Chapter 8
Connection Basins

8.1 Introduction

Until now, we presented and studied evolutions in positive time, or *forward evolutions*, and the associated concepts of (forward) viability kernels and basins. We were looking from the present to the future, without taking into account the past or the history of the evolution. In this chapter, we offer ways to extend this framework by studying evolutions from $-\infty$ to $+\infty$.

In order to achieve this goal, we have to properly define the meaning of an evolution arriving at a final time instead of evolutions starting at AN initial time. In other words, we consider not only the future, as we did until now, but also the past, and *histories*, defined as evolutions in the past.

For that purpose, we shall split in Sect. 8.2, p. 275 the evolution in forward time evolutions and backward time evolutions, or in negative times, going from 0 to $-\infty$. So, in Sect. 8.3, p. 279, we investigate the concept of bilateral viability kernel of an environment, the subset of initial states through which passes at least one evolution viable in this environment.

For instance, up to now, we have only considered *capture basins* of targets C viable in K, which are the subsets of initial states in K from which starts at least one evolution viable in K until it reaches the *target* C in finite time. In Sect. 8.4, p. 284, we shall consider the "backward" case when we consider another subset $B \subset K$, regarded as a *source*, and study the "reachable maps" from the source, subsets of final states in K at which arrives in finite time at least one evolution *viable* in K starting from the *source* B. This leads us to the concepts of reachable maps, detection tubes and Volterra inclusions in Sect. 8.4, p. 284.

Knowing how to arrive from a source and to reach a target, we are led to consider jointly problems when *in the same time* a source B and a target C are given: we shall introduce in Sect. 8.5, p. 291 the *connection basin*, the subset of elements $x \in K$ through which passes at least one viable evolution starting from the source B and arriving in finite time at target C. As a

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particular case, this leads us to consider evolutions connecting in finite time a state y to another state z by viable evolutions. The issue arises to select one such connecting evolution by optimizing an intertemporal criterion, as we did in Chap. 4, p. 125. For instance, in minimal time, this is the brachistochrone problem, or in minimal length, this is the problem of (viable) geodesics. It turns out that such optimal solutions can be obtained following a strategy going back to Eupalinos 2,500 years ago: start at the same time from both the initial and final states until the two evolutions meet in the middle. This is the reason we attribute the name of this genius to the optimization of evolutions connecting two states.

Actually, this is a particular case of the collision problem studied in Sect. 8.6, p. 298. In this case, two evolutions governed by two different evolutionary systems starting from two different point must collide at some future time. The set of pairs of initial states from which start two colliding evolutions is the collision kernel. Knowing it, we can select among the colliding evolutions the ones which optimize an intertemporal criteria.

We studied connection basins from a source to a target, but, if we regard them as two "cells", one initial, the other final, among a sequence of other cells, we investigate in Sect. 8.8, p. 302 how an evolution can visit a sequence of cells in a given order (see *Analyse qualitative*, [85, Dordan]).

We present here the important results of *Donald Saari* dealing with this issue. Actually, once the "visiting kernels" studied, we adapt Saari's theorems to the case of evolutionary systems. Given a finite sequence of cells, given any arbitrary infinite sequence of orders of visits of the cell, under Saari's assumption, one can always find one initial state from which at least one evolution will visit these cells in prescribed order. This is not a very quite and stable situation, which is another mathematical translation of the polysemous word "chaos", in the sense that "everything can happen". One can translate this vague "everything" in the following way: each cell is interpreted as *qualitative cell*. It describes the set of states sharing a given property characterizing this cell. In comparative economics as well as in qualitative physics, the issue is not so much to know precisely one evolution, but rather, to know what would be the qualitative consequences of at least one evolution starting from a given cell. If this cell is a viability kernel, it can be regarded as "qualitative equilibrium", because at least one evolution remains in the cell. Otherwise, outside of its viability kernel, all evolutions will leave this cell to enter another one. This fact means that the first cell is a qualitative cause of the other one. Saari's Theorem states that under its assumptions, whatever the sequence of properties, there is one possibility that each property "implies" the next one, in some weak but rigourously defined sense. Section 8.9, p. 305, explains how the concepts of invariance kernels and capture basin offer a conceptual basis for "non consistent" logics, involving time delays and some indeterminism. This issue is just presented but not developed in this book.

For simplicity, we assumed until now that the systems were time independent. This is not reasonable, and the concepts which we have met all along this book should hold true for time dependent systems, constraints and targets. This is the case, naturally, thanks a well-known standard "trick" allowing us to treat time-dependent evolutionary systems as time-independent ones. It consists in introducing an auxiliary "time variable" whose velocity is equal to one. So, Sect. 8.10, p. 309 is devoted to the implementation of this procedure, especially regarding capture basins and detection tubes.

Section 8.11, p. 316 grasps the following question: how much information about the current state is contained in past measurements of the states when the initial conditions are not accessible, but replaced by some observations on the past. The question boils down to this one: knowing a control system and a tube (obtained, for instance, as the set of states whose measures are at each instant in a set-valued map), can we recover the evolutions governed by this evolutionary system and *satisfying these past observations?* This is a question which motivated the concept of detector in a time dependent context, and which offers, as the Volterra inclusions (see Sect. 8.4.3, p. 289), quite interesting perspectives for future research on evolutions governed by an evolutionary system when the initial state is unknown.

8.2 Past and Future Evolutions

Until now, evolutions $x(\cdot) \in \mathcal{C}(0, +\infty; X)$ were meant to be "future" evolutions starting from $x(0)$ at "present" time 0, regarded as an initial time.

In order to avoid duplicating proofs of results, the idea is to split the "full evolution" $x(\cdot) \in \mathcal{C}(-\infty, +\infty; X)$ into two (future) evolutions:

1. the *"backward part"* $\overleftarrow{x}(\cdot) \in \mathcal{C}(0, +\infty; X)$ defined by
$$\forall\, t \geq 0, \quad \overleftarrow{x}(t) := x(-t) \in$$
2. the *"forward part"* $\overrightarrow{x}(\cdot) \in \mathcal{C}(0, +\infty; X)$ defined by
$$\forall\, t \geq 0, \quad \overrightarrow{x}(t) := x(t)$$

both defined on positive times. Observe that, for negative times,

$$\forall\, t \leq 0, \quad x(t) \; = \; \overrightarrow{x}(-t) \tag{8.1}$$

Conversely, knowing the forward part $\overrightarrow{x}(\cdot)$ and backward part $\overleftarrow{x}(\cdot)$ of a full evolution, we recover it by formula

$$x(t) \; := \begin{cases} \overleftarrow{x}(-t) \text{ if } t \leq 0 \\ \overrightarrow{x}(+t) \text{ if } t \geq 0 \end{cases} \tag{8.2}$$

The symmetry operation $x(\cdot) \mapsto \varsigma(x(\cdot))(\cdot)$ defined by

$$\varsigma(x(\cdot))(t) = x(-t)$$

is a bijection between the spaces $\mathcal{C}(0,+\infty;X)$ of evolutions and $\mathcal{C}(-\infty,0;X)$ of histories, as well as a bijection $x(\cdot) \in \mathcal{C}(-\infty,+\infty;X) \mapsto \varsigma(x(\cdot))(\cdot) \in \mathcal{C}(-\infty,+\infty;X)$. It is obviously idempotent: $\varsigma(\varsigma(x(\cdot)))(\cdot) = x(\cdot)$. Note that the symmetry operation is also denoted by $\overset{\vee}{x}(\cdot) := \varsigma(x(\cdot))(\cdot)$.

Definition 8.2.1 *[**Histories, Past and Future**] Functions* $x(\cdot) \in \mathcal{C}(-\infty,+\infty;X)$ *are called* "full evolutions"*. We reserve the term of (future)* "evolutions" *for functions* $x(\cdot) \in \mathcal{C}(0,+\infty;X)$*. The space* $\mathcal{C}(-\infty,0;X)$ *is the space of* "histories"*.* The history of a full evolution is the symmetry of its backward part and the backward part is the symmetry of its history.

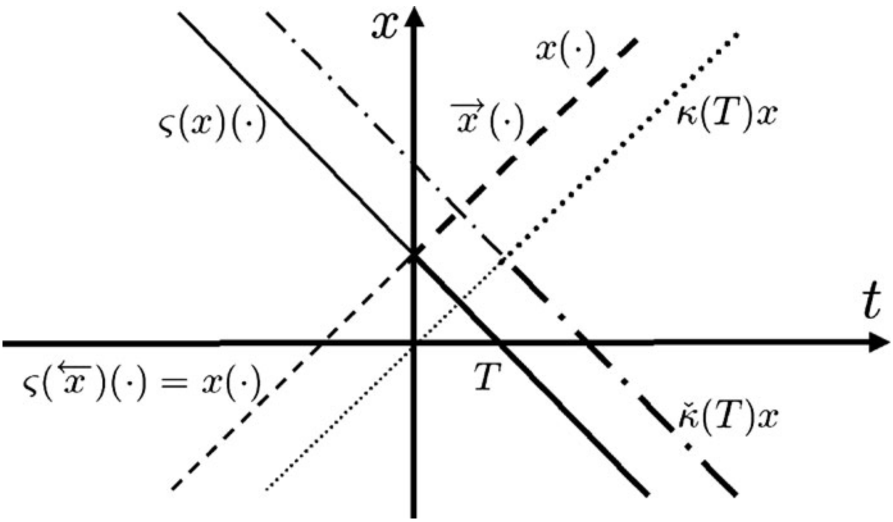

Fig. 8.1 Operations on Evolutions.
Symmetry and decomposition of a full evolution in the its backward and forward parts.

Recall that the translation $\kappa(T)x(\cdot) : \mathcal{C}(-\infty,+\infty;X) \mapsto \mathcal{C}(-\infty,+\infty;X)$ of an evolution $x(\cdot)$ is defined by $(\kappa(T)x(\cdot))(t) := x(t-T)$ (see Definition 2.8.1, p. 69). It is a translation to the right if T is positive and to the left if T is negative, satisfying $\kappa(T+S) = \kappa(T) \circ \kappa(S) = \kappa(S) \circ \kappa(T)$.

Definition 8.2.2 *[**Backward Shift of an Evolution**] The T*-backward shift operator $\overset{\vee}{\kappa}(T)$ *associates with any evolution* $x(\cdot)$ *its T*-backward shift evolution $\overset{\vee}{\kappa}(T)(x(\cdot))$ *defined by*

$$\forall t \in \mathbb{R},\ \overset{\vee}{\kappa}(T)(x(\cdot))(t) := x(T-t) \tag{8.3}$$

It is easy to observe that the operator $\overset{\vee}{\kappa}(T)$ the operator is idempotent:

$$\forall x(\cdot) \in \mathcal{C}(-\infty,+\infty;X),\ (\overset{\vee}{\kappa}(T)(\overset{\vee}{\kappa}(T)x(\cdot))) = x(\cdot)$$

and that $\overset{\vee}{\kappa}(T) := \kappa(T) \circ \varsigma = \varsigma \circ \kappa(-T)$.

Let $x(\cdot) : \mathbb{R} \mapsto X$ be a full evolution. Then for all $T \geq 0$, the restriction to $]-\infty,0]$ of the translation $\kappa(-T)(x(\cdot))(\cdot) \in \mathcal{C}(-\infty,0;X)$ can be regarded as *"encoding the history of the full evolution up to time T of the evolution $x(\cdot)$"*. The space $\mathcal{C}(-\infty,0;X)$ allows us to study the evolution of history dependent (or path dependent) systems governing the evolution $T \mapsto \kappa(-T)x(\cdot) \in \mathcal{C}(-\infty,0;X)$ of histories of evolutions. The terminology "path-dependent" is often used, in economics, in particular, but inadequately in the sense that paths are trajectories of evolutions.

A "full" evolutionary system $\mathcal{S} : X \mapsto \mathcal{C}(-\infty,+\infty;X)$ associates with any $x \in X$ an evolution $x(\cdot) \in \mathcal{C}(-\infty,+\infty;X)$ passing through x at time 0. Its backward system : $X \mapsto \mathcal{C}(-\infty,+\infty;X)$ is defined by

$$\overleftarrow{\mathcal{S}}(x) \ := \ \{\overleftarrow{x}(\cdot)\}_{x(\cdot)\in\mathcal{S}(x)}$$

We observe that for all $x(\cdot) \in \mathcal{S}(x)$,

$$\forall\, t \leq 0,\ \ x(t) \ = \ \overleftarrow{x}(-t) \text{ where } \overleftarrow{x}(\cdot) \in \overleftarrow{\mathcal{S}}(x)$$

Splitting evolutions allows us to decompose a full evolution passing through a given state at a given time into its backward and forward parts both governed by backward and forward evolutionary systems:

In particular, consider the system

$$\begin{cases} (i)\ \ x'(t) \ = \ f(x(t),u(t)) \\ (ii)\ u(t) \ \in \ U(x(t)) \end{cases} \tag{8.4}$$

The *backward system* $\overleftarrow{\mathcal{S}} : X \rightsquigarrow \mathcal{C}(0,+\infty;X)$ associates with any x the set of evolutions $\overleftarrow{x}(\cdot) \in \mathcal{C}(0,+\infty;X)$ governed by system

$$\begin{cases} (i)\ \ \overleftarrow{x}'(t) \ = \ -f(\overleftarrow{x}(t),\overleftarrow{u}(t)) \\ (ii)\ \overleftarrow{u}(t) \ \in \ U(\overleftarrow{x}(t)) \end{cases} \tag{8.5}$$

Lemma 8.2.3 *[**Splitting Full Evolutions of a Control System**] We denote by $\mathcal{S}(T,x)$ the subset of full solutions $x(\cdot)$ to system (8.4) passing through x at time T. We observe that a full evolution $x(\cdot)$ belongs to $\mathcal{S}(T,x)$ if and only if:*

1. *its forward part $\overrightarrow{x}(\cdot) := (\kappa(T)(x(\cdot)))(\cdot)$ at time T defined by $\kappa(T)(x(\cdot))(t) = x(t-T)$ is a solution to*

$$\overrightarrow{x}'(t) \ = \ f(\overrightarrow{x}(t), \overrightarrow{u}(t)) \text{ in which } \overrightarrow{u}(t) \ \in \ U(\overrightarrow{x}(t))$$

   satisfying $\overrightarrow{x}(0) \ = \ x$,

2. *its backward part $\overleftarrow{x}(\cdot) := (\overset{\vee}{\kappa}(T)x(\cdot))(\cdot)$ at time T defined by $(\overset{\vee}{\kappa}(T)x(\cdot))(t) = x(T-t)$ is a solution to differential inclusion*

$$\overleftarrow{x}'(t) \ = \ -f(\overleftarrow{x}(t), \overleftarrow{u}(t)) \text{ where } \overleftarrow{u}(t) \ \in \ U(\overleftarrow{x}(t))$$

   satisfying $\overleftarrow{x}(0) \ = \ x$.

Therefore, the full evolution $x(\cdot) \in \mathcal{S}(T,x)$ can be recovered from its backward and forward parts by formula

$$x(t) \ = \begin{cases} \overleftarrow{x}(T-t) \ \ (= (\overset{\vee}{\kappa}(T)\overleftarrow{x})(t)) \text{ if } t \leq T \\ \overrightarrow{x}(t-T) \ \ (= (\kappa(T)\overrightarrow{x})(t)) \ \text{ if } t \geq T \end{cases}$$

As a general rule in this chapter, all concepts introduced in the previous chapters (viable or locally viable evolutions, viability and invariance kernels, capture and absorption basins dealing with the forward part $\overrightarrow{x}(\cdot)$ of an evolution governed by the evolutionary system $\overrightarrow{\mathcal{S}}$ will be qualified of "forward" and those dealing with the backward part $\overleftarrow{x}(\cdot)$ governed by the backward system will be qualified of "backward", taking into account that both forward and backward evolutions are defined on $[0,+\infty[$.

As an example, we single out the concepts backward viability or invariance:

Definition 8.2.4 *[**Backward Viability and Invariance**] We shall say that a subset K is* backward viable (resp. invariant) *under $\mathcal{S}$ if for every $x \in K$, at least one backward evolution (resp. all backward evolutions) $\overleftarrow{x}(\cdot)$ starting from x is (resp. are) viable in K, or, equivalently, at least one evolution $x(\cdot)$ arriving at $x = x(t)$ at some finite time $t \geq 0$ is (resp. all evolutions are) viable in K.*

8.3 Bilateral Viability Kernels

Definition 8.3.1 *[**Bilateral Viability Kernel**] Let $B \subset K$ be a subset regarded as a source, $C \subset K$ be a subset regarded as a target and $\mathcal{S}$ be an evolutionary system. The* backward viability kernel $\text{Viab}_{\overleftarrow{\mathcal{S}}}(K, B)$ *(respectively, the* backward capture basin $\text{Capt}_{\overleftarrow{\mathcal{S}}}(K, B)$*) is the viability kernel (resp. capture basin) under the backward system $\overleftarrow{\mathcal{S}}$. The* bilateral viability kernel

$$\overleftrightarrow{\text{Viab}}_{\mathcal{S}}(K, (B, C)) \;=\; \text{Viab}_{\overleftarrow{\mathcal{S}}}(K, B) \cap \text{Viab}_{\mathcal{S}}(K, C)$$

of K between a source B and a target C is the subset of states $x \in K$ such that there exists one evolution $x(\cdot) \in \mathcal{C}(-\infty, +\infty; X)$ passing through $x = x(0)$ at time 0 and two times $\overleftarrow{T} \in [0, +\infty]$ and $\overrightarrow{T} \in [0, +\infty]$ such that $x(\cdot)$ is

1. *viable in K on $]-\infty, +\infty[$,*
2. *or viable in K on $[-\overleftarrow{T}, +\infty[$, with $x(-\overleftarrow{T}) \in B$,*
3. *or viable in K on $]-\infty, +\overrightarrow{T}]$ with $x(\overrightarrow{T}) \in C$,*
4. *or viable in K on $[-\overleftarrow{T}, +\overrightarrow{T}]$ with $x(-\overleftarrow{T}) \in B$ and $x(\overrightarrow{T}) \in C$.*

When $B = \emptyset$ and $C = \emptyset$ are empty,

$$\overleftrightarrow{\text{Viab}}_{\mathcal{S}}(K) := \overleftrightarrow{\text{Viab}}_{\mathcal{S}}(K, (\emptyset, \emptyset))$$

is the set of elements $x \in K$ through which passes one evolution at time 0 viable on $]-\infty, +\infty[$, called the bilateral viability kernel *of K.*

Observe that a closed subset K connecting a closed source B to a closed target C is both forward locally viable on $K \setminus C$ and backward locally viable on $K \setminus B$ under the evolutionary system (see Definition 2.13.1, p. 94 and Proposition 10.5.2, p. 400).

The *viability partition* of the environment K is made of the four following subsets:

- the bilateral viability kernel

$$\overleftrightarrow{\text{Viab}}_{\mathcal{S}}(K, (B, C)) \;=\; \text{Viab}_{\overleftarrow{\mathcal{S}}}(K, B) \cap \text{Viab}_{\mathcal{S}}(K, C)$$

- the complement $\text{Viab}_{\overleftarrow{\mathcal{S}}}(K, B) \backslash \text{Viab}_{\mathcal{S}}(K, C)$ of the forward viability kernel in the backward viability kernel,
- the complement $\text{Viab}_{\mathcal{S}}(K, C) \backslash \text{Viab}_{\overleftarrow{\mathcal{S}}}(K, B)$ of the backward viability kernel in the forward viability kernel,

- the complement $K \setminus (\mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K,B) \cup \mathrm{Viab}_{\mathcal{S}}(K,C))$.

The following statement describes the viability properties of evolutions starting in each of the subsets of this partition:

Theorem 8.3.2 ***[The Viability Partition of an Environment]*** *Let us consider the viability partition of the environment K under an evolutionary system $\mathcal{S}$:*

- *The bilateral viability kernel* $\overleftrightarrow{\mathrm{Viab}}_{\mathcal{S}}(K,(B,C))$ *is the set of initial states such that at least one evolution passing through it is bilaterally viable in K outside of B and C.*
- *The subset* $\mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K,B) \backslash \mathrm{Viab}_{\mathcal{S}}(K,C)$ *is the subset of initial states x from which* all evolutions $x(\cdot) \in \mathcal{S}(x)$ *leave K in finite time* $\tau_K(x(\cdot)) := \inf\{t \mid x(t) \notin K\}$ *and are viable in* $\mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K,B) \backslash \mathrm{Viab}_{\mathcal{S}}(K,C)$ *on the finite interval* $[0, \tau_K(x(\cdot))]$.
- *The subset* $\mathrm{Viab}_{\mathcal{S}}(K,C) \backslash \mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K,B)$ *is the subset of initial states x from which* all backward evolutions $\overleftarrow{x}(\cdot) \in \overleftarrow{\mathcal{S}}(x)$ *passing through x enter K in finite time* $\tau_K(\overleftarrow{x}(\cdot)) := \inf\{t \mid \overleftarrow{x}(t) \notin K\}$ *and are viable in* $\mathrm{Viab}_{\mathcal{S}}(K,C) \backslash \mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K,B)$ *on the finite interval* $[0, \tau_K(\overleftarrow{x}(\cdot))]$ *(see property (10.5.5)(iii), p. 410 of Theorem 2.15.4, p. 101).*
- *The set* $K \setminus (\mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K,B) \cup \mathrm{Viab}_{\mathcal{S}}(K,C))$ *is the subset of initial states x such that all evolutions passing through x are viable in* $K \setminus (\mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K,B) \cup \mathrm{Viab}_{\mathcal{S}}(K,C))$ *from the finite instant when it enters K to the finite instant when it leaves it.*

If furthermore, the subset $\mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K,B) \backslash \mathrm{Capt}_{\mathcal{S}}(K,C) \subset \mathrm{Int}(K)$, *then it is forward invariant.*

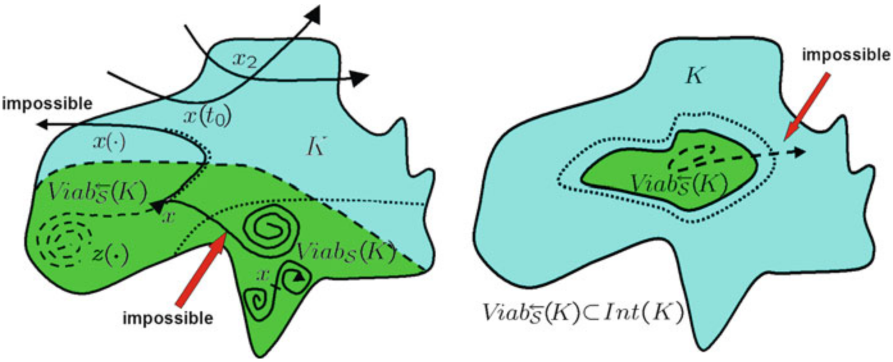

Fig. 8.2 Illustration of the Proof of Theorem 8.3.2.
Left. Proof of parts (2) and (4) of Theorem 8.3.2. Right. Proof of the last statement.

Proof. The first statement is obvious and the second and third ones are symmetric. Let us prove the second and fourth ones.

1. Let x belong to $\mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K,B)\backslash\mathrm{Viab}_{\mathcal{S}}(K,C)$ and $\overrightarrow{x}(\cdot) \in \mathcal{S}(x)$ be any evolution starting at x. It is viable in $\mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K,B)\backslash\mathrm{Viab}_{\mathcal{S}}(K,C)$ until it *must* leave $\mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K,B)$ at some time $t^\star \leq \tau_K(\overrightarrow{x}(\cdot))$, where $\tau_K(\overrightarrow{x}(\cdot)) := \inf\{t \mid \overrightarrow{x}(t) \notin K\}$ (which is finite because x does not belong to the forward viability kernel of K with target C). We observe that actually $\tau_K(\overrightarrow{x}(\cdot)) = \tau_K^\sharp(x)$. Otherwise, there would exist t_0 such that $\tau_K(\overrightarrow{x}(\cdot)) < t_0 \leq \tau_K^\sharp(x)$ where $\overrightarrow{x}(t_0) \in K\backslash\mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K,B)$. Let $\overleftarrow{z}(\cdot) \in \overleftarrow{\mathcal{S}}(x)$ be a backward evolution starting at x and viable in K on $[0,\overleftarrow{T}[$, where $\overleftarrow{T}$ is either infinite or finite, and in this case, $\overleftarrow{z}(\overleftarrow{T}) \in B$. Such an evolution exists since x belongs to the backward viability kernel $\mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K,B)$. The evolution $\overleftarrow{y}(\cdot)$ defined by
$$\overleftarrow{y}(t) \; := \begin{cases} \overrightarrow{x}(t_0 - t) \text{ if } t \in [0,t_0] \\ \overleftarrow{z}(t - t_0) \text{ if } t \in [t_0,\overleftarrow{T}] \end{cases}$$
would be a viable evolution of the backward evolutionary system starting at $\overleftarrow{y}(0) = \overrightarrow{x}(t_0) \in K\backslash\mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K,B)$. This would imply that $\overrightarrow{x}(t_0)$ belongs to the backward viability kernel, a contradiction. Hence $\overrightarrow{x}(\cdot)$ is viable in $\mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K,B)\backslash\mathrm{Viab}_{\mathcal{S}}(K,C)$ on the finite interval $[0,\tau_K(\overrightarrow{x}(\cdot))]$.
2. For the fourth subset of the viability partition, take any evolution $x(\cdot) \in \mathcal{S}(x)$. Let us set $S := \tau_K(\overleftarrow{x}(\cdot))$ and $T := \tau_K(x(\cdot))$. Then $x(\cdot)$ enters K in finite time $-S$, passes through x at time 0 and leaves K in finite time T. Its translation $y(\cdot) := (\kappa(S)x(\cdot))(\cdot) \in \mathcal{S}(x(-S))$ defined by $y(t) := x(t-S)$ is viable in the complement of $\mathrm{Viab}_{\mathcal{S}}(K,C)$ until it leaves K at time $T+S$. Then it is viable in the complement of $\mathrm{Viab}_{\mathcal{S}}(K,C)$. In the same way, the evolution $\overleftarrow{z}(\cdot) := (\overset{\vee}{\kappa}(T)x(\cdot))(\cdot) \in \overleftarrow{\mathcal{S}}(x(T))$ defined by $\overleftarrow{z}(t) := x(T-t)$ is viable in the complement of $\mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K,B)$. Then the evolution $x(\cdot) = (\kappa(-S)x)(\cdot) = (\overset{\vee}{\kappa}(T)\overleftarrow{z}(\cdot))(\cdot)$ is viable both in the complement of $\mathrm{Viab}_{\mathcal{S}}(K,C)$ and in the complement of $\mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K,B)$, and thus, in the complement $K \setminus (\mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K,B) \cup \mathrm{Viab}_{\mathcal{S}}(K,C))$.
3. If $\mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K,B)\backslash\mathrm{Capt}_{\mathcal{S}}(K,C) \subset \mathrm{Int}(K)$, then $\mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K,B)\backslash\mathrm{Capt}_{\mathcal{S}}(K,C)$ is forward invariant. Indeed, any evolution $\overrightarrow{x}(\cdot) \in \mathcal{S}(x)$ starting at $x \in \mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K,B) \setminus \mathrm{Capt}_{\mathcal{S}}(K,C)$ is viable in $\mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K,B) \setminus \mathrm{Capt}_{\mathcal{S}}(K,C)$. Otherwise, there would exist $t_0 > 0$ such that $\overrightarrow{x}(t_0) \in \mathrm{Int}(K)\backslash\mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K,B)$ because $\overrightarrow{x}(t) \notin \mathrm{Capt}_{\mathcal{S}}(K,C)$ since $\mathrm{Capt}_{\mathcal{S}}(K,C)$ is isolated. Associating with it the backward evolution $\overleftarrow{y}(\cdot)$ defined above, we would deduce that $\overrightarrow{x}(t_0) \in \mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K,B)$, a contradiction. Therefore, $\mathrm{Viab}_{\overleftarrow{\mathcal{S}}}(K,B)\backslash\mathrm{Capt}_{\mathcal{S}}(K,C)$ is forward invariant, and thus, contained in $\mathrm{Inv}_{\mathcal{S}}(K \setminus \mathrm{Capt}_{\mathcal{S}}(K,C))$. $\square$

We shall use the following consequence for localizing attractors, and in particular, Lorenz attractors (see Theorem 9.3.12, p. 352):

Proposition 8.3.3 ***[Localization of Backward Viability Kernels]*** *If the backward viability kernel of K is contained in the interior of K, then it is forward invariant and thus, contained in the invariance kernel of K.*

Proof. Take $B = C = \emptyset$, the statement ensues. □

8.3.1 Forward and Backward Viability Kernels under the Lorenz System

Usually, the attractor, defined as the union of limit sets of evolutions, is approximated by taking the union of the "tails of the trajectories" of the solutions that provides an illustration of the shape of the attractor, *although it is not the attractor*. Here, we use the viability kernel algorithm for computing the backward viability kernel, *which contains the attractor*.

Let us consider the Lorenz system (2.6), p. 57

$$\begin{cases} (i) & x'(t) = \sigma y(t) - \sigma x(t) \\ (ii) & y'(t) = rx(t) - y(t) - x(t)z(t) \\ (iii) & z'(t) = \ x(t)y(t) - bz(t) \end{cases}$$

(see of Sect. 2.4.2, p. 56).

Figure 8.3, p. 283 displays a numerical computation of the forward and backward viability kernel of the cube $K := [-30, +30] \times [-30, +30] \times [-0, +53]$ and of the cylinder $C := \{(x, y, z) \in [-100, +100] \times [-100, +100] \times [-20, +80] \mid y^2 + (z - r)^2 \leq 35^2\}$.

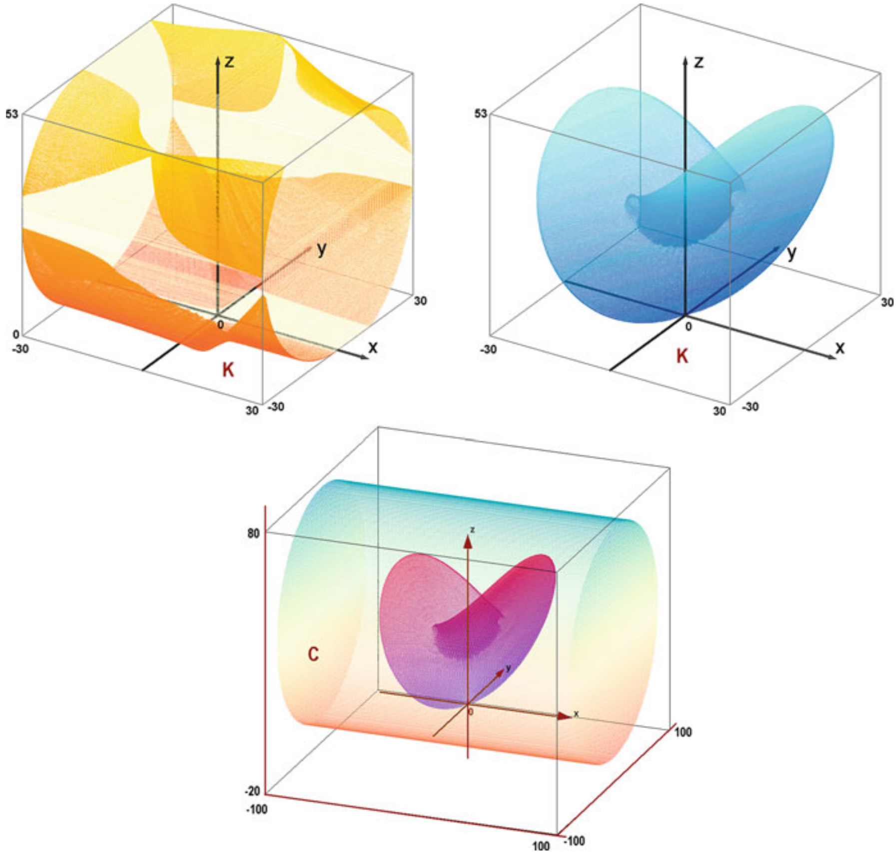

Fig. 8.3 Example of viability kernels for the forward and backward Lorenz systems when $\sigma > b+1$.
Up: *The figure displays both the forward viability kernel of the cube $K := [-30,+30] \times [-30,+30] \times [-0,+53]$ (left) and the backward viability kernel contained in it (right).* ***Down:****. The figure displays both the forward viability kernel of the cylinder $C := \{(x,y,z) \in [-100,+100] \times [-100,+100] \times [-20,+80] \mid y^2 + (z-r)^2 \leq 35^2\}$ which coincides with C itself, meaning that C is a viability domain, and the backward viability kernel contained in it which coincides with the backward viability kernel of the upper figure. Indeed, Proposition 8.3.3, p. 282 states that if the backward viability kernel is contained in the interior of K, the backward viability kernel is also contained in the forward viability kernel. The famous attractor is contained in the backward viability kernel.*

Since the backward viability kernel is contained in the interior of K, Proposition 8.3.3, p. 282 implies the following consequence:

Corollary 8.3.4 *[**Lorenz Attractor**] The Lorenz limit set is contained in this backward viability kernel.*

Usually, the attractor is approximated numerically by taking the union of the trajectories of the solutions that provides an idea of the shape of the attractor, *although it is not the attractor*. Here, we use the viability kernel algorithm for computing the backward viability kernel.

8.4 Detection Tubes

8.4.1 Reachable Maps

When $f : X \mapsto X$ is a Lipschitz single-valued map, it generates a deterministic evolutionary system $\mathcal{S}_f : X \mapsto \mathcal{C}(0,+\infty;X)$ associating with any initial state x the (unique) solution $x(\cdot) = \mathcal{S}_f(x)$ to the differential equation $x'(t) = f(x(t))$ starting at x. The single-valued map $t \mapsto \mathcal{S}_f(x)(t) = \{x(t)\}$ from $\mathbb{R}_+ \mapsto X$ is called the *flow* or *semi-group* associated with f. A flow exhibits the *semi-group property*

$$\forall t \geq s \geq 0,\;\; \mathcal{S}_f(x)(t) \;=\; \mathcal{S}_f(\mathcal{S}_f(x)(s))(t-s)$$

For deterministic systems, studying the dynamical system amounts to studying its associated flow or semi-group, even though when they are not necessarily associated with a dynamical system. Although this will no longer be the case for nondeterministic evolutionary systems, it is worth introducing the semigroup analogues, called "reachable maps" in the control literature:

Definition 8.4.1 *[**Reachable Maps and Tubes**] Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,+\infty;X)$ be an evolutionary system and $B \subset K \subset X$ be a source contained in the environment. Recall that $\mathcal{S}^K : K \rightsquigarrow \mathcal{C}(0,\infty;K)$ denotes the evolutionary system associating with any initial state $x \in K$ the subset of evolutions governed by $\mathcal{S}$ starting at x viable in K. The* reachable map (or set-valued flow) $\mathrm{Reach}^K_{\mathcal{S}}(\cdot;x)$ *viable in K is defined by*

$$\forall\, x \in X,\; \forall t \geq 0,\; \mathrm{Reach}^K_{\mathcal{S}}(t;x) := \{x(t)\}_{x(\cdot)\in\mathcal{S}^K(x)}$$

When $K := X$ is the whole space, we set $\mathrm{Reach}_{\mathcal{S}}(t;x) := \mathrm{Reach}^K_{\mathcal{S}}(t;x)$.
We associate with the source B the (viable) reachable tube $t \rightsquigarrow$

$\mathrm{Reach}_{\mathcal{S}}^{K}(t;B)$ *defined by*

$$\mathrm{Reach}_{\mathcal{S}}^{K}(t;B) := \left\{\mathrm{Reach}_{\mathcal{S}}^{K}(t;x)\right\}_{x\in B}$$

For simplifying the notations, we may drop the lower index $\mathcal{S}$ in the notation of reachable tubes, and mention it only when several systems are considered (the system and the backward system, for example).

We obtain the following properties:

Proposition 8.4.2 ***[The Semi-Group Property]*** *The reachable map* $t \rightsquigarrow \mathrm{Reach}_{\mathcal{S}}^{K}(t;x)$ *exhibits the* semi-group property*:*

$$\forall t \geq s \geq 0,\ \ \mathrm{Reach}_{\mathcal{S}}^{K}(t;x) = \mathrm{Reach}_{\mathcal{S}}^{K}(t-s;\mathrm{Reach}_{\mathcal{S}}^{K}(s;x))$$

Furthermore,

$$(\mathrm{Reach}_{\mathcal{S}}^{K}(t;\cdot))^{-1} \ := \ \mathrm{Reach}_{\overleftarrow{\mathcal{S}}}^{K}(t;\cdot)$$

Proof. The first statement is obvious. To say that $x \in \mathrm{Reach}_{\mathcal{S}}^{K}(T;y)$ means that there exists a viable evolution $x(\cdot) \in \mathcal{S}^{K}(x)$ such that $x(T) = y$. The evolution $\overleftarrow{y}(\cdot)$ defined by $\overleftarrow{y}(t) := x(T-t)$ belongs to $\overleftarrow{\mathcal{S}}(x)$, is viable in K on $[0,T]$ and $\overleftarrow{y}(T) = y$. This means that $y \in \mathrm{Reach}_{\overleftarrow{\mathcal{S}}}^{K}(T;x)$. $\square$

When a time-independent evolutionary system $\mathcal{S} : X \mapsto \mathcal{C}(0,+\infty;X)$ is deterministic, one can identify the (unique) evolution $x(\cdot) = \mathcal{S}(x)$ starting from x with the reachable (single-valued) map $t \in \mathbb{R} \mapsto \mathrm{Reach}_{\mathcal{S}}(t;x)$. This is why in many cases, the classical study of deterministic systems is reduced to the *flows* $t \in \mathbb{R} \mapsto \mathrm{Reach}_{\mathcal{S}}(t;x)$.

Important Remark: Reachable Maps and Evolutionary Systems. Even though (set-valued) reachable maps $\mathrm{Reach}_{\mathcal{S}}(\cdot;x)$ play an important role, *they no longer characterize a time-independent nondeterministic evolutionary system* $\mathcal{S}$: Knowing a state $y \in \mathrm{Reach}_{\mathcal{S}}(t;x)$, we know that an evolution starting from x passes through y at time t, but this does not guarantee that this evolution passes through any arbitrary $x_s \in \mathrm{Reach}_{\mathcal{S}}(s;x)$ at time s.

This is why, for nondeterministic evolutionary systems, the convenient general setting is to regard it as a set-valued map $\mathcal{S} : X \mapsto \mathcal{C}(0,+\infty;X)$ *instead of a set-valued semi-group or flow.*

The graph of the reachable tube is itself a capture basin under an auxiliary system, and thus, exhibits all the properties of capture basins:

Proposition 8.4.3 ***[Viability Characterization of Reachable Tubes]*** *Let us consider the backward auxiliary system*

$$\begin{cases} (i) & \tau'(t) = -1 \\ (ii) & x'(t) = -f(x(t), u(t)) \\ & \quad \text{where } u(t) \in U(x(t)) \end{cases} \tag{8.6}$$

and a source B. The graph of the viable reachable tube $\text{Reach}_{\mathcal{S}}^{K}((\cdot); B) : T \rightsquigarrow \text{Reach}_{\mathcal{S}}^{K}(T; B)$ *is the capture basin of $\mathbb{R}_+ \times K$ with target $\{0\} \times B$ under the auxiliary system (8.6), p. 286:*

$$\text{Graph}(\text{Reach}_{\mathcal{S}}^{K}(\cdot; B)) = \text{Capt}_{(8.6)}(\mathbb{R}_+ \times K, \{0\} \times B)$$

Proof. Indeed, to say that (T, x) belongs to the capture basin of target $\{0\} \times B$ viable in $\mathbb{R}_+ \times K$ under the auxiliary system (8.6) means that there exist an evolution $\overleftarrow{x}(\cdot)$ to the backward system (8.6), p. 286:

$$\begin{cases} (i) & \tau'(t) = -1 \\ (ii) & x'(t) = -f(x(t), u(t)) \\ & \quad \text{where } u(t) \in U(x(t)) \end{cases}$$

starting at $\overleftarrow{x}(0) := x$ and a time $t^* \geq 0$ such that

$$\begin{cases} (i) & \forall t \in [0, t^*], \ (T - t, \overleftarrow{x}(t)) \in \mathbb{R}_+ \times K \\ (ii) & \qquad\qquad (T - t^*, \overleftarrow{x}(t^*)) \in \{0\} \times B \end{cases}$$

The second condition means that $t^\star = T$ and that $\overleftarrow{x}(T)$ belongs to B. The first one means that for every $t \in [0, T]$, $\overleftarrow{x}(t) \in K$. This amounts to saying that the evolution $x(\cdot) := \overleftarrow{x}(T - \cdot)$ is a solution to system (8.4) starting at $\overleftarrow{x}(T) \in B$, satisfying $x(T) = x$ and

$$\forall t \in [0, T], \ x(t) \in K \qquad \square$$

8.4.2 Detection and Cournot Tubes

Definition 8.4.4 ***[Detection Basins and Tubes]*** *Let $B \subset K$ be a subset regarded as a source. The* detection basin $\text{Det}_{\mathcal{S}}(K, B)$ *is the subset of final*

states in K at which arrives in finite time at least one evolution viable in K starting from the source B*. The subset K is said to* detect B *if* $K = \mathrm{Det}_{\mathcal{S}}(K, B)$.
The T-detection basin $\mathrm{Det}_{\mathcal{S}}(K, B)(T)$ *is the subset of final states in K at which arrives* before T *at least one evolution viable in K starting from the* source B *and the set-valued maps* $T \rightsquigarrow \mathrm{Det}_{\mathcal{S}}(K, B)(T)$ *is called the* detection tube *of B.*

We first point-out the links between capture and detecting basins:

Lemma 8.4.5 *[**Capture and Detection Basins**] The (forward) detection basin $\mathrm{Det}_{\mathcal{S}}(K, B)$ of the source B under $\mathcal{S}$ is equal to the backward capture basin $\mathrm{Capt}_{\overleftarrow{\mathcal{S}}}(K, B)$ under $\overleftarrow{\mathcal{S}}$ of the source B regarded as a target:*

$$\mathrm{Det}_{\mathcal{S}}(K, B) \;=\; \mathrm{Capt}_{\overleftarrow{\mathcal{S}}}(K, B)$$

and thus, exhibits all the properties of the capture basins.

Proof. Indeed, to say that x belongs to $\mathrm{Det}_{\mathcal{S}}(K, B)$ amounts to saying that there exist an initial state $x_0 \in B$, an evolution $x(\cdot) \in \mathcal{S}(x_0)$ and some $T \geq 0$ such that $x(T) = x$ and $x(t) \in K$ for all $t \in [0, T]$. Then the evolution $\overleftarrow{x}(\cdot)$ defined by $\overleftarrow{x}(t) := x(T - t)$ is a solution to the backward system $\overleftarrow{\mathcal{S}}(x)$ starting at x and viable in K until time T when $\overleftarrow{x}(T) = x(0) = x_0 \in B$. This means that x belongs to $\mathrm{Capt}_{\overleftarrow{\mathcal{S}}}(K, B)$. □

The detection tube can be expressed in terms of "reachable maps":

Proposition 8.4.6 *[**Detection Tubes and Reachable maps**] Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ be an evolutionary system and $B \subset K \subset X$ be a source contained in the environment. Then the detection tube can be written in the form*

$$\mathrm{Det}_{\mathcal{S}}(K, B)(T) := \bigcup_{t \in [0,T]} \mathrm{Reach}^K_{\mathcal{S}}(t; B)$$

We also deduce from Theorem 4.3.2, p. 133 a viability characterization of the detection tube:

Proposition 8.4.7 ***[Viability Characterization of Detection Tubes]*** *Let us consider the backward auxiliary system (8.6), p. 286:*

$$\begin{cases} (i) & \tau'(t) = -1 \\ (ii) & x'(t) = -f(x(t), u(t)) \\ & \text{where } u(t) \in U(x(t)) \end{cases}$$

Then the graph of the viable-capturability tube $\mathrm{Det}_{\mathcal{S}}(K,B)(\cdot)$ *is the viable-capture basin of* $\mathbb{R}_+ \times B$ *viable in* $\mathbb{R}_+ \times K$ *under the system (8.6):*

$$\mathrm{Graph}(\mathrm{Det}_{\mathcal{S}}(K,B)(\cdot)) = \mathrm{Capt}_{(8.6)}(\mathbb{R}_+ \times K, \mathbb{R}_+ \times B)$$

Proof. The proof is analogous to the one of Proposition 8.4.3, p. 286. □

Detection tubes provide the set of final states at which arrive at least one evolution emanating from B. The question arises whether we can find the subset of these initial states. This is connected with a concept of uncertainty suggested by *Augustin Cournot* as the meeting of two independent causal series: *"A myriad partial series can coexist in time: they can meet, so that a single event, to the production of which several events took part, come from several distinct series of generating causes."* The search for causes amounts in this case to reversing time in the dynamics and to look for "retrodictions" (so to speak) instead of predictions.

We suggest to combine this Cournot approach uncertainty with the Darwinian view of contingent uncertainty for facing necessity (viability constraints) by introducing the concept of Cournot map.

Definition 8.4.8 ***[Cournot Map]*** *The* Cournot map $\mathrm{Cour}_{(\mathbf{K},B)} : \mathrm{Graph}(\mathbf{K}) \rightsquigarrow B$ *associates with any* $T \geq 0$ *and* $x \in \mathbf{K}(T)$ *the (possibly empty) subset* $\mathrm{Cour}_{(\mathbf{K},B)}(T,x)$ *of initial causes* $x_0 \in B$ *from which* $x := x(T)$ *can be reached by an* actual evolution $x(\cdot) \in \mathcal{S}(x_0)$ *viable in the tube:*

$$\forall t \in [0,T],\ x(t) \in \mathbf{K}(t) \tag{8.7}$$

At time T*, the state* x *is thus the result of past viable evolutions starting from all causal states* $x_0 \in \mathrm{Cour}_{(\mathbf{K},B)}(T,x)$*. The size of the set* $\mathrm{Cour}_{(\mathbf{K},B)}(T,x)$ *could be taken as* a measure of Cournot's concept of uncertainty *for the event* x *at time* T*.*

The set-valued map $T \rightsquigarrow \mathrm{Im}(\mathrm{Cour}_{(\mathbf{K},B)}(T,\cdot))$ *is decreasing*, refining the set of causal states of B from which at least one evolution has been selected through the tube $\mathbf{K}(\cdot)$ as time goes on.

We shall characterize the Cournot map as a viability kernel under an adequate auxiliary system.

Theorem 8.4.9 ***[Viability Characterization of Cournot Maps]*** *Let us set* $\boldsymbol{I}_B := \{(x,x)\}_{x\in B}$ *and introduce the auxiliary system*

$$\begin{cases} (i) & \tau'(t) \;=\; -1 \\ (ii) & x'(t) \;=\; -f(x(t),u(t)) \\ & \text{where } u(t) \;\in\; U(x(t)) \\ (iii) & y'(t) \;=\; 0 \end{cases} \tag{8.8}$$

The graph of the Cournot Map $\mathrm{Cour}_{(\mathbf{K},B)}$ *is given by the formula*

$$\mathrm{Graph}(\mathrm{Cour}_{(\mathbf{K},B)}) := \mathrm{Capt}_{(8.9)}(\mathrm{Graph}(\mathbf{K}\times B, \{0\}\times \boldsymbol{I}_B)) \tag{8.9}$$

Proof. To say that (T,x,x_0) belongs to $\mathrm{Capt}_{(8.9)}(\mathrm{Graph}(\mathbf{K}\times B,\{0\}\times \mathbf{I}_B))$ means that there exists an evolution $\overleftarrow{x}(\cdot)$ starting at x and a time $t^\star \geq 0$ such that

$$\begin{cases} (i) \;\; \forall t\in[0,t^\star],\; (T-t, \overleftarrow{x}(t), x_0) \in \mathrm{Graph}(\mathbf{K})\times B \\ (ii) \qquad\qquad\qquad\;\; (T-t^\star, \overleftarrow{x}(t^\star), x_0) \in \{0\}\times \mathbf{I}_B \end{cases}$$

The second condition means that $t^\star = T$ and that $\overleftarrow{x}(T) = x_0$ belongs to B. The first one means that for every $t\in[0,T]$, $\overleftarrow{x}(t)\in \mathbf{K}(T-t)$. This amounts to saying that the evolution $x(\cdot) := \overleftarrow{x}(T-\cdot)$ where x_0 belongs to B satisfies $x(T)=x$ and the viability conditions (8.7), i.e., that x_0 belongs to $\mathrm{Cour}_{(\mathbf{K},B)}(T,x)$. □

The issue is pursued in Sect. 13.8, p. 551 at the Hamilton-Jacobi level.

8.4.3 Volterra Inclusions

The standard paradigm of evolutionary system that we adopted is the initial-value (or Cauchy) problem. It assumes that the present is frozen, as well as the initial state from which start evolutions governed by an evolutionary system $\mathcal{S}$.

But the present time evolves, too, and consequences of earlier evolutions accumulate. Therefore, the questions of “gathering” present consequences of all earlier initial states arises.

There are two ways of mathematically translating this idea. The first one, the most familiar, is to take the *sum* of the number of these consequences: This leads to equations bearing the name of *Volterra*, of the form

$$\forall\, T \geq 0, \;\; x(T) \;=\; \int_0^T \theta(T-s; x(s))ds$$

A particular case is obtained for instance when the "kernel" $\theta(\cdot,\cdot)$ is itself the flow of a deterministic system $y'(t) = f(y(t))$. A solution $x(\cdot)$ to the Volterra equation, if it exists, provides at each ephemeral $T \geq 0$ the sum of the states obtained at time T from the state $x(s)$ at earlier time $T - s \in [0, T]$ of the solution by differential equation $y'(t) = f(y(t))$ starting at time 0 at a given initial state x. Then $\int_0^T \theta(T-s; x(s))ds$ denotes the sum of consequences at time T of a flow of earlier evolving initial conditions, for instance.

This is a typical situation that is met in traffic problems or in biological neuron networks. It is not enough to study the consequences of an initial condition, a vehicle or a neurotransmitter, since they arrive continuously at the entrance of the highway or of the neuron.

In the set-valued case, "gathering" the subsets of consequences at ephemeral time T of earlier initial conditions is mathematically translated by taking their *union*. Hence the map similar to the Volterra equation would be to find a tube $\mathbf{D} : t \rightsquigarrow \mathbf{D}(t)$ and to check whether it satisfies

$$\forall\, T \geq 0, \;\; \mathbf{D}(T) \;=\; \bigcup_{s\in[0,T]} \theta(T-s; \mathbf{D}(s))$$

where $(t, K) \mapsto \theta(t, K) \subset X$ is a set-valued "kernel".

The particular example of kernel is the reachable map $(t, K) \mapsto \mathrm{Reach}_{\mathcal{S}}^{K}(t, K)$, a solution to an initial value problem, in the spirit of *Cauchy*. Then, if a tube $\mathbf{D} : t \rightsquigarrow \mathbf{D}(t)$ is given, the set

$$\forall\, T \geq 0, \;\; \bigcup_{s\in[0,T]} \mathrm{Reach}_{\mathcal{S}}^{K}(T-s; \mathbf{D}(s))$$

of cumulated consequences gathers the consequences at time T of the evolutions at time T of evolutions starting at time $T - s$ from $\mathbf{D}(s)$. We shall prove that there exist solutions to the set-valued Volterra equation, that we shall call with a slight abuse of language, *Volterra inclusion*

$$\forall\, T \geq 0, \;\; \mathbf{D}(T) \;=\; \bigcup_{s\in[0,T]} \mathrm{Reach}_{\mathcal{S}}^{K}(T-s; \mathbf{D}(s)) \tag{8.10}$$

The reachable tube $\mathrm{Reach}_{\mathcal{S}}^{K}(\cdot; B)$ is obviously a solution to such a set-valued Volterra equation: This is nothing other than the semi-group property.

We shall see that this is the ***unique*** *viable tube satisfying the semi-group property contained in K and starting at B.*

We shall also prove that the *detection tube* $\mathrm{Det}_{\mathcal{S}}^{K}(\cdot, B)$ *is the* ***unique*** *viable tube solution to the set-valued Volterra equation "Volterra inclusion (8.10)" contained in K and starting at B.*

For that purpose, we have to slightly extend the concept of detection tube of subsets to detection tubes of tubes (see Theorem 8.10.6, p. 314).

8.5 Connection Basins and Eupalinian Kernels

8.5.1 Connection Basins

Let us consider an environment $K \subset X$ and an evolutionary system $\mathcal{S} : X \rightsquigarrow \mathcal{C}(-\infty, +\infty; X)$.

Definition 8.5.1 *[**Connection Basins**] Let $B \subset K$ be a subset regarded as a source, $C \subset K$ be a subset regarded as a target. The* connection basin $\mathrm{Conn}_{\mathcal{S}}(K, (B, C))$ *of K between B and C is the subset of states $x \in K$ through which passes at least one viable evolution starting from the source B and arriving in finite time at target C.*

The subset K is said to connect B *to* C *if* $K = \mathrm{Conn}_{\mathcal{S}}(K, (B, C))$.

We refer to Sect. 10.6.2, p. 413 for topological and other viability characterization of connection basins.

The set-valued map which associates with any $x \in \mathrm{Conn}_{\mathcal{S}}(K, (B, C))$ of the connection basin the pair $(x(\varpi_{(K,B)}(\overleftarrow{x}(\cdot)), \varpi_{(K,C)}(\overrightarrow{x}(\cdot)))) \in B \times C \subset K \times K$ of end-values of viable evolutions $x(\cdot)$ connecting B to C has, by definition, nonempty values.

The question arises whether we can invert this set-valued map: given a pair $(y, z) \in K \times K$, does there exist an evolution $x(\cdot)$ viable in K linking y to z in finite time in the sense where $x(0) = y$ and $x(T) = z$ for some finite time T? This is an instantiation of the problem of studying the connection basin $\mathrm{Conn}_{\mathcal{S}}(K, (\{y\}, \{z\}))$ when the pairs (y, z) range over the subset $\mathcal{E} \subset K \times K$ of pairs of end-values of viable evolutions $x(\cdot)$ connecting B to C.

In other words, the question boils down whether we can *a priori* know the subset $\mathcal{E} \subset K \times K$ of pairs (y, z) such that

$$\mathrm{Conn}_{\mathcal{S}}(K, (B, C)) \; = \; \bigcup_{(y,z)\in\mathcal{E}} \mathrm{Conn}_{\mathcal{S}}(K, (\{y\}, \{z\}))$$

Therefore, the study of connection basins amounts to finding this subset $\mathcal{E}$, that we shall call the *Eupalinian kernel* of K under $\mathcal{S}$, and to characterize it as capture basins of an auxiliary capturability problems.

8.5.1.1 Eupalinian Kernels

Eupalinos, a Greek engineer, excavated around 550 BC a 1,036 m. long tunnel 180 m. below Mount Kastro for building an aqueduct supplying Pythagoreion (then the capital of Samos) with water on orders of tyrant Polycrates. *He started to dig simultaneously the tunnel from both sides by two working teams who met in the center* of the channel and they had only 0.6 m. error. There is still no consensus on how he did it. However[1], this *"Eupalinian strategy"* has been used ever since for building famous tunnels (under the British Channel or the Mont-Blanc) or bridges: it consists in starting the construction at the same time from both end-points x and y and proceed until they meet, by continuously monitoring the progress of the construction.

Such models can also be used as mathematical metaphors in negotiation procedures when both actors start from opposite statements and try to reach a consensus by making mutual concessions step by step, continuously bridging the remaining gap.

This question arose in numerical analysis and control under the name of "shooting" methods, which, whenever the state is known at initial and final time, consists in integrating differential equations at the same time at both initial and final states and matching in the middle.

We suggest a mathematical metaphor for explaining such an Eupalinian strategy.

Definition 8.5.2 ***[Eupalinian Kernels]*** *Let* $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,+\infty;X)$ *and* $K \subset X$ *be an environment. We denote by* $\mathcal{S}^K(y,z) := \mathrm{Conn}_{\mathcal{S}}(K,(\{y\},\{z\}))$ *the set of* Eupalinian evolutions $x(\cdot)$ *governed by the evolutionary system* $\mathcal{S}$ *viable in* K *connecting* y *to* z*, i.e., the set of evolutions* $x(\cdot) \in \mathcal{S}(y)$ *such that there exists a finite time* $T \geq 0$ *satisfying* $x(T) = z$ *and, for all* $t \in [0,T]$*,* $x(t) \in K$*.*
The Eupalinian kernel $\mathcal{E} := \mathrm{Eup}_{\mathcal{S}}(K) \subset K \times K$ *is the subset of pairs* (y,z) *such that there exists at least one viable evolution* $x(\cdot) \in \mathcal{S}^K(y)$ *connecting* y *to* z *and viable in* K*.*

[1] The authors thank Hélène Frankowska for communicating them this historical information.

We can characterize the Eupalinian kernel as a capture basin:

Proposition 8.5.3 *[**Viability Characterization of Eupalinian Kernels**] Let us denote by* $\mathrm{Diag}(K) := \{(x,x)\}_{x \in K} \subset K \times K$ *the* diagonal of K*. The Eupalinian kernel* $\mathrm{Eup}_{\mathcal{S}}(K)$ *of* K *under the evolutionary system* $\mathcal{S}$ *associated with the system*

$$x'(t) \;=\; f(x(t), u(t)) \text{ where } u(t) \;\in\; U(x(t))$$

is the capture basin

$$\mathrm{Eup}_{\mathcal{S}}(K) \;=\; \mathrm{Capt}_{(8.11)}(K \times K, \mathrm{Diag}(K))$$

of the diagonal of K *viable in* $K \times K$ *under the auxiliary system*

$$\begin{cases} (i) \;\; y'(t) \;=\; f(y(t), u(t)) \;\text{ where } u(t) \;\in\; U(y(t)) \\ (ii) \; z'(t) = \; -f(z(t), v(t)) \text{ where } v(t) \;\in\; U(z(t)) \end{cases} \tag{8.11}$$

We "quantify" the concept of Eupalinian kernel with the concept of several Eupalinian intertemporal optimization problems. The domains of their value functions are the Eupalinian kernels, so that Proposition 8.5.3, p. 293 follows from the forthcoming Theorem 8.5.6. p. 295.

Since we shall minimize an intertemporal criterion involving controls, as we did in Chap. 4, p. 125, we denote by:

1. $\mathcal{P}^K(x)$ the set of state-control evolutions $(x(\cdot), u(\cdot))$ where $x(0) = x$ and regulated by the system.

$$x'(t) \;=\; f(x(t), u(t)) \text{ where } u(t) \;\in\; U(x(t))$$

   and viable in K
2. $\mathcal{P}^K(y,z)$ the set of state-control evolutions $(x(\cdot), u(\cdot))$ where $x(0) = y$ and $x(t^\star) = z$ for some finite time $t^\star$ governed by this and viable in K.

We introduce a cost function $\mathbf{c} : X \times X \mapsto \mathbb{R} \cup \{+\infty\}$ (regarded as a *connection cost*) and a Lagrangian $\mathbf{l} : (x,u) \rightsquigarrow \mathbf{l}(x,u)$.

We consider the Eupalinian optimization problem

$$\begin{cases} \mathbf{E}_{(\mathbf{c},\mathbf{l})}(y,z) \\ = \inf_{(x(\cdot),u(\cdot)) \in \mathcal{P}^K(y,z), t^\star \,\geq\, 0 \,\mid\, x(2t^\star) = z} \left(\mathbf{c}(x(t^\star), x(t^\star)) + \int_0^{2t^\star} \mathbf{l}(x(t), u(t)) dt \right) \end{cases}$$

- By taking $\mathbf{c} \equiv 0$ and $\mathbf{l}(x,u) \equiv 1$, we find the following Eupalinian minimal time problem:

Definition 8.5.4 *[**Eupalinian Distance and Brachistochrones**]* *The Eupalinian distance* $\epsilon_K(y,z)$

$$\epsilon_K(y,z) \ := \ 2 \inf_{(x(\cdot),u(\cdot))\in\mathcal{P}^K(y,z),\ t^\star\ \geq\ 0\ |\ x(2t^\star)=z} t^\star \ \in \ [0,+\infty[\qquad (8.12)$$

measures the minimal time needed for connecting the two states y and z by a evolution viable in K. *Let* $B \subset K$ *a source and* $C \subset K$ *be a target. The function*

$$\epsilon_K(B,C) \ := \ \inf_{y\in B,\ z\in C} \epsilon_K(y,z)$$

is called the Eupalinian distance *between the source* B *and the target* C. *Viable evolutions in* K *connecting* y *to* z *in* minimal time *are called* brachistochrones.

Their existence and computation was posed as a challenge by *Johann Bernoulli* in 1696, challenge met by *Isaac Newton*, *Jacob Bernoulli*, *Gottfried Leibnitz* and *Guillaume de L'Hopital* in a particular case.

- By taking $\mathbf{c} \equiv 0$ and $\mathbf{l}(x,u) = \|f(x,u)\|$, we obtain the *viable geodesic* connecting two states by a viable evolution in *minimal length function* $\gamma_K(x) : X \mapsto \mathbb{R}_+ \cup \{+\infty\}$ defined by

$$\gamma_K(x) \ := \ \inf_{(x(\cdot),u(\cdot))\in\mathcal{P}^K(x)} \int_0^{\infty} \|f(x(t),u(t))\|dt$$

(see Definition 4.4.1, p. 140).

Definition 8.5.5 *[**Geodesics**] We denote by* geodesic distance

$$\widehat{\gamma}_K(y,z) := \inf_{(x(\cdot),u(\cdot))\in\mathcal{P}^K(y,z),\ t^\star \geq 0 \ | \ x(2t^\star)=z} \int_0^{2t^\star} \|f(x(t),u(t))\|dt \tag{8.13}$$

measuring the minimal length needed for connecting the two states y and z by a evolution viable in K. *Any viable evolution* $(x(\cdot),u(\cdot)) \in \mathcal{P}^K(y,z)$ *achieving the minimum* $\widehat{\gamma}_K(y,z) := \int_0^{2t^\star} \|f(x(t),u(t))\|dt$ *is called a* viable geodesic. *The function*

$$\widetilde{\gamma}_K(B,C) := \inf_{y\in B,\ z\in C} \gamma_K(y,z)$$

is called the geodesic distance *between the source* B *and the target* C.

We shall prove that

Theorem 8.5.6 *[**Eupalinian Optimization Theorem**] Let us consider the auxiliary control system*

$$\begin{cases} (i) \quad y'(t) \ = \ f(y(t),u(t)) \text{ where } u(t) \ \in \ U(y(t)) \\ (ii) \ \ z'(t) \ = \ -f(z(t),v(t)) \text{ where } v(t) \ \in \ U(z(t)) \\ (iii) \ \lambda'(t) = -\mathbf{l}(y(t),u(t)) - \mathbf{l}(z(t),v(t)) \end{cases} \tag{8.14}$$

Then

$$\mathbf{E}_{(\mathbf{c},\mathbf{l})}(y,z) = \inf_{(y,z,\lambda)\in \mathrm{Capt}_{(8.14)}(K\times K\times\mathbb{R}_+,\mathcal{E}p(\mathbf{c})\cap(\mathrm{Diag}(K)\times\mathbb{R}_+))} \lambda$$

where $\mathrm{Diag}(K) := \{(x,x)\}_{x\in K} \subset K \times K$ *is the* diagonal of K.

Proof. Let $(y,z,\lambda) \in \mathrm{Capt}_{(8.14)}(K\times K\times\mathbb{R}_+,\mathcal{E}p(\mathbf{c})\cap(\mathrm{Diag}(K)\times\mathbb{R}_+))$ belong to the capture basin. This means that there exist one forward evolution $\overrightarrow{y}(\cdot) \in \mathcal{S}^K(y)$ viable in K, one backward evolution $\overleftarrow{z}(\cdot) \in \overleftarrow{\mathcal{S}}^K(z)$ viable in K, the evolution $\lambda(t) := \lambda - \int_0^t \mathbf{l}(\overrightarrow{y}(t),\overrightarrow{u}(t))dt - \int_0^t \mathbf{l}(\overleftarrow{z}(t),\overleftarrow{v}(t))dt$ and a time $t^\star$ such that:

- for all $t \in [0,t^\star]$, $\overrightarrow{y}(t) \in K$, $\overleftarrow{z}(t) \in K$,

$$\lambda - \int_0^t \mathbf{l}(\overrightarrow{y}(t), \overrightarrow{u}(t))dt - \int_0^t \mathbf{l}(\overleftarrow{z}(t), \overleftarrow{v}(t))dt \ \geq \ 0$$

- and $\overrightarrow{y}(t^\star) = \overleftarrow{z}(t^\star)$ and

$$\lambda - \int_0^{t^\star} \mathbf{l}(\overrightarrow{y}(t), \overrightarrow{u}(t))dt - \int_0^{t^\star} \mathbf{l}(\overleftarrow{z}(t), \overleftarrow{v}(t))dt \ \geq \ \mathbf{c}(\overrightarrow{y}(t^\star), \overleftarrow{z}(t^\star))$$

Let us introduce the evolution $x(t)$ defined by $x(t) := \overrightarrow{y}(t)$ for $t \in [0, t^\star]$ and $x(t) := \overleftarrow{z}(2t^\star - t)$ for $t \in [t^\star, 2t^\star]$. This evolution $x(\cdot)$ is continuous at $t^\star$ because $x(t^\star) = \overrightarrow{y}(t^\star) = \overleftarrow{z}(t^\star)$, belongs to $\mathcal{S}^K(y,z)$ since $x(0) = \overrightarrow{y}(0) = y$, $x(2t^\star) = \overleftarrow{z}(0) = z$ and is governed by the differential inclusion starting at y. Furthermore,

$$\begin{cases} \lambda - \left(\displaystyle\int_0^{2t^\star} \mathbf{l}(x(t), u(t))dt \right) \\ = \ \lambda - \left(\displaystyle\int_0^{t^\star} \mathbf{l}(\overrightarrow{y}(t), \overrightarrow{u}(t))dt + \int_0^{t^\star} \mathbf{l}(\overleftarrow{z}(t), \overleftarrow{v}(t))dt \right) \\ \geq \ \mathbf{c}(x(t^\star), x(t^\star)) \end{cases}$$

This means that there exist $x(\cdot) \in \mathcal{S}^K(y,z)$ and $t^\star \geq 0$ such that

$$\mathbf{c}(x(t^\star), x(t^\star)) + \int_0^{2t^\star} \mathbf{l}(x(t), u(t))dt \ \leq \ \lambda$$

This implies in particular that

$$\begin{cases} \mathbf{E}_{(\mathbf{c},\mathbf{l})}(y,z) \ := \displaystyle\inf_{(x(\cdot),u(\cdot)) \in \mathcal{P}^K(y,z),\ t^\star} \left(\mathbf{c}(x(t^\star), x(t^\star)) + \int_0^{2t^\star} \mathbf{l}(x(t), u(t))dt \right) \\ \leq \displaystyle\inf_{(y,z,\lambda) \in \mathrm{Capt}_{(8.14)}(\mathbb{R}_+ \times K \times K, \mathcal{E}p(\mathbf{c}) \cap (\mathbb{R}_+ \times \mathrm{Diag}(K)))} \lambda \end{cases}$$

For proving the opposite inequality, we associate with any $\varepsilon > 0$ an evolution $x_\varepsilon(\cdot) \in \mathcal{S}^K(y,z)$, a control $u_\varepsilon(\cdot)$ and $t_\varepsilon^\star \ \geq \ 0$ such that

$$\left(\mathbf{c}(x_\varepsilon(t_\varepsilon^\star), x_\varepsilon(t_\varepsilon^\star)) + \int_0^{2t_\varepsilon^\star} \mathbf{l}(x_\varepsilon(t), u_\varepsilon(t))dt \right) \ \leq \ \mathbf{E}_{(\mathbf{c},\mathbf{l})}(y,z) + \varepsilon$$

and the function

$$\lambda_\varepsilon(t) \ := \ \mathbf{E}_{(\mathbf{c},\mathbf{l})}(y,z) + \varepsilon - \int_0^{2t} \mathbf{l}(x_\varepsilon(t), u_\varepsilon(t))dt$$

Introducing the forward parts $\overrightarrow{y}_\varepsilon(t) := x_\varepsilon(t)$ and $\overrightarrow{u}_\varepsilon(t) := u_\varepsilon(t)$ for $t \in [0, t_\varepsilon^\star]$ and backward parts $\overleftarrow{z}_\varepsilon(t) := x_\varepsilon(2t_\varepsilon^\star - t)$ and $\overleftarrow{v}_\varepsilon(t) := u_\varepsilon(2t_\varepsilon^\star - t)$, we observe that $(\overrightarrow{y}_\varepsilon(t), \overleftarrow{z}_\varepsilon(t), \lambda_\varepsilon(t))$ is a solution to the auxiliary system (8.14) starting at $(y, z, \mathbf{E}_{(\mathbf{c},\mathbf{l})}(y,z)+\varepsilon)$, viable in $K \times K \times \mathbb{R}_+$ and satisfying

$$\begin{cases} \lambda_\varepsilon(t) \ := \ \mathbf{E}_{(\mathbf{c},\mathbf{l})}(y,z)+\varepsilon - \displaystyle\int_0^{2t_\varepsilon^\star} \mathbf{l}(x_\varepsilon(t), u_\varepsilon(t))dt \\ \geq \ \mathbf{c}(\overrightarrow{y}_\varepsilon(t_\varepsilon^\star), \overleftarrow{z}_\varepsilon(t_\varepsilon^\star)) \end{cases}$$

This implies that $(y, z, \mathbf{E}_{(\mathbf{c},\mathbf{l})}(y,z)+\varepsilon)$ belongs to the capture basin $\mathrm{Capt}_{(8.14)}(K \times K \times \mathbb{R}_+, \mathcal{E}p(\mathbf{c}) \cap (\mathrm{Diag}(K) \times \mathbb{R}_+))$. Hence

$$\inf_{(y,z,\lambda) \in \mathrm{Capt}_{(8.14)}(K\times K\times\mathbb{R}_+, \mathcal{E}p(\mathbf{c})\cap(\mathrm{Diag}(K)\times\mathbb{R}_+))} \lambda \ \leq \ \mathbf{E}_{(\mathbf{c},\mathbf{l})}(y,z)+\varepsilon$$

and it is enough to let ε converge to 0. □

Remark: Eupalinian Graphs. The Eupalinian kernel is a graph in the sense of "graph theory" where the points are regarded as "vertices" or "nodes", a set of pairs (y, z) connected by at least one evolution and, for a given intertemporal optimization problem, the set of "edges" or "arcs" $\mathbf{E}_{(\mathbf{c},\mathbf{l})}(y,z)$ linking y to z. □

Remark: The associated Hamilton-Jacobi-Bellman Equation. The tangential and normal characterizations of capture basins imply that the bilateral value function is the solution to the bilateral Hamilton-Jacobi-Bellman partial differential equation

$$\inf_{u \in U(x)} \left(\left\langle \frac{\partial \mathbf{E}}{\partial x}, f(x,u) \right\rangle + \mathbf{l}(x,u) \right) - \sup_{v \in U(y)} \left(\left\langle \frac{\partial \mathbf{E}}{\partial y}, f(y,v) \right\rangle - \mathbf{l}(y,v) \right) \ = \ 0 \tag{8.15}$$

in a generalized sense (see Chap. 17, p. 681) satisfying the *diagonal condition*

$$\forall\, x \in K, \ \ \mathbf{E}_{(\mathbf{c},\mathbf{l})}(x,x) \ = \ \mathbf{c}(x,x)$$

Even though the solution to this partial differential equation provides the Eupalinian value function, we do not need to approximate this partial differential equation for finding this Eupalinian value function since the Viability Kernel Algorithm provides it and the optimal Eupalinian evolutions. □

Remark: Regulation of Optimal Eupalinian Solutions. We introduce the two following forward and backward maps:

$$\begin{cases} (i) & \overrightarrow{R}(x,p) := \left\{ \overrightarrow{u} \in U(x) \mid \langle p, f(x,\overrightarrow{u})\rangle + \mathbf{l}(x,\overrightarrow{u}) \right. \\ & \left. \quad = \inf_{u\in U(x)} \left(\langle p, f(x,u)\rangle + \mathbf{l}(x,u)\right) \right\} \\ (ii) & \overleftarrow{R}(y,q) := \left\{ \overleftarrow{v} \in U(y) \mid \langle q, f(y,\overleftarrow{v})\rangle - \mathbf{l}(y,\overleftarrow{v}) \right. \\ & \left. \quad = \sup_{v\in U(y)} \left(\langle q, f(y,v)\rangle - \mathbf{l}(y,v)\right) \right\} \end{cases} \tag{8.16}$$

depending only on the dynamics f and U of the control system and of the transient cost function $\mathbf{l}$.

In order to find and regulate the optimal evolution, we plug into them the partial derivatives $p := \frac{\partial \mathbf{E}(x,y)}{\partial x}$ and $q := \frac{\partial \mathbf{E}(x,y)}{\partial y}$ of the bilateral value function (actually, when constraints K are involved or when the function $\mathbf{c}$ is only lower semicontinuous, the bilateral value function is lower semicontinuous and we have to replace the partial derivatives by subgradients $(p_x, q_y) \in \partial\mathbf{E}(x,y)$ of the bilateral value-function, as indicated in Chap. 17, p. 681).

Knowing the Eupalinian value function and its partial derivatives (or subgradients), one can thus derive from the results of Sect. 17.4.3, p. 704 that the optimal Eupalinian evolution $x(\cdot)$ linking y at time 0 and z at minimal time $2T$ is governed by the control system

$$\begin{cases} (i) & x'(t) = f(x(t), u(t)) \\ (ii) & \text{where} \\ & u(t) \in \begin{cases} \overrightarrow{R}\left(x(t), \dfrac{\partial \mathbf{E}(x(t), x(2T-t))}{\partial x}\right) & \text{if } t \in [0,T] \\ \overleftarrow{R}\left(x(t), \dfrac{\partial \mathbf{E}(x(2T-t), x(t))}{\partial y}\right) & \text{if } t \in [T,2T] \end{cases} \end{cases} \tag{8.17}$$

In other words, the controls regulating an optimal evolution linking y to z *"feed" both at current sate $x(t)$ and at state $x(2T-t)$ at time $2T-t$*, forward if $t \in [0,T]$ and backward if $t \in [T,2T]$. In other words, optimal evolutions can be governed by "forward and backward retroactions", keeping an eye on the current state and the other one on the state at another instant. In particular, the initial control depends upon both the initial and final states. □

8.6 Collision Kernels

Eupalinian kernels are particular cases of *collision kernels* associated with a pair of evolutionary systems denoted by $\mathcal{S}$ and $\mathcal{T}$ associated with control systems

$$\begin{cases} (i) & y'(t) = f(y(t), u(t)) \text{ where } u(t) \in U(y(t)) \\ (ii) & z'(t) = g(z(t), v(t)) \text{ where } v(t) \in V(z(t)) \end{cases} \tag{8.18}$$

Definition 8.6.1 *[**Collision Kernels**] Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,+\infty; X)$ and $\mathcal{T} : X \rightsquigarrow \mathcal{C}(0,+\infty; X)$ be two evolutionary systems, $K \subset X$ and $L \subset X$ be two intersecting environments. We denote by $\mathcal{S}^K(y) \times \mathcal{T}^L(z)$ the set of evolutions $(y(\cdot), z(\cdot)) \in \mathcal{S}^K(y) \times \mathcal{T}^L(z)$ governed by the pair of evolutionary systems $\mathcal{S}$ and $\mathcal{T}$ viable in $K \times L$. We say that they* collide *if there exists a finite* collision time *$t^\star \geq 0$ such that $y(t^\star) = z(t^\star) \in K \cap L$.*
The collision kernel $\mathrm{Coll}_{\mathcal{S},\mathcal{T}}(K, L) \subset K \times L$ *is the subset of pairs $(y, z) \in K \times L$ such that there exist at least two viable colliding evolutions $(y(\cdot), z(\cdot)) \in \mathcal{S}^K(y) \times \mathcal{T}^L(z)$.*

Remark. Eupalinian kernels are obtained when $g = -f$, $U = V$ and $L = K$, or, equivalently, when the evolutionary system $\mathcal{R} = \overleftarrow{\mathcal{S}}$ is the backward evolutionary system. $\square$

We can characterize the collision kernel as the capture basin of an auxiliary problem, so that it inherits the properties of capture basins:

Proposition 8.6.2 *[**Viability Characterization of Collision Kernels**] Recall that* $\mathrm{Diag}(K \cap L) := \{(x, x)\}_{x \in K \cap L}$ *denotes the* diagonal *of $K \cap L$. The collision kernel* $\mathrm{Coll}_{\mathcal{S},\mathcal{T}}(K, L)$ *of $K \cap L$ under the evolutionary systems $\mathcal{S}$ and $\mathcal{T}$ associated with the systems (8.18), p. 299 is the capture basin*

$$\mathrm{Coll}_{\mathcal{S},\mathcal{T}}(K, L) = \mathrm{Capt}_{(8.18)}(K \times L, \mathrm{Diag}(K \cap L))$$

of the diagonal of $K \cap L$ viable in $K \times L$ under the auxiliary system (8.18), p. 299.

We now "quantify" the concept of collision kernel with the concept of several collision intertemporal optimization problems. The domains of their value functions are the collision kernels, so that Proposition 8.6.2, p. 299 follows from Theorem 8.6.3, p. 300 below.

We introduce a cost function $\mathbf{c} : X \times X \mapsto \mathbb{R} \cup \{+\infty\}$ (regarded as a *collision cost*) and a Lagrangian $\mathbf{l} : (y, z, u, v) \rightsquigarrow \mathbf{l}(y, z, u, v)$.

The optimal viable collision problem consists in finding colliding viable evolutions $y(\cdot) \in \mathcal{S}^K(y)$ and $z(\cdot) \in \mathcal{T}^L(z)$ and a time $t^\star \geq 0$ minimizing

$$\begin{cases} \mathbf{W}_{(\mathbf{c},\mathbf{l})}(y, z) = \inf_{(y(\cdot),z(\cdot)) \in \mathcal{S}^K(y) \times \mathcal{T}^L(z),\, t^\star \mid y(t^\star) = z(t^\star)} \\ \left(\mathbf{c}(y(t^\star), z(t^\star)) + \int_0^{t^\star} \mathbf{l}(y(t), z(t), u(t), v(t)) dt\right) \end{cases}$$

By taking $\mathbf{c} \equiv 0$ and $\mathbf{l}(y, z, u, v) \equiv 1$, we find the problem of governing two evolutions in minimal time.

We shall prove that

Theorem 8.6.3 *[**Collision Optimization Theorem**] Let us consider the auxiliary control system*

$$\begin{cases} (i) \quad y'(t) \ = \ f(y(t), u(t)) \text{ where } u(t) \ \in \ U(y(t)) \\ (ii) \quad z'(t) \ = \ g(z(t), v(t)) \text{ where } v(t) \ \in \ V(z(t)) \\ (iii) \ \lambda'(t) = -\mathbf{l}(y(t), z(t), u(t), v(t)) \end{cases} \tag{8.19}$$

Then

$$\mathbf{W}_{(\mathbf{c},\mathbf{l})}(y, z) \ = \ \inf_{(y,z,\lambda) \in \mathrm{Capt}_{(8.19)}(K \times K \times \mathbb{R}_+, \mathcal{E}p(\mathbf{c}) \cap (\mathrm{Diag}(K) \times \mathbb{R}_+))} \lambda$$

where $\mathrm{Diag}(K) := \{(x, x)\}_{x \in K} \subset K \times K$ *is the* diagonal of K.

Proof. Let $(y, z, \lambda) \in \mathrm{Capt}_{(8.19)}(K \times L \times \mathbb{R}_+, \mathcal{E}p(\mathbf{c}) \cap (\mathrm{Diag}(K) \times \mathbb{R}_+))$ belong to the capture basin. This means that there exist one evolution $y(\cdot) \in \mathcal{S}^K(y)$ viable in K, one evolution $z(\cdot) \in \mathcal{T}^L(z)$ viable in L, the evolution $\lambda(t) := \lambda - \int_0^t \mathbf{l}(y(s), z(s), u(s), v(s))ds$ and a time $t^\star$ such that:

- for all $t \in [0, t^\star]$, $y(t) \in K$, $z(t) \in L$,

$$\lambda - \int_0^t \mathbf{l}(y(s), z(s), u(s), v(s))ds \ \geq \ 0$$

- $y(t^\star) = z(t^\star)$
- and

$$\lambda - \int_0^{t^\star} \mathbf{l}(y(s), z(s), u(s), v(s))ds \ \geq \ \mathbf{c}(y(t^\star), z(t^\star))$$

This implies that

$$\mathbf{W}_{(\mathbf{c},\mathbf{l})}(y, z) \ \leq \ \mathbf{c}(y(t^\star), z(t^\star)) + \int_0^{t^\star} \mathbf{l}(y(s), z(s), u(s), v(s))ds \ \leq \ \lambda$$

and thus, that

$$\mathbf{W}_{(\mathbf{c},\mathbf{l})}(y, z) \ \leq \ \inf_{(y,z,\lambda) \in \mathrm{Capt}_{(8.19)}(K \times K \times \mathbb{R}_+, \mathcal{E}p(\mathbf{c}) \cap (\mathrm{Diag}(K) \times \mathbb{R}_+))} \lambda$$

For proving the opposite inequality, we associate with any $\varepsilon > 0$ two colliding evolutions $y_\varepsilon(\cdot) \in \mathcal{S}^K(y)$ and $z_\varepsilon(\cdot) \in \mathcal{T}^L(z)$ at some time $t^\star_\varepsilon \geq 0$, controls $u_\varepsilon(\cdot)$ and $v_\varepsilon(\cdot)$ such that

$$\left(\mathbf{c}(y_\varepsilon(t^\star_\varepsilon), z_\varepsilon(t^\star_\varepsilon)) + \int_0^{t^\star_\varepsilon} \mathbf{l}(y_\varepsilon(t), z_\varepsilon(t), u_\varepsilon(t), v_\varepsilon(t))dt\right) \leq \mathbf{W}_{(\mathbf{c},\mathbf{l})}(y,z) + \varepsilon$$

and the function

$$\lambda_\varepsilon(t) := \mathbf{W}_{(\mathbf{c},\mathbf{l})}(y,z) + \varepsilon - \int_0^t \mathbf{l}(y_\varepsilon(s), z_\varepsilon(s), u_\varepsilon(s), v_\varepsilon(s))ds$$

By construction,

$$\lambda_\varepsilon(t^\star_\varepsilon) \geq \mathbf{c}(y(t^\star_\varepsilon), z(t^\star_\varepsilon)) \text{ and } y(t^\star_\varepsilon) = z(t^\star_\varepsilon)$$

This implies that $(y, z, \mathbf{W}_{(\mathbf{c},\mathbf{l})}(y,z) + \varepsilon)$ belongs to the capture basin $\mathrm{Capt}_{(8.19)}(K \times K \times \mathbb{R}_+, \mathcal{E}p(\mathbf{c}) \cap (\mathrm{Diag}(K) \times \mathbb{R}_+))$. Hence

$$\inf_{(y,z,\lambda)\in \mathrm{Capt}_{(8.19)}(K\times K\times\mathbb{R}_+, \mathcal{E}p(\mathbf{c})\cap(\mathrm{Diag}(K)\times\mathbb{R}_+))} \lambda \leq \mathbf{W}_{(\mathbf{c},\mathbf{l})}(y,z) + \varepsilon$$

and it is enough to let ε converge to 0. $\square$

8.7 Particular Solutions to a Differential Inclusion

Consider a pair of evolutionary systems $\mathcal{S}$ and $\mathcal{T}$ associated with control systems (8.18), p. 299:

$$\begin{cases} (i) \ \ y'(t) = f(y(t), u(t)) \text{ where } u(t) \in U(y(t)) \\ (ii) \ z'(t) = g(z(t), v(t)) \text{ where } v(t) \in V(z(t)) \end{cases}$$

We look for *common solutions* $x(\cdot)$ of these two evolutionary systems (8.18). Whenever the control system (8.18)(i) is simpler to solve than the differential inclusion (8.18)(ii), the solutions of which are interpreted as *"particular" solutions, one can regard such common solutions to (8.18)(i) and (8.18)(ii) as particular solutions to the differential inclusions (8.18)(i)* and (8.18)(ii).

For instance,

- taking $g(z,v) := 0$, the common solutions are equilibria of (8.18)(i),
- taking for $g(z,v) = v$ a constant velocity, then common solutions are affine functions of time t,
- taking for $g(z,v) = -mz$, then common solutions are exponential functions of time ze^{-mt}

and so on. The problem is to detect what are the initial states y which are equilibria, from which starts an affine evolution or from which starts an exponential solution.

In other words, finding particular solutions amounts to finding the set of the initial states from which common solutions do exist.

Lemma 8.7.1 *[**Extraction of Particular Solutions**] Denote by* $\mathrm{Diag}(X) := \{(x,x)\}_{x\in X}$ *the "diagonal" of* $X \times X$*. Then the set of points from which start common solutions to the control systems is the viability kernel* $\mathrm{Viab}_{(8.18)}(\mathrm{Diag}(X))$ *of the diagonal under (8.18).*

Proof. Indeed, to say that $x(\cdot) \in \mathcal{S}(x) \cap \mathcal{T}(x)$ is a common solution to control systems (8.18), p. 299 amounts to saying that the pair $(x(\cdot), x(\cdot))$ is a solution to system (8.18) viable in the diagonal $\mathrm{Diag}(K)$, so that (x,x) belongs to the viability kernel $\mathrm{Viab}_{(8.18)}(\mathrm{Diag}(X))$. Conversely, to say that (x,x) belongs to this viability kernel amounts to saying that there exist evolutions $(y(\cdot), z(\cdot)) \in \mathcal{S}(x) \times \mathcal{T}(y)$ viable in the diagonal $\mathrm{Diag}(X)$, so that, for all $t \geq 0$, $y(t) = z(t)$ is a common solution. $\square$

Being a viability kernel, the subset of initial states from which start particular evolutions inherits the properties of viability kernels.

8.8 Visiting Kernels and Chaos À la Saari

The fundamental problem of *qualitative analysis* (and in particular, of *qualitative physics* in computer sciences and *comparative statics* in economics) is the following: subsets $C_n \subset K$ are assumed to describe "qualitative properties", and thus are regarded as *qualitative cells* or, simply, cells. Examples of such qualitative cells are the *monotonic cells* (see Definition 9.2.14, p. 332). Given such an ordered sequence of overlapping qualitative cells, the question arises whether there exist viable evolutions visiting successively these qualitative cells in prescribed order. These are questions treated by *Donald Saari* that we partially cover here (see [182–184, Saari]).

We answer here the more specific question of the existence of such viable evolutions visiting not only finite sequences of cells, but also infinite sequences. Existence of viable visiting cells require some assumptions.

Definition 8.8.1 *[**Visiting Kernels**] Let us consider a sequence of nonempty closed subsets* $C_n \subset K$ *such that* $C_n \cap C_{n+1} \neq \emptyset$*. An evolution* $x(\cdot) \in \mathcal{S}(x)$ *is* visiting *the subsets* C_n *successively in the following sense: there exists a sequence of finite duration* $\tau_n \geq 0$ *such that, starting with*

t_0, for all $n \geq 0$,

$$\begin{cases} (i) \;\; t_{n+1} \;=\; t_n + \tau_n \\ (ii) \;\forall\, t \in [t_n, t_{n+1}], \;\; x(t) \in C_n \text{ and } x(t_{n+1}) \in C_{n+1} \end{cases} \tag{8.20}$$

The set $\mathrm{Vis}_{\mathcal{S}}(K, \overrightarrow{C})$ *of initial states* $x \in K$ *such that there exists an evolution* $x(\cdot) \in \mathcal{S}(x)$ *visiting successfully the cells* C_n *is called the* visiting kernel *of the sequence* $\overrightarrow{C}$ *of cells* C_n *viable in* K *under the evolutionary system* $\mathcal{S}$.
The T*-visiting kernel* $\mathrm{Vis}_{\mathcal{S}}(K, \overrightarrow{C})(T)$ *is the set of initial states from which starts at least one viable evolution visiting the cells with duration* τ_n *bounded by* T.

We begin by considering the case of a finite sequence of cells and a result due to *Donald Saari*:

Lemma 8.8.2 ***[Existence of Evolutions Visiting a Finite Number of Cells]*** *Let us consider a finite sequence of subsets* $C_n \subset K$ *(*$n = 0, \ldots, N$*) such that*

$$T \;:=\; \sup_{n=0,\ldots,N-1} \; \sup_{y \in C_{n+1}} \; \inf_{z \in C_n} \epsilon_{C_n}(y,z) \;<\; +\infty \tag{8.21}$$

where ϵ_{C_n} *is the* Eupalinian function *viable in* C_n *(see Definition 8.5.4, p. 294).*

Then, the T*-visiting kernel* $\mathrm{Vis}_{\mathcal{S}}(K, C_1, \ldots, C_N)(T)$ *of this finite sequence is not empty.*

Proof. We set

$$M_{N-1}^N := \mathrm{Capt}_{\mathcal{S}}(C_{N-1}, C_{N-1} \cap C_N)(T)$$

which is the subset of $x \in C_{N-1}$ such that there exist $\tau \in [0, T]$ and an evolution $x(\cdot) \in \mathcal{S}(x)$ viable in C_{N-1} on $[0, T]$ such that $x(\tau) \in C_N$. For $j = N-2, \ldots, 0$, we define recursively the cells:

$$M_j^N := \mathrm{Capt}_{\mathcal{S}}(C_j, C_j \cap M_{j+1}^N)(T)$$

which can be written

$$M_j^N = \{x \in C_j \mid \exists\, \tau_j \in [0,T],\, \exists\, x(\cdot) \in \mathcal{S}^{C_j}(x) \text{ such that } \; x(\tau_j) \in M_{j+1}^N\}$$

Therefore the set $\mathrm{Vis}_{\mathcal{S}}(K, C_0, \ldots, C_N)(T) = M_0^N$ is the set of initial states $x_0 \in C_0$ from which at least one solution will visit successively the cells $C_j,\ j = 0, \ldots, N$. $\square$

We shall prove here the existence of viable evolutions visiting a infinite number of cells C_n, although they require to take the limit, and therefore, to use theorems of Chap. 10, p. 375 (the proof can thus be omitted in a first reading).

Proposition 8.8.3 ***[Existence of Evolutions Visiting an Infinite Sequence of Cells]*** *Let K be a closed subset viable under an upper semicompact evolutionary system $\mathcal{S}$. We consider a sequence of compact subsets $C_n \subset K$ and we assume that*

$$T \;:=\; \sup_{n \,\geq\, 0} \sup_{y \in C_{n+1}} \inf_{z \in C_n} \epsilon_{C_n}(y, z) \;<\; +\infty \tag{8.22}$$

Then, the T-visiting kernel $\mathrm{Vis}_{\mathcal{S}}(K, \overrightarrow{C})(T)$ of this infinite sequence is not empty.

Proof. We shall prove that the intersection

$$K_\infty \;:=\; \bigcap_{n \,\geq\, 0} \mathrm{Vis}_{\mathcal{S}}(K, C_1, \ldots, C_n)(T) \;\subset\; \mathrm{Vis}_{\mathcal{S}}(K, \overrightarrow{C})(T)$$

is not empty and contained in the visiting kernel $\mathrm{Vis}_{\mathcal{S}}(K, \overrightarrow{C})(T)$. Lemma 8.8.2, p. 303 implies that the visiting kernels $\mathrm{Vis}_{\mathcal{S}}(K, C_1, \ldots, C_n)(T)$ are not empty, and closed since the evolutionary system is upper semicompact (see Theorem 10.3.14, p. 390). Since the family of subsets $\mathrm{Vis}_{\mathcal{S}}(K, C_1, \ldots, C_n)(T)$ form a decreasing family and since K is compact, the intersection K_∞ is nonempty.

It remains to prove that it is contained in the visiting kernel $\mathrm{Vis}_{\mathcal{S}}(K, \overrightarrow{C})(T)$. Let us take an initial state x in K_∞ and fix n. Hence there exist $x_n(\cdot) \in \mathcal{S}(x)$ and a sequence of $t_n^j \in [0, jT]$ such that

$$\forall j = 1, \ldots, n,\ x_n(t_n^j) \in M_j^n \subset C_j \text{ and } \forall t \in [t_n^{j-1}, t_n^j],\ x_n(t) \in C_j$$

Indeed, there exist $y_1(\cdot) \in \mathcal{S}(x)$ and $\tau_1^n \in [0, T]$ such that $y_1(\tau_1^n)$ belongs to M_1^n. We set $t_1^n := \tau_1^n$, $x_1^n = y_1(t_1^n)$ and $x_n(t) := y_1(t)$ on $[0, t_1^n]$.

Assume that we have built $x_n(\cdot)$ on the interval $[0, t_j^n]$ such that $x_n(t_j^n) \in M_j^n \subset C_j$ for $j = 1, \ldots, k$. Since $x_n(t_k^n)$ belongs to M_k^n, there exist $y_{k+1}(\cdot) \in \mathcal{S}(x_n(t_k^n))$ and $\tau_{k+1}^n \in [0, T]$ such that

$$y_{k+1}(\tau_{k+1}^n) \in M_{k+1}^n$$

We set

$$t_{k+1}^n := t_k^n + \tau_{k+1}^n \ \& \ x_n(t+\tau_k^n) := y_{k+1}(t)$$

on $[t_k^n, t_{k+1}^n]$. When $k = n$, we extend $x_n(\cdot)$ to $[t_n^n, +\infty[$ by any evolution starting at $x_n(t_n^n)$ at time t_n^n.

Since the evolutionary system is assumed to be upper semicompact, the Stability Theorem 10.3.3, p. 385 implies that a subsequence (again denoted $x_n(\cdot)$) of the sequence $x_n(\cdot) \in \mathcal{S}(x)$ converges (uniformly on compact intervals) to some evolution $x(\cdot) \in \mathcal{S}(x)$ starting at x. By extracting successive converging subsequences of $\tau_j^n \in [0,T]$ converging to τ_j when $n \geq j \to +\infty$ and setting $t_{j+1} := t_j + \tau_j$, we infer that $x(t_j) \in C_j$. □

As a consequence, we obtain an extension to evolutionary systems of a theorem on "chaos" due to *Donald Saari*:

Theorem 8.8.4 ***[Chaotic Behavior à la Saari]*** *Let K be a compact subset viable under an upper semicompact evolutionary system $\mathcal{S}$. We assume that K is covered by a family of closed subsets K_a ($a \in \mathcal{A}$) satisfying the following assumption:*

$$T := \sup_{a\in\mathcal{A}} \sup_{y\in K} \inf_{z\in K_a} \epsilon_{K_a}(y,z) < +\infty \tag{8.23}$$

Then, for any sequence $a_0, a_1, \ldots, a_n, \ldots$, there exists at least one evolution $x(\cdot) \in \mathcal{S}(x)$ and an increasing sequence of elements $t_j \geq 0$ such that for all $j \geq 0$, $\forall t \in [t_j, t_{j+1}]$, $x(t) \in K_{a_j}$ and $x(t_{j+1}) \in K_{a_{j+1}}$.

Proof. We associate with the sequence $a_0, a_1, \ldots, a_n, \ldots$ the sequence of subsets $C_n := K_{a_n}$ and we observe that assumption (8.22) of Proposition 8.8.3, p. 304 is satisfied. □

8.9 Contingent Temporal Logic

By identifying a subset $K \subset X$ with the subset of elements $x \in X$ satisfying the property $\mathcal{P}_K(x)$ of belonging to K, i.e., $\mathcal{P}_K(x)$ if and only if $x \in K$, we know that we *can identify implication and negation with inclusion and complementation*:

$$\begin{cases} (i) \ \ \mathcal{P}_K \Rightarrow \mathcal{P}_L \ (\mathcal{P}_K \text{ implies } \mathcal{P}_L) \text{ if and only if } K \subset L \\ (ii) \ \neg\mathcal{P}_K(x) \ (\text{not } \mathcal{P}_K(x)) \text{ if and only if } x \in \complement K \end{cases}$$

These axioms have been relaxed in many ways to define other logics. We adapt to the "temporal case" under uncertainty the concept of atypic logic introduced by Michel de Glas.

Taking into account time and uncertainty into logical operations requires an evolutionary system $\mathcal{S}$, associating with any x for instance the set $\mathcal{S}(x)$ of solutions $x(\cdot)$ to a differential inclusion $x' \in F(x)$ starting at x.

Definition 8.9.1 *[**Eventual Consequences**] Given an evolutionary system, we say that y is an* eventual consequence *of x – and write $y \succeq x$ – if there exist an evolution $x(\cdot) \in \mathcal{S}(x)$ starting from x and a time $T \geq 0$ such that $y = x(T)$ is reached by this evolution.*

This binary relation $y \succeq x$ is the contingent temporal preorder *associated with the evolutionary system $\mathcal{S}$,* temporal *because evolution is involved,* contingent *because this evolution is contingent.*

It is obvious that the binary relation $y \succeq x$ is:

1. **reflexive:** $x \succeq x$ and
2. **transitive:** if $y \succeq x$ and $z \succeq y$, then $z \succeq x$,

so that it is, by definition, a *preorder* on X.

Definition 8.9.2 *[**Contingent Temporal Implications and Falsification**] Let us consider an evolutionary system $\mathcal{S}$. We say that x:*

1. satisfies typically *($\overline{\wedge}\mathcal{P}_K(x)$) property $\mathcal{P}_K$ if **all eventual consequences** of x satisfy property $\mathcal{P}_K$:*
2. satisfies atypically *($\underline{\vee}\mathcal{P}_K(x)$) property $\mathcal{P}_K$ if **at least one eventual consequence** of x satisfies property $\mathcal{P}_K$,*
3. falsifies *($\exists\mathcal{P}_K(x)$) property $\mathcal{P}_K$ if **at least one eventual consequence** of x does not satisfy property $\mathcal{P}_K$,*

Falsification. *The contingent temporal preorder allows us to define a concept of falsification (in French,* réfutation*), which translates mathematically a weaker concept of negation which Karl Popper (1902–1994) made popular.*

These definitions can readily be formulated in terms of capture basins and invariance kernels:

Lemma 8.9.3 *[**Viability Formulation of Contingent Temporal Operations**] The formulas below relate logical operations to invariance kernels and capture basins:*

$$\begin{cases} (i) & \barwedge\mathcal{P}_K(x) \text{ if and only if } x \in \mathrm{Inv}(K) := \mathrm{Inv}(K, \emptyset) \\ & x \textit{ satisfies typically } \text{property } \mathcal{P}_K \\ (ii) & \veebar\mathcal{P}_K(x) \text{ if and only if } x \in \mathrm{Capt}(K) := \mathrm{Capt}(X, K) \\ & x \textit{ satisfies atypically } \text{property } \mathcal{P}_K \\ (iii) & \exists\mathcal{P}_K(x) \text{ if and only if } x \in \mathrm{Capt}(\complement(K)) \\ & x \textit{ falsifies } (\text{or } \textit{does not typically satisfies}) \text{ property } \mathcal{P}_K \end{cases}$$

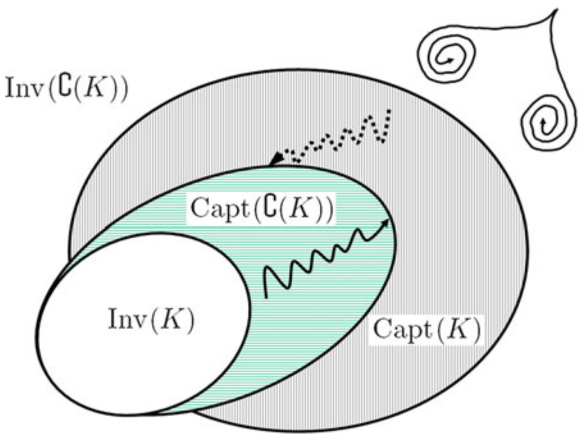

Fig. 8.4 Typical, atypical and falsifying elements.
This figure illustrates the above concepts: the subset K of elements satisfying property $\mathcal{P}_K$ is partitioned in the set $\mathrm{Inv}(K)$ *of* typical elements *satisfying property $\mathcal{P}_K$, and in the set* $\mathrm{Capt}(\complement(K))$ *of elements* falsifying *property $\mathcal{P}_K$. The capture basin* $\mathrm{Capt}(K)$ *of K is the set of* atypical elements *satisfying property $\mathcal{P}_K$.*

We now translate some elementary properties of capture basins and invariance kernels: For instance, contingent temporal logics are nonconsistent in the following sense:

Proposition 8.9.4 *[**Non-Consistency of Contingent Temporal Logics**] The contingent temporal logic is not consistent in the sense that:*

1. $\barwedge\mathcal{P}_K(x) \vee \exists\mathcal{P}_K(x)$ is always true, *2.* $\veebar\mathcal{P}_K(x) \wedge \exists\mathcal{P}_K(x)$ *may be true (or is not false):* $\veebar\mathcal{P}_K(x) \wedge \exists\mathcal{P}_K(x)$ *if and only if* x *both atypically satisfies and falsifies property* $\mathcal{P}_K$

3. *The falsification of the falsification of property* $\mathcal{P}_K$ *is the set of element satisfying extensively and intensively this property:*

$$\exists\exists\mathcal{P}_K(x) \;\Leftrightarrow\; \veebar\,\bar{\wedge}\,\mathcal{P}_K(x)$$

The relationships with conjunction and disjunction become

$$\begin{cases} (i) \;\; \exists(\mathcal{P}_{K_1} \wedge \mathcal{P}_{K2}) \text{ if and only if } \exists\mathcal{P}_{K_1} \vee \exists\mathcal{P}_{K_2} \\ (ii) \; \exists(\mathcal{P}_{K_1} \vee \mathcal{P}_{K2}) \text{ implies } \exists\mathcal{P}_{K_1} \wedge \exists\mathcal{P}_{K_2} \end{cases}$$

Definition 8.9.5 *[**Contingent Temporal Implications**] With an evolutionary system* $\mathcal{S}$*, we associate the following logical operations:*

1. Intensive contingent temporal implication

$$\mathcal{P}_K \rightrightarrows \mathcal{P}_L$$

*means that **all eventual consequences** of elements satisfying property* $\mathcal{P}_K$ *satisfy property* $\mathcal{P}_L$

2. Extensive contingent temporal implication

$$\mathcal{P}_K \Rrightarrow \mathcal{P}_L$$

*means that whenever **at least one eventual consequence** of an element satisfies property* $\mathcal{P}_K$*, it satisfies property* $\mathcal{P}_L$*.*

We observe that the *intensive and extensive contingent temporal implications imply the usual implication.*

Lemma 8.9.6 *[**Viability Characterization of Implications**] Extensive and intensive implications are respectively formulated in this way:*

$$\begin{cases} (i) \;\; \mathcal{P}_K \Rrightarrow \mathcal{P}_L \; (\mathcal{P}_K \text{ extensively } \textit{implies } \mathcal{P}_L) \textit{ if and only if } K \subset \mathrm{Inv}(L) \\ (ii) \; \mathcal{P}_K \rightrightarrows \mathcal{P}_L \; (\mathcal{P}_K \text{ intensively } \textit{implies } \mathcal{P}_L) \textit{ if and only if } \mathrm{Capt}(K) \subset L \end{cases}$$

and weak extensive and intensive implications defined respectively by

$$\begin{cases} (i) & \mathcal{P}_K \rightharpoonup \mathcal{P}_L \text{ } (\mathcal{P}_K \text{ weakly extensively } \textit{implies } \mathcal{P}_L) \textit{ if and only if} \\ & \mathrm{Capt}(K) \subset \mathrm{Capt}(L) \\ (ii) & \mathcal{P}_K \rightharpoondown \mathcal{P}_L \text{ } (\mathcal{P}_K \text{ weakly intensively } \textit{implies } \mathcal{P}_L) \textit{ if and only if} \\ & \mathrm{Inv}(K) \subset \mathrm{Inv}(L) \end{cases}$$

We infer the following

Proposition 8.9.7 *[**Contraposition Properties**] The following statements are equivalent:*

1. *property $\mathcal{P}_K$ intensively implies $\mathcal{P}_L$:*

$$\mathcal{P}_K \rightrightarrows \mathcal{P}_L$$

2. *negation of property $\mathcal{P}_L$ extensively implies the negation of property $\mathcal{P}_K$:*

$$\neg\mathcal{P}_L \Rrightarrow \neg\mathcal{P}_K$$

3. *falsification of property $\mathcal{P}_L$ implies the negation of property $\mathcal{P}_K$:*

$$\dashv\mathcal{P}_L \Rightarrow \neg\mathcal{P}_K$$

8.10 Time Dependent Evolutionary Systems

8.10.1 Links Between Time Dependent and Independent Systems

Consider the time-dependent system

$$x'(t) = f(t, x(t), u(t)) \text{ where } u(t) \in U(t, x(t))$$

Definition 8.10.1 *[**Time-Dependent Systems**] When the dynamics*

$$\begin{cases} (i) & x'(t) = f(t, x(t), u(t)) \\ (ii) & u(t) \in U(t, x(t)) \end{cases} \tag{8.24}$$

of a system depend upon the time, we denote by $\mathcal{S} : \mathbb{R}\times X \rightsquigarrow \mathcal{C}(-\infty, +\infty; X)$ the time-dependent evolutionary system *associating with any (T, x) the set* *of evolutions* $x(\cdot) \in \mathcal{C}(-\infty,+\infty;X)$ *governed by this time-dependent system passing through* x *at time* T*:* $x(T) = x$*. Whenever* $\mathbf{K} : t \rightsquigarrow K(t)$ *is a tube, we denote by* $\mathcal{S}^{\mathbf{K}}(x)$ *the set of evolutions* $x(\cdot) \in \mathcal{S}(x)$ *such that*

$$\forall\, t \geq 0,\ \ x(t)\ \in\ K(t)$$

Splitting evolutions allows us to decompose a full evolution passing through a given state at present time 0 into its backward and forward parts both governed by backward and forward evolutionary systems:

The *backward time-dependent system* $\overleftarrow{\mathcal{S}} : \mathbb{R} \times X \rightsquigarrow \mathcal{C}(-\infty,+\infty;X)$ associates with any (T,x) the set of evolutions $x(\cdot) \in \mathcal{C}(-\infty,+\infty;X)$ passing through x at time T: $x(T) = x$ and governed by

$$\begin{cases} (i) \ \ \overleftarrow{x}'(t) = -f(-t, \overleftarrow{x}(t), \overleftarrow{u}(t)) \\ (ii) \ \overleftarrow{u}(t) \in U(-t, \overleftarrow{x}(t)) \end{cases} \tag{8.25}$$

We observe that $x(\cdot) \in \mathcal{S}(T,x)$ if and only if:

1. its forward part $\overrightarrow{x}(\cdot) := \kappa(T)(x(\cdot))(\cdot)$ at time T defined by $\kappa(T)(x(\cdot))(t) = x(t-T)$ is a solution to differential inclusion
$$\overrightarrow{x}'(t)\ =\ f(T+t, \overrightarrow{x}(t), \overrightarrow{u}(t)) \text{ where } \overrightarrow{u}(t) \in U(T+t, \overrightarrow{x}(t))$$
satisfying $\overrightarrow{x}(0)\ =\ x$.
2. its backward part $\overleftarrow{x}(\cdot) := (\overset{\vee}{\kappa}\,(T)x(\cdot))(\cdot)$ at time T defined by $(\overset{\vee}{\kappa}\,(T)x(\cdot))(t) = x(T-t)$ is a solution to differential inclusion
$$\overleftarrow{x}'(t)\ =\ f(T-t, \overleftarrow{x}(t), \overleftarrow{x}(t)) \text{ where } \overleftarrow{u}(t) \in U(T-t, \overleftarrow{x}(t))$$
satisfying $\overleftarrow{x}(0)\ =\ x$.

This implies that when the system is time-independent, the *backward time-independent system* $\overleftarrow{\mathcal{S}} : X \rightsquigarrow \mathcal{C}(0,+\infty;X)$ associates with any $x \in X$ the set of evolutions $\overleftarrow{x}(\cdot) \in \mathcal{C}(0,+\infty;X)$ passing through x at time T: $x(T) = x$ and governed by (8.25), p. 310, which boils down to

$$\begin{cases} (i) \ \ \overleftarrow{x}'(t)\ =\ -f(\overleftarrow{x}(t), \overleftarrow{u}(t)) \\ (ii) \ \overleftarrow{u}(t)\ \in\ U(\overleftarrow{x}(t)) \end{cases}$$

This also allows us to introducing an auxiliary "time variable" τ the absolute value of the velocity of which is equal to one: $+1$ for forward time, -1 for backward time (time to horizon, time to maturity in finance, etc.).

Definition 8.10.2 ***[Time-Independent Auxiliary System]*** *We associate with the time-dependent evolutionary system (8.24), p. 309 the* time-independent auxiliary system $\mathcal{A}_{\mathcal{S}}$ *associated with:*

$$\begin{cases} (i) \ \ \tau'(t) = 1 \\ (ii) \ x'(t) = f(\tau(t), x(t), u(t)) \\ \qquad \text{where } u(t) \in U(\tau(t), x(t)) \end{cases} \tag{8.26}$$

and the backward time-independent auxiliary system $\mathcal{A}_{\overleftarrow{\mathcal{S}}}$ *associated with*

$$\begin{cases} (i) \ \ \tau'(t) = -1 \\ (ii) \ x'(t) = -f(\tau(t), x(t), u(t)) \\ \qquad \text{where } u(t) \in U(\tau(t), x(t)) \end{cases} \tag{8.27}$$

We can reformulate the evolutions of the time-dependent system in terms of evolutions of the auxiliary time-independent systems $\mathcal{A}_{\mathcal{S}}$:

Lemma 8.10.3 ***[Links Between Evolutionary Systems and their Auxiliary Systems]***

1. *an evolution* $x_+(\cdot) \in \mathcal{S}(T, x)$ *is a solution to system (8.24), p. 309 starting from* x *at time* T *and defined on the interval* $[T, +\infty[$ *if and only if* $x_+(\cdot) = \kappa(T)\overrightarrow{x}(\cdot)$ *where* $(\overrightarrow{\tau}(\cdot), \overrightarrow{x}(\cdot))$ *is a solution of the auxiliary time-independent system (8.26) starting at* (T, x).
2. *an evolution* $x_-(\cdot) \in \mathcal{S}(T, x)$ *is a solution to system (8.24) arriving at* x *at time* T *and defined on the interval* $]-\infty, T]$ *if and only if* $x_-(\cdot) = \overset{\vee}{\kappa}(T)\overleftarrow{x(\cdot)}(\cdot)$ *where* $(\overleftarrow{\tau}(\cdot), \overleftarrow{x}(\cdot))$ *is a solution to the backward auxiliary time-independent system (8.27) starting at* (T, x).

In other words, an evolution $x(\cdot) \in \mathcal{S}(T, x)$ governed by time-dependent system (8.24), p. 309

$$x'(t) = f(t, x(t), u(t)) \text{ where } u(t) \in U(t, x(t))$$

can be split in the form

$$x(t) \ := \begin{cases} \overleftarrow{x}(-t) \text{ if } t \leq 0 \\ \overrightarrow{x}(t) \quad \text{if } t \geq 0 \end{cases}$$

where $\overleftarrow{x}(\cdot) \in \mathcal{A}_{\overleftarrow{\mathcal{S}}}(T, x)$ and $\overrightarrow{x}(\cdot) \in \mathcal{A}_{\mathcal{S}}(T, x)$.

Proof. Indeed, let $x_+(\cdot)$ satisfying $x_+(T) = x$ and $x'_+(t) = f(t, x_+(t), u_+(t))$. Therefore, $\overrightarrow{x}(\cdot) := \kappa(-T)x_+(\cdot)$ defined by $\overrightarrow{x}(t) := x_+(t+T)$ satisfies $\overrightarrow{x}(0) := x_+(T) = x$ and $\overrightarrow{x}'(t) := x'_+(t+T) = f(t+T, x'_+(t+T), u_+(t+T)) =$

$f(\overrightarrow{\tau}(t), \overrightarrow{x}(t), \overrightarrow{u}(t))$ where $\overrightarrow{\tau}(t) := t + T$. This means that $(\overrightarrow{\tau}(\cdot), \overrightarrow{x}(\cdot))$ is a solution of the auxiliary time-independent system (8.26) starting at (T, x).

In the same way, let $x_-(\cdot)$ satisfying $x_-(T) = x$ and $x'_-(t) = f(t, x_-(t), u_-(t))$. Therefore, $\overleftarrow{x}(\cdot) := (\overset{\vee}{\kappa}(T)x_-(\cdot))(\cdot)$ defined by $\overleftarrow{x}(t) := x_-(T - t)$ satisfies $\overleftarrow{x}(0) := x_-(T) = x$ and $\overleftarrow{x}'(t) := -x'_-(T - t) = f(T - t, x'_-(T - t), u_-(T - t)) = f(\overleftarrow{\tau}(t), \overleftarrow{x}(t), \overleftarrow{u}(t))$ where $\overleftarrow{\tau}(t) := T - t$. This means that $(\overleftarrow{\tau}(\cdot), \overleftarrow{x}(\cdot))$ is a solution of the backward auxiliary time-independent system (8.27) starting at (T, x). $\square$

Consequently, we just have to transfer the properties of the forward and backward systems of time-independent systems in forward time for obtaining the properties of time-dependent systems.

8.10.2 Reachable Maps and Detectors

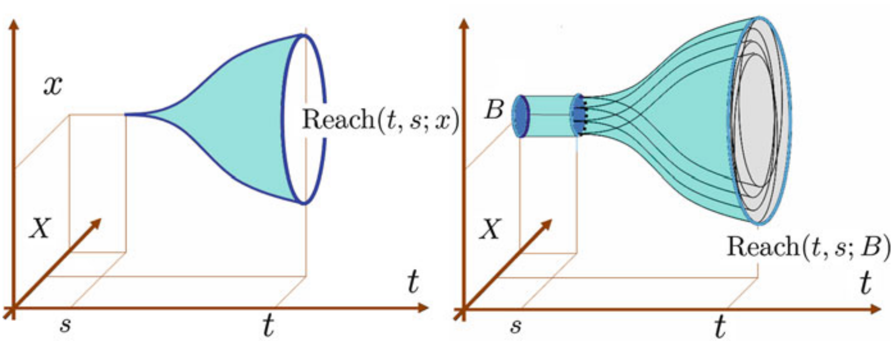

Fig. 8.5 Reachable Maps.
Left: Illustration of the reachable map Reach$(t, s; x)$ *associated with a point* x *between the times* s *and* t. *Right: Illustration of the reachable tube* Reach$(t, s; B)$ *associated with a set* B *between the times* s *and* t.

Definition 8.10.4 *[**Viable Reachable Tubes**] Let us consider a tube* $\mathbf{K}$ *regarded as the tube of time-dependent environments* $\mathbf{K}(t)$ *and a source* $B \subset X$. *Let* $\mathcal{S}^{\mathbf{K}} : \mathbb{R} \times X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ *be the evolutionary system associated with (8.24), p. 309*

$$x'(t) = f(t, x(t), u(t)) \text{ where } u(t) \in U(t, x(t))$$

the set of evolutions viable in the tube $\mathbf{K}$. *The* viable reachable map $\text{Reach}_{\mathcal{S}}^{\mathbf{K}}((\cdot), s; x) : t \rightsquigarrow \text{Reach}_{\mathcal{S}}^{\mathbf{K}}(t, s; x)$ *associating with any* $x \in \mathbf{K}(s)$ *the* *set of $x(t)$ when $x(\cdot) \in \mathcal{S}^{\mathbf{K}}(s,x)$ ranges over the set of evolutions starting from x at time $s \leq t$ and viable in the tube:*

$$\forall\, x \in X,\ \forall t \geq s \geq 0,\ \mathrm{Reach}_{\mathcal{S}}^{\mathbf{K}}(t,s;B) \ := \ \{x(t)\}_{x(\cdot)\in\mathcal{S}^{\mathbf{K}}(s,B)}$$

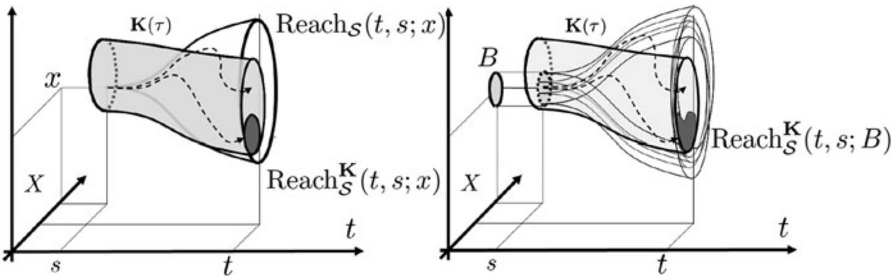

Fig. 8.6 Reachable Tubes.
Left: Illustration of $\mathrm{Reach}_{\mathcal{S}}^{\mathbf{K}}(t,s;x)$ *as defined in Definition 8.10.4. It depicts the reachable tube* $\mathrm{Reach}_{\mathcal{S}}(t,s;x)$ *without constraints and the tube representing the evolving environment* $\mathbf{K}(\cdot)$*. The dark area at time t is the viable reachable set* $\mathrm{Reach}_{\mathcal{S}}^{\mathbf{K}}(t,s;B)$*. It is contained in the intersection of the constrained tube and the reachable tube without constraints. Right: Illustration of* $\mathrm{Reach}_{\mathcal{S}}^{\mathbf{K}}(t,s;B)$*, obtained by taking the union of the tubes* $\mathrm{Reach}_{\mathcal{S}}^{\mathbf{K}}(t,s;x)$ *when x ranges over B.*

Definition 8.10.5 *[**Detectors**] Consider an evolutionary system $\mathcal{S} : \mathbb{R} \times X \rightsquigarrow \mathcal{C}(-\infty,+\infty;X)$ and two tubes $\mathbf{K}(\cdot) : t \rightsquigarrow \mathbf{K}(t)$ and $\mathbf{B}(\cdot) : t \rightsquigarrow \mathbf{B}(t) \subset \mathbf{K}(t)$. The* detector $\mathrm{Det}_{\mathcal{S}}(\mathbf{K},\mathbf{B}) : \mathbb{R}_+ \rightsquigarrow X$ *associates with any $T \geq 0$ the (possibly empty) subset* $\mathrm{Det}_{\mathcal{S}}(\mathbf{K},\mathbf{B})(T)$ *of states $x \in \mathbf{K}(T)$ which can be reached by at least one evolution $x(\cdot)$ starting at some finite earlier time $\tau \leq T$ from $\mathbf{B}(\tau)$ and* viable *on the interval $[\tau, T]$ in the sense that*

$$\forall t \in [\tau, T],\ x(t) \in \mathbf{K}(t) \tag{8.28}$$

In other words, it is defined by formula:

$$\mathrm{Det}_{\mathcal{S}}(\mathbf{K},\mathbf{B})(T) \ := \ \bigcup_{s \leq T} \mathrm{Reach}_{\mathcal{S}}^{\mathbf{K}}(T,s;\mathbf{B}(s)) \tag{8.29}$$

Observe that when the system is time-independent, the formula for detectors boils down to

$$\mathrm{Det}_{\mathcal{S}}(\mathbf{K},\mathbf{B})(T) \;=\; \bigcup_{0\le s\le T} \mathrm{Reach}_{\mathcal{S}}^{\mathbf{K}}(T-s,0;\mathbf{B}(s))$$

By taking for tube $\mathbf{B}_\emptyset(\cdot)$ the tube defined by

$$\mathbf{B}_\emptyset(t) := \begin{cases} B \text{ if } t=0 \\ \emptyset \;\text{ if } t>0 \end{cases}$$

we recognize the viable reachable tube

$$\mathrm{Det}_{\mathcal{S}}(\mathbf{K},\mathbf{B}_\emptyset)(T) \;:=\; \mathrm{Reach}_{\mathcal{S}}^{\mathbf{K}}(T,0;B)$$

and by taking the constant tube $\mathbf{B}_0 : t \rightsquigarrow B$, we obtain

$$\mathrm{Det}_{\mathcal{S}}(\mathbf{K},\mathbf{B}_0)(T) \;:=\; \mathrm{Det}_{\mathcal{S}}(\mathbf{K},B)(T)$$

An illustration of a detector is shown in Fig. 8.7. This figures relates to Theorem 8.10.6 below.

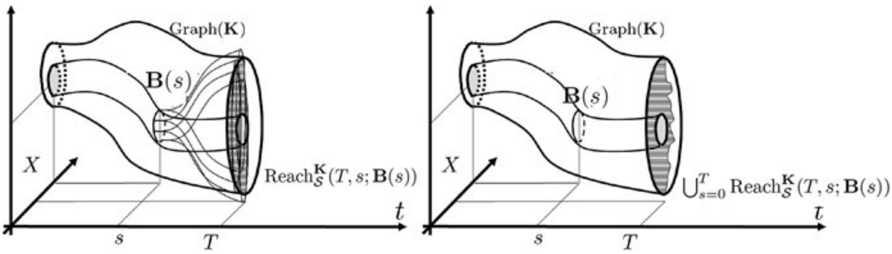

Fig. 8.7 Detectors.
Left: Illustration of $\mathrm{Reach}_{\mathcal{S}}^{\mathbf{K}}(T,s;\mathbf{B}(s))$. ***Right:*** *Illustration of* $\bigcup_{s=0}^{T}\mathrm{Reach}_{\mathcal{S}}^{\mathbf{K}}$ $(T,s;\mathbf{B}(s))$ *which is the detector* $\mathrm{Det}_{\mathcal{S}}(\mathbf{K},\mathbf{B})(T)$.

As for the viable reachable map, the graph of the detector is the viability kernel of the graph of the tube $\mathbf{K}(\cdot)$ with the target chosen to be the graph of the source tube $\mathbf{B}(\cdot)$ under the auxiliary evolutionary system (8.27).

Theorem 8.10.6 ***[Viability Characterization of Detectors]*** *The graph of the detector* $\mathrm{Det}_{\mathcal{S}}(\mathbf{K},\mathbf{B})$ *is the capture basin of the target* $\mathrm{Graph}(\mathbf{B})$ *viable in the graph* $\mathrm{Graph}(\mathbf{K})$ *under the auxiliary system (8.27):*

$$\mathrm{Graph}(\mathrm{Det}_{\mathcal{S}}(\mathbf{K},\mathbf{B})) = \mathrm{Capt}_{(8.27)}(\mathrm{Graph}(\mathbf{K}),\mathrm{Graph}(\mathbf{B}))$$

Furthermore, the detector is the unique tube $\mathbf{D}$ *between the tubes* $\mathbf{B}$ *and* $\mathbf{K}$ *satisfying the "bilateral fixed tube" property:*

$$\mathbf{D}(T) \;=\; \bigcup_{s\leq T} \mathrm{Reach}_{\mathcal{S}}^{\mathbf{D}}(T,s;\mathbf{B}(s))$$

and the Volterra property

$$\mathbf{D}(T) \;=\; \bigcup_{s\leq T} \mathrm{Reach}_{\mathcal{S}}^{\mathbf{K}}(T,s;\mathbf{D}(s)) \tag{8.30}$$

Proof. Indeed, to say that (T,x) belongs to the capture basin of target $\mathrm{Graph}(\mathbf{B})$ viable in $\mathrm{Graph}(\mathbf{K})$ under the auxiliary system (8.27) means that there exist an evolution $\overleftarrow{x}(\cdot)$ to the backward system

$$\begin{cases} (i) & \overleftarrow{x}'(t) = -f(T-t, \overleftarrow{x}(t), \overleftarrow{u}(t)) \\ (ii) & \overleftarrow{u}(t) \in U(T-t, \overleftarrow{x}(t)) \end{cases}$$

starting at $\overleftarrow{x}(0) := x$ and a time $t^* \geq 0$ such that

$$\begin{cases} (i) & \forall t \in [0,t^*],\ (T-t, \overleftarrow{x}(t)) \in \mathrm{Graph}(\mathbf{K}) \\ (ii) & \qquad\qquad (T-t^*, \overleftarrow{x}(t^*)) \in \mathrm{Graph}(\mathbf{B}) \end{cases}$$

The second condition means that $\overleftarrow{x}(t^\star)$ belongs to $\mathbf{B}(T-t^\star)$. The first one means that for every $t \in [t^\star, T]$, $\overleftarrow{x}(t) \in \mathbf{K}(T-t)$. This amounts to saying that the evolution $x(\cdot) := \overset{\vee}{\kappa}(T)\overleftarrow{x}(\cdot) = \overleftarrow{x}(T-\cdot)$ is a solution to the parameterized system (8.24), p. 309

$$x'(t) = f(t,x(t),u(t)) \text{ where } u(t) \in U(t,x(t))$$

starting at $\overleftarrow{x}(T-t^\star) \in \mathbf{B}(T-t^\star)$, satisfying $x(T) = x$ and

$$\forall t \in [T-t^\star, T],\ x(t) \in \mathbf{K}(t)$$

Setting $s^\star := T - t^\star$, this means that $x \in \mathrm{Det}_{\mathcal{S}}(\mathbf{K},\mathbf{B})(T)$. Hence $x \in \mathrm{Reach}_{\mathcal{S}}^{\mathbf{K}}(T,s^\star;x(s^\star))$. This proves formula (8.29).

Theorem 10.2.5 implies that the graph of the detector is the unique graph $\mathrm{Graph}(\mathbf{D})$ of a set-valued map $\mathbf{D}$ between $\mathrm{Graph}(\mathbf{B})$ and $\mathrm{Graph}(\mathbf{K})$ satisfying

$$\begin{cases} \mathrm{Graph}(\mathbf{D}) \;=\; \mathrm{Capt}_{(8.27)}(\mathrm{Graph}(\mathbf{D}), \mathrm{Graph}(\mathbf{B})) \\ = \; \mathrm{Capt}_{(8.27)}(\mathrm{Graph}(\mathbf{K}), \mathrm{Graph}(\mathbf{D})) \end{cases}$$

and thus formula (8.30). $\square$

We shall extend this concept of detection tubes to travel time tubes useful in transportation engineering or in population dynamics.

We do not provide here more illustrations of straightforward adaptations to the time-dependent case of other results gathered in this book to time-independent case.

8.11 Observation, Measurements and Identification Tubes

Detectors are extensively used in control theory, under various names, motivated by different problems dealing with observations, measurements and questions revolving around these issues.

For instance, there are situations when the initial state is not known: We only know the evolutionary system, associated, for instance, with a time-dependent control system

$$\begin{cases} (i) \;\; x'(t) = f(t, x(t), u(t)) \\ (ii) \; u(t) \in U(t, x(t)) \end{cases} \tag{8.31}$$

The question arises to compensate for the ignorance of initial conditions. Among various ways to do it, we investigate here the case when we have access to some observation $y(t) = h(x(t))$ up to a given present time T, where $h : X \mapsto Y$ is regarded as a *measurement map* (or a *sensor map*, an *observation map*). In other words, we do not have direct access to the state $x(t)$ of the system, but to some measurements of observations $y(t) \in Y$ of the state.

The questions arises whether we can find at each present time T an evolution $x(\cdot)$ governed by control system (8.31) satisfying

$$\forall t \in [0, T], \;\; y(t) \; = \; h(x(t))$$

More generally, we can take into account "contingent noise" in the measurements, and assume instead that the measurement map is a set-valued map $H : X \rightsquigarrow Y$ associates with the

$$\forall t \in [0, T], \; y(t) \; \in \; H(x(t)) \tag{8.32}$$

In summary, we have to detect evolutions governed by an evolutionary system satisfying the "time-dependent viability conditions" (8.32) on each time interval $[0, T]$. This answers questions raised by *David Delchamps* in 1989:

Information Contained in Past Measurements. *David Delchamps* (see *State Space and Input-Output Linear Systems*, [78, Delchamps]) regards measurements *as a deterministic memoryless entity that gives us a limited* *amount of information about the states.* The problem is formulated as follows: *How much information about the current state is contained in a long record of past (quantized) measurements of the system's output? Furthermore, how can the inputs to the system be manipulated so as to make the system's output record more informative about the state evolution than might appear possible based on a cursory appraisal?*

We shall study this problem in a more general setting, since condition (8.32) can be written in the form

$$\forall t \in [0, T], \; x(t) \in \mathbf{K}(t) := H^{-1}(y(t))$$

Since the solutions we will bring to this problem depends upon the "tube" $t \rightsquigarrow \mathbf{K}(t)$ and not on the fact that it is derived from a measurement map, this is in this context that we shall look for the set $\text{Reach}^{\mathbf{K}}_{(8.4)}(T, \mathbf{K}(0))$ of sates $x(T)$ where $x(\cdot)$ is an evolution governed by (8.32) satisfying the "time-dependent viability conditions" (8.32).

Furthermore, we may need also to regulate the evolutions satisfying the above viability property by a regulation law associating with the observations $y(t)$ up to time T the controls $u(t)$ performing such a task. This is a solution to the *parameter identification* problem where controls are regarded as state-dependent parameters to be identified, problems also called "inverse problems" (see Sect. 10.9, p. 427).

8.11.1 Anticipation Tubes

Another example of tube $\mathbf{K}(t)$ is provided not only by past measurements, but also by taking into account expectations made at each instant t for future dates $s := t + a$, $a \geq 0$. For each current time $t \geq 0$, we assume not only that we know (through measurements, for instance) that the state $x(t)$ belongs to a subset $P(t)$, but also that from $x(t)$ starts a prediction $a \mapsto x(t; a)$ made at time t, solution to a differential inclusion $\frac{d}{da}x(t; a) \in G(t; x(a))$ (parameterized by time t) satisfying constraints of the form

$$\forall a \geq 0, \; x(t; a) \in P(t)$$

In other words, we take for tube the viability kernel

$$\mathbf{K}(t) := \text{Viab}_{G(t,\cdot)}(P(t))$$

Taking such a tube $\mathbf{K}(\cdot)$ as an evolving environment means that the decision maker involves at each instant t *predictions* on the future viewed at time t concerning future evolutions $a \mapsto x(t;a)$ governed by an anticipated dynamical system $\frac{d}{da}x(t;a) \in G(t;x(a))$ viable on anticipated constraints $P(t)$. These anticipations are taken into account in the viability kernel $\mathbf{K}(t)$ only, but are not implemented in the differential inclusion $x'(t) \in F(t,x(t))$. When the dynamics depend upon the past, we have to study viability problems under historical differential inclusions.