of used package collected by the collectors. Also the quantity discarded from the recycling center (expanded distribution center) to the disposal place should be less than or equal to the quantity discarded of the recycling center (expanded distribution center), and the total quantity of bottles including recycled bottles and new bottles should be used to produce new products. For one distribution center, the inflow should be equal to the outflow, so we have the following (5.96-5.100) constraints,

$$\sum_{t\in T}x_{kt}^{CR} + \sum_{j\in J}x_{kj}^{CD} \le \bar{\tilde{R}}_k, k\in K, \tag{5.96}$$

$$\sum_{i\in I}x_{ti}^{RP} \le (1-\alpha_t^R)\sum_{k\in K}x_{kt}^{CR}, t\in T, \tag{5.97}$$

$$\sum_{i\in I}x_{ji}^{DP} \le (1-\alpha_j^D)\sum_{k\in K}x_{kj}^{CD}, j\in J, \tag{5.98}$$

$$\sum_{t\in T}x_{ti}^{RP} + \sum_{j\in J}x_{ji}^{DP} + x_i = \sum_{j\in J}x_{ij}^{PD}, i\in I, \tag{5.99}$$

$$\sum_{j\in J} x_{ij}^{PD} = \sum_{k\in K} x_{jk}^{DC}, j\in J. \tag{5.100}$$

(2) Capacity constraints:
There are some limits on capacity of the recycling centers and the expanded distribution centers, so we have constraint (5.101) and (5.102),

$$\sum_{k\in K} x_{kt}^{CR} \le y_t^R Q_t^R, t\in T, \tag{5.101}$$

$$\sum_{k\in K} x_{kj}^{CD} \le y_j^D Q_j^D, j\in J. \tag{5.102}$$

(3) Number constraints:
Before setting up a network, because of capital or other reasons, the decision maker will give the numbers of recycling centers and expanded distribution centers, so we have the following (5.103) and (5.104) constraints,

$$\sum_{t\in T} y_t^R \le N^R, \tag{5.103}$$

$$\sum_{j\in J} y_j^D \le N^D. \tag{5.104}$$

(4) Time constraints:
For every wholesaler, the total transport time is required to be under a time limit, so we have constraint (5.105),

$$\sum_{j\in J} \bar{\tilde{T}}_{jk} y_{jk}^{DC} \le T_k^{Lim}, k\in K. \tag{5.105}$$

(5) Logical constraints:
In order to describe some non-negative variables and 0-1 variables in the model, we present constraint (5.106) and (5.107),

$$x_{ij}^{PD}, x_{jk}^{DC}, x_{kt}^{CR}, x_{kj}^{CD}, x_{ti}^{RP}, x_{ji}^{DP}, x_i \ge 0, i\in I, j\in J, k\in K, t\in T, \tag{5.106}$$

$$y_t^R, y_j^D, y_{jk}^{DC} = \{0,1\}, j\in J, k\in K, t\in T. \tag{5.107}$$

5.6.2 Uncertain Linear Multi-objective Model

It's obvious that the above model is non-linear, because the fifth and the seventh part exist in the first objective function. In order to simplify it, we changed it to an uncertain linear multi-objective model by adding the constraints (5.108)-(5.110).

$$e_k^- = \bar{\tilde{D}}_k - \sum_{j\in J} x_{jk}^{DC}, k\in K, \tag{5.108}$$

$$e_k^+ = \sum_{j\in J} x_{jk}^{DC} - \bar{\tilde{D}}_k, k \in K, \tag{5.109}$$

$$e_k^-, e_k^+ \geq 0. \tag{5.110}$$

We proposed the Fu-Ro linear multi-objective model for integrated logistics as follows:

$$\begin{aligned}
&\min C = \\
&\sum_{i\in I}\sum_{j\in J} x_{ij}^{PD}\bar{\tilde{C}}_{ij} + \sum_{j\in J}\sum_{k\in K} x_{jk}^{DC}\bar{\tilde{C}}_{jk} + \sum_{k\in K}\sum_{t\in T} x_{kt}^{CR}\bar{\tilde{C}}_{kt} + \sum_{t\in T}\sum_{i\in I} x_{ti}^{RP}\bar{\tilde{C}}_{ti} + \sum_{k\in K}\sum_{j\in J} x_{kj}^{CD}\bar{\tilde{C}}_{kj} \\
&+ \sum_{j\in J}\sum_{i\in I} x_{ji}^{DP}\bar{\tilde{C}}_{ji} + \sum_{t\in T} y_t^R F_t^R + \sum_{j\in J} y_j^D F_j^D + \sum_{k\in K}\sum_{t\in T} x_{kt}^{CR} V_t^R + \sum_{k\in K}\sum_{j\in J} x_{kj}^{CD} V_j^D + \sum_{j\in J} x_i P_i \\
&+ \sum_{k\in K} U_k^l e_k^- + \sum_{k\in K} U_k^e e_k^+ + \sum_{k\in K}\sum_{t\in T} x_{kt}^{CR}\alpha_t^R U^P + \sum_{k\in K}\sum_{j\in J} x_{kj}^{CD}\alpha_j^D U^P \\
&\min T = \\
&\sum_{t\in T}\sum_{k\in K} \bar{\tilde{T}}_{kt} y_t^R + \sum_{t\in T}(T_t^R \sum_{k\in K} x_{kt}^{CR}) + \sum_{i\in I}\sum_{t\in T} \bar{\tilde{T}}_{ti} y_t^R + \sum_{j\in J}\sum_{k\in K} \bar{\tilde{T}}_{kj} y_j^D + \sum_{j\in J}(T_j^D \sum_{k\in K} x_{kj}^{CD}) \\
&+ \sum_{i\in I}\sum_{j\in J} \bar{\tilde{T}}_{ji} y_j^D \\
&s.t.\begin{cases}
\sum_{t\in T} x_{kt}^{CR} + \sum_{j\in J} x_{kj}^{CD} \leq \bar{\tilde{R}}_k, k \in K \\
\sum_{i\in I} x_{ti}^{RP} \leq (1-\alpha_t^R)\sum_{k\in K} x_{kt}^{CR}, t \in T \\
\sum_{i\in I} x_{ji}^{DP} \leq (1-\alpha_j^D)\sum_{k\in K} x_{kj}^{CD}, j \in J \\
\sum_{t\in T} x_{ti}^{RP} + \sum_{j\in J} x_{ji}^{DP} + x_i = \sum_{j\in J} x_{ij}^{PD}, i \in I \\
\sum_{j\in J} x_{ij}^{PD} = \sum_{k\in K} x_{jk}^{DC}, j \in J \\
\sum_{k\in K} x_{kt}^{CR} \leq y_t^R Q_t^R, t \in T \\
\sum_{k\in K} x_{kj}^{CD} \leq y_j^D Q_j^D, j \in J \\
\sum_{t\in T} y_t^R \leq N^R \\
\sum_{j\in J} y_j^D \leq N^D \\
\sum_{j\in J} \bar{\tilde{T}}_{jk} y_{jk}^{DC} \leq T_k^{Lim}, k \in K \\
e_k^- = \bar{\tilde{D}}_k - \sum_{j\in J} x_{jk}^{DC}, k \in K \\
e_k^+ = \sum_{j\in J} x_{jk}^{DC} - \bar{\tilde{D}}_k, k \in K \\
x_{ij}^{PD}, x_{jk}^{DC}, x_{kt}^{CR}, x_{kj}^{CD}, x_{ti}^{RP}, x_{ji}^{DP}, x_i, e_k^-, e_k^+ \geq 0, i \in I, j \in J, k \in K, t \in T \\
y_t^R, y_j^D, y_{jk}^{DC} = \{0,1\}, j \in J, k \in K, t \in T.
\end{cases}
\end{aligned} \tag{5.111}$$

The model we proposed is actually a Fu-Ro two-objective linear model, and both of the two objectives are needed for optimization. These two objectives are uncomparable, and there exists inconsistency between them. When we want to reduce the transportation time, but have a large number of recycling centers, we could reduce the number of recycling centers, so it will reduce the cost of building these centers, but the transportation time will inevitably rise.

Since the model (5.111) is including Fu-Ro variables, we need to use the Fu-Ro expected value operator to handle the objective functions and Fu-Ro chance-constrained operator to deal with the constraints.

5.6.3 Application to Beer Company

The beer company Lan Ma was set up in the year 2000 and is located in Xi'an in China's Shanxi province and it has developed successfully for the years of its operation. This enterprise has 2 production plants in Xian Yang and 3 distribution centers, each with the responsibility for a section of Shanxi - Guan Zhong, Shan Bei and Shan Nan. There are 5 main wholesalers and they are located in Wei Nan, Shang Luo, Han Zhong, An Kang and Yan An.

This company wants to establish integrated logistics through building up recycling centers or expanding the existing distribution centers, and integrate the forward logistics and reverse logistics to a loop logistics network which has the abilities of production, distribution, recycle and reuse. So we used this model to help the company to program an integrated logistics network.

At present, according to the survey results, there are four options which could be used to establish new recycling centers, and all three existing distribution centers could be expanded. The alternative locations are Zhou Zhi, Pu Cheng, Zha Shui and Hua Xian. The largest processing capacities of these 4 places are 20000, 23000, 15000, and 27000. The fixed construction costs are 12.5, 16.5, 10 and 19.5(*10000RMB). We suppose the discard proportions are all 0.2 and they want to build 3 recycling centers at the most. We also could expand the 3 distribution centers to process the recycled packages, the expanding costs are 6.6, 5.4 and 7(*10000RMB), their capacities are 11000, 9000, and 12000, the discard proportions are all 0.2. The company has requested that we expand 2 at the most. The price of a new bottle is 0.7(RMB). The other data are as follows.

Table 5.7 Amount of recycling and demand

Wholesaler	Recycling amount	Demand
Wein	$(\xi_1, 100, 100)_{LR}$, $\xi_1 \vdash$([8000,10000],[8500,9500])	$(\xi_6, 100, 100)_{LR}$, $\xi_6 \vdash$([10000,12000],[10500,11500])
Shangl	$(\xi_2, 50, 50)_{LR}$, $\xi_2 \vdash$([6000,7000],[6250,6750])	$(\xi_7, 100, 100)_{LR}$, $\xi_7 \vdash$([7000,8000],[7250,7750])
Hanzh	$(\xi_3, 100, 100)_{LR}$, $\xi_3 \vdash$([12000,14000],[12500,13500])	$(\xi_8, 100, 100)_{LR}$, $\xi_8 \vdash$([14000,16000],[14500,15500])
Ank	$(\xi_4, 50, 50)_{LR}$, $\xi_4 \vdash$([10000,11000],[10250,10750])	$(\xi_9, 100, 100)_{LR}$, $\xi_9 \vdash$([11000,13000],[11500,12500])
Yan an	$(\xi_5, 100, 100)_{LR}$, $\xi_5 \vdash$([16000,18000],[16500,17500])	$(\xi_{10}, 100, 100)_{LR}$, $\xi_{10} \vdash$([18000,20000],[18500,19500])

Transport cost, and time are triangular fuzzy numbers with the left and the right spread 0.02 and 0.1, the middle value of the triangular fuzzy variable are rough variables which shown in following Table. 5.9

We introduced the above data of the company into the proposed model, and got the integrated logistics network model for this Lan Ma beer company. After solving it, we can provide some advice to help the leader make strategic decisions about constructing the integrated logistics network system.

Table 5.8 Default processing cost and time limit (h) of every wholesaler

Wholesaler	Wein	Shangl	Hanzh	Ank	Yan an
Cost of short supply	1.2	1	1.1	1	1.5
Cost of excessive supply	1.8	1.9	1.6	1.3	1.5
Time limit	4	4	3.5	4	3.5
Default cost	3000	3500	4000	3000	4500

Table 5.9 The expected value of transport cost, and time (h) from collectors to recyclers

	Wein	Shangl	Hanzh	Ank	Yan an
Zhouzh	0.1 2.2	0.12 3.8	0.1 4.2	0.05 2.7	0.12 2.9
Puch	0.13 2.9	0.15 3.0	0.06 4.0	0.11 4.5	.08 3.3
Zhash	0.11 3.5	0.15 4.5	0.08 2.5	0.13 5.0	0.2 2.9
Huax	0.12 2.8	0.1 3.2	0.19 4.5	0.1 4.0	0.11 3.0

Table 5.10 The expected value of transport cost, and time (h) from collectors to distributors (recycled bottles)

	Wein	Shangl	Hanzh	Ank	Yan an
Guanzh	0.08 3.5	0.15 4.0	0.06 2.5	0.12 2.0	0.10 4.5
Shanb	0.10 2.5	0.08 2.0	0.10 4.5	0.12 3.5	0.08 5.0
Shann	0.11 4.0	0.10 2.5	0.08 3.0	0.13 3.5	0.11 3

Table 5.11 The expected value of transport cost, and time (h) from distributors to wholesalers (products)

	Wein	Shangl	Hanzh	Ank	Yan an
Guanzh	0.23 3.5	0.31 4.0	0.15 2.5	0.17 2.0	0.3 4.5
Shanb	0.17 2.5	0.2 2.0	0.15 4.5	0.18 3.5	0.27 5.0
Shann	0.13 4.0	0.25 2.5	0.22 3.0	0.18 3.5	0.26 3

Table 5.12 Transport cost and time from producers to distributors (recycled bottles)

	Guanzh	Shanb	Shann
Plant 1	0.1 2.5	0.08 1.0	0.15 2.0
Plant 2	0.15 2.0	0.2 2.5	0.08 1.5

Table 5.13 Transport cost and time from distributors to producers (products)

	Guanzh	Shanb	Shann
Plant 1	0.3 2.5	0.25 1.0	0.35 2.0
Plant 2	0.35 2.0	0.4 2.5	0.2 1.5

Table 5.14 The expected value of transport cost from recyclers to producers

	Zhouzh	Puch	Zhash	Huax
Plant 1	0.17 1.5	0.13 2.0	0.12 2.5	0.15 1.5
Plant 2	0.1 2.5	0.16 1.0	0.11 2.0	0.08 2.5

Table 5.15 Processing cost, processing time (s) and disposal cost of recyclers

	Zhouzh	Puch	Zhash	Huax
Processing cost	0.25	0.2	0.23	0.18
Processing time	3.0	2.5	3.5	2.0
Disposal cost	0.2	0.15	0.18	0.13

Table 5.16 The expected value of processing cost, time (s) and disposal cost of distributors

	Guanzh	Shanb	Shann
Processing cost	0.28	0.22	0.25
Processing time	5.0	6.0	4.0
Disposal cost	0.2	0.15	0.18

We use the expected value operator and the chance operator to tackle the fuzzy rough objectives and the fuzzy rough constraint, and used the Fu-Ro simulation-based GA to solve this problem under the predetermined confidence level (0.8, 0.8); the corresponding parameters are 100 genetic generation iteration, the population of every generation is 10, the crossover rate 0.3 and the mutation rate is 0.2.

After a run of a genetic algorithm computer program, we obtained the following satisfactory solution: the optimal value of the objective function is Z*=456253 (RMB), T*=2200.3 (hour) and the value of the corresponding location variables are in Table. 5.17.

Table 5.17 Location decision

Zhouzh	Puch	Zhash	Huax	Guanzh	Shanb	Shann
0	1	1	0	1	1	0

Then we could do some sensitivity analysis: we adjusted the weights of these two objectives, and the solutions of the integrated logistics network problem are shown in Table. 5.18.

It shows that small changes in the weights do not significantly influence the location results, and the result is satisfactory to the decision maker of this company. On all accounts, we offered this strategy for Lan Ma beer company - establish the recycling centers in Pu Cheng and Zha Shui, and expand the Guan Zhong and Shan Bei distribution centers. If we consider a given budget, with regard to the number of recycling centers built and the distribution centers expanded, we make the following observations. First, when the location cost factor increases, i.e., the recycling center

Table 5.18 The results of TS (Appr=0.8, Pos=0.8)

w_C	w_T	C^*	T^*	Zhouzh	Puch	Zhash	Huax	Guanzh	Shanb	Shann
0.7	0.3	455353	2289.5	0	1	1	0	1	1	0
0.6	0.4	456253	2200.3	0	1	1	0	1	1	0
0.5	0.5	456696	2186.3	0	1	1	0	1	1	0

location costs increase relative to other costs, the number of opened recycling centers decreases. Second, when the transportation costs between facilities increase, the number of opened recycling centers also decreases. However, since the total cost and time are often conflicting, the handling of multi-objective programming is dependent on the decision-maker's objective. Generally, the solution to this problem often is a balance of multiple objectives.