Chapter 2
Viability and Capturability

2.1 Introduction

This rather long chapter is the central one. It is aimed at allowing the reader to grasp enough concepts and statements of the principal results proved later on in the book to read directly and independently most of the chapters of the book: Chaps. 4, p. 125 and 6, p. 199 for qualitative applications, Chap. 8, p. 273 and 9, p. 319 for other quantitative concepts, Chaps. 7, p. 247 and 15, p. 603 for social science illustrations, Chaps. 3, p. 105, 14, p. 563 and 16, p. 631 in engineering, even though, here and there, some results require statements proved in the mathematical Chaps. 10, p. 375 and 11, p. 437.

This chapter defines and reviews the basic concepts: evolutions and their properties, in Sect. 2.2, p. 45, and next, several sections providing examples of evolutionary systems. We begin by the simple single-valued discrete systems and differential equations.

We next introduce parameters in the dynamics of the systems, and, among these parameters, distinguish constant coefficients, controls, regulons and tyches (Sect. 2.5, p. 58) according to the roles they will play. They motivate the introduction of controlled discrete and continuous time systems. All these systems generate *evolutionary systems* defined in Sect. 2.8, p. 68, the abstract level where it is convenient to study the viability and capturability properties of the evolutions they govern, presented in detail in Chap. 10, p. 375.

Next, we review the concepts of viability kernels and capture basins, under discrete time controlled systems in Sect. 2.9, p. 71. In this framework, the computation of the regulation map is easy and straightforward. We present the viability kernel algorithm in Sect. 2.9.2, p. 74 and use it to compute the Julia sets and Fatou dusts in Sect. 2.9.3, p. 75 and show in Sect. 2.9.4, p. 79 how the celebrated fractals are related to viability kernels under the class of discrete disconnected systems.

J.-P. Aubin et al., *Viability Theory*, DOI 10.1007/978-3-642-16684-6_2,

Viability kernels and capture basins under continuous time controlled systems are the topics of Sect. 2.10, p. 85. We illustrate the concepts of viability kernels by computing the viability kernel under the (backward) Lorenz system, since we shall prove in Chap. 9, p. 319 that it contains the famous Lorenz attractor.

Viability kernels and capture basins single out initial states from which **at least** one discrete evolution is viable forever or until it captures a target. We are also interested to the invariance kernels and absorption basins made of initial states from which **all** evolutions are viable forever or until they capture a target. One mathematical reason is that these concepts are, in some sense, "dual" to the concepts of viability kernels and capture basins respectively, and that they will play a fundamental role for characterizing them. The other motivation is the study of "tychastic" systems where the parameters are tyches (perturbations, disturbances, etc.) which translate one kind of uncertainty without statistical regularity, since tyches are neither under the control of an actor nor chosen to regulate the system.

We also address computational issues:

- viability kernel and capture basin algorithms for computing viability kernels and capture basins under discrete system in Sect. 2.9.2, p. 74,
- and next, discretization issues in Sect. 2.14, p. 96.

For computing viability kernels and capture basins under continuous time controlled systems, we proceed in two steps. First, approximate the continuous time controlled systems by discretized time controlled systems, so that viability kernels and capture basins under discretized systems converge to the viability kernels and capture basins under the continuous time evolutionary system, and next, compute the viability kernels and capture basins under the discretized time controlled systems by the viability kernel and capture basin algorithms.

This section just mentions these problems and shows how the (trivial) characterization of viability kernels and capture basins under discretized controlled systems gives rise to the tangential conditions characterizing them under the continuous time controlled system, studied in Chap. 11, p. 437.

The chapter ends with a "viability survival kit" in Sect. 2.15, p. 98, which summarizes the most important statements necessary to apply viability theory without proving them. They are classified in three categories:

- At the most general level, where simple and important results and are valid without any assumption,
- At the level of evolutionary systems, where viability kernels and capture basins are characterized in terms of local viability properties and backward invariance,
- At the level of control systems, where viability kernels and capture basins are characterized in terms of "tangential characterization" which allows us to define and study the regulation map.

2.2 Evolutions

Let X denote the *state space* of the system. Evolutions describe the behavior of the state of the system as a function of time.

Definition 2.2.1 ***[Evolutions and their Trajectories]*** *The time t ranges over a set $\mathbb{T}$ that is in most cases,*

1. *either the* ***discrete time set*** *of times $j \in \mathbb{T} := \mathbb{N} := \{0, \ldots, +\infty\{$ ranging over the set of positive integers $j \in \mathbb{N}$,*
2. *or the* ***continuous time set*** *of times $t \in \mathbb{T} := \mathbb{R}_+ := [0, \ldots, +\infty[$ ranging over the set of positive real numbers or scalars $t \in \mathbb{R}_+$.*

Therefore, evolutions are functions $x(\cdot) : t \in \mathbb{T} \mapsto x(t) \in X$ *describing the evolution of the state $x(t)$. The* trajectory (or orbit) *of an evolution $x(\cdot)$ is the subset $\{x(t)\}_{t \in \mathbb{T}} \subset X$ of states $x(t)$ when t ranges over $\mathbb{T}$.*

Warning: The terminology "trajectory" is often used as a synonym of evolution, but inadequately: *a trajectory is the range of an evolution.*

Unfortunately, for discrete time evolutions, tradition imposes upon us to regard discrete evolutions as *sequences* and to use the notation $\overrightarrow{x} : j \in \mathbb{N} \mapsto x_j := x(j) \in X$. We shall use this notation when we deal explicitly with discrete time. We use the notation $x(\cdot) : t \in \mathbb{R}_+ \mapsto x(t) \in X$ for continuous time evolutions and whenever the results we mention are valid for both continuous and discrete times. It should be obvious from the context whether $x(\cdot)$ denotes an evolution when time ranges over either discrete $\mathbb{T} := \mathbb{N}_+$ time set or continuous $\mathbb{T} := \mathbb{R}_+$ time set.

The choice between these two representations of time is not easy. The "natural" one, which appears the simplest for non mathematicians, is the choice of the set $\mathbb{T} := \mathbb{N}_+$ of discrete times. It has drawbacks, though. On one hand, it may be difficult to find a common *time scale* for the different components of the state variables of the state space of a given type of models. On the other hand, by doing so, we deprive ourselves from the concepts of velocity, acceleration and other dynamical concepts introduced by Isaac Newton (1642–1727), that are not well taken into account by discrete time systems as well as of the many results of the differential and integral calculus gathered for more than four centuries since the invention of the infinitesimal calculus by Gottfried Leibniz (1646–1716). Therefore, the choice between these two representations of times being impossible, we have to investigate both discrete and continuous time systems. Actually, the results dealing with viability kernels and capture basins use the same proofs, as we shall see in Chap. 10, p. 375. Only the viability characterization becomes dramatically simpler, not to say trivial, in the case of discrete systems.

Note however that for computational purposes, we shall approximate continuous time systems by discrete time ones where the time scale becomes infinitesimal.

Warning: *Viability properties of the discrete analogues of continuous-time systems can be drastically different*: we shall see on the simple example of the *Verhulst* logistic equation that the interval $[0, 1]$ is invariant under the continuous system

$$x'(t) = rx(t)(1 - x(t))$$

for all $r \geq 0$ whereas the viability kernel of $[0, 1]$ under its discrete counterpart

$$x_{n+1} = rx_n(1 - x_n)$$

is a Cantor subset of $[0, 1]$ when $r > 4$. Properties of discrete counterparts of continuous time dynamical systems can be different from their discretizations. These discretizations, under the assumptions of adequate convergence theorems, share the same properties than the continuous time systems.

We shall assume most of the time that:

1. the state space is a finite dimensional vector space $X := \mathbb{R}^n$,
2. evolutions are *continuous* functions $x(\cdot) : t \in \mathbb{R}_+ \mapsto x(t) \in X$ describing the evolution of the state $x(t)$.

We denote the space of continuous evolutions $x(\cdot)$ by $\mathcal{C}(0, \infty; X)$.

Some evolutions, mainly motivated by physics, are classical: *equilibria and periodic evolutions*. But these properties are not necessarily adequate for problems arising in economics, biology, cognitive sciences and other domains involving living beings. Hence we add the concept of evolutions *viable in a environment or capturing a target in finite time* to the list of properties satisfied by evolutions.

2.2.1 Stationary and Periodic Evolutions

We focus our attention to specific properties of evolutions, denoting by $\mathcal{H} \subset \mathcal{C}(0, \infty; X)$ the subset of evolutions satisfying these properties. For instance, the most common are stationary ones and periodic ones:

Definition 2.2.2 *[Stationary and Periodic Evolutions]*

1. *The subset $\mathcal{X} \subset \mathcal{C}(0, \infty; X)$ of* stationary evolutions *is the subset of evolutions $x : t \mapsto x$ when x ranges over the state space X.*

2. *The subset* $\mathcal{P}_T(X)$ *of* T-periodic evolutions *is the subset of evolutions* $x(\cdot) \in \mathcal{C}(0, \infty; X)$ *such that,* $\forall t \geq 0,\ x(t+T) = x(t)$*.*

Stationary evolutions are periodic evolutions for all periods T.

Stationary and periodic evolutions have been a central topic of investigation in dynamical systems motivated by physical sciences. Indeed, the brain, maybe because it uses periodic evolutions of neurotransmitters through subsets of synapses, has evolved to recognize periodic evolutions, in particular those surrounding us in daily life (circadian clocks associated with the light of the sun). Their extensive study is perfectly legitimate in physical sciences, as well as their new developments (bifurcations, catastrophes, dealing with the dependence of equilibria in terms of a parameter, and chaos, investigating the absence of continuous dependence of evolution(s) with respect to the initial states, for instance). However, even though we shall study evolutions regulated by constant parameters, bifurcations are quite difficult to observe, as it was pointed out in Sect. 3.3 of the book *Introduction to nonlinear systems and chaos* by *Stephen Wiggins*:

11 *[**On the Interpretation and Application of Bifurcation Diagrams: A Word of Caution**] At this point, we have seen enough examples so that it should be clear that the term bifurcation refers to the phenomenon of a system exhibiting qualitatively new dynamical behavior as parameters are varied. However, the phrase "as parameters are varied" deserves careful consideration [...] In all of our analysis thus far the parameters have been constant. The point is that we cannot think of the parameter as varying in time, even though this is what happens in practice. Dynamical systems having parameters that change in time (no matter how slowly!) and that pass through bifurcation values often exhibit behavior that is very different from the analogous situation where the parameters are constant.*

The situation in which *coefficients* are kept constant is familiar in physics, but, in engineering as well as in economic and biological sciences, they may have to vary with time, playing the roles of controls in engineering, of *regulons* in social and biological sciences, or *tyches*, when they play the role of random variables when uncertainty does not obey statistical regularity, as we shall see in Sect. 2.5, p. 58.

Insofar as physical sciences privilege the study of stability or chaotic behavior around *attractors* (see Definition 9.3.8, p. 349) and their *attraction basins* (see Definition 9.3.3, p. 347), the thorough study of *transient evolutions* have been neglected, although they pervade economic, social and biological sciences.

2.2.2 Transient Evolutions


Theory of Games and Economic Behavior. John von Neumann (1903–1957) and Oskar Morgenstern (1902–1976) concluded the first chapter of their monograph *"Theory of Games and Economic Behavior"* (1944) by these words: *Our theory is thoroughly static. A dynamic theory would unquestionably be more complete and therefore, preferable. But there is ample evidence from other branches of science that it is futile to try to build one as long as the static side is not thoroughly understood. [...] Finally, let us note a point at which the theory of social phenomena will presumably take a very definite turn away from the existing patterns of mathematical physics. This is, of course, only a surmise on a subject where much uncertainty and obscurity prevail [...] A dynamic theory, when one is found, will probably describe the changes in terms of simpler concepts.*

Unfortunately, the concept of equilibrium is polysemous. The mathematical one, which we adopt here, expresses stationary – time independent – evolution, that is, no evolution. The concept of equilibrium used by von Neumann and Morgenstern is indeed this static concept, derived from the concept of general equilibrium introduced by Léon Walras (1834–1910) in his book *Éléments d'économie politique pure* (1873) as an equilibrium (stationary point) of his tâtonnement process.

Another meaning results from the articulation between dynamics and *viability constraints*: This means here that, starting from any initial state satisfying these constraints, **at least** one evolution satisfies these constraints at each instant (such an evolution is called *viable*). An equilibrium can be viable or not, these two issues are independent of each other.

The fact that many scarcity constraints in economics are presented in terms of "balance", such as the balance of payments, may contribute to the misunderstanding. Indeed, the image of a balance conveys both the concept of equalization of opposite forces, hence of constraints, and the resulting stationarity – often called "stability", again, an ambivalent word connoting too many different meanings.

This is also the case in biology, since Claude Bernard (1813-1878) introduced the notion of constancy of inner milieu (constance du milieu intérieur). In 1898 he wrote: *Life results form the encounter of organisms and milieu, [...] we cannot understand it with organisms only, or with milieu only.* This idea was taken up under the name of "homeostasis" by Walter Cannon (1871-1945) in his book *Bodily changes in pain, hunger, fear and rage* (1915). This is again the case in ecology and environmental studies, as well as in many domains of social and human sciences when organisms adapt or not to several forms of viability constraints.

2.2.3 Viable and Capturing Evolutions

Investigating evolutionary problems, in particular those involving living beings, should start with identifying the constraints bearing on the variables which cannot – or should not – be violated. Therefore, if a subset $K \subset X$ represents or describes an environment, we mainly consider evolutions $x(\cdot)$ *viable in the environment* $K \subset X$ in the sense that

$$\forall t \geq 0, \;\; x(t) \in K \tag{2.1}$$

or *capturing the target* C in the sense that they are viable in K until they reach the target C *in finite time*:

$$\exists\, T \geq 0 \;\text{ such that } \begin{cases} x(T) \in C \\ \forall t \in [0,T], \; x(t) \; \in \; K \end{cases} \tag{2.2}$$

We complement Definition 6, p. 15 with the following notations:

Definition 2.2.3 *[**Viable and Capturing Evolutions**] The subset of evolutions* viable *in K is denoted by*

$$\mathcal{V}(K) := \{x(\cdot) \mid \forall t \geq 0, \;\; x(t) \in K\} \tag{2.3}$$

and the subset of evolutions capturing *the target C by*

$$\mathcal{K}(K,C) := \{x(\cdot) \mid \exists\, T \geq 0 \text{ such that } x(T) \in C \;\&\; \forall t \in [0,T], \; x(t) \; \in \; K\} \tag{2.4}$$

We also denote by

$$\mathcal{V}(K,C) := \mathcal{V}(K) \cup \mathcal{K}(K,C) \tag{2.5}$$

the set of evolutions viable in K outside C, *i.e. that are viable in K forever or until they reach the target C in finite time.*

Example: The first examples of such environments used in control theory were vector (affine) subsets because, historically, analytical formulas could be obtained. Nonlinear control theory used first geometrical methods, which required *smooth equality constraints*, yielding environments of the form

$$K := \{x \mid g(x) = 0\} \;\text{ where } g : X := \mathbb{R}^c \mapsto Y := \mathbb{R}^b \; (b < c) \;\text{ is smooth}$$

Subsets such as smooth manifolds (Klein bottle, for instance), having no boundaries, the viability and invariance problems were evacuated. This is no longer the case when the environment is defined by inequality constraints, even smooth ones, yielding subsets of the form

$$K := \{x \mid g(x) \leq 0\}$$

the boundary of which is a proper subset. Subsets of the form

$$K := \{x \in L \mid g(x) \in M\}$$

where $L \subset X$, $g : X \mapsto Y$ and where $M \subset Y$ are typical environments encountered in mathematical economics. It is for such cases that mathematical difficulties appeared, triggering viability theory.

Constrained subsets in economics and biology are generally not smooth. The question arose to build a theory and forge new tools that did require neither the smoothness nor the convexity of the environments. *Set-valued analysis*, motivated in part by these viability and capturability issues, provided such tools.

Remark. These constraints can depend on time (time-dependent constraints), as we shall see in Chap. 8, p. 273, upon the state, the history (or the path) of the evolution of the state. *Morphological equations* are kind of differential equations governing the evolution of the constrained state $K(t)$ and can be paired with evolutions of the state. The issues are dealt in [23, Aubin]. □

Remark. We shall introduce other families of evolutions, such as the viable evolutions *permanent* in a cell $C \subset K$ of *fluctuating* around C which are introduced in bio-mathematics (see Definition 9.2.1, p. 322). In "qualitative physics", a sequence of tasks or objectives is described by a family of subsets regarded as qualitative cells. We shall investigate the problem of finding evolutions visiting these cells in a prescribed order (see Sect. 8.8, p. 302). □

These constraints have to be confronted with evolutions. It is now time to describe how these evolutions are produced and to design mathematical translations of several evolutionary mechanisms.

2.3 Discrete Systems

Discrete evolutionary systems can be defined on any metric state space X.

12 ***Examples of State Spaces for Discrete Systems:***

1. *When* $X := \mathbb{R}^d$, *we take any of the equivalent vector space metrics for which the addition and the multiplication by scalars is continuous.*
2. *When* $X_\rho := \rho\mathbb{Z}^d$ *is a grid with step size* ρ, *we take the discrete topology, defined by* $d(x, x) := 0$ *and* $d(x, y) := 1$ *whenever* $x \neq y$. *A sequence of elements* $x_n \in X$ *converges to* x *if there exists an integer* N *such that for any* $n \geq N$, $x_n = x$, *any subset is both closed and open, the compacts are finite subsets. Any single-valued map from* $X := \mathbb{Z}^d$ *to some space* E *is continuous.*

Deterministic discrete systems, defined by

$$\forall j \geq 0,\ x_{j+1} = \varphi(x_j) \text{ where } \varphi : x \in X \mapsto \varphi(x) \in X$$

are the simplest ones to formulate, but not necessarily the easiest ones to investigate.

Definition 2.3.1 *[**Evolutionary Systems associated with Discrete Systems**] Let* X *be any metric space and* $\varphi : X \mapsto X$ *be the single-valued map associating with any state* $x \in X$ *its successor* $\varphi(x) \in X$.
The space of discrete evolutions $\overrightarrow{x} := \{x_j\}_{j\in\mathbb{N}}$ *is denoted by* $X^{\mathbb{N}}$. *The* evolutionary system $\mathcal{S}_\varphi : X \mapsto X^{\mathbb{N}}$ *associated with the map* $\varphi : x \in X \mapsto \varphi(x) \in X$ *associates with any* $x \in X$ *the set* $\mathcal{S}_\varphi(x)$ *of discrete evolutions* $\overrightarrow{x}$ *starting at* $x_0 = x$ *and governed by the discrete system*

$$\forall j \geq 0,\ x_{j+1} = \varphi(x_j)$$

An equilibrium of a discrete dynamical system is a stationary evolution governed by this system.

An equilibrium $\overline{x} \in X$ (stationary point) of an evolution $\overrightarrow{x}$ governed by the discrete system $x_{j+1} = \varphi(x_j)$ is a *fixed point* of the map φ, i.e., a solution to the equation $\varphi(\overline{x}) = \overline{x}$. There are two families of Fixed Point Theorems based:

1. either on the simple Banach–Cacciopoli–Picard Contraction Mapping Theorem in complete metric spaces,
2. or on the very deep and difficult 1910 Brouwer Fixed Point Theorem on convex compact subsets, the cornerstone of nonlinear analysis.

Example: The Quadratic Map The quadratic map φ associates with $x \in [0,1]$ the element $\varphi(x) = rx(1-x) \in \mathbb{R}$, governing the celebrated *discrete logistic system* $x_{j+1} = rx_j(1-x_j)$. The fixed points of φ are 0 and $c := \frac{r-1}{r}$, which is smaller than 1. We also observe that $\varphi(0) = \varphi(1) = 0$ so that the successor of 1 is the equilibrium 0.

For $K := [0,1] \subset \mathbb{R}$ to be a state space under this discrete logistic system, we need φ to map $K := [0,1]$ to itself, i.e., that $r \leq 4$. Otherwise, for $r > 4$, the roots of the equation $\varphi(x) = 1$ are equal to $a := \frac{1}{2} - \frac{\sqrt{r^2-4r}}{2r}$ and $b := \frac{1}{2} + \frac{\sqrt{r^2-4r}}{2r}$, where $b < c$. We denote by $d \in [0,a]$ the other root of the equation $\varphi(d) = c$. Therefore, for any $x \in]a,b[$, $\varphi(x) > 1$.

A way to overcome this difficulty is to *associate with the single-valued* $\varphi : [0,1] \mapsto \mathbb{R}$ *the set-valued map* $\Phi : [0,1] \rightsquigarrow [0,1]$ *defined by* $\Phi(x) := \varphi(x)$ when $x \in [0,a]$ and $x \in [b,1]$ and $\Phi(x) := \emptyset$ when $x \in]a,b[$. Let us set

$$\omega^\flat(y) := \frac{1}{2} - \frac{\sqrt{r^2-4ry}}{2r} \text{ and } \omega^\sharp(y) := \frac{1}{2} + \frac{\sqrt{r^2-4ry}}{2r}$$

The inverse Φ^{-1} is defined by

$$\Phi^{-1}(y) := \left(\omega^\flat(y), \omega^\sharp(y)\right)$$

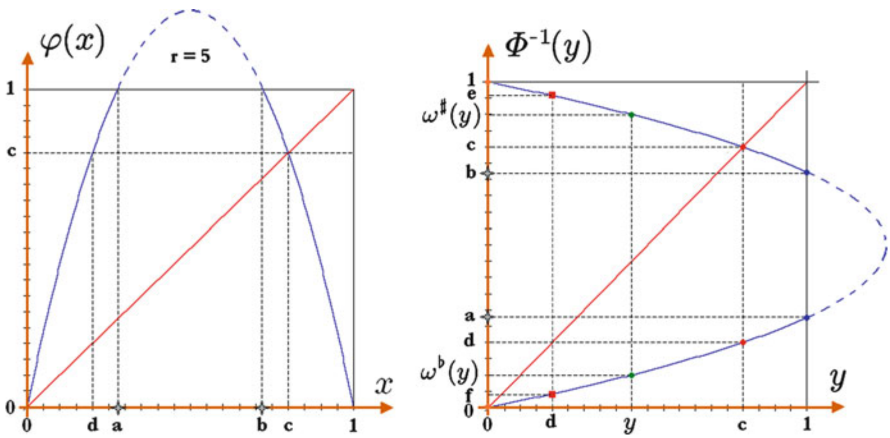


Fig. 2.1 Discrete Logistic System.
The graph of the function $x \mapsto \varphi(x) := rx(1-x)$ *for* $r = 5$ *is displayed as a function* $\varphi : [0,1] \mapsto \mathbb{R}$ *as a set-valued map* $\Phi : [0,1] \rightsquigarrow [0,1]$ *associating with any* $x \in [a,b]$ *the empty set. Equilibria are the abscissas of points of the intersection of the graph* Graph(φ) *of* φ *and of the bisectrix. We observe that* 0 *and the point* c *(to the* right *of* b*) are equilibria. On the right, the graph of the inverse is displayed, with its two branches.*

The predecessors $\Phi^{-1}(0)$ and $\Phi^{-1}(c)$ of equilibria 0 and c are initial states of viable discrete evolutions because, starting from them, the equilibria are their successors, from which the evolution remains in the interval forever. They are made of $\omega^\sharp(0) = 1$ and of $c_1 := \omega^\flat(c)$. In the same way, the four predecessors $\Phi^{-2}(0) := \Phi^{-1}(\Phi^{-1}(0)) = \{\omega^\flat(1) = a, \omega^\sharp(1) = b\}$ and $\Phi^{-2}(c)$ are initial states of viable evolutions, since, after two iterations, we obtain the two equilibria from which the evolution remains in the interval forever. And so on: The subsets $\Phi^{-p}(0)$ and $\Phi^{-p}(c)$ are made of initial states from which start evolutions which reach the two equilibria after p iterations, and thus, which are viable in K. They belong to the *viability kernel* of K (see Definition 2.9.1, p. 71 below). This study will resume in Sect. 2.9.4, p. 79.

2.4 Differential Equations

2.4.1 Determinism and Predictability

We begin by the simplest class of continuous time evolutionary systems, which are associated with differential equations

$$x'(t) = f(x(t))$$

where $f : X \mapsto X$ is the single-valued map associating with any state $x \in X$ its velocity $f(x) \in X$.

Definition 2.4.1 *[**Evolutionary Systems associated with Differential Equations**] Let $f : X \mapsto X$ be the single-valued map associating with any state $x \in X$ its velocity $f(x) \in X$.*
The evolutionary system *$\mathcal{S}_f : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ defined by $f : X \mapsto X$ is the set-valued map associating with any $x \in X$ the set $\mathcal{S}_f(x)$ of evolutions $x(\cdot)$ starting at x and governed by differential equation*

$$x'(t) = f(x(t))$$

The evolutionary system is said to be deterministic *if $\mathcal{S}_f : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ is single-valued. An equilibrium of a differential equation is a stationary solution of this equation.*

An equilibrium $\overline{x}$ (stationary point) of a differential equation $x'(t) = f(x(t))$ being a constant evolution, its velocity is equal to 0, so that it *is characterized as a solution to the equation $f(\overline{x}) = 0$.*

The evolutionary system $\mathcal{S}_f$ associated with the single-valued map f is a priori a set-valued map, taking:

1. nonempty values $\mathcal{S}_f(x)$ whenever there exists a solution to the differential equation starting at x, guaranteed by (local) existence theorems (the Peano Theorem, when f is continuous),
2. at most one value $\mathcal{S}_f(x)$ whenever uniqueness of the solution starting at x is guaranteed. There are many sufficient conditions guaranteeing the uniqueness: f is Lipschitz, by the Cauchy–Lipschitz Theorem, or f is monotone in the sense that there exists a constant $\lambda \in \mathbb{R}$ such that

$$\forall x, y \in X, \;\; \langle f(x) - f(y), x - y \rangle \;\leq\; \lambda \|x - y\|^2$$

(we shall not review other uniqueness conditions here.)

Since the study of such equations, linear and nonlinear, has, for a long time, been a favorite topic among mathematicians, the study of dynamical systems has for a long time focussed on equilibria: existence, uniqueness, stability, which are investigated in Chap. 9, p. 319.

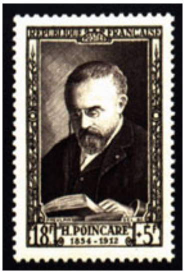
Existence and uniqueness of solutions to a differential equation was identified with the mathematical description of determinism by many scientists after the 1796 book *L'Exposition du système du monde* and the 1814 book *Essai philosophique sur les probabilités* by *Pierre Simon de Laplace* (1749–1827):

Determinism and predictability. *"We must regard the present state of the universe as the effect of its anterior state and not as the cause of the state which follows. An intelligence which, at a given instant, would know all the forces of which the nature is animated and the respective situation of the beings of which it is made of, if by the way it was wide enough to subject these data to analysis, wouldembrace in a unique formula the movements of the largest bodies of the universe and those of the lightest atom: Nothing would be uncertain for it, and the future, as for the past, would present at its eyes." Does it imply what is meant by* "predictability"*?*

Although a differential equation assigns a unique velocity to each state, this does not imply that the associated evolutionary system $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ is *deterministic*, in the sense that it is univoque (single-valued). It may happen that several evolutions governed by a differential equation start from a same initial state. *Valentin–Joseph Boussinesq* (1842–1929) used this lack of uniqueness property of solutions to a differential equation starting at some initial state (that he called "bifurcation", with a different meaning that this word has now, as in Box 11, p. 47) to propose that the multivalued character of evolutions governed by a univoque mechanism describes the evolution of living beings.

The lack of uniqueness of some differential equations does not allows us to regard differential equations as a model for a deterministic evolution. Determinism can be translated by evolutionary systems which associate with any initial state one and only one evolution.

But even when a differential equation generates a deterministic evolutionary system, Laplace's enthusiasm was questioned by *Henri Poincaré* in his study of the evolution of the three-body problem, a simplified version of the evolution of the solar system in his famous 1887 essay. He observed in his 1908 book "*La science et l'hypothèse*" that tiny differences of initial conditions implied widely divergent positions after some time:



Predictions. *"If we knew exactly the laws of Nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment. But even if it was the case that the natural laws had no longer any secret for us, we could still know the situation approximately. If that enabled us to predict the succeeding situation with the same approximation, that is all we require, and we should say that the phenomenon has been predicted, that it is governed by the laws. But it is not always so; it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter.* Prediction becomes impossible and we obtain a fortuitous phenomenon.*"*

The sensitive dependence on initial conditions is one prerequisite of "chaotic" behavior of evolutions, resurrected, because, two centuries earlier, even before Laplace, *Paul Henri Thiry, Baron d'Holbach* wrote in one of his wonderful books, the 1770 *Système de la nature*:

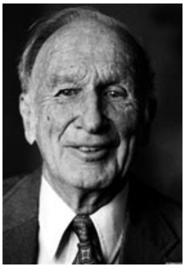
Holbach. *"Finally, if everything in nature is linked to everything, if all motions are born from each other although they communicate secretely to each other unseen from us, we must hold for certain that there is no cause small enough or remote enough which sometimes does not bring about the largest and the closest effects on us. The first elements of athunderstorm may gather in the arid plains of Lybia, then will come to us with the winds, make our weather heavier, alter the moods and the passions of a man of influence, deciding the fate of several nations".*

Some nonlinear *differential equations* produce *chaotic* behavior, quite *unstable*, sensitive to initial conditions and producing fluctuating evolutions (see Definition 9.2.1, p. 322). However, for many problems arising in biological, cognitive, social and economic sciences, we face a completely *orthogonal situation*, governed by *differential inclusions, regulated or controlled systems, tychastic or stochastic systems*, but producing evolutions as *regular or stable* (in a very loose sense) as possible for the sake of adaptation and viability required for life.

2.4.2 Example of Differential Equations: The Lorenz System

Since uncertainty is the underlying theme of this book, we propose to investigate the Lorenz system of differential equations, which is deterministic, but unpredictable in practice, as a simple example to test results presented in this book.

Studying a simplified meteorological model made of a system of three differential equations, the meteorologist *Edward Lorenz* discovered by chance (the famous *serendipity*) in the beginning of the 1960s that for certain parameters for which the system has three non stable equilibria, the "limit set" was quite strange, "chaotic" in the sense that many evolutions governed by this system "fluctuate", approach one equilibrium while circling around it, then suddenly leave away toward another equilibrium around which it turns again, and so on. In other words, this behavior is strange in the sense that the limit set of an evolution is not a trajectory of a periodic solution.[1]



Predictability: Does the flap of a butterfly's wing in Brazil set off a tornado in Texas? *After* Henri Poincaré *who discovered the lack of predictability of evolutions of the three-body problem, Lorenz presented in 1979 a lecture to the American Association for the Advancement of Sciences with the above famous title.*

Lorenz introduced the following variables:

1. x, proportional to the intensity of convective motion,

[1] As in the case of two-dimensional systems, thanks to the Poincaré–Bendixon Theorem.

2. y, proportional to the temperature difference between ascending and descending currents,
3. z, proportional to the distortion (from linearity) of the vertical temperature profile.

Their evolution is governed by the following system of differential equations:

$$\begin{cases} (i) \quad x'(t) = \sigma y(t) - \sigma x(t) \\ (ii) \quad y'(t) = rx(t) - y(t) - x(t)z(t) \\ (iii) \; z'(t) = \; x(t)y(t) - bz(t) \end{cases} \tag{2.6}$$

where the positive parameters σ and b satisfy $\sigma > b+1$ and r is the normalized Rayleigh number.

We observe that the vertical axis $(0, 0, z)_{z\in\mathbb{R}}$ is a symmetry axis, which is also the viability kernel of the hyperplane $(0, y, z)$ under the Lorenz system, from which the solutions boil down to the exponentials $(0, 0, ze^{-bt})$.

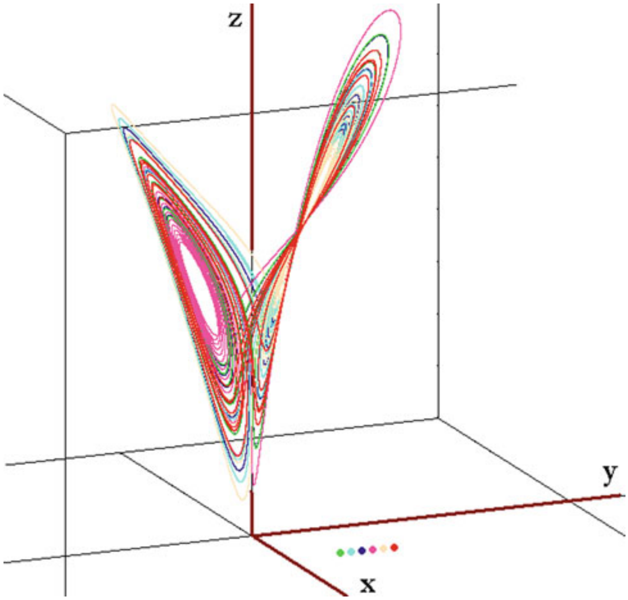


Fig. 2.2 Trajectories of six evolutions of the Lorenz system. *starting from initial conditions* $(i, 50, 0)$, $i = 0, \ldots, 5$. *Only the part of the trajectories from step times ranging between* 190 *and* 200 *are shown for clarity.*

If $r \in]0, 1[$, then 0 is an asymptotically stable equilibrium. If $r = 1$, the equilibrium 0 is "neutrally stable". When $r > 1$, the equilibrium 0 becomes unstable and two more equilibria appear:

$$e_1 := \left(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1\right) \ \& \ e_2 := \left(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1\right)$$

Setting $r^\star := \dfrac{\sigma(\sigma+b+3)}{\sigma-b-1}$, these equilibria are stable when $r^\star > 1$ and unstable when $r > r^\star$. We take $\sigma = 10$, $b = \frac{8}{3}$ and $r = 28$ in the numerical experiments.

2.5 Regulons and Tyches

For physical and engineering systems controlled in an optimal way by optimal controls, agents are decision makers or identified actors having access to the controls of the system.

For systems involving living beings, agents interfering with the evolutionary mechanisms are often *myopic, conservative, lazy and opportunistic*, from molecules to (wo)men, exhibiting some *contingent* freedom to choose among some *regulons* (regulatory parameters) to govern evolutions.

In both cases, controls and regulons may have to protect themselves against *tychastic* uncertainty, obeying no statistical regularity.

These features are translated by adding to state variables other ones, parameters, among which (constant) coefficients, controls, regulons and tyches. These different names describe the different questions concerning their role in the dynamics of the system.

In other words, the *state* of the system evolves according to evolutionary laws involving *parameters*, which may in their turn depend on *observation variables* of the states:

Definition 2.5.1 *[Classification of Variables]*

1. states *of the system;*
2. parameters*, involved in the law of evolution of the states;*
3. values, indicators *which provide some information on the system, such as exit functions, minimal time functions, minimal length functions, Lyapunov functions, value functions in optimal control, value of a portfolio, monetary mass, congestion traffic, etc.;*
4. observations *on the states, such as measurements, information, predictions, etc., given or built.*

We distinguish several categories of parameters, according to the existence or the absence of an actor (controller, agent, decision-maker, etc.) acting on them on one hand, or the degree of knowledge or control on the other hand, and to explain their role:

Definition 2.5.2 ***[Classification of Parameters]*** *Parameters can be classified in the following way:*

1. constant coefficients *which may vary, but not in time (see Box 11, p. 47),*
2. *parameters under the command of an identified actor, called* controls *(or decision parameters),*
3. *parameters evolving independently of an identified actor, which can themselves be divided in two classes:*
   a. Regulons or regulation controls,
   b. Tyches, perturbations, disturbances, random events.

These parameters participate in different ways to the general concept of uncertainty. *A given system can depend only on controls, and is called a* controlled system, *or on regulons, and is called a* regulated system *or on tyches, and is called a* tychastic system. *It also can involve two or three of these parameters: for instance, if it involves controls and tyches, it is called a* tychastic controlled system, *and, in the case of regulons and tyches, a* tychastic regulated system.

The study of parameterized systems thus depends on the interpretation of the parameters, either regarded as controls and regulons on one hand, or as tyches or random variables on the other.

1. In *control theory*, it is common for parameters to evolve in order to solve some specific requirements (optimality, viability, reachability) by **at least one evolution** governed by an identified actor (agent, decision-maker, etc.). Control theory is also known under the names of *automatics*, and, when dealing with mechanical systems, *robotics*. Another word, *Cybernetics*, from the Greek *kubernesis*, "control", "govern", as it was suggested first by André Ampère (1775–1836), and later, by *Norbert Wiener* (1894-1964) in his famous book *Cybernetics or Control and Communication in the Animal and the Machine* published in 1948, is by now unfortunately no longer currently used by American specialists of control theory. In physics and engineering, the actors are well identified and their purpose clearly defined, so that only state, control and observation variables matter.
2. In biological, cognitive, social and economic sciences, these parameters *are not under the control* of an identified and consensual agent involved in the evolutionary mechanism governing the evolutions of the state of the system. In those so called "soft sciences" involving uncertain evolutions of systems (organizations, organisms, organs, etc.) of living beings, the situation is more complex, because the identification of actors governing the evolution of parameters is more questionable, so that we regard in this case these parameters as regulons (for regulatory parameters).

13 *Examples of States and Regulons*

Field	***State***	***Regulon***	***Viability***	***Actors***
Economics	*Physical goods*	*Fiduciary goods*	*Economic scarcity*	*Economic agents*
Genetics	*Phenotype*	*Genotype*	*Viability or homeostasis*	*Bio-mechanical metabolism*
Sociology	*Psychological state*	*Cultural codes*	*Sociability*	*Individual actors*
Cognitive sciences	*Sensorimotor states*	*Conceptual codes*	*Adaptiveness*	*Organims*

The main question raised about controlled systems is to find "optimal controls" optimizing an intertemporal criteria, or other objectives, to which we devote Chap. 4, p. 125. The questions raised about regulated system deal with "inertia principle" (keep the regulon constant as long as viability is not at stakes), inertia functions, heavy evolutions, etc., which are exposed in Chap. 6, p. 199.

3. However, even control theory has to take into account some uncertainty (disturbances, perturbations, etc.) that we summarize under the name of *tyches*. Tyches describe uncertainties played by an indifferent, maybe hostile, Nature.

Tyche. *Uncertainty without statistical regularity can be translated mathematically by parameters on which actors, agents, decision makers, etc. have no controls. These parameters are often perturbations, disturbances (as in "robust control" or "differential games against nature") or more generally,* tyches *(meaning "chance" in classical Greek, from the Goddess Tyche) ranging over a state-dependent* tychastic map. *They could have be called "random variables" if this terminology were not already preempted by probabilists.*

This is why we borrow the term of *tychastic evolution* to *Charles Peirce* who introduced it in 1893 under the title *evolutionary love*:

Tychastic evolution. *"Three modes of evolution have thus been brought before us: evolution by fortuitous variation, evolution by mechanical necessity, and evolution by creative love. We may term them tychastic evolution, or tychasm, anancastic evolution, or anancasm, and agapastic evolution, or agapasm." In this paper, Peirce associates the concept of* anancastic evolution *with the Greek concept of necessity,* ananke, *anticipating the "chance and necessity" framework that motivated viability theory.*

When parameters represent tyches (disturbances, perturbations, etc.), we are interested in "robust" control of the system in the sense that **all evolutions** of the evolutionary system starting from a given initial state satisfy a given evolutionary property.

Fortune *fortuitously* left its role to randomness, originating in the French *"randon"*, from the verb *"randir"*, sharing the same root than the English "to run" and the German *rennen*. When running too fast, one looses the control of himself, the race becomes a poor "random walk", bumping over *scandala* (stones scattered on the way) and falling down, *cadere* in Latin, a matter of *chance* since it is the etymology of this word. Hazard was imported by William of Tyre from the crusades from Palestine castle named after a dice game, *az zahr*. Now dice, in Latin, is *alea*, famed after Julius Caesar's *alea jacta est*, which was actually thrown out the English language: chance and hazard took in this language the meaning of danger, itself from Latin *dominarium*. Being used in probability, the word *random* had to be complemented by *tyche* for describing evolutions without statistical regularity prone to extreme events.

Zhu Xi (1130–1200), 朱 熹 one of the most important unorthodox neo-Confucian of the Song dynasty, suggested that *"if you want to treat everything, and as changes are infinite, it is difficult to predict, it must, according to circumstances, react to changes (literally,* "follow, opportunity, reaction, change*)*, instead of a priori action.

The four ideograms *follow*, *opportunity*, *reaction*, *change*:

随 机 应 变

are combined to express in Chinese:

1. by the first half, *"follow, opportunity"*, 随 机, the concept of randomness or stochasticity,
2. while Shi Shuzhong has proposed that the second half, *"reaction, change"*, 应 变; translate the concept of tychasticity,
3. and *"no, necessary"*, 未 定, translates contingent.

2.6 Discrete Nondeterministic Systems

Here, the time set is $\mathbb{N}$, the state space is any metric set X and the evolutionary space is the space $X^{\mathbb{N}}$ of sequences $\overrightarrow{x} := \{x_j\}_{j\in\mathbb{N}}$ of elements $x_j \in X$. The space of parameters (controls, regulons or tyches) is another set denoted by $\mathcal{U}$. The evolutionary system is defined by the discrete parameterized system (φ, U) where:

1. $\varphi : X \times \mathcal{U} \mapsto X$ is a map associating with any state-parameter pair (x, u) the successor $\varphi(x, u)$,
2. $U : X \rightsquigarrow \mathcal{U}$ is a set-valued map associating with any state x a set $U(x)$ of parameters *feeding back* on the state x.

Definition 2.6.1 ***[Discrete Systems with State-Dependent Parameters]*** *A discrete parameterized system $\Phi := \varphi(\cdot, U(\cdot))$ defines the evolutionary system $\mathcal{S}_{\Phi} : X \rightsquigarrow X^{\mathbb{N}}$ in the following way: for any $x \in X$, $\mathcal{S}_{\Phi}(x)$ is the set of sequences $\overrightarrow{x}$ governed by*

$$\begin{cases} (i) \;\; x_{j+1} = \varphi(x_j, u_j) \\ (ii) \; u_j \in U(x_j) \end{cases} \tag{2.7}$$

starting from x.

When the parameter space is reduced to a singleton, we find discrete equations $x_{j+1} = \varphi(x_j)$ as a particular case. They generate deterministic evolutionary systems $\mathcal{S}_{\varphi} : X \mapsto X^{\mathbb{N}}$.

Setting

$$\Phi(x) := \varphi(x, U(x)) = \{\varphi(x, u)\}_{u\in U(x)}$$

the subset of all available successors $\varphi(x, u)$ at x when u ranges over the set of parameters allows us to treat these dynamical systems as difference inclusions:

Definition 2.6.2 ***[Difference Inclusions]*** *Let $\Phi(x) := \varphi(x, U(x))$ denote the set of velocities of the parameterized system. The evolutions $\overrightarrow{x}$ governed by the parameterized system*

$$\begin{cases} (i) \;\; x_{j+1} = \varphi(x_j, u_j) \\ (ii) \; u_j \in U(x_j) \end{cases} \tag{2.8}$$

are governed by the difference inclusion

$$x_{j+1} \in \Phi(x_j) \tag{2.9}$$

and conversely.

An equilibrium of a difference inclusion is a stationary solution of this inclusion.

Actually, any difference inclusion $x_{j+1} \in \Phi(x_j)$ can be regarded as a parameterized system (φ, U) by taking $\varphi(x, u) := u$ and $U(x) := \Phi(x)$.

Selections of the set-valued map U are *retroactions* (see Definition 1.7, p. 18) governing specific evolutions. Among them, we single out the following

- **The Slow Retroaction**. It is associated with a given fixed element $a \in X$ (for instance, the origin in the case of a finite dimensional vector space). We set
$$u^{\circ}(x) := \left\{ y \in U(x) \mid d(a, y) = \inf_{z \in U(x)} d(a, z) \right\}$$
The evolutions governed by the dynamical system
$$\forall n \geq 0,\ x_{n+1} \in \varphi(x_n, u^{\circ}(x_n))$$
are called *slow evolutions*, i.e., evolutions associated with parameters remaining as close as possible to the given element a. In the case of a finite dimensional vector space, *slow evolutions are evolutions associated with controls with minimal norm.*
- **The Heavy Retroaction**. We denote by $\mathcal{P}(\mathcal{U})$ the hyperspace (see Definition 18.3.3, p. 720) of all subsets of U. Consider the set-valued map $\mathbb{S} : \mathcal{P}(\mathcal{U}) \times \mathcal{U} \rightsquigarrow \mathcal{U}$ associating with any pair (A, u) the subset $\mathbb{S}(A, u) := \{v \in A \mid d(u, v) = \inf_{w \in A} d(u, w)\}$ of "best approximations of u by elements of A". The evolutions governed by the dynamical system
$$\forall n \geq 0,\ x_{n+1} \in \varphi(x_n, \mathbf{s}(U(x_n), u_{n-1}))$$
are called *heavy evolutions.*
This amounts to taking at time n a regulon $u_n \in s(U(x_n), u_{n-1})$ as close as possible to the regulon u_{n-1} chosen at the preceding step. If such a regulon u_{n-1} belongs to $U(x_n)$, it can be kept at the present step n. This is in this sense that the selection $\mathbf{s}(U(x), u)$ provides a heavy solution, since the regulons are kept constant during the evolution as long as the viability is not at stakes.
For instance, when the state space is a finite dimensional vector space X supplied with a scalar product and when the subsets $U(x)$ are closed and convex, the projection theorem implies that the map $\mathbf{s}(U(x), u)$ is single-valued.

2.7 Retroactions of Parameterized Dynamical Systems

2.7.1 Parameterized Dynamical Systems

The space of parameters (controls, regulons or tyches) is another finite dimensional vector space $\mathcal{U} := \mathbb{R}^c$.

Definition 2.7.1 *[**Evolutionary Systems associated with Control Systems**] We introduce the following notation*

1. $f : X \times \mathcal{U} \mapsto X$ *is a map associating the* velocity $f(x, u)$ *of the state* x *with any state-control pair* (x, u)*,*
2. $U : X \rightsquigarrow \mathcal{U}$ *is a set-valued map associating a set* $U(x)$ *of controls* feeding back *on the state* x*.*

The evolutionary system $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ *defined by the control system* (f, U) *is the set-valued map associating with any* $x \in X$ *the set* $\mathcal{S}(x)$ *of evolutions* $x(\cdot)$ *governed by the control (or regulated) system*

$$\begin{cases} (i) \;\; x'(t) \; = \; f(x(t), u(t)) \\ (ii) \; u(t) \; \in \; U(x(t)) \end{cases} \tag{2.10}$$

starting from x*.*

Remark. Differential equation (2.10)(i) is an "input-output map" associating an output-state with an input-control. Inclusion (2.10)(ii) associates input-controls with output-states, *"feeds back"* the system (the a priori feedback relation is set-valued, otherwise, we just obtain a differential equation). See Figure 1.4, p. 14. □

Remark. We have to give a meaning to the differential equation $x'(t) = f(x(t), u(t))$ and inclusion $u(t) \in U(x(t))$ in system (2.10). Since the parameters are not specified, this system is not valid for any $t \geq 0$, but only for *"almost all"* $t \geq 0$ (see Theorems 19.2.3, p.771 and 19.4.3, p.783). We delay the consequences of the Viability Theorem with such mathematical property. The technical explanations are relegated to Chap. 19, p.769 because they are not really used in the rest of the book. By using graphical derivatives $Dx(t)(1)$ instead of the usual derivatives, the "differential" equation $Dx(t)(1) \ni f(x(t), u(t))$ providing the same evolutions holds true for any $t \geq 0$ (see Proposition 19.4.5, p.787). □

2.7.2 Retroactions

In control theory, open and closed loop controls, feedbacks or retroactions provide the central concepts of cybernetics and general systems theory:

Definition 2.7.2 *[**Retroactions**] Retroactions are single-valued maps* $\widetilde{u} : (t,x) \in \mathbb{R}_+ \times X \mapsto \widetilde{u}(t,x) \in \mathcal{U}$ *that are plugged as inputs in the differential equation*

$$x'(t) = f(x(t), \widetilde{u}(t, x(t))) \tag{2.11}$$

In control theory, state-independent retroactions $t \mapsto \widetilde{u}(t,x) := u(t)$ *are called* open loop controls *whereas time-independent retroactions* $x \mapsto \widetilde{u}(t,x) := \widetilde{u}(x)$ *are called* closed loop controls *or* feedbacks*. See Figure 1.7, p. 18.*

The class $\widetilde{\mathcal{U}}$ in which retroactions are taken must be consistent with the properties of the parameterized system so that

- the differential equations $x'(t) = f(x(t), \widetilde{u}(t, x(t)))$ have solutions[2],
- for every $t \geq 0$, $\widetilde{u}(t,x) \in U(x)$.

When no state-dependent constraints bear on the controls, i.e., when $U(x) = U$ does not depend on the state x, then open loop controls can be used to parameterize the evolutions $S(x, u(\cdot))(\cdot)$ governed by differential equations (2.10)(i).

This is no longer the case when the constraints on the controls depend on the state. In this case, we parameterize the evolutions of control system (2.10) by closed loop controls or retroactions.

Inclusion (2.10)(ii), which associates input-controls with output-states, "feeds back" the system in a set-valued way. Retroactions can be used to parameterize the evolutionary system spanned by the parameterized system (f, U): with any retroaction $\widetilde{u}$ we associate the evolutionary system $S(\cdot, \widetilde{u})$ generated by the differential equation

$$x'(t) = f(x(t), \widetilde{u}(t, x(t)))$$

Whenever a class $\widetilde{\mathcal{U}}$ has been chosen, we observe the following

$$\{S(x, \widetilde{u})\}_{\widetilde{u} \in \widetilde{\mathcal{U}}} \subset \mathcal{S}(x)$$

[2] Open loop controls can be only measurable. When the map $f : X \times \mathcal{U} \mapsto X$ is continuous, the Carathéodory Theorem states that differential equation (2.10)(i) has solutions even when the open loop control is just measurable, (and thus, can be discontinuous as a function of time). It is the case whenever they take their values in a finite set, in which case they are usually called "bang-bang controls" in the control terminology.

The evolutionary system can be parameterized by a feedback class $\widetilde{\mathcal{U}}$ if equality holds true.

The choice of an adequate class $\widetilde{\mathcal{U}}$ of feedbacks regulating specific evolutions satisfying required properties is often an important issue. Finding them may be a difficult problem to solve. Even though one could solve this problem, computing or using a feedback in a class too large may not be desirable whenever feedbacks are required to belong to a class of specific maps (constant maps, time-dependent polynomials, etc.). Another issue concerns the use of a prescribed class of retroactions and to "combine" them to construct new feedbacks for answering some questions, viability or capturability, for instance. This issue is dealt with in Chap. 11, p. 437.

Remark. For one-dimensional systems, retroactions are classified in positive retroactions, when the phenomenon is "amplified", and negative ones in the opposite case. They were introduced in 1885 by French physiologist *Charles-Edouard Brown-Séquard* under the nicer names *"dynamogenic" and "inhibitive" retroactions respectively.* □

The concepts of retroaction and feedback play a central role in control theory, for building servomechanisms, and then, later, in all versions of the "theory of systems" born from the influence of the mathematics of their time on biology, as the Austrian biologist *Ludwig von Bertalanffy* (1901-1972) in his book *Das biologische Weltbild* published in 1950, and after *Jan Smuts* (1870-1950) in his 1926 *Holism and evolution*. The fact that not only effects resulted from causes, but that also effects retroacted on causes, "closing" a system, has had a great influence in many fields.

2.7.3 Differential Inclusions

In the early times of (linear) control theory, the set-valued map was assumed to be constant ($U(\cdot) = U$) and even the parameter set U was taken to be equal to the entire vector space $\mathcal{U} := \mathbb{R}^c$. In this case, the parameterized system is a system of parameterized differential equations, so that the theory of (linear) differential equations could be used.

The questions arose to consider the case of state-dependent constraints bearing on the controls. For example, set-valued maps of the form $U(x) := \prod_{j=1}^{m}[a_j(x), b_j(x)]$ summarize state-dependent constraints of the form:

$$\forall t \geq 0,\ \forall j = 1, \ldots, m,\ a_j(x(t)) \leq u_j(t) \leq b_j(x(t))$$

When the constraints bearing on the parameters (controls, regulons, tyches) are state dependent, we can no longer use differential equations. We must appeal to the theory of differential inclusions, initiated in the early

1930's by *André Marchaud* and *Sanislas Zaremba*, and next, by the Polish and Russian schools, around *Tadeusz Ważewski* and *Alexei Filippov*, who laid the foundations of the mathematical theory of differential inclusions after the 1950's.

Indeed, denoting by

$$F(x) := f(x, U(x)) = \{f(x, u)\}_{u \in U(x)}$$

the subset of all available velocities $f(x, u)$ at x when u ranges over the set of parameters, we observe the following:

Lemma 2.7.3 *[**Differential Inclusions**] Let $F(x) := f(x, U(x))$ denote the set of velocities of the parameterized system. The evolutions $x(\cdot)$ governed by the parameterized system*

$$\begin{cases} (i) \;\; x'(t) = f(x(t), u(t)) \\ (ii) \; u(t) \in U(x(t)) \end{cases} \tag{2.12}$$

are governed by the differential inclusion

$$x'(t) \in F(x(t)) \tag{2.13}$$

and conversely.

An equilibrium of a differential inclusion is a stationary solution of this inclusion.

By taking $f(x, u) := u$ and $U(x) := F(x)$, any differential inclusion $x'(t) \in F(x(t))$ appears as a parameterized system (f, U) parameterized by its velocities. *Whenever we do not need to write the controls explicitely, it is simpler to consider a parameterized system as a differential inclusion.* Most theorems on differential equations can be adapted to differential inclusions (some of them, the basic ones, are indeed more difficult to prove), but they are by now available.

However, there are examples of differential inclusions without solutions, such as the simplest one:

Example of Differential Inclusion Without Solution: The constrained set is $K := [a, b]$ and the subsets of velocities are singletons except at one point $c \in]a, b[$, where $F(x) := \{-1, 1\}$:

$$F(x) := \begin{cases} +1 & \text{if } x \in [a, c[\\ -1 \text{ or } +1 & \text{if } x = c \\ -1 & \text{if } x \in]c, b] \end{cases}$$

No evolution can start from c. Observe that this is no longer a counter-example when $F(c) := [-1, +1]$, since in this case c is an equilibrium, since its velocity 0 belongs to $F(c)$.

Remark. Although a differential inclusion assigns several velocities to a same states, this does not imply that the associated evolutionary system is non deterministic. It may happen for certain classes of differential inclusions. This is the case for instance when there exists a constant $\lambda \in \mathbb{R}$ such that

$$\forall x, y \in X, \ \forall u \in F(x), \ \forall v \in F(y), \ \ \langle u - v, x - y \rangle \leq \lambda \|x - y\|^2$$

because in this case evolutions starting from each initial state, if any, are unique. $\square$

For discrete dynamical systems, the single-valuedness of the dynamics $\varphi : X \mapsto X$ is equivalent to the single-valuedness of the associated evolutionary system $\mathcal{S}_\varphi : X \mapsto X^{\mathbb{N}}$. This is no longer the case for continuous time dynamical systems:

Warning: *The deterministic character of an evolutionary system generated by a parameterized system is a concept different from the set-valued character of the map F. What matters is that the evolutionary system $\mathcal{S}$ associated with the parameterized system is single-valued (deterministic evolutionary systems) or set-valued (nondeterministic evolutionary systems).*

It is the case, for instance, for set-valued maps F which are *monotone set-valued maps* in the sense that

$$\forall \ y \in \mathrm{Dom}(F), \ \forall \ u \ \in \ U(x), \ y \ \in \ V(y), \ \ \langle u - v, x - y \rangle \ \leq \ 0$$

2.8 Evolutionary Systems

Therefore, we shall study general evolutionary systems defined as set-valued maps $X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ satisfying given requirements listed below. For continuous time evolutionary systems, the state space X is a finite dimensional vector space for most examples. However, besides the characterization of *regulation maps*, which are specific for control systems, many theorems are true even in cases when the evolutionary system is not generated by control systems or differential inclusions, and for infinite dimensional vector spaces X.

Other Examples of State Spaces:

1. When $\mathrm{X} := \mathcal{C}(-\infty, 0; X)$ is the space of evolution histories (see Chap. 12 of the first edition of [18, Aubin]), we supply it with the metrizable compact convergence topology,
2. When X is a space of spatial functions when one deals with partial differential inclusions or distributed control systems, we endow it with its natural topology for which it is a complete metrizable spaces,
3. When $X := \mathcal{K}(\mathbb{R}^d)$ is the set of nonempty compact subsets of the vector space $\mathbb{R}^d$, we use the Pompeiu–Hausdorff topology (morphological and mutational equations, presented in [23, Aubin]).

The algebraic structures of the state space appear to be much less relevant in the study of evolutionary systems. Only the following algebraic operations on the evolutionary spaces $\mathcal{C}(0, +\infty; X)$ are used in the properties of viability kernels and capture basins:

Definition 2.8.1 ***[Translations and Concatenations]***

1. ***Translation*** *Let $x(\cdot) : \mathbb{R}_+ \mapsto X$ be an evolution. For all $T \geq 0$, the translation (to the left) $\kappa(-T)(x(\cdot))$ of the evolution $x(\cdot)$ is defined by $\kappa(-T)(x(\cdot))(t) := x(t+T)$ and t the translation (to the right) $\kappa(+T)(x(\cdot))(t) := x(t-T)$,*
2. ***Concatenation*** *Let $x(\cdot) : \mathbb{R}_+ \mapsto X$ and $y(\cdot) : \mathbb{R}_+ \mapsto X$ be two evolutions. For all $T \geq 0$, the concatenation $(x(\cdot) \diamond_T y(\cdot))(\cdot)$ of the evolutions $x(\cdot)$ and $y(\cdot)$ at time T is defined by*

$$(x(\cdot) \diamond_T y(\cdot))(t) := \begin{cases} x(t) & \text{if } t \in [0,T] \\ \kappa(+T)(y(\cdot))(t) := y(t-T) & \text{if } t \geq T \end{cases} \tag{2.14}$$

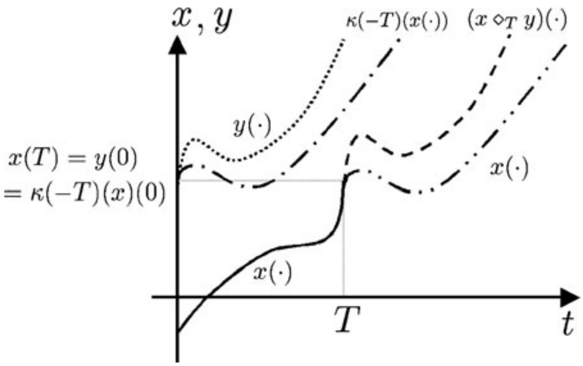


Fig. 2.3 Translations and Concatenations.
• *plain (—): $x(\cdot)$ for $t \in [0,T]$;*
• *dash dot dot ($- \cdot \cdot$): $x(\cdot)$ for $t \geq T$;*

- *dot* $(\cdots)$: $y(\cdot)$;
- *dash dot* $(-\cdot)$: $\kappa(-T)(x(\cdot))$;
- *dashed* $(-\ -)$: $(x \diamond_T y)(\cdot)$.

$x(\cdot)$ is thus the union of the plain and the dash dot dot. The concatenation $x(\cdot) \diamond_T y(\cdot)$ of x and y is the union of the plain and the dashed.

The concatenation $(x(\cdot) \diamond_T y(\cdot))(\cdot)$ of two continuous evolutions at time T is continuous if $x(T) = y(0)$. We also observe that $(x(\cdot) \diamond_0 y(\cdot))(\cdot) = y(\cdot)$, that $\forall T \geq S \geq 0,\ \ (\kappa(-S)(x(\cdot) \diamond_T y(\cdot))) \ = \ (\kappa(-S)x(\cdot)) \diamond_{T-S} y(\cdot)$ and thus, that

$$\forall T \geq 0,\ \ \kappa(-T)(x(\cdot) \diamond_T y(\cdot)) \ = \ y(\cdot)$$

The adaptation of these definitions to discrete time evolutions is obvious:

$$\begin{cases} (i)\ \ \kappa(-N)(\overrightarrow{x})_j := x_{j+N} \\ (ii)\ (\overrightarrow{x} \diamond_N \overrightarrow{y})_j \ := \begin{cases} x_j & \text{if } 0 \leq j < N \\ y_{j-N} & \text{if } j \geq N \end{cases} \end{cases} \tag{2.15}$$

We shall use only the following properties of evolutionary systems:

Definition 2.8.2 ***[Evolutionary Systems]*** *Let us consider a set-valued map $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ associating with each initial state $x \in X$ a (possibly empty) subset of* evolutions *$x(\cdot) \in \mathcal{S}(x)$ starting from x in the sense that $x(0) = x$. It is said to be an* evolutionary system *if it satisfies*

1. *the* translation property*: Let $x(\cdot) \in \mathcal{S}(x)$. Then for all $T \geq 0$, the translation $\kappa(-T)(x(\cdot))$ of the evolution $x(\cdot)$ belongs to $\mathcal{S}(x(T))$,*
2. *the* concatenation property*: Let $x(\cdot) \in \mathcal{S}(x)$. Then for every $T \geq 0$ and $y(\cdot) \in \mathcal{S}(x(T))$, the concatenation $(x(\cdot) \diamond_T y(\cdot))(\cdot)$ belongs to $\mathcal{S}(x)$.*

The evolutionary system is said to be deterministic *if $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ is single-valued.*

There are several ways for describing continuity of the evolutionary system $x \rightsquigarrow \mathcal{S}(x)$ with respect to the initial state, regarded as stability property: Stability means generally that the solution of a problem depends continuously upon its data or parameters. Here, for differential inclusions, the data are usually and principally the initial states, but can also be other parameters involved in the right hand side of the differential inclusion. We shall introduce them later, when we shall study the topological properties of the viability kernels and capture basins (See Sect. 10.3.2, p. 387 of Chap. 10, p. 375).

2.9 Viability Kernels and Capture Basins for Discrete Time Systems

2.9.1 Definitions

Definition 6, p. 15 can be adapted to discrete evolution $\overrightarrow{x}$: it is *viable in a subset* $K \subset X$ (an environment) if:

$$\forall n \geq 0, \quad x_n \in K \tag{2.16}$$

and *capture* a target C if it is viable in K until it reaches the target C in *finite time*:

$$\exists\, N \geq 0 \text{ such that} \begin{cases} x_N \in C \\ \forall n \leq N, \ x_N \ \in \ K \end{cases} \tag{2.17}$$

Consider a set-valued map $\Phi : X \rightsquigarrow X$ from a metric space X to itself, governing the evolution $\overrightarrow{x} : n \mapsto x_n$ defined by

$$\forall j \geq 0, \ x_{j+1} \in \Phi(x_j)$$

and the associated evolutionary system $\mathcal{S}_\Phi : X \rightsquigarrow X^{\mathbb{N}}$ associating with any $x \in X$ the set of evolutions $\overrightarrow{x}$ of solutions to the above discrete system starting at x. Replacing the space $\mathcal{C}(0, +\infty; X)$ of continuous time-dependent functions by the space $X^{\mathbb{N}}$ of discrete-time dependent functions (sequences) and making the necessary adjustments in definitions, we can still regard $\mathcal{S}_\Phi$ as an evolutionary system from X to $X^{\mathbb{N}}$.

The viability kernels $\text{Viab}_{(\varphi,U)}(K, C) := \text{Viab}_\Phi(K, C) := \text{Viab}_{\mathcal{S}_\Phi}(K, C)$ and the invariance kernels $\text{Inv}_\Phi(K, C) := \text{Inv}_{\mathcal{S}_\Phi}(K, C)$ are defined in the very same way:

Definition 2.9.1 ***[Viability Kernel under a Discrete System]*** *Let $K \subset X$ be an environment and $C \subset K$ a target.*

The subset $\text{Viab}_\Phi(K, C)$ *of initial states $x_0 \in K$ such that* ***at least one evolution*** *$\overrightarrow{x} \in \mathcal{S}_\Phi(x_0)$ starting at x_0 is viable in K for all $n \geq 1$ or viable in K until it reaches C in finite time is called the* viability kernel of K with target C *under $\mathcal{S}$.*

When the target $C = \emptyset$ is the empty set, we say that $\text{Viab}_\Phi(K) = \text{Viab}_\Phi(K, \emptyset)$ *is the* viability kernel *of K.*

The subset $\text{Capt}_\Phi(K, C)$ *of initial states $x_0 \in K$ such that* ***at least one evolution*** *$\overrightarrow{x} \in \mathcal{S}_\Phi(x_0)$ starting at x_0 is viable in K until it reaches C in finite time is called the* capture basin of C viable in K *under $\mathcal{S}_\Phi$.*

We say that

1. *a subset* K *is* viable outside the target $C \subset K$ *under the discrete system* $\mathcal{S}_\Phi$ *if* $K = \mathrm{Viab}_\Phi(K, C)$ *and that* K *is* viable *under* $\mathcal{S}_\Phi$ *if* $K = \mathrm{Viab}_\Phi(K)$,
2. *that* C *is* isolated *in* K *if* $C = \mathrm{Viab}_\Phi(K, C)$,
3. *that* K *is a* repeller *if* $\mathrm{Viab}_\Phi(K) = \emptyset$, *i.e. if the empty set is isolated in* K.

We introduce the discrete invariance kernels and absorption basins:

Definition 2.9.2 ***[Invariance Kernel under a Discrete System]*** *Let* $K \subset X$ *be a environment and* $C \subset K$ *a target.*

The subset $\mathrm{Inv}_\Phi(K, C) := \mathrm{Inv}_{\mathcal{S}_\Phi}(K, C)$ *of initial states* $x_0 \in K$ *such that* ***all evolutions*** $\overrightarrow{x} \in \mathcal{S}_\Phi(x_0)$ *starting at* x_0 *are viable in* K *for all* $n \geq 1$ *or viable in* K *until they reach* C *in finite time is called the* discrete invariance kernel of K with target C *under* $\mathcal{S}_\Phi$.

When the target $C = \emptyset$ *is the empty set, we say that* $\mathrm{Inv}_\Phi(K) := \mathrm{Inv}_\Phi(K, \emptyset)$ *is the* discrete invariance kernel *of* K.

The subset $\mathrm{Abs}_\Phi(K, C)$ *of initial states* $x_0 \in K$ *such that* ***all evolutions*** $\overrightarrow{x} \in \mathcal{S}_\Phi(x_0)$ *starting at* x_0 *are viable in* K *until they reach* C *in finite time is called the* absorption basin of C invariant in K *under* $\mathcal{S}_\Phi$.

We say that

1. *a subset* K *is* invariant outside a target $C \subset K$ *under the discrete system* $\mathcal{S}_\Phi$ *if* $K := \mathrm{Inv}_\Phi(K, C)$ *and that* K *is* invariant *under* $\mathcal{S}_\Phi$ *if* $K = \mathrm{Inv}_\Phi(K)$,
2. *that* C *is* separated *in* K *if* $C = \mathrm{Inv}_\Phi(K, C)$.

In the discrete-time case, the following characterization of viability and invariance of K with a target $C \subset K$ is a tautology:

Theorem 2.9.3 ***[The Discrete Viability and Invariance Characterization]*** *Let* $K \subset X$ *and* $C \subset K$ *be two subsets and* $\Phi : K \rightsquigarrow X$ *govern the evolution of the discrete system. Then the two following statements are equivalent*

1. K *is viable outside* C *under* Φ *if and only if*

$$\forall x \in K \backslash C, \ \Phi(x) \cap K \neq \emptyset \tag{2.18}$$

2. K *is invariant outside* C *under* Φ *if and only if*

$$\forall x \in K \backslash C, \ \Phi(x) \subset K \tag{2.19}$$

Unfortunately, the analogous characterization is much more difficult in the case of continuous time control systems, where the proofs of the statements require almost all fundamental theorems of functional analysis to be proved (see Chap. 19, p.769).

Remark. The fact that the above characterizations of viability and invariance in terms of (2.18) and (2.19) are trivial does not imply that using them is necessarily an easy task: Proving that $\Phi(x) \cap K$ is not empty or that $\Phi(x) \subset K$ can be difficult and requires some sophisticated theorems of nonlinear analysis mentioned in Chap. 9, p. 319. We shall meet the same obstacles – but compounded – when using the Viability Theorem 11.3.4, p. 455 and Invariance Theorem 11.3.7, p. 457 for continuous time systems. □

For discrete systems $x_{j+1} \in \Phi(x_j) := \varphi(x_j, U(x_j))$, it is also easy to construct the regulation map R_K governing viable evolutions in the viability kernel:

Definition 2.9.4 *[**Regulation Maps**] Let (φ, U) be a discrete parameterized system, K be an environment and $C \subset K$ be a target. The* regulation map *R_K is defined on the viability kernel of K by* $\forall x \in \mathrm{Viab}_{(\varphi,U)}(K,C) \setminus C$,

$$R_K(x) := \{u \in U(x) \text{ such that } \varphi(x,u) \in \mathrm{Viab}_{(\varphi,U)}(K,C)\} \tag{2.20}$$

The regulation map is computed from the discrete parameterized system (φ, U), the environment K and the target $C \subset K$.

For discrete-time parameterized systems (φ, U), all evolutions governed by the discrete parameterized subsystem (φ, R_K) are viable in the viability kernel of K with target C. Unfortunately, this important property is no longer necessarily true for continuous-time systems.

Theorem 2.9.5 *[**Invariance Property of Regulation Maps**] The regulation map R_K satisfies*

$$\mathrm{Viab}_{(\varphi,U)}(K,C) \;=\; \mathrm{Inv}_{(\varphi,R_K)}(K,C)$$

All other submaps $P \subset R_K$ also satisfy

$$\mathrm{Viab}_{(\varphi,U)}(K,C) \;=\; \mathrm{Inv}_{(\varphi,P)}(K,C) \tag{2.21}$$

The regulation map is the largest map satisfying this property.

Proof. Theorem 2.9.3, p. 72 and Definition 2.9.4, p. 73 imply that the regulation map R_K satisfy

$$\mathrm{Viab}_{(\varphi,U)}(K,C) = \mathrm{Inv}_{(\varphi,R_K)}(K,C)$$

1. If $Q \subset R_K \subset U$ is a set-valued map defined on $\mathrm{Viab}_{(\varphi,U)}(K,C)$, then inclusions

$$\begin{cases} \mathrm{Viab}_{(\varphi,U)}(K,C) = \mathrm{Inv}_{(\varphi,R_K)}(K,C) \subset \mathrm{Inv}_{(\varphi,Q)}(K,C) \\ \subset \mathrm{Viab}_{(\varphi,Q)}(K,C) \subset \mathrm{Viab}_{(\varphi,R_K)}(K,C) \subset \mathrm{Viab}_{(\varphi,U)}(K,C) \end{cases}$$

   imply that all the subsets coincide, and in particular, that $\mathrm{Inv}_{(\varphi,Q)}(K,C) = \mathrm{Viab}_{(\varphi,U)}(K,C)$.
2. The regulation map R_K is the largest one by construction satisfying (2.21), p. 73, because if a set-valued map $P \supset R_K$ is strictly larger than R_K, then there would exist an element $(x_0,u_0) \in \mathrm{Graph}(P) \setminus \mathrm{Graph}(R_K)$, i.e., such that $\varphi(x_0,u_0) \notin \mathrm{Viab}_{(\varphi,U)}(K,C)$. But since $\mathrm{Inv}_{(\varphi,P)}(K,C) \subset \mathrm{Viab}_{(\varphi,U)}(K,C)$, all elements $\varphi(x,u)$ when $u \in P(x_0)$ belong to $\mathrm{Viab}_{(\varphi,U)}(K,C)$, a contradiction. □

2.9.2 Viability Kernel Algorithms

For evolutionary systems associated with discrete dynamical inclusions and control systems, the *Viability Kernel Algorithm* and the *Capture Basin Algorithm* devised by Patrick Saint-Pierre allow us to

1. compute the viability kernel of an environment or the capture basin of a target under a control system,
2. compute the graph of the regulation map governing the evolutions viable in the environment, forever or until they reach the target in finite time.

This algorithm *manipulates subsets instead of functions*, and is part of the emerging field of *"set-valued numerical analysis"*.

The viability kernel algorithm provides the exact subset of initial states of the state space from which at least one evolution of the discrete system remains in the constrained set, forever or until it reaches the target in finite time, *without computing these evolutions.*

However, viable evolutions can be obtained from any state in the viability kernel or the capture basin by using the regulation map. The viability kernel algorithms provide the regulation map by computing their graphs, which are also subsets.

This regulation map allows us to "tame" evolutions to maintain them in the viability kernel or the capture basin. Otherwise, using the initial dynamical system instead of the regulation map, evolutions may quickly leave the environment, above all for systems which are sensitive to initial states, such as the Lorenz system.

Consequently, viability kernel and capture basin algorithms face the same "dimensionality curse" than algorithms for solving partial differential equations or other "grid" algorithms. They manipulate indeed "tables" of points in the state space, which become very large when the dimension of the state space is larger than 4 or 5. At the end of the process, the graph of the regulation map can be recovered and stored in the state-control space, which requires higher dimensions. Once the graph of the regulation map is stored, it is then easy to pilot evolutions which are viable forever or until they reach their target.

Despite these shortcomings, the viability kernel algorithms present some advantage over the simulation methods, known under the name of *shooting methods*. These methods compute the evolutions starting at each point and check whether or not at least one evolution satisfies the required properties. They need much less memory space, but demand a considerable amount of time, because, the number of initial states of the environment is high, and second, in the case of controlled systems, the set of evolutions starting from a given initial state becomes huge.

On the other hand, viability properties and other properties of this type, such as asymptotic properties, *cannot be checked on computers*. For instance, one cannot verify whether an evolution is viable forever, since computers provide evolutions defined on a finite number of time steps.

Nothing guarantees that the *finite* time chosen to stop the computation of the solution is large enough to check whether a property bearing on the whole evolution is valid. Such property can be satisfied for a given number of times, without implying that it still holds true later on, above all for systems, like the Lorenz one, which are sensitive to initial conditions.

Finally, starting from an initial state in the viability kernel or the capture basin, shooting methods use solvers which do not take into consideration the corrections *for imposing the viability of the solution*, for instance. Since the initial state is only an approximation of the viability kernel, the absence of these corrections does not allow us to "tame" evolutions which then may leave the environment, and very quickly for systems which are sensitive to initial states, such as the Lorenz system or the discrete time dynamics related to Julia sets.

2.9.3 Julia and Mandelbrot Sets

Studies of dynamical systems (bifurcations, chaos, catastrophe) focus on the dependence on some properties of specific classes of dynamical systems of constant parameters u (which, in contrary to the control case, are not allowed to evolve with time): The idea is to study a given property in terms of the parameter u of a discrete dynamical system $x_{j+1} = \varphi(x_j, u)$ where u is a parameter ranging over a subset $\mathcal{U}$.

Benoît Mandelbrot introduced in the late 1970's the Mandelbrot sets and functions in his investigation of the fractal dimension of subsets:

Definition 2.9.6 ***[The Mandelbrot Function]*** *For a discrete dynamical system $x_{j+1} = \varphi(x_j, u)$ where u is a parameter ranging over a subset $\mathcal{U}$, the* Mandelbrot function $\mu : X \times \mathcal{U} \mapsto \mathbb{R}_+ \cup \{+\infty\}$ *associates with any pair (x, u) the scalar*

$$\mu(x,u) := \sup_{j \geq 0} \|x_j\|$$

where $x_{j+1} = \varphi(x_j, u)$ and $x_0 = x$.
The subset $K_u := \mathrm{Viab}_\varphi(B(0,1))$ is the filled-in Julia set *and its boundary $J_u := \partial K_u$ the* Julia set.

The Mandelbrot function μ is characterized through the viability kernel of an auxiliary system:

Lemma 2.9.7 ***[Viability Characterization of the Mandelbrot Function]*** *Let us associate with the map φ the following map $\Phi : X \times \mathcal{U} \times \mathbb{R} \mapsto X \times \mathcal{U} \times \mathbb{R}$ defined by $\Phi(x,u,y) = (\varphi(x,u), u, y)$. Consider the subset*

$$K := \{(x,u,y) \in X \times \mathcal{U} \times \mathbb{R} \mid \|x\| \ \leq y\}$$

Then the Mandelbrot function is characterized by the formula

$$\mu(x,u) = \inf_{(x,u,y) \in \mathrm{Viab}_\Phi(K)} y$$

or, equivalently,

$$\mu(x,u) \leq y \text{ if and only if } x \in \mathrm{Viab}_{\varphi(\cdot,u)}(B(0,y))$$

Proof. Indeed, to say that (x,u,y) belongs to the viability kernel of $K := \{(x,u,y) \mid \|x\| \ \leq y\}$ means that the solution (x_j, u, y) to the auxiliary system satisfies

$$\forall j \geq 0, \ \|x_j\| \leq y$$

i.e., that $\mu(x,u) \leq y$. $\square$

This story was initiated by *Pierre Fatou* and *Gaston Julia*:

Pierre Fatou and Gaston Julia. *Pierre Fatou [1878-1929] and Gaston Julia [1893-1978] studied in depth the iterates of complex function*
$z \mapsto z^2 + u$,
or, equivalently, of the map
$$(x, y) \mapsto \varphi(x, y) := (x^2 - y^2 + a, 2xy + b)$$
when $z = x + iy$ *and* $u = a + ib$.
The subset $K_u := \mathrm{Viab}_\varphi(B(0, 1))$ *is the* filled-in Julia set *for this specific map* φ *and its boundary* $J_u := \partial K_u$ *the* Julia set. *The subsets whose filled-in Julia sets have empty interior are called* Fatou dust.

Therefore, the viability kernel algorithm allows us to compute the Julia sets, offering an alternative to "shooting methods". These shooting methods compute solutions of the discrete system starting from various initial states and check whether a given property is satisfied or not. Here, this property is the viability of the evolution *for a finite number of times.* Instead, the Viability Kernel Algorithm provides the set of initial states from which at least one evolution is viable forever, *without computing all evolutions to check whether one of them satisfies it.*

Furthermore, the regulation map built by the Viability Kernel Algorithm *provides evolutions which are effectively viable in the viability kernel.* This is a property that shooting methods cannot provide:

- first, because the viability kernel is not known precisely, but only approximatively,
- and second, because even if we know that the initial state belongs to the viability kernel, the evolution governed by such a program is not necessarily viable

The reason why this happens is that programs computing evolutions are independent of the viability problem. They do not make the corrections at each step guaranteeing that the new state is (approximatively) viable in K, contrary to the one computed with Viability Kernel Algorithm.

The more so in this case, since this system is sensitive to initial data.

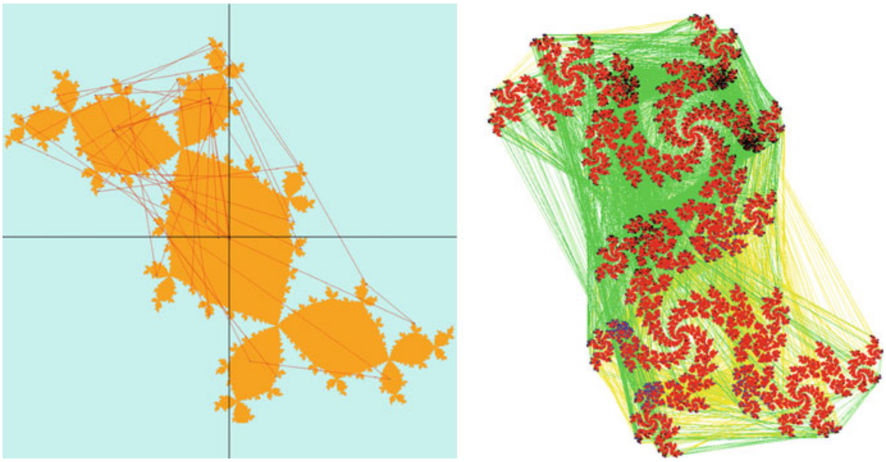


Fig. 2.4 Julia sets: Douady Rabbit and Fatou Dust.
We refer to Figure 2.5 for the computation of its boundary, the Julia set, thanks to Theorem 9.2.18. Even for discrete systems, the round-off errors do not allow the discrete evolution to remain in filled-in Julia set, which is the viability kernel of the ball, whereas the viability kernel algorithm provides both the filled-in Julia set, its boundary and evolutions which remain in the Julia set.

In 1982, a deep theorem by *Adrien Douady* and *Hubbard* states that K_u is connected if and only if $\mu(0, u)$ is finite.

Theorem 9.2.18, p. 339 states that the Julia set is the viability kernel of $K \setminus C$ if and only if it is viable and C absorbs the interior of K_u. In this case, the Viability Kernel Algorithm also provides the Julia set J_u by computing the viability kernel of $K \setminus C$.

We illustrate this fact by computing the viability kernel of the complement of a ball $B(0, \alpha) \subset K$ in K whenever the interior of K_u is not empty (we took $u = -0.202 - 0.787i$). We compute the viability kernel for $\alpha :=$ 0.10, 0.12, 0.14 and 0.16, and we observed that this viability kernel is equal to the boundary for $\alpha = 0.16$. In this case, the ball $B(0, 0.16)$ is absorbing the interior of the filled-in Julia set. The resulting computations can be seen in Figures 2.4 and 2.5.

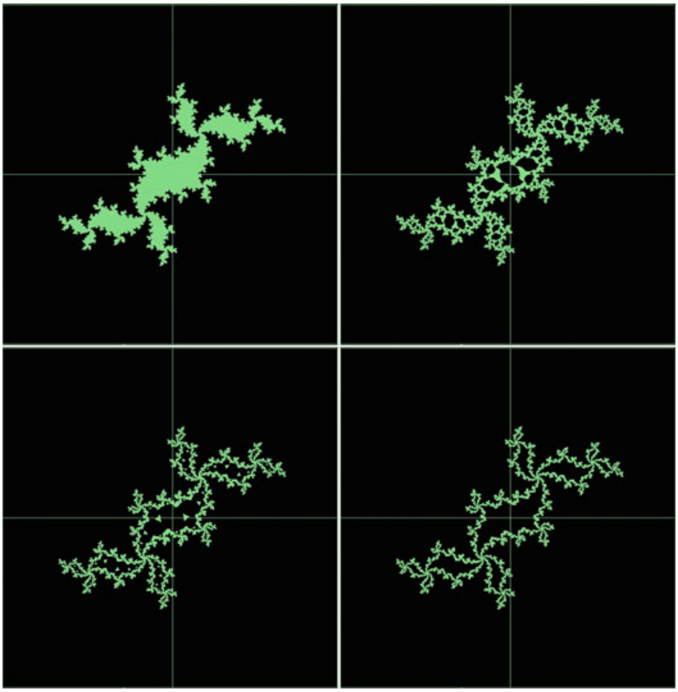


Fig. 2.5 Computation of the Julia set.
The figure on the left is the filled-in Julia set K_u *with* $u = -0.202 - 0.787i$, *which is the viability kernel of the unit ball. The other figures display the viability kernels of* $K \setminus B(0, \alpha)$ *for* $\alpha := 0.10,\ 0.12,\ 0.14$ *and* 0.16. *We obtain the Julia set for* $\alpha = 0.16$. *Theorem 9.2.18, p. 339 states that the ball* $B(0, 0.16)$ *absorbs the interior of the filled-in Julia set.*

2.9.4 Viability Kernels under Disconnected Discrete Systems and Fractals

If Φ is disconnecting, the viability kernel is a Cantor set, with further properties (self similarity, fractal dimension). Recall that Φ^{-1} denotes the inverse of Φ.

Definition 2.9.8 *[**Hutchinson Maps**] A set-valued map* Φ *is said to be* disconnecting *on a subset* K *if there exists a finite number* p *of functions* $\alpha_i : K \mapsto X$ *such that*

$$\forall x \in K,\ \Phi^{-1}(x) := \bigcup_{i=1}^{p} \alpha_i(x)$$

and such that there exist constants $\lambda_i \in]0,1[$ *satisfying: for each subset* $C \subset K$*,*

$$\begin{cases} (i) \quad \forall i = 1,\ldots,p,\ \alpha_i(C) \ \subset\ C \ (\alpha_i \text{ is } \textit{antiextensive}) \\ (ii) \ \ \forall i \neq j,\ \alpha_i(C) \cap \alpha_j(C) = \emptyset \\ (iii) \ \forall i = 1,\ldots,p,\ \mathrm{diam}(\alpha_i(C)) \leq \lambda_i \mathrm{diam}(C) \end{cases}$$

If the functions $\alpha_i : K \mapsto K$ *are contractions with Lipschitz constants* $\lambda_i \in]0,1[$*, then* Φ^{-1} *is called an* Hutchinson map *(introduced in 1981 by* John Hutchinson *and also called an* iterated function system *by* Michael Barnsley*.)*

We now define Cantor sets:

Definition 2.9.9 *[**Cantor Sets**] A subset* K *is said to be*

1. perfect *if it is closed and if each of its elements is a limit of other elements of* K*,*
2. totally disconnected *if it contains no nonempty open subset,*
3. *a* Cantor set *if it is non-empty compact, totally disconnected and perfect.*

The famous Cantor Theorem states:

Theorem 2.9.10 *[**The Cantor Theorem**] The viability kernel of a compact set under a disconnecting map is an uncountable Cantor set.*

The Cantor set is a viability kernel and the Viability Kernel Algorithm is the celebrated construction procedure of the Cantor set.

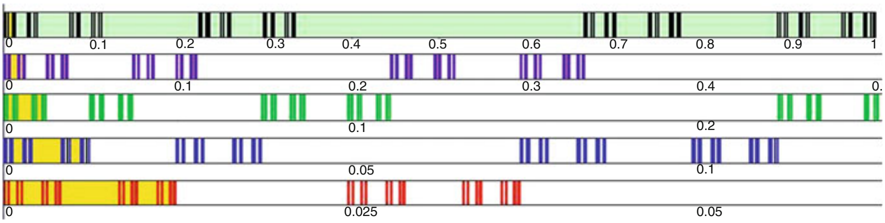


Fig. 2.6 Example: Cantor Ternary Map.

Corollary 2.9.11 *[**The Cantor Ternary Set**] The Cantor ternary set* $\mathbf{C}$ *is the viability kernel of the interval* $[0,1]$ *under the* Cantor Ternary Map Φ *defined on* $K := [0,1] \subset \mathbb{R}$ *by*

$$\Phi(x) := (3x, 3(1-x))$$

The Cantor Ternary Set is a self similar (see Definition 2.9.14, p. 84), symmetric, uncountable Cantor set with fractal dimension $\frac{\log 2}{\log 3}$ *(see Definition 2.9.13, p. 83) and satisfies* $\mathbf{C} = \alpha_1(\mathbf{C}) \cup \alpha_2(\mathbf{C})$ *and* $\alpha_1(\mathbf{C}) \cap \alpha_2(\mathbf{C}) = \emptyset$.

Proof. The *Cantor Ternary Map* is disconnecting because

$$\Phi^{-1}(x) := \left(\alpha_1(x) := \frac{x}{3}, \alpha_2(x) := 1 - \frac{x}{3}\right)$$

so that $\alpha_1(K) = \left[0, \frac{1}{3}\right]$ and $\alpha_2(K) = \left[\frac{2}{3}, 1\right]$ and that the α_i's are antiextensive contractions of constant $\frac{1}{3}$. $\square$

Example: Quadratic Map In Sect. 2.3, p. 50, we associated with the quadratic map $\varphi(x) := 5x(1-x)$ the set-valued map $\Phi : [0,1] \rightsquigarrow [0,1]$ defined by $\Phi(x) := \varphi(x)$ when $x \in [0,a]$ and $x \in [b,1]$ and $\varphi(x) := \emptyset$ when $x \in]a,b[$, where $a := \dfrac{1}{2} - \dfrac{\sqrt{5}}{10}$ and $b := \dfrac{1}{2} + \dfrac{\sqrt{5}}{10}$ are the roots of the equation $\varphi(x) = 1$.

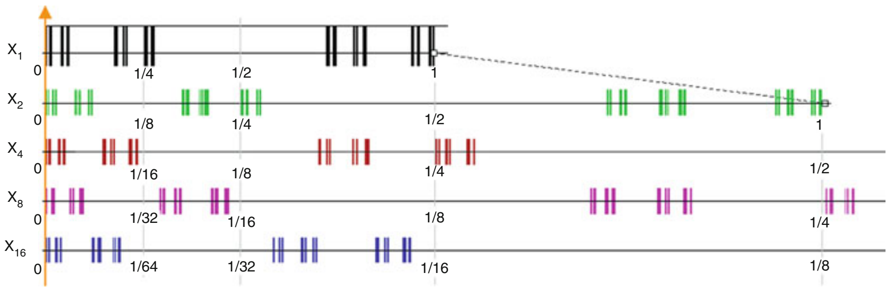


Fig. 2.7 Viability Kernel under the Quadratic Map.
The viability kernel of the interval $[0,1]$ *under the quadratic map* Φ *associated with the map* $\varphi(x) := \{5x(1-x)\}$ *is an uncountable, symmetric Cantor set.*

The interval $[0,1]$ *is viable under the Verhulst logistic differential equation* $x'(t) = rx(t)(1 - x(t))$ *whereas its viability kernel is a Cantor set for its discrete counterpart* $x_{n+1} = rx_{n+1}(1 - x_{n+1})$ *when* $r > 4$.

Proof. Indeed, in this case, the inverse Φ^{-1} is defined by

$$\Phi^{-1}(y) := \left(\omega^{\flat}(y), \omega^{\sharp}(y)\right)$$

where we set

$$\omega^{\flat}(y) := \frac{1}{2} - \frac{\sqrt{r^2 - 4ry}}{2r} \text{ and } \omega^{\sharp}(y) := \frac{1}{2} + \frac{\sqrt{r^2 - 4ry}}{2r}$$

(see Sect. 2.3.1, p. 51). □

The intervals $\omega^{\flat}(K) = \left[0, \left(\frac{1}{2} - \frac{\sqrt{r^2-4r}}{2r}\right)\right]$ and $\omega^{\sharp}(K) = \left[\left(\frac{1}{2} + \frac{\sqrt{r^2-4r}}{2r}\right), 1\right]$ are disjoint intervals which do not cover $[0,1]$. The maps $\omega^{\flat}$ and $\omega^{\sharp}$ are antiextensive and contractions.

We know that the interval $[0,1]$ is viable under the *Verhulst* logistic equation, whereas for $r > 4$, we saw that the discrete viability kernel is a Cantor subset of $[0,1]$. But $[0,1]$ is still viable under the discretizations of the *Verhulst* logistic equation:

Proposition 2.9.12 ***[Discretization of the Verhulst Logistic Equation]*** *The interval* $[0,1]$ *is viable under the explicit discretization* Φ_h *of the* Verhulst *logistic equation, defined by*

$$\Phi_h(x) := rhx\left(\frac{1 + rh}{rh} - x\right)$$

Proof. Indeed, Φ_h is surjective from $[0,1]$ to $[0,1]$, and thus, $[0,1]$ is viable under Φ_h: Starting from $x_0 \in [0,1]$, the discrete evolution $\overrightarrow{x}$ defined by

$$x_{n+1} = rhx_n\left(\frac{1 + rh}{rh} - x_n\right)$$

remains in K. □

This is an example illustrating the danger of using "discrete analogues" of continuous time differential equations instead of their discretizations. The latter share the same properties than the differential equation (under adequate assumptions), whereas discrete analogues may not share them. This is the case for the quadratic map, the prototype of maps producing chaos, analogues of the Verhulst logistic equation.

Example: Sierpinski Gasket

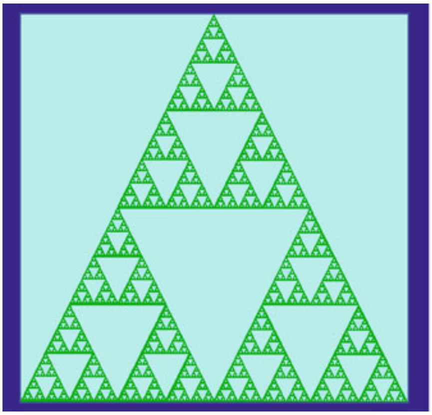


Fig. 2.8 The Sierpinski Gasket.
The Sierpinski Gasket is the viability kernel of the square $[0,1]^2$ *under the discrete map associating with each pair* (x,y) *the subset* $\Phi(x,y) := \left\{(2x,2y),(2x-1,2y),(2x-\frac{1}{2},2y-1)\right\}$ *of 3 elements. Since this map is disconnecting, the Sierpinski Gasket is a self similar, uncountable Cantor set with fractal dimension* $\frac{\log 3}{\log 2}$ *(*left figure*), named from* Waclaw Sierpinski *(1882–1969) (*right figure*).*

2.9.4.1 Fractal Dimension of Self-Similar Sets

Some viability kernels under discrete disconnecting maps have a fractal dimension that we now define:

Definition 2.9.13 *[**Fractal Dimension**] Let* $K \subset \mathbb{R}^d$ *be a subset of* $\mathbb{R}^d$ *and* $\nu_K(\varepsilon)$ *the smallest number of* ε*-cubes* $\varepsilon[-1,+1]^d$ *needed to cover the subset* K*. If the limit*

$$\dim(K) := \lim_{\varepsilon \mapsto 0+} \frac{\log(\nu_K(\varepsilon))}{\log\left(\frac{1}{\varepsilon}\right)}$$

exists and is not an integer, it is called the fractal dimension *of* K*.*

To say that K has a fractal dimension $\delta := \dim(K)$ means that the smallest number $\nu_K(\varepsilon)$ of ε-cubes needed to cover K behaves like $\frac{a}{\varepsilon^\delta}$ for some constant $a > 0$.

Actually, it is enough to take subsequences $\varepsilon_n := \lambda^n$ where $0 < \lambda < 1$ converging to 0 when $n \to +\infty$, so that

$$\dim(K) := \lim_{n \to +\infty} \frac{\log\left(\nu_K(\lambda^n)\right)}{n \log\left(\frac{1}{\lambda}\right)}$$

Definition 2.9.14 *[**Self-Similar Sets**] Functions α_i are called* similarities *if*

$$\forall x,\ y \in K,\ d(\alpha_i(x), \alpha_i(y)) = \lambda_i d(x,y)$$

Let Φ be a disconnecting map associated with p similarities α_i.

A subset $K_\propto$ is said to be self-similar under Φ if

$$K_\propto = \bigcup_{i=1}^{p} \alpha_i(K_\propto) \text{ and the subsets } \alpha_i(K_\propto) \text{ are pairwise disjoint.}$$

For example,

1. the Cantor set is self-similar:

$$\mathbf{C} = \alpha_1(\mathbf{C}) \cup \alpha_2(\mathbf{C})$$

It is the union of two similarities of constant $\frac{1}{3}$,

2. the Sierpinski gasket is self-similar[3]:

$$\mathbf{S} = \Phi^{-1}(\mathbf{S}) = \bigcup_{i=1}^{3} \alpha_i(\mathbf{S})$$

It is the union of three similarities of constant $\frac{1}{2}$.

[3] Actually, the subsets are not pairwise disjoint, but the above results hold true when the intersections $\alpha_i(\mathbf{C}) \cap \alpha_j(\mathbf{C})$ are manifolds of dimension strictly smaller than the dimension of the vector space.

Lemma 2.9.15 *[**Fractal Dimension of Self-Similar Sets**] If the p similarities α_i have the same contraction rate $\lambda < 1$, then the fractal dimension of a self-similar set $K_\propto = \bigcup_{i=1}^{p} \alpha_i(K_\propto)$ is equal to*

$$\dim(K_\propto) = \frac{\log(p)}{\log\left(\frac{1}{\lambda}\right)}$$

Consequently,

1. The fractal dimension of the Cantor set is equal to $\frac{\log 2}{\log 3}$: $p = 2$ and $\lambda = \frac{1}{3}$,
2. The fractal dimension of the Sierpinski gasket is equal to $\frac{\log 3}{\log 2}$: $p = 3$ and $\lambda = \frac{1}{2}$.

2.10 Viability Kernels and Capture Basins for Continuous Time Systems

Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ denote the evolutionary system associated with parameterized dynamical system (2.10) and $\mathcal{H} \subset \mathcal{C}(0, \infty; X)$ be a subset of evolutions sharing a given set of properties.

2.10.1 Definitions

When the parameterized system is regarded as a control system, we single out the *inverse image* (see Definition 18.3.3, p. 720) of $\mathcal{H}$ under the evolutionary system:

Definition 2.10.1 *[**Inverse Image under an Evolutionary System**] Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ denote an evolutionary system and $\mathcal{H} \subset \mathcal{C}(0, \infty; X)$ a subset of evolutions sharing a given set of properties. The set*

$$\mathcal{S}^{-1}(\mathcal{H}) := \{x \in X \mid \mathcal{S}(x) \cap \mathcal{H} \neq \emptyset\} \tag{2.22}$$

*of initial states $x \in X$ from which starts **at least one evolution** $x(\cdot) \in \mathcal{S}(x)$ satisfying the property $\mathcal{H}$ is the* inverse image *of $\mathcal{H}$ under $\mathcal{S}$.*

For instance, taking for set $\mathcal{H} := \mathcal{X}$ defined as the set of stationary evolutions, we obtain the set of all equilibria x of the evolutionary system: at least one evolution $x(\cdot) \in \mathcal{S}(x)$ remains constant and equal to x. In the same way, taking for set $\mathcal{H} := \mathcal{P}_T(X)$ the set of T-periodic evolutions, we obtain the set of points through which passes at least one T-periodic evolution of the evolutionary system.

When we take $\mathcal{H} := \mathcal{V}(K, C)$ to be the set of evolutions viable in a constrained subset $K \subset X$ outside a target $C \subset K$ (see 2.5, p. 49), we obtain the *viability kernel* $\mathrm{Viab}_{\mathcal{S}}(K, C)$ of K with target C:

Definition 2.10.2 ***[Viability Kernel and Capture Basin]*** *Let $K \subset X$ be a environment and $C \subset K$ be a target.*

1. *The subset* $\mathrm{Viab}_{\mathcal{S}}(K, C)$ *of initial states $x_0 \in K$ such that* ***at least one evolution*** *$x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 is viable in K for all $t \geq 0$ or viable in K until it reaches C in finite time is called the* viability kernel of K with target C *under $\mathcal{S}$.*
   When the target $C = \emptyset$ is the empty set, we say that $\mathrm{Viab}_{\mathcal{S}}(K) := \mathrm{Viab}_{\mathcal{S}}(K, \emptyset)$ *is the* viability kernel *of K. We set* $\mathrm{Capt}_{\mathcal{S}}(K, \emptyset) = \emptyset$.
2. *The subset* $\mathrm{Capt}_{\mathcal{S}}(K, C)$ *of initial states $x_0 \in K$ such that* ***at least one evolution*** *$x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 is viable in K until it reaches C in finite time is called the* capture basin of C viable in K *under $\mathcal{S}$. When $K = X$ is the whole space, we say that* $\mathrm{Capt}_{\mathcal{S}}(C) := \mathrm{Capt}_{\mathcal{S}}(X, C)$ *is the* capture basin *of C. (see Figure 5.2, p. 182)*

We say that

1. *a subset K is* viable *under $\mathcal{S}$ if $K = \mathrm{Viab}_{\mathcal{S}}(K)$,*
2. *K is* viable outside the target $C \subset K$ *under the evolutionary system $\mathcal{S}$ if $K = \mathrm{Viab}_{\mathcal{S}}(K, C)$,*
3. *C is* isolated *in K if $C = \mathrm{Viab}_{\mathcal{S}}(K, C)$,*
4. *K is a* repeller *if $\mathrm{Viab}_{\mathcal{S}}(K) = \emptyset$, i.e., if the empty set is isolated in K.*

Remark: Trapping Set. A connected closed viable subset is sometimes called a *trapping set*. In the framework of differential equations, *Henri Poincaré* introduced the concept of *shadow* (in French, *ombre*) of K, which is the set of initial points of K from which (all) evolutions leave K in finite time. It is thus equal to the complement $K \backslash \mathrm{Viab}_{\mathcal{S}}(K)$ of the viability kernel of K in K. □

Remark. Theorem 9.3.13, p. 353 provides sufficient conditions (the environment K is compact and backward viable, the evolutionary system is upper semicompact) for the viability kernel to be nonempty.

Another interesting case is the one when the viability kernel $\mathrm{Viab}_{\mathcal{S}}(K) \subset \mathrm{Int}(K)$ of K is contained in the interior of K (in this case, $\mathrm{Viab}_{\mathcal{S}}(K)$) is said to be *source* of K (see Definition 9.2.3, p. 323). □

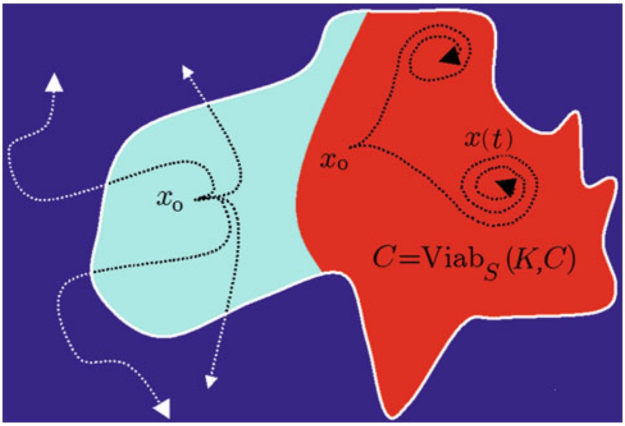


Fig. 2.9 Viability Outside a Target and Isolated Target.
If C is isolated, all evolutions starting in K outside of C are viable outside C before leaving K in finite time.

2.10.2 Viability Kernels under the Lorenz System

We resume our study of the Lorenz system (2.6), p. 57 initiated in Sect. 2.4.2, p. 56.

We provide the viability kernel of the cube $[-\alpha, +\alpha] \times [-\beta, +\beta] \times [-\gamma, +\gamma]$ under the Lorenz system (2.6), p. 57 and the backward Lorenz system

$$\begin{cases} (i) & x'(t) = -\sigma y(t) + \sigma x(t) \\ (ii) & y'(t) = -rx(t) + y(t) + x(t)z(t) \\ (iii) & z'(t) = \; -x(t)y(t) + bz(t) \end{cases}$$

We call "backward viability kernel" the viability kernel under the backward system.

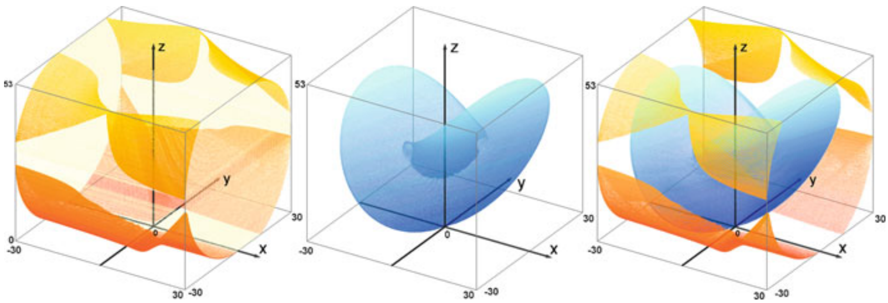


Fig. 2.10 Viability Kernels of a Cube K under Forward and Backward Lorenz Systems.
The figure displays the forward viability kernel of the cube K (left), the backward viability kernel (center) and the superposition of the two (right). We take $\sigma > b + 1$, Proposition 8.3.3, p. 282 implies that whenever the viability kernel of the backward system is contained in the interior of K, the backward viability kernel is contained in the forward viability kernel. Proposition 9.3.11, p. 351 implies that the famous Lorenz attractors *(see Definition 9.3.8, p. 349) is contained in the backward viability kernel.*

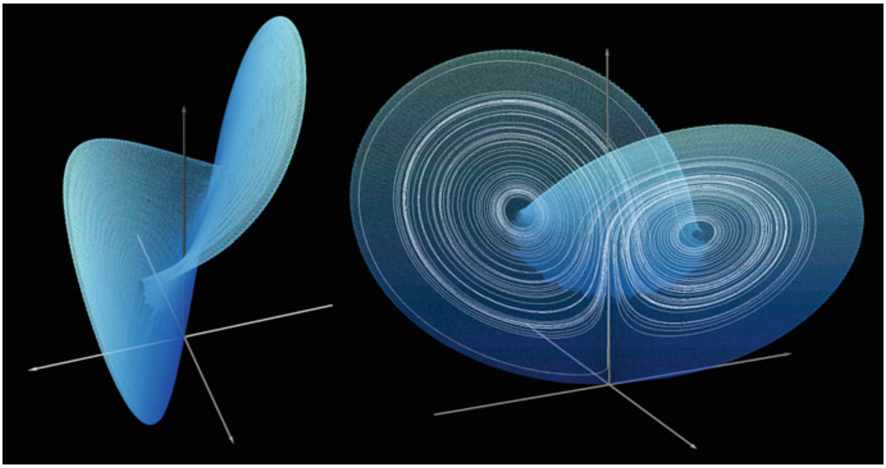


Fig. 2.11 Backward Viability Kernel and Viable Evolution.
This figure displays another view of the backward viability kernel and a viable evolution. They are computed with the viability kernel algorithm.

2.11 Invariance Kernel under a Tychastic System

The questions involved in the concepts of viability kernels and capture basins ask only of the existence of an evolution satisfying the viability or the viability/capturability issue. In the case of parameterized systems, this lead to the interpretation of the parameter as a control or a regulon. When the parameters are regarded as tyches, disturbances, perturbations, etc., the questions are dual: they require that all evolutions satisfy the viability or the viability/capturability issue.

We then introduce the "dual" concept of *invariance kernel and absorption basin*:

Here, we regard the parameterized system

$$x'(t) \ = \ f(x(t), v(t)) \text{ where } v(t) \ \in \ V(x(t)) \tag{2.23}$$

where $v(t)$ is no longer a control or a regulon, but a tyche, where the set of tyches is $\mathcal{V}$ and where $V : X \rightsquigarrow \mathcal{V}$ is a *tychastic map* as a *tychastic system.* Although this system is formally the same that control system (1.1), p. 14

$$x'(t) = f(x(t), u(t)) \text{ where } u(t) \ \in \ U(x(t))$$

the questions asked are different: We no longer check whether a given property is satisfied by **at least** one evolution governed by the control or regulated system, but by **all** evolutions governed by the tychastic system.

When the parameterized system is regarded as a tychastic system, it is natural to consider the *core* (see Definition 18.3.3, p. 720) of a set of evolutions under a tychastic system:

Definition 2.11.1 *[**Core under an Evolutionary System**] Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ denote an evolutionary system and $\mathcal{H} \subset \mathcal{C}(0, \infty; X)$ a subset of evolutions sharing a given property. The set*

$$\mathcal{S}^{\ominus 1}(\mathcal{H}) := \{x \in X \mid \mathcal{S}(x) \ \subset \ \mathcal{H}\} \tag{2.24}$$

*of initial states $x \in X$ from which **all evolutions** $x(\cdot) \in \mathcal{S}(x)$ satisfy the property $\mathcal{H}$ is called the* core *of $\mathcal{H}$ under $\mathcal{S}$.*

Taking $\mathcal{H} := \mathcal{V}(K, C)$, we obtain the *invariance kernel* $\mathrm{Inv}_{\mathcal{S}}(K, C)$ of K with target C:

Definition 2.11.2 *[**Invariance Kernel and Absorption Basin**] Let $K \subset X$ be a environment and $C \subset K$ be a target.*

1. *The subset* $\mathrm{Inv}_{\mathcal{S}}(K, C)$ *of initial states $x_0 \in K$ such that **all evolutions** $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 are viable in K for all $t \geq 0$ or viable in K until they reach C in finite time is called the* invariance kernel of K with target C *under $\mathcal{S}$.*
   When the target $C = \emptyset$ is the empty set, we say that $\mathrm{Inv}_{\mathcal{S}}(K) := \mathrm{Inv}_{\mathcal{S}}(K, \emptyset)$ *is the* invariance kernel *of K.*
2. *The subset* $\mathrm{Abs}_{\mathcal{S}}(K, C)$ *of initial states $x_0 \in K$ such that **all evolutions** $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 are viable in K until they reach C in finite time is called the* absorption basin of C invariant in K *under $\mathcal{S}$.*
   When $K = X$ is the whole space, we say that $\mathrm{Abs}_{\mathcal{S}}(X, C)$ *is the* absorption basin *of C.*

We say that

1. *a subset K is* invariant *under $\mathcal{S}$ if $K = \mathrm{Inv}_{\mathcal{S}}(K)$,*
2. *K is* invariant outside a target $C \subset K$ *under the evolutionary system $\mathcal{S}$ if $K = \mathrm{Inv}_{\mathcal{S}}(K, C)$,*
3. *C is* separated *in K if $C = \mathrm{Inv}_{\mathcal{S}}(K, C)$.*

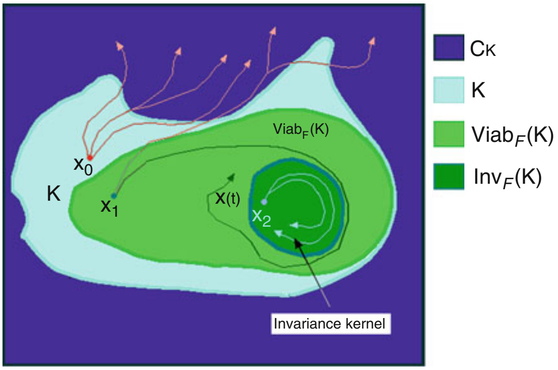


Fig. 2.12 Figure of an Invariance Kernel.
A state x_2 belongs to the invariance kernel of the environment K under an evolutionary system if ***all*** *the evolutions starting from it are viable in K forever. Starting from a state $x_1 \in K$ outside the invariance kernel,* ***at least*** *one evolution leaves the environment in finite time.*

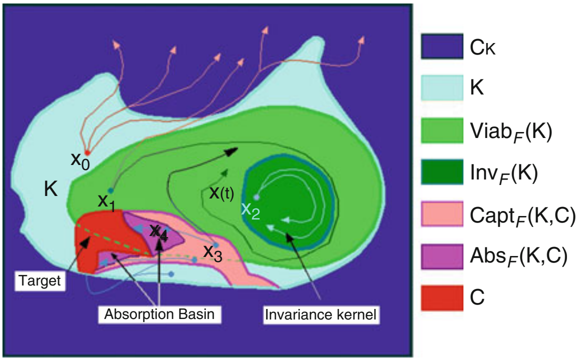


Fig. 2.13 Figure of an Absorption Basin.
All *evolutions starting from a state x_4 in the absorption basin of the target C invariant in the environment K are viable in K until they reach C in finite time.* ***At least*** *one evolution starting from $x_3 \in K$ outside the absorption basin remains viable outside the target C forever or until it leaves K.*

These are four of the main concepts used by viability theory. Other definitions, motivations and comments are given in Chap. 2, p. 43, their general properties in Chap. 10, p. 375, whereas their characterization in terms of tangential conditions are presented in Chap. 11, p. 437. Many other subsets of interest of initial conditions from which at least one or all evolution(s) satisfies(y) more and more complicated interesting properties will be introduced all along the book. They all are combinations in various ways of these basic kernels and basins.

For instance, *tychastic control systems* (or *dynamical games*) involve both regulons and tyches in the dynamics. Tyches describe uncertainties played by an indifferent, maybe hostile, Nature. Regulons are chosen among the available ones by the system in order *to adapt its evolutions regardless of the tyches*. We introduce the concept of *tychastic (or guaranteed) viability kernel*, which is the subset of initial states from which *there exists a regulon such that, for all tyches*, the associated evolutions are *viable* in the environment forever.

The set of initial states from which *there exists a regulon such that, for all tyches*, the associated evolutions reach the target *in finite time* before possibly violating the constraints is called the *tychastic (or guaranteed) absorption basin* of the target invariant in the environment.

Remark: Semi-permeability. We deduce from the definitions that from any $x \in \mathrm{Viab}_{\mathcal{S}}(K,C) \setminus \mathrm{Inv}_{\mathcal{S}}(K,C)$,

1. *there exists at least one evolution which is viable in* $\mathrm{Viab}_{\mathcal{S}}(K,C)$ *until it may reach the target* C,
2. *there exists at least one evolution which leaves* $\mathrm{Viab}_{\mathcal{S}}(K,C)$ *in finite time, and is viable in* $K \setminus C$ *until it leaves* K *in finite time.*

The latter property is a *semi-permeability property*:

1. the boundary of the invariance kernel separates the set of initial states from which all evolutions are viable in K until they may reach the target from the set of initial states satisfying the above property,
2. the boundary of the viability kernel separates the set of initial states from which there exists at least two different evolutions satisfying the above property from the set of initial states from which all evolutions are viable in $K \setminus C$ as long as it is viable in K.

Therefore $x \in \mathrm{Viab}_{\mathcal{S}}(K,C) \setminus \mathrm{Inv}_{\mathcal{S}}(K,C)$ is the set where some uncertainty about viability prevails, Outside the viability kernel, only one property is shared by **all** evolutions starting from an initial state: either they are viable in K until they may reach the target, or they leave C in finite time and are viable in $K \setminus C$ until they leave K in finite time.

The Quincampoix Barrier Theorems 10.5.19, p. 409 and 10.6.4, p. 413 provide precise statements of the properties of the boundaries of the viability and invariance kernels. □

2.12 Links between Kernels and Basins

Viability kernels and absorption basins are linked to each other by complementarity, as well as invariance kernels and capture basins:

Definition 2.12.1 ***[Complement of a Subset]*** *The complement of the subset $C \subset K$ in K is the set $K\backslash C := K \cap \complement C$ of elements $x \in K$ not belonging to C. When $K := X$ is the whole space, we set $\complement\, C := X\backslash C$. Observe that*

$$K\backslash C = \complement C\backslash\complement K \text{ and } \complement(K \setminus C) = C \cup \complement K$$

The following useful consequences relating the kernels and basins follow readily from the definitions:

Lemma 2.12.2 ***[Complements of Kernels and Basins]*** *Kernels and Basins are exchanged by complementarity:*

$$\begin{cases} (i) \;\; \complement\mathrm{Viab}_{\mathcal{S}}(K,C) = \mathrm{Abs}_{\mathcal{S}}(\complement C, \complement K) \\ (ii) \; \complement\mathrm{Capt}_{\mathcal{S}}(K,C) = \mathrm{Inv}_{\mathcal{S}}(\complement C, \complement K) \end{cases} \quad (2.25)$$

Remark. This would suggest that only two of these four concepts would suffice. However, we would like these kernels and basins to be closed under adequate assumption, and for that purpose, we need the four concepts, since the complement of a closed subset is open. But every statement related to the closedness property of these kernels and basins provide corresponding results on openness properties of their complements, as we shall see in Sect. 10.3.2 p. 387. □

The next result concerns the a priori futile or subtle differences between viability kernels with targets (concept proposed by Marc Quincampoix) and capture basins:

Lemma 2.12.3 *[**Comparison between Viability Kernels with Targets and Capture Basins**] The viability kernel of K with target C and the capture basin of C viable in K are related by formulas*

$$\mathrm{Viab}_{\mathcal{S}}(K, C) = \mathrm{Viab}_{\mathcal{S}}(K \setminus C) \cup \mathrm{Capt}_{\mathcal{S}}(K, C) \tag{2.26}$$

Hence the viability kernel with target C coincides with the capture basin of C viable in K if $\mathrm{Viab}_{\mathcal{S}}(K \setminus C) = \emptyset$, i.e., if $K \setminus C$ is a repeller. This is particularly the case when the viability kernel $\mathrm{Viab}_{\mathcal{S}}(K)$ of K is contained in the target C, and more so, when K itself is a repeller.

Proof. Actually, we shall prove that

$$\begin{cases} (i) \ \ \mathrm{Viab}_{\mathcal{S}}(K, C) \setminus \mathrm{Capt}_{\mathcal{S}}(K, C) \ \subset \ \mathrm{Viab}_{\mathcal{S}}(K \setminus C) \\ (ii) \ \mathrm{Viab}_{\mathcal{S}}(K, C) \setminus \mathrm{Viab}_{\mathcal{S}}(K \setminus C) \ \subset \ \mathrm{Capt}_{\mathcal{S}}(K, C) \end{cases}$$

Indeed, inclusion $\mathrm{Viab}_{\mathcal{S}}(K \setminus C) \cup \mathrm{Capt}_{\mathcal{S}}(K, C) \subset \mathrm{Viab}_{\mathcal{S}}(K, C)$ being obvious, the opposite inclusion is implied by, for instance,

$$\mathrm{Viab}_{\mathcal{S}}(K, C) \setminus \mathrm{Capt}_{\mathcal{S}}(K, C) \ \subset \ \mathrm{Viab}_{\mathcal{S}}(K \setminus C) \tag{2.27}$$

because

$$\begin{cases} \mathrm{Viab}_{\mathcal{S}}(K, C) \ = \ \mathrm{Capt}_{\mathcal{S}}(K, C) \cup (\mathrm{Viab}_{\mathcal{S}}(K, C) \setminus \mathrm{Capt}_{\mathcal{S}}(K, C)) \\ \subset \ \mathrm{Viab}_{\mathcal{S}}(K \setminus C) \cup \mathrm{Capt}_{\mathcal{S}}(K, C) \end{cases}$$

For proving the first formula

$$\mathrm{Viab}_{\mathcal{S}}(K, C) \setminus \mathrm{Capt}_{\mathcal{S}}(K, C) \ \subset \ \mathrm{Viab}_{\mathcal{S}}(K \setminus C) \tag{2.28}$$

we observe that Lemma 2.12.2, p. 92 implies that $\mathrm{Viab}_{\mathcal{S}}(K, C) \setminus \mathrm{Capt}_{\mathcal{S}}(K, C) = \mathrm{Viab}_{\mathcal{S}}(K, C) \cap \mathrm{Inv}_{\mathcal{S}}(\complement C, \complement K)$ by formula (2.25)(i). Take any $x \in \mathrm{Viab}_{\mathcal{S}}(K, C) \cap \mathrm{Inv}_{\mathcal{S}}(\complement C, \complement K)$. Since $x \in \mathrm{Viab}_{\mathcal{S}}(K, C)$, there exists at least one evolution $x(\cdot) \in \mathcal{S}(x)$ either viable in K forever or reaching C in finite time. But since $x \in \mathrm{Inv}_{\mathcal{S}}(\complement C, \complement K)$, all evolutions starting from x are viable in $\complement C$ forever or until they leave K in finite time. Hence the evolution $x(\cdot)$ cannot reach C in finite time, and thus, is viable in K forever, hence cannot leave K in finite time, and thus is viable in $\complement C$, and consequently, in $K \setminus C$.

Next, let us prove inclusion $\mathrm{Viab}_{\mathcal{S}}(K, C) \setminus \mathrm{Viab}_{\mathcal{S}}(K \setminus C) \subset \mathrm{Capt}_{\mathcal{S}}(K, C)$. Lemma 2.12.2, p. 92 implies that $\complement \mathrm{Viab}_{\mathcal{S}}(K \setminus C) = \mathrm{Abs}_{\mathcal{S}}(X, C \cup \complement K)$. Therefore, for any $x \in \mathrm{Viab}_{\mathcal{S}}(K, C) \setminus \mathrm{Viab}_{\mathcal{S}}(K \setminus C) = \mathrm{Viab}_{\mathcal{S}}(K, C) \cap \mathrm{Abs}_{\mathcal{S}}(X, C \cup \complement K)$, there exists an evolution $x(\cdot) \in \mathcal{S}(x)$ viable in K forever or until a time $t^\star < +\infty$ when $x(t^\star) \in C$, and all evolutions starting at x

either leave K in finite time or reach C in finite time. Hence, $x(\cdot)$ being forbidden to leave K in finite time, must reach the target in finite time. □

Lemma 2.12.4 ***[Partition of the Viability Kernel with Targets]*** *The following equalities hold true:*

$$\mathrm{Capt}_{\mathcal{S}}(K,C) \cap \mathrm{Inv}_{\mathcal{S}}(K \setminus C) = \mathrm{Abs}_{\mathcal{S}}(K,C) \cap \mathrm{Viab}_{\mathcal{S}}(K \setminus C) = \emptyset$$

Therefore, equality $\mathrm{Viab}_{\mathcal{S}}(K\backslash C) = \mathrm{Inv}_{\mathcal{S}}(K\backslash C)$ *implies that* $\mathrm{Viab}_{\mathcal{S}}(K \setminus C)$ *and* $\mathrm{Capt}_{\mathcal{S}}(K,C)$ *form a partition of* $\mathrm{Viab}_{\mathcal{S}}(K,C)$.

For invariance kernels, we obtain:

Lemma 2.12.5 ***[Comparison between Invariance Kernels with Targets and Absorption Basins]*** *The invariance kernel of K with target C and the absorption basin of C viable in K coincide whenever $K \setminus C$ is a repeller.*

Proof. We still observe that the invariance kernel $\mathrm{Inv}_{\mathcal{S}}(K,C)$ of K with target C coincides with the absorption basin $\mathrm{Abs}_{\mathcal{S}}(K,C)$ of C invariant in K whenever the viability kernel $\mathrm{Viab}_{\mathcal{S}}(K\backslash C)$ is empty. □

Therefore, *the concepts of viability and of invariance kernels with a target allow us to study both the viability and invariance kernels of a closed subset and the capture and absorption basins of a target.*

Remark: Stochastic and Tychastic Properties. There are natural and deeper mathematical links between viability and capturability properties under stochastic and tychastic systems. A whole book could be devoted to this topic. We just develop in this one few remarks in Sect. 10.10, p. 433. □

2.13 Local Viability and Invariance

We introduce the weaker concepts of *local* viability and invariance:

Definition 2.13.1 ***[Local Viability and Invariance]***
Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,\infty;X)$ be an evolutionary system and a subset $K \subset X$.

1. *A subset K is said to be* locally viable *under $\mathcal{S}$ if from any initial state $x \in K$* there exists at least one evolution $x(\cdot) \in \mathcal{S}(x)$ *and a strictly positive time $T_{x(\cdot)} > 0$ such that $x(\cdot)$ is viable in K on the nonempty interval $[0, T_{x(\cdot)}[$ (it is thus* viable *if $T_{x(\cdot)} = +\infty$),*
2. *A subset K is said to be* locally invariant *under $\mathcal{S}$ if from any initial state $x \in K$ and* for any evolution $x(\cdot) \in \mathcal{S}(x)$*, there exists a strictly positive $T_{x(\cdot)} > 0$ such that $x(\cdot)$ is viable in K on the nonempty interval $[0, T_{x(\cdot)}[$ (it is thus* invariant *if $T_{x(\cdot)} = +\infty$).*

The (local) viability property of viability kernels and invariance property of invariance kernels are particular cases of viability property of inverse images of sets of evolutions and invariance property of their cores when the sets of evolutions are (locally) stable under translations. Local viability kernels are studied in Sect. 10.4.3, p. 396. For the time, we provide a family of examples of subsets (locally) viable and invariant subsets built from subsets of evolutions stable (or invariant) under translation.

Definition 2.13.2 *[**Stability Under Translation**] A subset $\mathcal{H} \subset \mathcal{C}(0, \infty; X)$ of evolutions is* locally stable under translation *if for every $x(\cdot) \in \mathcal{H}$, there exists $T_{x(\cdot)} > 0$ such that for every $t \in [0, T_{x(\cdot)}[$, the translation $\kappa(-t)(x(\cdot))(\cdot)$ belongs to $\mathcal{H}$. It is said to be* stable under translation *if we can always take $T_{x(\cdot)} = +\infty$.*

Inverse images (resp. cores) of subsets of evolutions stable under translation (resp. concatenation) are viable (resp. invariant) subsets:

Proposition 2.13.3 *[**Viability of Inverse Images and Invariance of Cores**] Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ be an evolutionary system and $\mathcal{H} \subset \mathcal{C}(0, \infty; X)$ be a subset of evolutions. If $\mathcal{H}$ is (locally) stable under translation, then*

1. *its inverse image $\mathcal{S}^{-1}(\mathcal{H}) := \{x \in X \mid \mathcal{S}(x) \cap \mathcal{H}\}$ under $\mathcal{S}$ is (locally) viable,*
2. *its core $\mathcal{S}^{\ominus 1}(\mathcal{H}) := \{x \in X \mid \mathcal{S}(x) \subset \mathcal{H}\}$ under $\mathcal{S}$ is (locally) invariant.*

(See Definition 18.3.3, p. 720).

Proof. 1. The (local) translation property of $\mathcal{S}$ implies the (local) viability of the inverse image $\mathcal{S}^{-1}(\mathcal{H})$. Take $x_0 \in \mathcal{S}^{-1}(\mathcal{H})$ and prove that there exists an evolution $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 viable in $\mathcal{S}^{-1}(\mathcal{H})$ on some interval $[0, T_{x(\cdot)}]$. Indeed, there exists an evolution $x(\cdot) \in \mathcal{S}(x_0) \cap \mathcal{H}$ and

$T_{x(\cdot)} > 0$ such that for every $t \in [0, T_{x(\cdot)}[$, the translation $\kappa(-t)(x(\cdot))(\cdot)$ belongs to $\mathcal{H}$. It also belongs to $\mathcal{S}(x(t))$ thanks to the translation property of evolutionary systems. Therefore $x(t)$ does belong to $\mathcal{S}^{-1}(\mathcal{H})$ for every $t \in [0, T_{x(\cdot)}[$.

2. The concatenation property of $\mathcal{S}$ implies the local invariance of the core $\mathcal{S}^{\ominus 1}(\mathcal{H})$. Take $x_0 \in \mathcal{S}^{\ominus 1}(\mathcal{H})$ and prove that for all evolutions $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0, there exists $T_{x(\cdot)}$ such that $x(\cdot)$ is viable in $\mathcal{S}^{\ominus 1}(\mathcal{H})$ on the interval $[0, T_{x(\cdot)}]$. Indeed, take any such evolution $x(\cdot) \in \mathcal{S}(x_0)$ which belongs to $\mathcal{H}$ by definition. Thus there exists $T_{x(\cdot)} > 0$ such that for every $t \in [0, T_{x(\cdot)}]$, the translation $\kappa(-t)(x(\cdot))(\cdot)$ belongs to $\mathcal{H}$. Take any $t \in [0, T_{x(\cdot)}[$ and any evolution $y(\cdot) \in \mathcal{S}(x(t))$. Hence the t-concatenation $(x \diamond_t y)(\cdot)$ belongs to $\mathcal{S}(x_0)$ by definition of evolutionary systems, and thus to $\mathcal{H}$ because $x_0 \in \mathcal{S}^{\ominus 1}(\mathcal{H})$. Since $\mathcal{H}$ is locally stable under translation, we deduce that $y(\cdot) = (\kappa(-t)((x \diamond_t y(\cdot))))(\cdot)$ also belongs to $\mathcal{H}$. Since this holds true for every any evolution $y(\cdot) \in \mathcal{S}(x(t))$, we infer that $x(t) \in \mathcal{S}^{\ominus 1}(\mathcal{H})$. $\square$

The study of local viability is continued in Sect. 10.4.3, p. 396.

2.14 Discretization Issues

The task for achieving this objective is divided in two different problems:

1. Approximate the continuous problem by discretized problem (in time) and digitalized on a grid (in state) by *difference inclusions* on *digitalized sets*. Most of the time, the real mathematical difficulties come from the proof of the convergence theorems stating that the limits of the solutions to the approximate discretized/digitalized problems converge (in an adequate sense) to solutions to the original continuous-time problem.
2. Compute the viability kernel or the capture basin of the discretized/digitalized problem with a specific algorithm, also providing the viable evolutions, as mentioned in Sect. 2.9.2, p. 74.

Let h denote the time discretization step. There are many more or less sophisticated ways to discretize a continuous parameterized system (f, U) by a discrete one (ϕ_h, U). The simplest way is to choose the explicit scheme $\phi_h(x, u) := x + hf(x, u)$. Indeed, the discretized system can be written as

$$\frac{x_{j+1} - x_j}{h} = f(x_j, u_j) \text{ where } u_j \in U(x_j)$$

The simplest way to digitalize a vector space $X := \mathbb{R}^d$ is to embed a (regular) *grid*[4] $X_\rho := \rho\mathbb{Z}^d$ in X. Points of the grid are of the form $x :=$

[4] supplied with the metric $d(x, y)$ equal to 0 if $x = y$ and to 1 if $x \neq y$.

$(\rho n_i)_{i=1,\ldots,n}$ where for all $i = 1, \ldots, n$, n_i ranges over the set $\mathbb{Z}$ of positive or negative integers.

We cannot define the above discrete system on the grid X_ρ, because there is no reason why for any $x \in X_\rho$, $\phi_h(x, u)$ would belong to the grid X_ρ. Let us denote by $B := [-1, +1]^d$ the unit square ball of X^d. One way to overcome this difficulty is to "add" the set $\rho B = [-\rho, +\rho]^d$ to $\phi_h(x, u)$. Setting $\lambda A + \mu B := \{\lambda x + \mu y\}_{x \in A,\, y \in B}$ when $A \subset X$ and $B \subset X$ are nonempty subsets of a vector space X, we obtain the following example:

Definition 2.14.1 *[**Explicit Discrete/Digital Approximation**] Parameterized control systems*

$$x'(t) \;=\; f(x(t), u(t)) \text{ where } u(t) \in U(x(t))$$

can be approximated by discrete/digital parameterized systems

$$\frac{x_{j+1} - x_j}{h} \in f(x_j, u_j) + \rho B \text{ where } u_j \in U(x_j)$$

which is a discrete system $x_{j+1} \in \Phi_{h,\rho}(x_j)$ on X_ρ where

$$\Phi_{h,\rho}(x) \;:=\; x + h f(x, U(x)) + \rho h B$$

We can also use implicit difference schemes:

Definition 2.14.2 *[**Implicit Discrete/Digital Approximation**] Parameterized control systems*

$$x'(t) \;=\; f(x(t), u(t)) \text{ where } u(t) \in U(x(t))$$

can be approximated by discrete/digital parameterized systems

$$\frac{x_{j+1} - x_j}{h} \in f(x_{j+1}, u_{j+1}) \text{ where } u_{j+1} \in U(x_{j+1})$$

which is a discrete system $x_{j+1} \in \Psi_{h,\rho}(x_j)$ on X_ρ where

$$\Psi_{h,\rho}(x) \;:=\; (\mathbf{I} - h f(\cdot, U(\cdot)))^{-1}(x) + \rho h B$$

Characterization Theorem 2.9.3, p. 72 of viability and invariance under discrete systems, applied to the explicit discretization of control systems, indicates how tangential conditions for characterizing viability and invariance under control systems did emerge:

Lemma 2.14.3 *[**Discretized Regulation Map**] Let us introduce the discretized* regulation map $R_{K_{h,\rho}}$ *defined by*

$$\forall x \in K, \ \ R_{K_{h,\rho}}(x) \ := \ \left\{ u \in U(x) \ \text{ such that } \ f(x,u) \ \in \ \frac{K-x}{h} + \rho B \right\} \tag{2.29}$$

Then K is viable (resp. invariant) under the discretized system if and only if $\forall\, x \in K, \ R_{K_{h,\rho}}(x) \neq \emptyset$ (resp. $\forall\, x \in K, \ R_{K_{h,\rho}}(x) = U(x)$).

For proving viability and invariance theorems in Chap. 11, p. 437, we shall take the limit in the results of the above lemma, and in particular, for the kind of "difference quotient" $\dfrac{K-x}{h}$, if one is allowed to say so.

14 *[**How Tangential Conditions Emerge**] The Bouligand-Severi tangent cone $T_K(x)$ (see Definition 11.2.1, p. 442) to K at $x \in K$ is the upper limit in the sense of Painlevé-Kuratowski of $\frac{K-x}{h}$ when $h \to 0$: $f(x,u)$ is the limit of elements v_n such that $x + h_n v_n \in K$ for some $h_n \to 0+$, i.e., of velocities $v_n \in \dfrac{K-x}{h_n}$.*
Consequently, "taking the limit", formally (for the time), we obtain the emergence of the (continuous-time) tangential condition

$$\forall x \in K, \ \ R_K(x) \ := \ \{u \in U(x) \ \text{ such that } \ f(x,u) \ \in \ T_K(x)\} \tag{2.30}$$

where $T_K(x)$ is the Bouligand-Severi tangent cone to K at $x \in K$ (see Definition 11.2.1, p.442).

This tangential condition will play a crucial role for characterizing viability and invariance properties for continuous-time systems in Chap. 11, p. 437.

2.15 A Viability Survival Kit

The mathematical properties of viability and invariance kernels and capture and absorption basins are presented in detail in Chap. 10 p. 375 for evolutionary systems and in Chap. 11, p. 437 for differential inclusions and control systems, where we can take advantage of tangential conditions involving tangent cones to the environments. This section presents few selected statements that are most often used, restricted to viability kernels and capture basins only. Three categories of statements are presented:

- The first one provides characterizations of viability kernels and capture bilateral fixed points, which are simple, important and are valid without any assumption.
- The second one provides characterizations in terms of local viability properties and backward invariance, involving topological assumptions on the evolutionary systems.
- The third one characterizes viability kernels and capture basins under differential inclusions in terms of tangential conditions, which furnishes the regulation map allowing to pilot viable evolutions (and optimal evolutions in the case of optimal control problems).

2.15.1 Bilateral Fixed Point Characterization

We consider the maps $(K, C) \mapsto \mathrm{Viab}(K, C)$ and $(K, C) \mapsto \mathrm{Capt}(K, C)$. The properties of these maps provide fixed point characterizations of viability kernels of the maps $K \mapsto \mathrm{Viab}(K, C)$ and $C \mapsto \mathrm{Viab}(K, C)$ and fixed point characterizations of capture basins of the maps $K \mapsto \mathrm{Capt}(K, C)$ and $C \mapsto \mathrm{Capt}(K, C)$. We refer to Definition 2.10.2, p.86 for the definitions of viable and isolated subsets.

Theorem 2.15.1 *[**The Fundamental Characterization of Viability Kernels**] Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ be an evolutionary system and $K \subset X$ be a environment. The viability kernel $\mathrm{Viab}_{\mathcal{S}}(K) := \mathrm{Viab}_{\mathcal{S}}(K, \emptyset)$ of K (see Definition 2.10.2, p. 86) is the **unique** subset D contained in K that is both*

1. ***viable** in K (and is the largest viable subset $D \subset K$ contained in K),*
2. ***isolated** in K (and is the smallest subset $D \subset K$ isolated in K):*

i.e., the bilateral fixed point

$$\mathrm{Viab}_{\mathcal{S}}(\mathrm{Viab}_{\mathcal{S}}(K)) = \mathrm{Viab}_{\mathcal{S}}(K) = \mathrm{Viab}_{\mathcal{S}}(K, \mathrm{Viab}_{\mathcal{S}}(K)) \tag{2.31}$$

For capture basins, we shall prove

Theorem 2.15.2 *[**The Fundamental Characterization of Capture Basins**] Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ be an evolutionary system, $K \subset X$ be an environment and $C \subset K$ be a nonempty target. The capture basin*

$\mathrm{Capt}_{\mathcal{S}}(K, C)$ *of* C *viable in* K *(see Definition 2.10.2, p. 86) is the* ***unique*** *subset* D *between* C *and* K *that is both*

1. ***viable outside*** C *(and is the largest subset* $D \subset K$ *viable outside* C*),*
2. *satisfying* $\mathrm{Capt}_{\mathcal{S}}(K, C) = \mathrm{Capt}_{\mathcal{S}}(K, \mathrm{Capt}_{\mathcal{S}}(K, C))$ *(and is the smallest subset* $D \supset C$ *to do so):*

i.e., the bilateral fixed point

$$\mathrm{Capt}_{\mathcal{S}}(\mathrm{Capt}_{\mathcal{S}}(K, C), C) = \mathrm{Capt}_{\mathcal{S}}(K, C) = \mathrm{Capt}_{\mathcal{S}}(K, \mathrm{Capt}_{\mathcal{S}}(K, C)) \tag{2.32}$$

2.15.2 Viability Characterization

However important Theorems 2.15.1, p. 99 and 2.15.2, p. 99 are, isolated subsets are difficult to characterize, in contrast to viable or locally viable subsets (see Definition 2.13.1, p. 94). It happens that isolated subsets are, under adequate assumptions, backward invariant (see Sect. 10.5.2, p. 401). Hence we shall introduce the concept of *backward evolutionary system* (see Definition 8.2.1, p. 276) and the concept of *backward invariance*, i.e., of invariance with respect to the backward evolutionary system (see Definition 8.2.4, p. 278). Characterizing viability kernels and capture basins in terms of forward viability and backward invariance allows us to use the results on viability and invariance.

Definition 2.15.3 *[**Backward Relative Invariance**] A subset* $C \subset K$ *is* backward invariant relatively to K *under* $\mathcal{S}$ *if for every* $x \in C$*, for every* $t_0 \in]0, +\infty[$*, for all evolutions* $x(\cdot)$ *arriving at* x *at time* t_0 *such that there exists* $s \in [0, t_0[$ *such that* $x(\cdot)$ *is viable in* K *on the interval* $[s, t_0]$*, then* $x(\cdot)$ *is viable in* C *on the same interval.*

If K is itself backward invariant, any subset backward invariant relatively to K is actually backward invariant.

Viability results hold true whenever the evolutionary system is upper semicontinuous (see Definitions 18.4.3, p. 729).

Using the concept of backward invariance, we provide a further characterization of viability kernels and capture basins:

Theorem 2.15.4 *[**Characterization of Viability Kernels**] Let us assume that $\mathcal{S}$ is upper semicompact and that the subset K is closed. The viability kernel* $\mathrm{Viab}_{\mathcal{S}}(K)$ *of a subset K under $\mathcal{S}$ is the **unique** closed subset $D \subset K$ satisfying*

$$\begin{cases} (i) & D \text{ is } \textit{viable} \text{ under } \mathcal{S} \\ (ii) & D \textit{ is baclward invariant} \text{ under } \mathcal{S} \\ (iii) & K \backslash D \text{ is a } \textit{repeller} \text{ under } \mathcal{S}. \end{cases} \tag{2.33}$$

For capture basins, we obtain

Theorem 2.15.5 *[**Characterization of Capture Basins**] Let us assume that $\mathcal{S}$ is upper semicompact, that the environment $K \subset X$ and the target $C \subset K$ are closed subsets satisfying*

1. K *is* backward invariant
2. $K \setminus C$ *is a* repeller *(*$\mathrm{Viab}_{\mathcal{S}}(K \setminus C) = \emptyset$*)*

Then the viable capture basin $\mathrm{Capt}_{\mathcal{S}}(K, C)$ *is the **unique** closed subset D satisfying $C \subset D \subset K$ and*

$$\begin{cases} (i) & D\backslash C \text{ is } \textit{locally viable} \text{ under } \mathcal{S} \\ (ii) & D \textit{ is relatively baclward invariant with respect to } K \text{ under } \mathcal{S}. \end{cases} \tag{2.34}$$

2.15.3 Tangential Characterization

These theorems, which are valid for any evolutionary system, paved the way to go one step further when the evolutionary system is associated with a differential inclusion (and control systems, as we shall see in Sect. 11.3.1, p. 453). We mentioned, in the case of discrete systems, how tangential conditions (2.30), p. 98 did emerge when we characterized viable and invariance (see Box 14, p. 98). Actually, we shall use the closed convex hull $T_K^{\star\star}(x)$ of the tangent cone $T_K(x)$ (see Definition 11.2.1, p. 442) for this purpose.

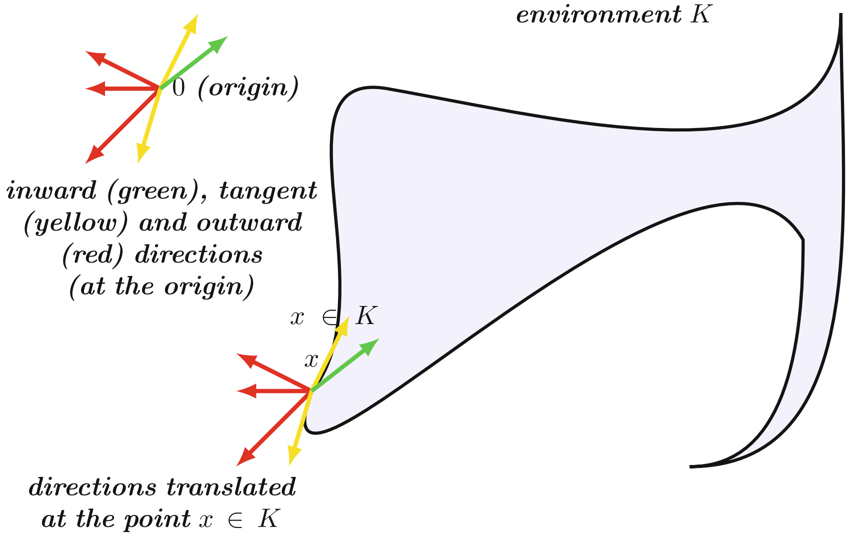


Fig. 2.14 Schematic Representation of Tangent Cones.
We represent the environment K, an element $x \in K$ and the origin. Six vectors v are depicted: one which points inward K, and thus tangent to K, two tangent vectors which are not inward and three outward vectors. Their translations at x belong to K for the inward vector, "almost" belong to K for the two tangent and not inward vectors (see Definition 11.2.1, p. 442) and belong to the complement of K for the three outward vectors.

Not only Viability and Invariance Theorems provide characterizations of viability kernels and capture basins, but also the *regulation map* $R_D \subset F$ which governs viable evolutions:

Definition 2.15.6 *[**Regulation Map**] Let us consider three subsets $C \subset D \subset K$ (where the target C may be empty) and a set-valued map $F : X \rightsquigarrow X$.*

The set-valued map $R_D : x \in D \rightsquigarrow F(x) \cap T_D^{\star\star}(x) \subset X$ is called the regulation map of F on $D \setminus C$ if

$$\forall\, x \in D \setminus C, \;\; R_D(x) \; := \; F(x) \cap T_D^{\star\star}(x) \; \neq \; \emptyset \tag{2.35}$$

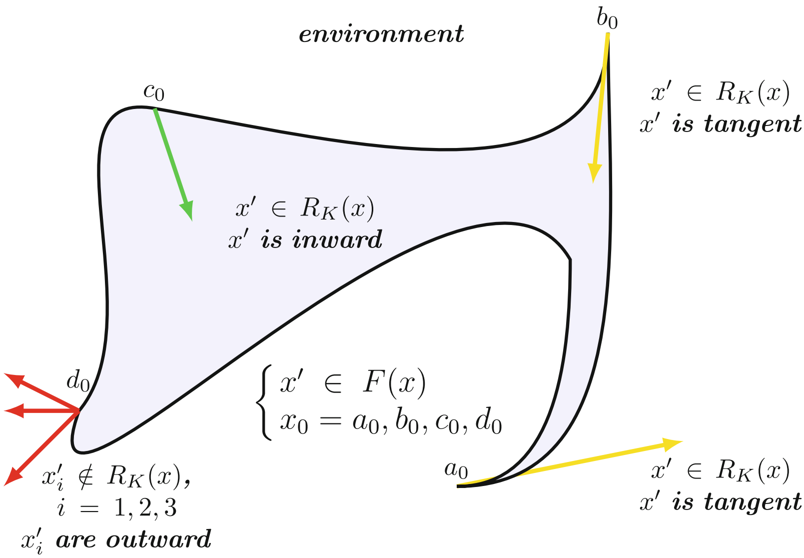


Fig. 2.15 Schematic Illustration of the Regulation Map.
In this scheme, we describe four situations at elements a_0, b_0, c_0 and $d_0 \in K$. At a_0 and b_0, the right hand side of the differential inclusions contains tangent velocities to K, so that we can expect an evolution to be viable. At c_0, this hope is even more justified because the velocity points in the interior of K. Finally, at d_0, all velocities point outward K, and it is intuitive that all evolutions leave K instantaneously. The viability theorem states that these intuition and hopes are correct for any closed subset K and for Marchaud maps.

The Viability and Invariance Theorems imply that

Theorem 2.15.7 *[**Tangential Characterization of Viability Kernels**] Let us assume that F is Marchaud (see Definition 10.3.2, p. 384) and that the subset K is closed. The viability kernel $\mathrm{Viab}_{\mathcal{S}}(K)$ of a subset K under $\mathcal{S}$ is the **largest** closed subset $D \subset K$ satisfying*

$$\forall x \in D, \quad R_D(x) := F(x) \cap T_D^{\star\star}(x) \neq \emptyset \tag{2.36}$$

Furthermore, for every $x \in D$, there exists at least one evolution $x(\cdot) \in \mathcal{S}(x)$ viable in D and all evolutions $x(\cdot) \in \mathcal{S}(x)$ viable in D are governed by the differential inclusion

$$x'(t) \in R_D(x(t))$$

For capture basins, we obtain

Theorem 2.15.8 *[**Tangential Characterization of Capture Basins**] Let us assume that F is Marchaud, that the environment $K \subset X$ and the target $C \subset K$ are closed subsets such that $K \setminus C$ is a repeller ($\mathrm{Viab}_F(K \backslash C) = \emptyset$). Then the viable-capture basin $\mathrm{Capt}_{\mathcal{S}}(K, C)$ is the **largest** closed subset D satisfying $C \subset D \subset K$ and*

$$\forall x \in D \backslash C, \quad F(x) \cap T_D^{\star\star}(x) \neq \emptyset$$

Furthermore, for every $x \in D$, there exists at least one evolution $x(\cdot) \in \mathcal{S}(x)$ viable in D until it reaches the target C and all evolutions $x(\cdot) \in \mathcal{S}(x)$ viable in D until they reach the target C are governed by the differential inclusion

$$x'(t) \in R_D(x(t))$$

Further important properties hold true when the set-valued map F is Lipschitz (see Definition 10.3.5, p. 385).

Theorem 2.15.9 *[**Characterization of Viability Kernels**] Let us assume that (f, U) is both Marchaud and Lipschitz and that the subset K is closed. The viability kernel $\mathrm{Viab}_F(K)$ of a subset K under $\mathcal{S}$ is the **unique** closed subset $D \subset K$ satisfying*

- *$K \setminus D$ is a repeller;*
- *and the* Frankowska property*:*

$$\begin{cases} (i) \ \forall\, x \in D, \ \ F(x) \cap T_D^{\star\star}(x) \neq \emptyset \\ (ii) \ \forall\, x \in D \cap \mathrm{Int}(K), \ \ -F(x) \subset T_D^{\star\star}(x) \\ (ii) \ \forall\, x \in D \cap \partial K, \ \ -F(x) \cap T_K^{\star\star}(x) = -F(x) \cap T_D^{\star\star}(x) \end{cases} \tag{2.37}$$

For capture basins, we obtain

Theorem 2.15.10 *[**Characterization of Capture Basins**] Let us assume that (f, U) is Marchaud and Lipschitz and that the environment $K \subset X$ and the target $C \subset K$ are closed subsets such that $K \setminus C$ is a repeller ($\mathrm{Viab}_F(K \backslash C) = \emptyset$). Then the viable-capture basin $\mathrm{Capt}_F(K, C)$ is the **unique** closed subset D satisfying*

- *$C \subset D \subset K$,*
- *and the* Frankowska property *(2.37), p. 104.*