Chapter 13
Viability Solutions to Hamilton–Jacobi Equations

13.1 Introduction

This chapter presents the viability approach to a class of Hamilton–Jacobi equations. We assume not only that the solution depends on time, but on "structured" or "causal" variables. They include age-structured Hamilton–Jacobi–McKendrick equations, useful in population dynamics as well as in transport management (the age variable being replaced by the travel time), as well as Hamilton–Jacobi–Cournot equation, where the "structured" or "causal" variable is the initial state of the underlying control system. Chapters 14, p. 563 and 15, p. 603 apply the results of this chapter to transportation management, finance and economics.

They also are a particular case of Hamilton–Jacobi–Bellman equations of optimal control problems and more generally, intertemporal optimization problems, studied in Chap. 14, p. 563. We already presented in Chap. 4, p. 125 examples of such control problems: minimal time and exit functions, minimal length functions, Lyapunov functions, safety and transgression functions, etc., are value functions. We illustrated the "viability approach" showing that the epigraphs of these functions are the viability kernels or capture basins of epigraphical environments and targets under an adequately defined "characteristic system" (see the epigraphical miracle mentioned in Sect. 4.12.2, p. 172). We only alluded to the fact that they were solutions to Hamilton–Jacobi–Bellman equations.

In this chapter, we start instead with a class of Hamilton–Jacobi equations *the Hamiltonian of which is convex with respect to the gradient of the solution.* We shall prove that their "solution" is the value function of an intertemporal optimization of evolutions governed by a *hidden* underlying control system, where:

1. the hidden controls, called "celerities", range over the state space,

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2. the optimality criterion involves the Lagrangian associated with the Hamiltonian by the Fenchel Transform,
3. the evolutions are governed by a specific class of "epigraphical control system" involving the epigraph of the Lagrangian,
4. the map regulating optimal evolutions is associated with the gradient of the solution.

Which solution? We define the concept of "viability solution" *solving at once all the above related problems*. They are "constructive solutions", in the sense that their epigraph is *defined* as the viability kernel or capture basin of an epigraphical environment and target. They inherit their properties which are enough to "translate" in the language of partial differential equations and control theory.

13.2 From Duality to Trinity

Let $t \geq 0$ denotes the time, $x \in X := \mathbb{R}^n$ the *state variable* and $d \in \mathcal{D} \subset \mathbb{R}^m$ the *causal variable* or *structuring variable* of the system.

We introduce:

1. a *causal map* $\varphi : d \in \mathbb{R}^m \mapsto \varphi(d) := (\varphi_i(d))_{i=1,\ldots,d} \in \mathbb{R}^d$ depending only on the causal variable d (assumed single-valued for simplicity of the presentation),
2. a *Hamiltonian* function

$$(d, x, p) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbf{l}^\star(d, x; p) \in \mathbb{R} \cup \{+\infty\}$$

convex with respect to the *"costate" variable* $p \in \mathbb{R}^n$,
3. a *viability constraint* function $(d, x) \mapsto \mathbf{k}(d, x) \in \mathbb{R} \cup \{+\infty\}$ such that $\mathbf{k}(d, x) < +\infty$ implies that $d \in \mathcal{D}$,
4. an *internal condition* function $(d, x) \mapsto \mathbf{c}(d, x) \in \mathbb{R} \cup \{+\infty\}$ such that

$$\mathbf{k}(d, x) \;\leq\; \mathbf{c}(d, x)$$

The terminology is motivated by the *asymmetric* role played by the two variables d and x, since φ *only* depends on the causal variable d whereas the Hamiltonian $\mathbf{l}^\star$ may depend on *both* causal and state variables. Hijacking the terminology used in population dynamics, we say that the causal variables *structure* the system.

1. **The Macroscopic Approach**

The *macroscopic* description of the system requires us to look for the *viable solution* V to the *structured Hamilton–Jacobi equation*

$$\sum_{i=1}^{m} \left\langle \frac{\partial V(d,x)}{\partial d_i}, \varphi_i(d) \right\rangle + \mathbf{l}^\star \left(d, x; \frac{\partial V(d,x)}{\partial x} \right) \ = \ 0 \tag{13.1}$$

satisfying inequalities

$$\mathbf{k}(d,x) \ \leq \ V(d,x) \ \leq \ \mathbf{c}(d,x) \tag{13.2}$$

Examples of Structured Hamilton–Jacobi Equations

(a) *Hamilton–Jacobi equations.* By taking $d := t \in \mathcal{D} := \mathbb{R}_+$ describing time and $\varphi(t) = 1$, we obtain the usual Hamilton–Jacobi partial differential equation

$$\frac{\partial V(t,x)}{\partial t} + \mathbf{l}^\star \left(t, x; \frac{\partial V(t,x)}{\partial x} \right) \ = \ 0$$

(b) *Hamilton–Jacobi–McKendrick equations.* By taking $d := (t,a) \in \mathcal{D} := \mathbb{R}_+^2$ describing time and age (in population dynamics) or travel time (in traffic problems for instance) and taking $\varphi(t,a) = (1,1)$, we obtain Hamilton–Jacobi–McKendrick partial differential equations

$$\frac{\partial V(t,a,x)}{\partial t} + \frac{\partial V(t,a,x)}{\partial a} + \mathbf{l}^\star \left(t, a, x; \frac{\partial V(t,a,x)}{\partial x} \right) \ = \ 0$$

(c) Taking $d := (t,b)$ and $\varphi(t,b) := (1, \psi(t,b))$, we obtain the following partial differential equation

$$\frac{\partial V(t,b,x)}{\partial t} + \left\langle \frac{\partial V(t,b,x)}{\partial b}, \psi(t,b) \right\rangle + \mathbf{l}^\star \left(t, b, x; \frac{\partial V(t,b,x)}{\partial x} \right) \ = \ 0$$

(d) *Hamilton–Jacobi–Cournot equations.* We introduce an other causal variable χ and set $\psi(d,\chi) := (\varphi(d), 0)$, independent of the first causal variable χ. we obtain structured Hamilton–Jacobi partial differential equations

$$\sum_{i=1}^{m} \left\langle \frac{\partial V(d,x)}{\partial d_i}, \varphi_i(d) \right\rangle + \mathbf{l}^\star \left(d, x; \frac{\partial V(d,\chi,x)}{\partial x} \right) \ = \ 0$$

structured *also* by *constant parameters* χ. They are solutions to the same partial differential equation (13.1), p. 525, but are subjected to conditions

$$\mathbf{k}(d,\chi,x) \ \leq \ V(d,\chi,x) \ \leq \ \mathbf{c}(d,\chi,x)$$

depending on χ.

An important example is provided by Hamilton–Jacobi–Cournot partial differential equations parameterized by specific parameters $\chi \in \mathbb{R}^n$ regarded as *initial conditions* and requiring that $\mathbf{c}(d,\chi,x) = +\infty$ whenever $x \neq \chi$ (see Definition 8.4.8, p. 288) and Sect. 13.8, p. 551. □

Examples of Internal and Viability Functions

(a) An internal condition function $\mathbf{c} : (d,x) \mapsto \mathbf{c}(d,x) \in \mathbb{R} \cup \{+\infty\}$ becomes a *boundary condition* function if $\mathbf{c}(d,x) = +\infty$ whenever $d \in \mathrm{Int}(\mathcal{D})$.

(b) Consider the case when the environment in which ranges the state variable x is described by *viability environments* $K(d) \subset X$ of the state variable depending on the structural variable d. This constraint is taken into consideration by *requiring that the constraint function* $\mathbf{k}$ *satisfies* $\mathbf{k}(d,x) = +\infty$ *for all* $x \notin K(d)$ (since this implies automatically that $V(t,x) = +\infty$ whenever $d \notin K(d)$).

Here, we define the concept of "viability solution" *which always exists and can be computed by viability algorithms.* This is a *constructive* approach allowing us to derive some known and new properties of the viability solution from the tools of viability theory (dealing with sets instead of functions). Regarding functions as their epigraphs, we bypass the regularity issues to arrive directly to the concept of Barron–Jensen/Frankowska viscosity solutions. We above all take into account viability constraints and extend classical boundary conditions to other "internal" conditions.

2. **The Variational Approach**

The link between Hamilton–Jacobi equations and the associated variational problem relies on the *Legendre–Fenchel transform* $(d,x,u) \mapsto \mathbf{l}(d,x;u)$ of the Hamiltonian defined by

$$\mathbf{l}(d,x;u) \ := \ \sup_p [\langle p,u\rangle - \mathbf{l}^\star(d,x;p)]$$

the *Lagrangian.* We denote by

$$F(d,x) \ := \ \{u \text{ such that } \ell(d,x;u) \ +\infty\} \tag{13.3}$$

the domain of the Lagrangian ℓ. It defines a set-valued map $F : (d,x) \rightsquigarrow F(d,x)$ which will be the right hand side of the differential inclusion governing the evolutions of the microsystem.

Optimal evolutions achieve the minimum in the variational principle

$$V(d,x) \ := \ \inf_{x(\cdot)} \left(\mathbf{c}(d(0),x(0)) + \int_0^{t^\sharp} \mathbf{l}(d(t),x(t),x'(t))dt \right) \tag{13.4}$$

among all *viable* evolutions $x(\cdot)$ starting at initial time 0 and arriving at x at terminal time $t^\sharp \leq t$ when $d(t^\sharp) = d$. The function V is called the "valuation function" (and not the classical value function which depends upon current time t whereas the valuation function depends upon the terminal time).

They satisfy the dynamic programming equation: $\forall\, t \in [0, t^\sharp]$,

$$V(d,x) \;=\; V(d(t),x(t)) + \int_t^{t^\sharp} \mathbf{l}(d(\tau),x(\tau),x'(\tau))d\tau \tag{13.5}$$

3. **The Microscopic Approach**

The main purpose of this study is not only to prove that the *viability solution* is the *unique* solution to this partial differential equation in an adequate generalized (weak) sense, but also to *uncover a hidden dual microscopic* equivalent problem allowing us to characterize and *compute* from the solution V the *retroaction map* $(d,x) \rightsquigarrow R(d,x)$ governing *optimal evolutions* of the state through the *microscopic* regulation of structured-viable evolutions $(d(\cdot),x(\cdot))$ satisfying, for any given $d \in \mathcal{D}$ and x and for some $t^\sharp \geq 0$,

$$\left\{\begin{array}{l} (i)\;\; d(t^\sharp) \;=\; d \text{ and } x(t^\sharp) \;=\; x \;\; \text{(terminal condition)} \\ (ii)\; \forall\, t \in [0,t^\sharp],\;\; x'(t) \;\in\; R(d(t),x(t)) \;\; \text{(retroaction law)} \end{array}\right. \tag{13.6}$$

In other words, the macroscopic problem (looking for a macroscopic function V) and microscopic problem (looking for the regulation of evolutions $x(\cdot)$) are "dual".

4. **The Viability Solution**

The links between the microscopic, macroscopic and variational approaches are due to a "matchmaker", the "viability solution", which

(a) coincides with the viable solution to the macroscopic structured Hamilton–Jacobi partial differential equation (13.1), p. 525, satisfying the boundary condition (13.2), p. 525,
(b) provides the *retroaction law* microsystem (13.6), p. 527 governing optimal viable evolutions of the state variable to a any given terminal state in optimal time,
(c) Is equal to the valuation function (13.4), p. 526, of the associated intertemporal optimization problem.

13.3 Statement of the Problem

13.3.1 Lagrangian and Hamiltonian

Recall that the *epigraph* of an *extended function* $\mathbf{v} : X \mapsto \mathbb{R} \cup \{+\infty\}$ is the set of pairs $(x,y) \in X \times \mathbb{R}$ such that $\mathbf{v}(x) \leq y$. An extended function is called *lower semicontinuous* if *its epigraph is closed.*

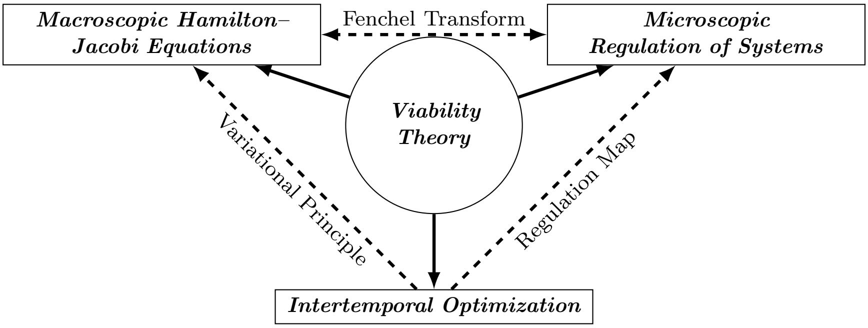

Fig. 13.1 From duality to trinity. This diagram describes the three problems under investigation: the macroscopic approach through first-order partial differential equations, the microscopic version dealing with the regulation of an underlying control system and the intertemporal optimization problem. The links relating optimization problems to Hamilton–Jacobi–Bellman equations and the regulation of control systems has been extensively studied. The tools of viability theory allow us to show that the viability solution solves these three problems at once.

The Lagrangian $\mathbf{l} : (d,x;u) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbf{l}(d,x;u) \in \mathbb{R} \cup \{+\infty\}$ is assumed once and for all to be *a nontrivial lower semicontinuous function convex with respect to* u: $\mathbf{l}(d,x;\cdot) : u \in x \mapsto \mathbf{l}(d,x;u) \in \mathbb{R} \cup \{+\infty\}$ is convex. Here, x is regarded as the state and u as a *celerity*. For instance, u is a velocity (in mechanics), a transaction (in economics and finance) or a *celerity* in general systems.

We also introduce the *costate* or *dual variable* $p \in \mathbb{R}^{n^\star}$ the dual of $\mathbb{R}^n$ and the duality product $\langle p, u \rangle := p(u)$. For instance, p is regarded as a force, a price or a density, and the duality product $\langle p, u \rangle$ is a power, a value or a flux respectively.

The *conjugate function* $\mathbf{l}^\star(d,x;\cdot)$ is defined on costate variables by

$$\forall\, p \in \mathbb{R}^n,\ \ \mathbf{l}^\star(d,x;p) \ := \ \sup_u [\langle p, u \rangle - \mathbf{l}(d,x;u)]$$

The *conjugate function* $\mathbf{l}^\star(d,x;\cdot)$ *is always a non trivial lower semicontinuous convex function* satisfying the *Fenchel inequality*

$$\langle p, u \rangle \ \leq \ \mathbf{l}(d,x;u) + \mathbf{l}^\star(d,x;p)$$

The biconjugate satisfies $\mathbf{l}^{\star\star}(d,x;u) \leq \mathbf{l}(d,x;u)$. The main result of convex analysis states that $\mathbf{l}^{\star\star}(d,x;\cdot) = \mathbf{l}(d,x;\cdot)$ *if and only if* $\mathbf{l}(d,x;\cdot)$ *is convex, lower semicontinuous and nontrivial.*

The *Legendre property* of the *Fenchel transform* $\mathbf{l}(d,x;\cdot) \mapsto \mathbf{l}^\star(d,x;\cdot)$ implies that the *subdifferentials*

$$\partial_u \mathbf{l}(d,x;u) \ := \ \partial_u \mathbf{l}(d,x;(\cdot)(u))$$

and

$$\partial_p \mathbf{l}^\star(d,x;p) \ := \ \partial_p \mathbf{l}^\star(d,x;(\cdot)(p))$$

of the lower semicontinuous convex functions $\mathbf{l}$ and $\mathbf{l}^\star$ are *defined* by the following equivalent conditions:

$$\begin{cases} (i) \quad \langle p,u\rangle \ = \ \mathbf{l}(d,x;u)+\mathbf{l}^\star(d,x;p) \\ (ii) \ \ p \ \in \ \partial_u \mathbf{l}(d,x;u) \\ (iii) \ u \ \in \ \partial_p \mathbf{l}^\star(d,x;p) \end{cases} \tag{13.7}$$

The two equalities (13.7)(ii) and (iii) describe the Legendre property: the inverse of the subdifferential map $u \rightsquigarrow \partial_u \mathbf{l}(d,x;u)$ is the subdifferential map $p \rightsquigarrow \partial_p \mathbf{l}^\star(d,x;p)$.

If the functions $\mathbf{l}$ or $\mathbf{l}^\star$ are differentiable in the classical sense, then

$$\begin{cases} \partial_u \mathbf{l}(d,x;u) \ = \ \left\{ \dfrac{\partial}{\partial u}\mathbf{l}(d,x;u) \right\} \\ \partial_p \mathbf{l}^\star(d,x;p) \ = \ \left\{ \dfrac{\partial}{\partial p}\mathbf{l}^\star(d,x;p) \right\} \end{cases}$$

At this point, we have to make assumptions under which viability properties hold true. In our specific settings, we need to make assumptions either on the Lagrangian $\mathbf{l}$ or on the Hamiltonian $\mathbf{l}^\star$ to fit the Marchaud requirement.

Definition 13.3.1 *[**Marchaud Functions**] We shall say that*

1. *a Lagrangian* $(d,x,u) \mapsto \mathbf{l}(d,x;u) \ \in \ \mathbb{R}\cup\{+\infty\}$ *is* Marchaud *if it is a lower semicontinuous function convex with respect to* u *and if there exists a finite positive constant* $c>0$ *such that*

$$\begin{cases} \text{Dom}(\mathbf{l}(d,x;\cdot)) \ \subset \ c(\|x\|+\|d\|+1)B \text{ and is closed} \\ \forall \ u \ \in \ \text{Dom}(\mathbf{l}(d,x;\cdot)), \quad 0 \le \ \mathbf{l}(d,x;u) \ \le \ c(\|x\|+\|d\|+1) \end{cases} \tag{13.8}$$

2. *a Hamiltonian* $(d,x;p) \mapsto \mathbf{l}^\star(d,x;p) \ \in \ \mathbb{R}\cup\{+\infty\}$ *is* Marchaud *if it is convex and lower semicontinuous with respect to* u*, upper semicontinuous with respect to* (d,x) *and if there exist finite positive constants* c *and* c_0 *such that, for all* $p\in\mathbb{R}^n$*,*

$$\begin{cases} \sigma_{\text{Dom}(\mathbf{l})}(d,x;p) - c(\|x\|+\|d\|+1) \\ \le \ \mathbf{l}^\star(d,x;p) \ \le \ c(\|x\|+\|d\|+1)\|p\|_\star \end{cases}$$

where $\sigma_K(\cdot)$ *denotes the* support function *of* K *(see Definition 18.2.3, p. 715).*

Lemma 18.7.4, p. 757 states that the Lagrangian $\mathbf{l}$ is Marchaud if the Hamiltonian $\mathbf{l}^\star$ is Marchaud. If the Lagrangian is Marchaud and continuous with respect to (d, x), then the Hamiltonian is Marchaud.

13.3.2 The Viability Solution

Knowing the Hamiltonian $\mathbf{l}^\star$, the viability constraint function $\mathbf{k}$ and the internal condition function $\mathbf{c}$, we define the structured Hamilton–Jacobi problem:

Definition 13.3.2 *[**Structured Hamilton–Jacobi Problem**] A function $(d, x) \mapsto V(d, x)$ is said to be a solution to the structured Hamilton–Jacobi equation if*

$$\left\langle \frac{\partial V(d,x)}{\partial d}, \varphi(d) \right\rangle + \mathbf{l}^\star \left(d, x; \frac{\partial V(d,x)}{\partial x} \right) \; = \; 0 \tag{13.9}$$

and a solution to the structured Hamilton–Jacobi problem if, furthermore, it satisfies the conditions

$$\mathbf{k}(d, x) \; \leq \; V(d, x) \; \leq \; \mathbf{c}(d, x) \tag{13.10}$$

We introduce the *structured characteristic system* defined by

$$\begin{cases} (i) \;\; \delta'(t) \; = \; -\varphi(\delta(t)) \\ (ii) \; (\xi'(t), \eta'(t)) \; \in \; -\mathcal{E}p(\mathbf{l}(\delta(t), \xi(t); \cdot)) \end{cases} \tag{13.11}$$

The characterization of the solution to the structured Hamilton–Jacobi problem states that its epigraph is the *viable-capture basin* of the epigraph of $\mathbf{c}$, viable in the epigraph of $\mathbf{k}$, under the structured characteristic system defined by (13.11), p. 530.

Definition 13.3.3 *[**Viability Solution**] The viability solution V to structured Hamilton–Jacobi problem 13.3.2, p. 530 is defined by*

$$V(d, x) \; := \; \inf_{(d,x,y) \in \mathrm{Capt}_{(13.10)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))} y \tag{13.12}$$

It may seem strange at first glance to solve a well known partial differential equation by a solution of an auxiliary and seemingly artificial viability problem.

Defining (lower semicontinuous) functions through their (closed) epigraphs allows us to treat the functions $\mathbf{l}$, $\mathbf{k}$, $\mathbf{c}$ and the viability solution V as subsets, bypassing and avoiding the pointwise characterization of partial differential equations familiar in classical analysis.

But this also allows us to just apply results surveyed and summarized in the "viability survival kit" (Sect. 2.15, p. 98) and proved in Chaps. 10, p. 375 and 11, p. 437 based on the fundamental viability and invariance theorems at the simpler level of set-valued analysis (with much less notations).

Above all, viable-capture basins and their regulation maps can be computed numerically by software using viability algorithms.

The translation of the properties of viable-capture basins in terms of structured problems provides without technical difficulties the properties we shall uncover.

The definition of the viability solution does not involve the concept of derivatives, a strange way for defining solutions to partial differential equations. Actually, it is known that the solution to the structured Hamilton–Jacobi is not differentiable. The lack of regularity happens whenever viability constraints are involved: in this case, the most we can expect is that *the solution is only lower semicontinuous*. However, it is possible to give a meaning to lower semicontinuous solutions to structured Hamilton–Jacobi equation (13.9), p. 530, by weakening the concepts of gradients in the sense of nonsmooth analysis. The viability solutions then becomes a "solution" to this partial differential equation in the sense of Barron–Jensen/Frankowska viscosity solution (Theorem 13.10.3, p. 560).

13.4 Variational Principle

13.4.1 Lagrangian Microsystems

We shall assume that the causal map φ is Lipschitz (or more generally, monotone) for guaranteeing the *uniqueness* of the solution $\delta(\cdot)$ to differential equation

$$\delta'(t) \ = \ -\varphi(\delta(t))$$

starting from any given initial state $d \in \mathcal{D}$.

We further assume that the subset $\mathcal{D}$ is closed and is a *repeller* under $-\varphi$ in the sense that for any $d \in \mathcal{D}$, the evolution $\delta(\cdot)$ leaves $\mathcal{D}$ in finite time

$$\tau^{\sharp}(d) \ := \ \inf_{\delta(t) \in \complement \mathcal{D}} t$$

at $\delta(\tau^{\sharp}(d)) \in \partial\mathcal{D}$.

This is the case for usual Hamilton–Jacobi equations when $d := t \in \mathcal{D} := \mathbb{R}_+$ and $\varphi(t) = 1$: in this case $\tau^\sharp_{\mathcal{D}}(t) = t$, as well as for the Hamilton–Jacobi–McKendrick equations when $d := (t, a) \in \mathcal{D} := \mathbb{R}^2_+$ and $\varphi(t, a) = (1, 1)$: in this case $\tau^\sharp_{\mathcal{D}}(t, a) = a$ if $t \geq a \geq 0$ and $\tau^\sharp_{\mathcal{D}}(t, a) = t$ if $a \geq t \geq 0$.

In summary, whenever we mention d, we attach to it either the unique evolution $t \mapsto \delta(t)$ governed by $\delta'(t) = -\varphi(\delta(t))$ starting from d at initial time 0 or the unique evolution $t \mapsto d(t) := \delta(t^\sharp - t)$ governed by $d'(t) = \varphi(d(t))$ and arriving at d at time $t^\sharp$, *without mentioning it explicitly.*

We associate with the Lagrangian, the causal map and the viability constraint function $\mathbf{k}$ the microsystem governing viable evolutions of the state:

Definition 13.4.1 ***[Microsystem]*** *We denote by*

$$F(d, x) \ := \ \{u \text{ such that } \mathbf{l}(d, x; u) \ < \ +\infty\} \tag{13.13}$$

the domain of the Lagrangian $\mathbf{l}$ *and by* $\mathcal{A}_{\mathbf{k}}(t^\sharp; d, x)$ *the set of evolutions* $x(\cdot)$ *governed by the system*

$$x'(t) \ \in \ F(d(t), x(t)) \tag{13.14}$$

"viable" in the sense that

$$\sup_{t \in [0, t^\sharp]} \mathbf{k}(d(t), x(t)) \ < \ +\infty$$

and arriving at x *at time* $t^\sharp$ *when* $d(t^\sharp) = d$.
When the function $\mathbf{k}$ *is associated with a viability environment* $d \rightsquigarrow K(d)$ *by its indicator (see Definition 18.6.1, p. 743)*

$$\mathbf{k}(d, x) \ := \ \psi_{K(d)}(x) = \psi_{\text{Graph}(K)}(d, x)$$

then the viable evolutions are the ones satisfying

$$\forall\, t \in [0, t^\sharp], \ \ x(t) \ \in \ K(d(t))$$

13.4.2 The Variational Principle

13.4.2.1 Case of Boundary-Value Problems

We first assume (for simplicity of the formula) that the internal condition is actually a boundary condition: $\mathbf{c}(d, x) = +\infty$ whenever $d \in \text{Int}(\mathcal{D})$.

The *valuation function* of the intertemporal optimization problem is defined by

$$U(d,x) \ := \inf_{x(\cdot)\in\mathcal{A}_{\mathbf{k}}(\tau^{\sharp}(d);d,x)} \left(\mathbf{c}(d(0),x(0)) + \int_0^{\tau^{\sharp}(d)} \mathbf{l}(d(t),x(t),x'(t))dt\right) \tag{13.15}$$

Theorem 13.4.2 *[**The Viability Solution Solves the Variational Problem**] The viability solution V defined by (13.12), p. 530:*

$$U(d,x) \ = \ V(d,x) \ := \inf_{(d,x,y)\in \mathrm{Capt}_{(13.10)}(\mathcal{E}p(\mathbf{k}),\mathcal{E}p(\mathbf{c}))} y$$

is equal the valuation function U of the variational problem (13.15), p. 533.

13.4.2.2 General Case

In the general case, the statement of the intertemporal optimization problem is more intricate and requires further notations:

Theorem 13.4.3 *[**The Viability Solution Solves the Variational Problem**] We associate with the function $\mathbf{c}$ the functional $\mathbf{J_c}$ defined by*

$$\begin{cases} \mathbf{J_c}(t^{\sharp};x(\cdot))(d,x) \\ := \ \mathbf{c}(d(0),x(0)) + \displaystyle\int_0^{t^{\sharp}} \mathbf{l}(d(\tau),x(\tau),x'(\tau))d\tau \end{cases}$$

with the function $\mathbf{k}$ the functional $I_{\mathbf{k}}$ defined by

$$\begin{cases} \mathbf{I_k}(t^{\sharp};x(\cdot))(d,x) := \\ \displaystyle\sup_{s\in[0,t^{\sharp}]} \left(\mathbf{k}(d(s),x(s)) + \int_s^{t^{\sharp}} \mathbf{l}(d(\tau),x(\tau),x'(\tau))d\tau\right) \end{cases}$$

and with both functions $\mathbf{k}$ and $\mathbf{c}$ the functionals defined by

$$\begin{cases} \mathbf{L_{(k,c)}}(t^{\sharp};x(\cdot))(d,x) \ := \ \max(\mathbf{I_k}(t^{\sharp};x(\cdot))(d,x), \mathbf{J_c}(t^{\sharp};x(\cdot))(d,x)) \\ \mathbf{M_{(k,c)}}(t^{\sharp};d,x) \ := \ \inf_{x(\cdot)\in\mathcal{A}_{\mathbf{k}}(t^{\sharp};d,x)} \mathbf{L_{(k,c)}}(t^{\sharp};x(\cdot))(d,x) \end{cases}$$

Hence the viability solution V is equal the valuation function U of the variational problem defined by

$$\begin{cases} U(d,x) \;=\; \inf_{t^\sharp\in[0,\tau^\sharp(d)]}\mathbf{M}_{(\mathbf{k},\mathbf{c})}(t^\sharp;d,x) \\ = \;\inf_{t^\sharp\in[0,\tau^\sharp(d)]}\inf_{x(\cdot)\in\mathcal{A}_{\mathbf{k}}(t^\sharp;d,x)}\mathbf{L}_{(\mathbf{k},\mathbf{c})}(t^\sharp;x(\cdot))(d,x) \end{cases} \tag{13.16}$$

Proof. By Definition 2.10.2, p. 86 of viable-capture basins, to say that (d,x,y) belongs to the viable-capture basin $\mathrm{Capt}_{(13.10)}(\mathcal{E}p(\mathbf{k}),\mathcal{E}p(\mathbf{c}))$ means that there exist some $t^\sharp \geq 0$ and a measurable function $\upsilon(\cdot) : [0,t^\sharp] \mapsto \mathrm{Dom}(\mathbf{l})$ such that

$$t \in [0,t^\sharp] \mapsto (\delta(t),\xi(t),\eta(t)) \;\in\; \mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^n$$

is a solution to (13.11)

$$\begin{cases} (i) \quad \delta'(t) \;=\; -\psi(\delta(t)) \\ (ii) \;\; \xi'(t) \;=\; -\upsilon(t) \\ (iii)\; \eta'(t) \;=\; -\mathbf{l}(\delta(t),\xi(t);\upsilon(t)) \end{cases}$$

starting at x viable in the epigraph of $\mathbf{k}$ until time $t^\sharp \leq \tau^\sharp(d)$ when it belongs to the epigraph of the function $\mathbf{c}$, where

$$\begin{cases} (i) \;\; \xi(t) \;:=\; x - \displaystyle\int_0^t \upsilon(\tau)d\tau \\ (ii)\; \eta(t) \;\leq\; \eta_0(t) \;:=\; y - \displaystyle\int_0^t \mathbf{l}(\delta(\tau),\xi(\tau),\upsilon(\tau))d\tau \end{cases}$$

This implies that $t^\sharp \leq \tau^\sharp(d)$, that

$$\begin{cases} (i) \;\; \mathbf{c}\left(\delta(t^\sharp),\xi(t^\sharp)\right) \;\leq\; \eta(t^\sharp) \;\leq\; y - \displaystyle\int_0^{t^\sharp} \mathbf{l}(\delta(\tau),\xi(\tau),\upsilon(\tau))d\tau \;=\; \eta_0(t^\sharp) \\ (ii)\; \forall\, s \;\in\; [0,t^\sharp], \\ \qquad \mathbf{k}(\delta(s),\xi(s)) \;\leq\; \eta(s) \;\leq\; y - \displaystyle\int_0^s \mathbf{l}(\delta(\tau),\xi(\tau),\upsilon(\tau))d\tau \;=\; \eta_0(s) \end{cases} \tag{13.17}$$

In the particular case of boundary conditions when $\mathbf{c}(d,x) = +\infty$ whenever $d \in \mathrm{Int}(\mathcal{D})$, we obtain $t^\sharp = \tau^\sharp(d)$.
We introduce the set-valued map $\mathcal{F}$ defined by $\mathcal{F}$ defined by

$$\mathcal{F}(d,x) \;:=\; \mathcal{E}p(\mathbf{l}(d,x;\cdot)) \cap [(c(\|x\|+\|d\|+1)B \times [-c(\|x\|+\|d\|+1),0])] \tag{13.18}$$

which has nonempty values.

Introducing the system

$$\begin{cases} (i) \;\; \delta'(t) \;=\; -\varphi(\delta(t)) \\ (ii)\; (\xi'(t),\eta'(t)) \;\in\; -\mathcal{F}(\delta(t),\xi(t)) \end{cases} \tag{13.19}$$

this implies that $\text{Capt}_{(13.10)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})) \subset \text{Capt}_{(13.17)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))$. Since $\text{Graph}(\mathbf{l}) \subset \mathcal{E}p(\mathbf{l})$, the converse is true, so that equality

$$\text{Capt}_{(13.10)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})) \ = \ \text{Capt}_{(13.17)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})) \tag{13.20}$$

ensues.

Inequalities (13.17), p. 534 imply that

$$\sup_{s \in [0, t^\sharp]} \left(\mathbf{k}\left(\delta(s), \xi(s)\right) + \int_0^s \mathbf{l}(\delta(\tau), \xi(\tau), \upsilon(\tau)) d\tau \right) \ \leq \ y$$

The target condition implies that

$$\mathbf{c}\left(\delta(t^\sharp), \xi(t^\sharp)\right) + \int_0^{t^\sharp} \mathbf{l}(\delta(\tau), \xi(\tau), \upsilon(\tau)) d\tau \ \leq \ y$$

Let us set

$$\overleftarrow{\mathbf{J}}_{\mathbf{c}}(t^\sharp; \upsilon(\cdot))(d, x) := \ \mathbf{c}(\delta(t^\sharp), \xi(t^\sharp)) + \int_0^{t^\sharp} \mathbf{l}(\delta(\tau), \xi(\tau), \upsilon(\tau)) d\tau$$

and

$$\begin{cases} \overleftarrow{\mathbf{I}}_{\mathbf{k}}(t^\sharp; \upsilon(\cdot))(d, x) := \\ \sup_{s \in [0, t^\sharp]} \left(\mathbf{k}(\delta(s), \xi(s)) + \int_0^s \mathbf{l}(\delta(\tau), \xi(\tau), \upsilon(\tau)) d\tau \right) \end{cases}$$

We posit

$$\begin{cases} \overleftarrow{\mathbf{L}}_{(\mathbf{k},\mathbf{c})}(t^\sharp; \upsilon(\cdot))(d, x) \\ := \ \max(\overleftarrow{\mathbf{I}}_{\mathbf{k}}(t^\sharp; x(\cdot), \upsilon(\cdot))(d, x), \overleftarrow{\mathbf{J}}_{\mathbf{c}}(t^\sharp; \upsilon(\cdot))(d, x)) \end{cases}$$

We have proved that the viability and capturability conditions imply that there exist $t^\sharp \in [0, \tau^\sharp(d)]$ and $\upsilon(\cdot)$ such that

$$\overleftarrow{\mathbf{L}}_{(\mathbf{k},\mathbf{c})}(t^\sharp; \upsilon(\cdot))(d, x) \ \leq \ y \tag{13.21}$$

Therefore, setting

$$U(d, x) \ := \ \inf_{(t^\sharp; \upsilon(\cdot))} \overleftarrow{\mathbf{L}}_{(\mathbf{k},\mathbf{c})}(t^\sharp; \upsilon(\cdot))(d, x)$$

the viability solution $V(d, x) := \inf_{(d,x,y) \in \text{Capt}_{(13.10)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))} y$ defined by (13.12), p. 530, satisfies inequality $U(d, x) \leq V(d, x)$.

For proving the opposite inequality, take any $\varepsilon > 0$. Then there exist $\upsilon_\varepsilon(\cdot)$ and $t_\varepsilon^\sharp \in [0, \tau^\sharp(d)]$ such that

$$\overleftarrow{\mathbf{L}}_{(\mathbf{k},\mathbf{c})}(t_\varepsilon^\sharp; (\xi_\varepsilon(\cdot), \upsilon_\varepsilon(\cdot)))(d, x) \ \leq \ U(d, x) + \varepsilon$$

Setting

$\xi_\varepsilon(t) := x - \int_0^t \upsilon_\varepsilon(\tau)d\tau$ and

$$\eta_\varepsilon(t) \ := \ U(d,x) - \int_0^t \mathbf{l}(\delta(\tau), \xi_\varepsilon(\tau), \upsilon_\varepsilon(\tau))d\tau - \varepsilon$$

we observe that $(\delta(\cdot), x_\varepsilon(\cdot), \eta_\varepsilon(\cdot))$ is a solution to (13.11), p. 530 starting at $(d, x, U(d,x) - \varepsilon)$, reaching the epigraph of $\mathbf{c}$ at time $t_\varepsilon^\sharp$ and viable in $\mathcal{E}p(\mathbf{k})$ on $[0, t_\varepsilon^\sharp]$. Therefore $(d, x, U(d,x) - \varepsilon)$ belongs to $\text{Capt}_{(13.10)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))$, and thus $U(d,x) - \varepsilon \geq V(d,x)$.

Letting ε converge to 0 implies that $U(d,x) \geq V(d,x)$ so that equality ensues.

Finally, setting $d(t) := \delta(t^\sharp - t)$, $x(t) := \xi(t^\sharp - t)$, $y(t) := \eta(t^\sharp - t)$, $u(t) := \upsilon(t^\sharp - t)$, $\mathbf{L}_{(\mathbf{k},\mathbf{c})}(t^\sharp; x(\cdot))(d,x) \ := \ \overleftarrow{\mathbf{L}}_{(\mathbf{k},\mathbf{c})}(t^\sharp; \upsilon(\cdot))(d,x)$, etc., we deduce that $x(\cdot) \in \mathcal{A}_\mathbf{k}(t^\sharp; d, x)$ is a solution arriving at x at time $t^\sharp$ and starting from $x(0) = \xi(t^\sharp)$ at time $t^\sharp$. Since $x'(t) = -\xi'(t^\sharp - t) = \upsilon(t^\sharp - t) = u(t)$, we infer that

$$\begin{cases} \mathbf{J}_\mathbf{c}(t^\sharp; x(\cdot))(d,x) \\ := \ \mathbf{c}(d(0), x(0)) + \displaystyle\int_0^{t^\sharp} \mathbf{l}(d(\tau), x(\tau), x'(\tau))d\tau \\ \\ \mathbf{I}_\mathbf{k}(t^\sharp; x(\cdot))(d,x) := \\ \displaystyle\sup_{s \in [0, t^\sharp]} \left(\mathbf{k}(d(s), x(s)) + \int_{t^\sharp - s}^{t^\sharp} \mathbf{l}(d(\tau), x(\tau), x'(\tau))d\tau \right) \end{cases}$$

Hence we have proved that

$$V(d,x) \ = \ \inf_{t^\sharp \in [0, \tau^\sharp(d)]} \ \inf_{x(\cdot) \in \mathcal{A}_\mathbf{k}(t^\sharp; d, x)} \mathbf{L}_{(\mathbf{k},\mathbf{c})}(t^\sharp; x(\cdot))(d,x)$$

is the valuation function of the intertemporal optimization problem. □

Remark. Theorem 13.4.2, p. 533, follows from Theorem 13.4.3, p. 533 because assumption $\mathbf{c}(d,x) = +\infty$ whenever $d \in \text{Int}(\mathcal{D})$ implies that $t^\sharp = \tau^\sharp(d)$. □

Theorem 13.4.4 ***[Continuity Properties of the Viability Solution]*** *Assume that the Lagrangian is Marchaud. Then the viability solution V defined by (13.12), p. 530, is lower semicontinuous and its epigraph is equal to the viable-capture basin:*

$$\mathcal{E}p(V) \ = \ \text{Capt}_{(13.10)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))$$

Proof. Recalling inequality (13.8), p. 529 involved in the definition of Marchaud Lagrangian, we introduce the set-valued map $\mathcal{F}$ defined by (13.18), p. 534, *which has nonempty values* and the system (13.19), p. 534:

$$\begin{cases} (i) \ \ \delta'(t) \ = \ -\varphi(\delta(t)) \\ (ii) \ (\xi'(t), \eta'(t)) \ \in \ -\mathcal{F}(\delta(t), \xi(t)) \end{cases}$$

1. Inclusions $\text{Graph}(\mathbf{l}) \subset \mathcal{F} \subset \mathcal{E}p(\mathbf{l})$ imply

   $$\begin{cases} \text{Capt}_{(13.10)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})) \ \subset \ \text{Capt}_{(13.17)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})) \\ \subset \ \text{Capt}_{(13.10)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})) \end{cases}$$

   and equality (13.20), p. 535 implies that $\text{Capt}_{(13.10)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})) = \text{Capt}_{(13.10)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))$.
   Hence, these three viable-capture basins coincide:

   $$\begin{cases} \text{Capt}_{(13.10)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})) \ = \ \text{Capt}_{(13.17)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})) \\ = \ \text{Capt}_{(13.10)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})) \end{cases} \tag{13.22}$$

2. The differential inclusion (13.19), p. 534 is Marchaud. Indeed, the graph of $(d,x) \rightsquigarrow \mathcal{E}p(\mathbf{l}(d,x;\cdot))$ is closed because it is the epigraph of the Lagrangian $\mathbf{l}$ is lower semicontinuous. Its values are convex since the Lagrangian is convex with respect to u. The set-valued map $\mathcal{F}$ being its intersection with $(d,x) \rightsquigarrow c(\|x\|+\|d\|+1)B \times [-c(\|x\|+\|d\|+1), 0]$ has linear growth. Consequently, the intersection $\mathcal{F}$ has a closed graph, convex values and linear growth, i.e., is a Marchaud set-valued map (see Definition 10.3.2, p. 384).
3. Hence Viability Theorem 2.15.5, p. 101 implies that the viable-capture basin $\text{Capt}_{(13.10)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))$ is closed. This implies in particular that $(d, x, V(d,x))$ belongs to this viable-capture basin, which, then, coincides with the epigraph of V. Being closed, the viability solution is lower semicontinuous. □

A voluminous literature is devoted to regularity theorems providing sufficient conditions for the viability solution to be continuous, Lipschitz, semi-concave, differentiable in such and such sense.

This study does not deal with the regularity properties of the viability solutions, but focuses on the existence of optimal evolutions and on microsystem regulating them.

13.5 Viability Implies Optimality

We now deduce from the viability theorems that there exists an optimal solution to the variational problem, actually, that all viable evolutions are optimal and satisfy the dynamic programming equations under viability constraints.

13.5.1 Optimal Evolutions

Theorem 13.5.1 *[**Viable and Optimal evolutions**] Assume that the Lagrangian is Marchaud. For any $(d, x) \in \mathrm{Dom}(V)$, there* exist $t^\sharp \in [0, \tau^\sharp(d)]$ *such that $d(t^\sharp) = d$ and one evolution $\overline{x}(\cdot) \in \mathcal{A}_{\mathbf{k}}(t^\sharp; d, x)$ such that*

$$V(d, x) \;=\; \mathbf{L}_{(\mathbf{k},\mathbf{c})}(t^\sharp; \overline{x}(\cdot))(d, x)$$

achieves the minimum in the intertemporal optimization problem.

Proof. Consider *any* solution $t \mapsto (\delta(t), \xi(t), \eta(t))$ to the system (13.11) starting at $(d, x, V(d, x))$ *viable* in $\mathcal{E}p(\mathbf{k})$ until it reaches $\mathcal{E}p(\mathbf{c})$ at some finite time $t^\sharp$. At least one of such evolutions does exist since $(d, x, V(d, x))$ belongs to viable-capture basin $\mathrm{Capt}_{(13.10)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))$ thanks to Theorem 13.4.4, p. 536.

It is associated with a control $\upsilon(\cdot)$ satisfying

$$\eta(t) \;\le\; \eta_0(t) \;:=\; V(d, x) - \int_0^t \mathbf{l}(\delta(\tau), \xi(\tau), \upsilon(\tau))d\tau$$

Inequality (13.21), p. 535, with $y = V(d, x)$, implies that

$$\overleftarrow{\mathbf{L}}_{(\mathbf{k},\mathbf{c})}(t^\sharp; \upsilon(\cdot))(d, x) \;\le\; V(d, x)$$

and thus that $(t^\sharp; \upsilon(\cdot))$ is optimal.

Therefore, setting $d(t) := \delta(t^\sharp - t)$, $x(t) := \xi(t^\sharp - t)$, etc., we infer that there exist $t^\sharp$ and $(d(\cdot), x(\cdot)) \in \mathcal{A}_{\mathbf{k}}(t^\sharp; d, x)$ such that

$$V(d, x) \;=\; \overleftarrow{\mathbf{L}}_{(\mathbf{k},\mathbf{c})}(t^\sharp; \upsilon(\cdot))(d, x) \;=\; \mathbf{L}_{(\mathbf{k},\mathbf{c})}(t^\sharp; x(\cdot))(d, x)$$

achieves the minimum in the intertemporal optimization problem. □

Summary. Denoting by

$$\mathbb{T}_{(\mathbf{k},\mathbf{c})}(d, x) \;:=\; \inf\left\{t \in [0, \tau^\sharp(d)] \text{ such that } V(d, x) \;=\; \mathbf{M}_{(\mathbf{k},\mathbf{c})}(t; d, x)\right\}$$

the *initial time map* and by

$$\mathcal{O}_{\mathbf{k}}(t^\sharp; d, x) \ := \ \left\{ x(\cdot) \text{ such that } \mathbf{L}_{(\mathbf{k},\mathbf{c})}(t^\sharp; x(\cdot))(d, x) \ := \ \mathbf{M}_{(\mathbf{k},\mathbf{c})}(t; d, x) \right\}$$

the *optimal map*, then the search of optimal evolutions splits into two steps:

1. Take $t^\sharp := \mathbb{T}_{(\mathbf{k},\mathbf{c})}(d, x)$.
2. Choose any evolution $\overline{x}(\cdot) \in \mathcal{O}_{\mathbf{k}}(t^\sharp; d, x)$

For boundary condition functions $\mathbf{c}$ such that $\mathbf{c}(d, x) = +\infty$ whenever $d \in \text{Int}(\mathcal{D})$, then $t^\sharp := \tau^\sharp(d)$. □

13.5.2 Dynamic Programming under Viability Constraints

Optimal evolutions satisfy the dynamic programming principle:

Theorem 13.5.2 *[**Dynamic Programming under Viability Constraints**] We assume that the Lagrangian* $\mathbf{l}$ *is Marchaud and that the function* $\mathbf{k}$ *is continuous in its domain. Consider an optimal evolution* $\overline{x}(\cdot) \in \mathcal{A}_{\mathbf{k}}(t^\sharp; d, x)$*. Let* $\overline{\kappa} \in [0, t^\sharp]$ *be the first time when*

$$\mathbf{k}(d(\overline{\kappa}), \overline{x}(\overline{\kappa})) + \int_{\overline{\kappa}}^{t^\sharp} \mathbf{l}(d(\tau), \overline{x}(\tau), \overline{x}'(\tau))d\tau \ = \ V(d, x) \qquad (13.23)$$

Set $\kappa^\sharp := \min(t^\sharp, \overline{\kappa})$*. Then* $\overline{x}(\cdot)$ *satisfies the* dynamic programming equation*:*

$$\forall\, t \in [\kappa^\sharp, t^\sharp], \ \ V(d(t), \overline{x}(t)) + \int_t^{t^\sharp} \mathbf{l}(d(\tau), \overline{x}(\tau), \overline{x}'(\tau))d\tau \ = \ V(d, x) \quad (13.24)$$

In particular, in the case without constraints $\mathbf{k}(d, x) = -\infty$*,* $\kappa^\sharp = 0$ *and the dynamic programming equation holds on the interval* $[0, t^\sharp]$*.*

Proof. By Theorem 2.15.2, p. 99, we know that the viable-capture basin $\text{Capt}_{(13.10)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))$ of the epigraph of $\mathbf{c}$ under the auxiliary system (13.11), p. 530 is the unique bilateral fixed point

$$\text{Capt}_{(13.10)}(\mathcal{E}p(V), \mathcal{E}p(\mathbf{c})) \ = \ \mathcal{E}p(V) \ = \ \text{Capt}_{(13.10)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(V))$$

1. Let $(d, x, V(d, x))$ belong to the viable-capture basin $\text{Capt}_{(13.10)}(\mathcal{E}p(V), \mathcal{E}p(\mathbf{c})) = \mathcal{E}p(V)$ of the epigraph of $\mathbf{c}$ under the auxiliary system (13.11), p. 530. There exist $t^\sharp \in [0, \tau^\sharp(d)]$ and $\overline{v}(\cdot)$ such that

$$t \mapsto (\delta(t), \overline{\xi}(t), \overline{v}(t))$$

is viable in the epigraph of V until it reaches the epigraph of $\mathbf{c}$ at time $t^\sharp$. Then the proof of Theorem 13.4.3, p. 533 implies that

$$\forall s \in [0, t^\sharp], \;\; V(\delta(s), \overline{\xi}(s)) + \int_0^s \mathbf{l}(\delta(\tau), \overline{\xi}(\tau), \overline{v}(\tau))d\tau \;\le\; V(d, x) \quad (13.25)$$

2. We shall deduce the opposite inequality from the second fixed point property $\mathcal{E}p(V) = \text{Capt}_{(13.10)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(V))$. The assumption that $\mathcal{D}$ is a repeller under $-\varphi$ implies that $\mathcal{E}p(\mathbf{k})$ is also a repeller under structured characteristic system (13.11), p. 530, and thus $\mathcal{E}p(V) = \text{Viab}_{(13.10)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(V))$. Hence

$$\complement(\text{Capt}_{(13.10)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(V))) \;=\; \text{Abs}_{(13.10)}(\complement\mathcal{E}p(V), \complement\mathcal{E}p(\mathbf{k}))$$

We set

$$\complement\mathcal{E}p(V) \;:=\; \{(d, x, y) \text{ such that } y \;<V(d, x)\} \;=: \overset{\circ}{\mathcal{H}yp}\,(V)$$

For any $\varepsilon > 0$, $(d, x, V(d, x) - \varepsilon)$ does not belong to $\mathcal{E}p(V) = \text{Capt}_{(13.10)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(V))$, so that

$$(d, x, V(d, x) - \varepsilon) \;\in\; \text{Abs}_{(13.10)}(\overset{\circ}{\mathcal{H}yp}\,(V), \overset{\circ}{\mathcal{H}yp}\,(\mathbf{k}))$$

Therefore, for any $v(\cdot)$, there exists $\kappa_\varepsilon \le \tau^\sharp(d)$ such that $(\delta(\kappa_\varepsilon), \xi(\kappa_\varepsilon), \eta(\kappa_\varepsilon))$ reaches $\mathcal{E}p(\mathbf{k})$ and leaves it before eventually reaching $\mathcal{E}p(\mathbf{c})$. Hence κ_ε is defined by

$$\mathbf{k}(\delta(\kappa_\varepsilon), \xi(\kappa_\varepsilon)) + \int_0^{\kappa_\varepsilon} \mathbf{l}(\delta(\tau), \xi(\tau), v(\tau))d\tau \;=\; V(d, x) - \varepsilon$$

and, consequently,

$$\forall\, s \in [0, \kappa_\varepsilon], \;\; V(d, x) - \varepsilon \;\le\; V(\delta(s), \xi(s)) + \int_0^s \mathbf{l}(\delta(\tau), \xi(\tau), v(\tau))d\tau$$

Since $\kappa_\varepsilon \;\le\; \tau^\sharp(d) < +\infty$, a subsequence (again denoted by) κ_ε converges to some $\kappa \;\le\; \tau^\sharp(d)$ when $\varepsilon \to 0+$. The function $\mathbf{k}$ being continuous by assumption, we deduce that

$$\mathbf{k}(\delta(\kappa), \xi(\kappa)) + \int_0^{\kappa} \mathbf{l}(\delta(\tau), \xi(\tau), v(\tau))d\tau \;=\; V(d, x)$$

and that,

$$\forall\, s \in [0,\kappa],\;\; V(d,x) \;\leq\; V(\delta(s),\xi(s)) + \int_0^s \mathbf{l}(d(\tau),\xi(\tau),\upsilon(\tau))d\tau \quad (13.26)$$

This inequality holds in particular for the above viable evolution $(\delta(\cdot),\overline{\xi}(\cdot),\overline{\eta}(\cdot))$ on the interval $[0,t^{\sharp}]$ so that inequalities (13.25), p. 540 and (13.26), p. 541 imply equality

$$\forall\, s \in [0,\min(\kappa,t^{\sharp})],\;\; V(d,x) \;=\; V(\delta(s),\overline{\xi}(s)) + \int_0^s \mathbf{l}(d(\tau),\overline{\xi}(\tau),\overline{\upsilon}(\tau))d\tau$$

ensues.

We derive the conclusion (13.24), p. 539 by setting $\overline{x}(t) := \overline{\xi}(t^{\sharp}-t)$ and $\overline{\kappa} := t^{\sharp}-\kappa$ satisfying (13.23), p. 539. □

13.6 Regulation of Optimal Evolutions

It is not enough to know the existence of optimal evolutions: the question arises whether we can compute it. For that purpose, we shall carve in the set-valued map $(d,x) \mapsto F(d,x)$ governing the evolution $x(\cdot)$ through differential inclusion $x'(t) \in F(d(t),x(t))$ a regulation map $(d,x) \mapsto R_V(d,x) \subset F(d,x)$ piloting optimal viable evolutions by differential inclusion $x'(t) \in R_V(d(t),x(t))$ until it reaches the terminal state x at optimal time $t^{\sharp}$.

The regulation map is characterized by the viability solution by a formula which uses the fact that the viability solution V is also the solution of the structured Hamilton–Jacobi inequality, which holds true for a Marchaud Lagrangian.

When V is any differentiable function, the regulation map associated with it is defined by:

$$R_V(d,x) \;:=\; \left\{ \begin{array}{l} \upsilon \in F(d,x) \text{ such that} \\ \left\langle \dfrac{\partial V(d,x)}{\partial d}, \varphi(d) \right\rangle + \left\langle \dfrac{\partial V(d,x)}{\partial x}, \upsilon \right\rangle - \mathbf{l}(d,x;\upsilon) \;\geq\; 0 \end{array} \right\}$$

Remark: Lax–Oleinik Formula. Observe that if the Lagrangian $\mathbf{l}$ is Marchaud and if the function V satisfies the Hamilton–Jacobi inequality

$$\left\langle \frac{\partial V(d,x)}{\partial d}, \varphi(d) \right\rangle + \mathbf{l}\left(d,x;\frac{V(d,x)}{dx}\right) \;\geq\; 0$$

then the Legendre property (13.7), p. 529 of the Fenchel transform implies the *generalized Lax–Oleinik formula*

$$\partial \mathbf{l}\left(d, x; \frac{V(d,x)}{dx}\right) \in R_V(d,x)$$

Indeed, if we assume further that $R_V(d,x) = \{r(d,x)\}$ is a singleton and that the Lagrangian is differentiable with respect to u, this implies that

$$\frac{\partial V(d,x)}{\partial x} = \frac{d}{du}\mathbf{l}(d,x; r(d,x))$$

which is the *Lax–Oleinik formula*. General formulas (13.68), p. 561 relating the regulation map associated with the viability solution and its partial derivatives (or subdifferentials) with respect to state x are obtained under (much) stronger assumptions. They are consequences of the proof that the viability solution is the unique Barron–Jensen/Frankowska viscosity solution of the Hamilton–Jacobi. We postpone it to Sect. 13.10, p. 557 since we do not need this purely mathematical result for studying the regulation of optimal viable evolutions. □

When V is no longer differentiable, we consider its *epiderivative* $D_\uparrow^{\star\star}V(d,x)$ of V defined

$$\mathcal{E}p(D_\uparrow^{\star\star}V(d,x)) := T^{\star\star}_{\mathcal{E}p(V)}(d,x,V(d,x))$$

which is a directional derivative $(\delta, \upsilon) \mapsto D_\uparrow^{\star\star}V(d,x)(\delta,\upsilon) \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ convex and lower semicontinuous instead of being linear (and continuous).

Definition 13.6.1 ***[Regulation Map Associated with a Function]*** *The regulation map R_V associated with a function V is defined by*

$$\begin{cases} R_V(d,x) := \\ \left\{\upsilon \in F(d,x) \text{ such that } D_\uparrow^{\star\star}V(d,x)(-\varphi(d),-\upsilon) + \mathbf{l}(d,x;u) \le 0\right\} \end{cases} \tag{13.27}$$

We begin by proving that the regulation map associated with the viability solution has nonempty values:

Theorem 13.6.2 ***[Regulation Map of the Viability Solution]*** *If the Lagrangian $(d,p,\upsilon) \rightsquigarrow \mathbf{l}(d,x;\upsilon)$ is Marchaud, then the viability solution V defined by (13.12), p. 530 is the smallest lower semicontinuous function satisfying conditions (13.10), p. 530 and*

$$\inf_{v \in F(d,x)} \left(D_{\uparrow}^{\star\star} V(d,x)(-\varphi(d), -v) + \mathbf{l}(d,x;v) \right) \le 0$$

is the contingent solution *(introduced by Hélène Frankowska) such that, whenever* $V(d,x) < \mathbf{c}(d,x)$*, the value* $R_V(d,x)$ *of the regulation map associated with the viability solution* V *is not empty.*

Proof. Since the Hamiltonian is Marchaud, so is the Lagrangian. Actually, the theorem remains true under the weaker assumption that the Lagrangian $\mathbf{l}$ is Marchaud. So is the set-valued map $\mathcal{F}$ defined by (13.18), p. 534 and, by (13.22), p. 537, the viable-capture basin $\mathcal{E}p(V) := \text{Capt}_{(13.10)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))$ is equal to $\text{Capt}_{(13.17)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})) = \text{Capt}_{(13.10)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))$. It is then the largest closed subset between $\mathcal{E}p(\mathbf{c})$ and $\mathcal{E}p(\mathbf{k})$, locally viable in $\mathcal{E}p(V) \setminus \mathcal{E}p(\mathbf{c})$ thanks to Viability Theorem 11.4.6, p. 463, which also states that, whenever $(d, x, V(d,x)) \in \mathcal{E}p(V) \setminus \mathcal{E}p(\mathbf{c})$, i.e., whenever $V(d,x) < \mathbf{c}(d,x)$, there exists some $v \in \text{Dom}(\mathbf{l}(d,x;\cdot))$ such that

$$(-\varphi(d), -v, -\mathbf{l}(d,x;v)) \in T_{\mathcal{E}p(V)}(d, x, V(d,x))$$

Definition 2.15.6, p. 102 of the regulation map R for general differential inclusions and Definition 13.6.1, p. 542 imply that such v belongs to $R_V(d,x)$. □

Optimal evolutions do exist thanks to Theorem 13.4.3, p. 533. The question asked is how to regulate them. The first answer is provided by

Theorem 13.6.3 ***[Regulation of Optimal Evolutions]*** *If the Lagrangian* $(d,p,v) \rightsquigarrow \mathbf{l}(d,x;v)$ *is Marchaud, viable optimal evolutions* $x(\cdot) \in \mathcal{O}_{\mathbf{k}}(t^\sharp; d, x)$ *when* $t^\sharp \in \mathbb{T}_{(\mathbf{k},\mathbf{c})}(d,x)$ *is the optimal time are regulated by differential inclusion*

$$\forall\, t \in [0, t^\sharp[, \quad x'(t) \in R_V(d(t), x(t))$$

and satisfy the terminal condition

$$d(t^\sharp) = d \text{ and } x(t^\sharp) = x$$

Proof. The proof of Theorem 13.5.1, p. 538 states that evolutions $t \mapsto (\delta(t), \xi(t), \eta(t))$ starting from $(d, x, V(d,x)) \in \mathcal{E}p(V)$ viable in $\mathcal{E}p(\mathbf{k})$ until they reach $\mathcal{E}p(\mathbf{c})$ at time $t^\sharp$ are optimal and regulated by

$$\forall\, t \in [0, t^\sharp], \quad v(t) \in R(\delta(t), \xi(t))$$

thanks to Viability Theorem 11.4.6, p. 463. By setting $d(t) := \delta(t^\sharp - t)$, $x(t) := \xi(t^\sharp - t)$ and $x'(t) := v(t^\sharp - t)$, this is equivalent to saying that optimal viable evolutions $x(\cdot) \in \mathcal{O}_{\mathbf{k}}(t^\sharp; d, x)$ are regulated by the differential inclusion

$$x'(t) \in R_V(d(t), x(t))$$

and satisfy the terminal condition $d(t^\sharp) = d$ and $x(t^\sharp) = x$. □

We also know that

$$\mathbf{c}(d(0), x(0)) + \int_0^{t^\sharp} \mathbf{l}(d(\tau), x(\tau), x'(\tau))d\tau \le V(d,x)$$

The initial causal variable $d(0) := \delta(t^\sharp)$ being known, the initial state $x(0)$ may be derived from the above formula which requires the knowledge of the evolution $x(\cdot)$ arriving at x and regulated by the regulation map R.

13.7 Aggregation of Hamiltonians

13.7.1 Aggregation

For simplicity, we consider problems without viability constraints. We introduce $j = 1, \ldots, J$ copies of the state space $Y := \mathbb{R}^n$ and the product space $X := \prod_{j=1}^{J} \mathbb{R}^n = \mathbb{R}^{Jn}$.

For each $j = 1, \ldots, n$, we introduce:

1. a lower semicontinuous convex Hamiltonian $\mathbf{l}_j^\star : \mathcal{D} \times Y \mapsto \mathbb{R} \cup \{+\infty\}$;
2. a function $\mathbf{c}_j : \mathcal{D} \times Y \mapsto \mathbb{R} \cup \{+\infty\}$

with which we associate the solutions to the *decentralized* Hamilton–Jacobi problems

$$\begin{cases} \left\langle \dfrac{\partial V_j(d, x_j)}{\partial d}, \varphi(d) \right\rangle + \mathbf{l}_j^\star \left(d, \dfrac{\partial V_j(d, x_j)}{\partial x_j} \right) = 0 \text{ on } \mathbb{R}^m \times \mathbb{R}^q \\ \text{satisfying } V_j(d, x_j) \le \mathbf{c}_j(d, x_j) \end{cases} \tag{13.28}$$

We are now looking for the solution $W : \mathcal{D} \times Y \mapsto \mathbb{R} \cup \{+\infty\}$ to the *centralized* Hamilton–Jacobi problem

$$\begin{cases} \left\langle \frac{\partial W(d,y)}{\partial d}, \varphi(d) \right\rangle + \sum_{j=1}^{J} \mathbf{l}_j^{\star}\left(d, \frac{\partial W(d,y)}{\partial y}\right) \ = \ 0 \text{ on } \mathcal{D} \times \mathbb{R}^n \\ \text{satisfying } W(d,y) \ \leq \ \mathbf{c}(d,y) \end{cases} \tag{13.29}$$

The link between the solution W of the centralized problem and solutions V_j to decentralized Hamilton–Jacobi problems is provided by the operation of inf-convolution (see Definition 18.8.1, p. 762): The *inf-convolution* $\bigstar_{j=1}^{J}\mathbf{v}_j : X \mapsto \mathbb{R} \cup \{+\infty\}$ of functions $\mathbf{v}_j$ is defined by

$$\mathcal{E}p(\bigstar_{j=1}^{J}\mathbf{v}_j) \ := \ \sum_{j=1}^{J} \mathcal{E}p(\mathbf{v}_j) \text{ (Minkowski sum of subsets)} \tag{13.30}$$

Lemma 18.8.3, p. 763 states that whenever

$$0 \ \in \ \text{Int}\left(\{(p,\ldots,p)_{p \in Y^{\star}}\} + \prod_{j=1}^{J} \text{Dom}(\mathbf{v}_j^{\star})\right)$$

holds true, then there exist J elements $\overline{x}_j \in X_j$ such that

$$\sum_{j=1}^{J} \overline{x}_j \ = \ x \text{ and } \mathbf{v}(x) \ = \ \sum_{j=1}^{J} \mathbf{v}_j(\overline{x}_j) \tag{13.31}$$

In the case of two functions, we obtain

$$(\mathbf{u} \star \mathbf{v})(y) \ = \ \inf_{z}(\mathbf{u}(z) + \mathbf{v}(y-z))$$

from which the name of the operation is derived (when $\inf_y$ is replaced by $\int_y$ for the usual convolution in analysis).

We consider the inf-convolutions of the functions $\mathbf{c}_j$ and V_j:

$$\mathbf{c}(d,y) \ := \ \bigstar_{j=1}^{J}\mathbf{c}_j(y) \text{ and } W(d,y) \ := \ \bigstar_{j=1}^{J} V_j(y) \tag{13.32}$$

The natural question is to know whether, whenever the function $\mathbf{c}$ is the inf-convolution of the functions $\mathbf{c}_i$, the solution $W(d,y)$ of the centralized Hamilton–Jacobi problem (13.29), p. 545, is the inf-convolution of the solutions $V_j(d,x_j)$ of the decentralized Hamilton–Jacobi problem (13.29), p. 545.

The answer is positive under adequate assumptions.

More important is the second conclusion relating the regulation maps of the centralized and decentralized regulation maps $R_W(d,y)$ and $R_{V_j}(d,x_j)$. We shall prove that a centralized control v belongs to the centralized regulation map $R_W(d,y)$ at y if and only if there exist decentralized states x_j and controls u_j belonging to the regulation maps $R_{V_j}(d,x_j)$ at x_j satisfying

$$\sum_{j=1}^{J} x_j \;=\; y \text{ and } \sum_{j=1}^{J} u_j \;=\; v \tag{13.33}$$

Therefore, for any (d, y), an evolution of the centralized associated problem $y(\cdot) \in \mathcal{A}(t^\sharp(d); d, y)$ regulated by $y'(t) \in R_W(d(t), y(t))$ can be written in the form

$$y(t) \;=\; \sum_{j=1}^{J} x_j(t)$$

where the decentralized (optimal) evolutions $x_j(\cdot) \in \mathcal{A}_j(t^\sharp(d); d, x_j)$ regulated by $x_j'(t) \in R_{V_j}(d(t), x_j(t))$.

These results are quite intricate and it is both convenient and useful to cover other problems to deduce them from an abstract problem by setting

$$A : x = (x_1, \ldots, x_J) \;\in\; \mathbb{R}^q := \mathbb{R}^{Jn} \mapsto Ax \;:=\; \sum_{j=1}^{J} x_j \;\in\; \mathbb{R}^n$$

and

$$\mathbf{m}^\star(d; q) \;=\; \sum_{j=1}^{J} \mathbf{l}_j^\star(d; q) \;\text{ and } \mathbf{m}(d; v) \;=\; \bigstar_{j=1}^{J} \mathbf{l}_j(d; v)$$

13.7.2 Composition of a Hamiltonian by a Linear Operator

Consider then the more general problem for arbitrary state space $\mathbb{R}^q$, with (much) higher dimension $q \geq n$ and a linear operator $A \in \mathcal{L}(\mathbb{R}^q, \mathbb{R}^n)$, the transpose of which is and its transpose $A^\star \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^q)$.

We shall assume throughout this section that the Hamiltonian $(d, p) \mapsto \mathbf{l}^\star(d; p)$ and the Lagrangian $(d, u) \mapsto \mathbf{l}(d; u)$ *do not depend upon the state variable* x.

Definition 13.7.1 ***[Composition of a Hamiltonian by a Linear Operator]*** *Let us consider a Hamiltonian* $(d, p) \mapsto \mathbf{l}^\star(d; p)$ *and linear operator* $A \in \mathcal{L}(\mathbb{R}^q, \mathbb{R}^n)$*. The composed Hamiltonian* $m^\star : (d, q) \in \mathbb{R}^m \times \mathbb{R}^n \mapsto m^\star(d, q) \in \mathbb{R} \cup \{+\infty\}$ *of* $\mathbf{l}^\star$ *by* $A^\star$ *is defined by*

$$m^\star(d, q) \;:=\; \mathbf{l}^\star(d, A^\star q)$$

and we denote by $m(d, y) := \mathbf{l}^{\star\star}$ *its biconjugate.*

We also introduce two internal condition functions:

1. $\mathbf{c} : (d,x) \in \mathbb{R}^m \times \mathbb{R}^q \mapsto \mathbf{c}(d,x) \in \mathbb{R} \cup \{+\infty\}$
2. $\mathbf{b} : (d,y) \in \mathbb{R}^m \times \mathbb{R}^n \mapsto \mathbf{b}(d,y) \in \mathbb{R} \cup \{+\infty\}$

related by

$$\mathbf{b}(d,y) := \inf_{Ax=y} \mathbf{c}(d,x)$$

Theorem 18.2.5, p. 715 states that the assumptions: there exists a constant $c < +\infty$ such that

$$\begin{cases} (i) \ \ \forall\,(d,u) \in \mathrm{Dom}(\mathbf{l}), \forall \nu \in \mathbb{R}^n, \exists\, \mu \in \mathrm{Dom}(D_{\uparrow}\mathbf{l}(d,u)) \cap c\|\nu\| \text{ and } A\nu = \mu \\ (ii) \ \forall\,(d,x) \in \mathrm{Dom}(\mathbf{c}), \forall \nu \in \mathbb{R}^n, \exists\, \mu \in \mathrm{Dom}(D_{\uparrow}\mathbf{c}(d,x)) \cap c\|\nu\| \text{ and } A\nu = \mu \end{cases} \tag{13.34}$$

imply that the infimum is achieved in formulas

$$\begin{cases} (i) \ \ m(d,v) \ = \ \min_{Au=v} \mathbf{l}(d,u) \\ \qquad\ \text{is the conjugate function of } m^{\star}(d,p) := \mathbf{l}^{\star}(d, A^{\star} q) \\ (ii) \ \mathbf{b}(d,y) \ = \ \min_{Ax=y} \mathbf{c}(d,x) \end{cases}$$

Theorem 13.7.2 ***[Link between Solutions of the Hamilton–Jacobi Equations]*** *We consider the two structured Hamilton–Jacobi problems*

$$\begin{cases} \left\langle \dfrac{\partial V(d,x)}{\partial d}, \varphi(d) \right\rangle + \mathbf{l}^{\star}\left(d, \dfrac{\partial V(d,x)}{\partial x}\right) \ = \ 0 \text{ on } \mathbb{R}^m \times \mathbb{R}^q \\ \text{satisfying } V(d,x) \ \leq \ \mathbf{c}(d,x) \end{cases} \tag{13.35}$$

and

$$\begin{cases} \left\langle \dfrac{\partial W(d,y)}{\partial d}, \varphi(d) \right\rangle + \mathbf{l}^{\star}\left(d, A^{\star}\dfrac{\partial W(d,y)}{\partial y}\right) \ = \ 0 \text{ on } \mathbb{R}^m \times \mathbb{R}^n \\ \text{satisfying } W(d,y) \ \leq \ \mathbf{b}(d,y) \end{cases} \tag{13.36}$$

Assume furthermore that the constraint qualification assumptions (13.34), p. 547 hold true. Then their viability solutions are related by formula

$$W(d,y) \ = \ \inf_{Ax=y} V(d,x) \tag{13.37}$$

Proof. Let us consider the operator $(\mathbf{1} \times A \times \mathbf{1}) : (d,x,\lambda) \in \mathcal{D} \times X \times \mathbb{R}_+ \mapsto (d, Ax, \lambda) \in \mathcal{D} \times Y \times \mathbb{R}_+$ and the two following characteristic systems

$$\begin{cases} (i) \ \ \delta'(t) \ = \ -\varphi(\delta(t)) \\ (ii) \ (\eta'(t), \zeta'(t)) \ \in \ -\mathcal{E}p(m(\delta(t);\cdot)) \ = \ -(\mathbb{I} \times A \times \mathbb{I})\mathcal{E}p(\mathbf{l}(\delta(t);\cdot)) \end{cases} \tag{13.38}$$

governing the evolution of $\eta(t) \in \mathbb{R}^n$ and

$$\begin{cases} (i) \ \ \delta'(t) \ = \ -\varphi(\delta(t)) \\ (ii) \ (\xi'(t), \zeta'(t)) \ \in \ -\mathcal{E}p(\mathbf{l}(\delta(t);\cdot)) \end{cases} \tag{13.39}$$

governing the evolution of $\xi(t) \in \mathbb{R}^q$. We introduce the viable-capture basins defining the epigraphs of the viability solutions

$$\begin{cases} (i) \ \ \text{Capt}_{(13.37)}(\mathcal{E}p(\mathbf{b})) \ = \ \mathcal{E}p(W) \text{ where } \mathcal{E}p(\mathbf{b}) \ = \ (\mathbb{I} \times A \times \mathbb{I})\mathcal{E}p(\mathbf{c}) \\ (ii) \ \text{Capt}_{(13.38)}(\mathcal{E}p(\mathbf{c})) \ = \ \mathcal{E}p(V) \end{cases} \tag{13.40}$$

We shall prove that

$$(\mathbb{I}\times A\times\mathbb{I})\text{Capt}_{(13.38)}(\mathcal{E}p(\mathbf{c}))=\text{Capt}_{(13.37)}((\mathbb{I}\times A\times\mathbb{I})\mathcal{E}p(\mathbf{c}))=\text{Capt}_{(13.37)}(\mathcal{E}p(\mathbf{b}))$$

1. *Proof of inequality* $W(d, Ax) \le V(d, x)$. This inequality is always true. Take any $(d, x, V(d, x)) \in \text{Capt}_{(13.37)}(\mathcal{E}p(\mathbf{c}))$. There exist $t \mapsto \nu(t)$ and $t^\sharp \ge 0$ such that

$$\mathbf{c}\left(\delta(t^\sharp), x - \int_0^{t^\sharp} \nu(t)dt\right) + \int_0^{t^\sharp} \mathbf{l}(\delta(t); \nu(t))dt \ \le \ V(d, x)$$

   Since $\mathbf{b}(d, Ax) \le \mathbf{c}(d, x)$ and $m(d, A\nu) \le \mathbf{l}(d, \nu)$, we infer that, setting $\mu(t) := A\nu(t)$,

$$\mathbf{b}\left(\delta(t^\sharp), Ax - \int_0^{t^\sharp} \mu(t)dt\right) + \int_0^{t^\sharp} m(\delta(t); \mu(t))dt \ \le \ V(d, x)$$

   and thus, that $(d, Ax, V(d, x))$ belongs to $\text{Capt}_{(13.37)}(\mathcal{E}p(\mathbf{b})) = \mathcal{E}p(W)$. This implies that $W(d, Ax) \le V(d, x)$ and that

$$(\mathbb{I} \times A \times \mathbb{I})\text{Capt}_{(13.38)}(\mathcal{E}p(\mathbf{c})) \ \subset \ \text{Capt}_{(13.37)}((\mathbb{I} \times A \times \mathbb{I})\mathcal{E}p(\mathbf{c}))$$

2. *Proof of inequality*: $\forall\ y, \ \ \exists\ \ x$ such that $Ax = y$ and $V(d, x) \le W(d, y)$. Take any $(d, y, W(d, y)) \in \text{Capt}_{(13.38)}(\mathcal{E}p(\mathbf{b}))$. There exist an integrable function $t \mapsto \mu(t)$ and $t^\sharp \ge 0$ such that

$$\mathbf{b}\left(\delta(t^\sharp), x - \int_0^{t^\sharp} \mu(t)dt\right) + \int_0^{t^\sharp} m(\delta(t); \mu(t))dt \ \le \ W(d, y)$$

   Theorem 18.4.14, p. 734 and assumption (13.34)(i), p. 547 imply that the subset

$$\Phi(d,v) \; := \; \{u \text{ such that } Au = v \text{ and } \mathbf{l}(d;u) \leq m(d;v)\}$$

is not empty and the set-valued map Φ has a closed graph. The Measurable Selection Theorem (see for instance Theorem 8.1.13, p. 308 of *Set-Valued Analysis*, [27, Aubin & Frankowska]) implies that we can associate with the measurable function $\mu(\cdot)$ a measurable function $\nu(\cdot)$ such that, for almost all t, $\nu(t) \in \Phi(\delta(t), \mu(t))$. Furthermore, Theorem 18.4.14, p. 734 and assumption (13.34)(ii), p. 547 imply that we can associate with $y - \int_0^{t^\sharp} \mu(t)dt$ an element z such that $Az = y - \int_0^{t^\sharp} \mu(t)dt$ and $\mathbf{c}(z) = y - \int_0^{t^\sharp} \mu(t)dt$. Setting $x := z + \int_0^{t^\sharp} \nu(t)dt$, we have proved that

$$\begin{cases} \mathbf{c}\left(\delta(t^\sharp), x - \int_0^{t^\sharp} \nu(t)dt\right) + \int_0^{t^\sharp} \mathbf{l}(\delta(t);\nu(t))dt \\ = \; \mathbf{b}\left(\delta(t^\sharp), y - \int_0^{t^\sharp} \mu(t)dt\right) + \int_0^{t^\sharp} \mathbf{l}(\delta(t);\mu(t))dt \; \leq \; W(d,y) \end{cases}$$

Therefore, $(d,y,W(d,y)) = (\mathbb{I} \times A \times \mathbb{I})(d,x,W(d,y))$ where $(d,x,W(d,y))$ belongs to $\text{Capt}_{(13.38)}(\mathcal{E}p(\mathbf{c})) \; = \; \mathcal{E}p(W)$. Consequently, $(d,y,W(d,y)) \in (\mathbb{I} \times A \times \mathbb{I})\mathcal{E}p(V)$, and thus, $V(d,x) \leq W(d,y)$ and

$$\text{Capt}_{(13.37)}((\mathbb{I} \times A \times \mathbb{I})\mathcal{E}p(\mathbf{c})) \; \subset \; (\mathbb{I} \times A \times \mathbb{I})\text{Capt}_{(13.38)}(\mathcal{E}p(\mathbf{c}))$$

These two inequalities imply that $V(d,x) \leq W(d,y) = W(d,Ax) \leq V(d,x)$, so that

$$W(d,y) \; = \; \min_{Ax=y} V(d,x)$$

and $\text{Capt}_{(13.37)}(\mathbb{I} \times A \times \mathbb{I})(\mathcal{E}p(\mathbf{c})) = (\mathbb{I} \times A \times \mathbb{I})\text{Capt}_{(13.38)}(\mathcal{E}p(\mathbf{c}))$. □

We now compare the two regulation maps:

Theorem 13.7.3 ***[Links between the Regulation Maps]*** *We posit the assumptions of Theorem 13.7.2, p. 547. Consider the two regulation maps*

$$R_W(d,y) \; := \left\{ \begin{array}{l} \mu \in \text{Dom}(m(d,y;\cdot)) \text{ such that} \\ \left\langle \frac{\partial W(d,y)}{\partial d}, \varphi(d) \right\rangle + \left\langle \frac{\partial W(d,y)}{\partial y}, \mu \right\rangle - m(d,y;\mu) \; \geq \; 0 \end{array} \right\}$$

and

$$R_V(d,x) \; := \left\{ \begin{array}{l} \nu \in \text{Dom}(\mathbf{l}(d,x;\cdot)) \text{ such that} \\ \left\langle \frac{\partial V(d,x)}{\partial d}, \varphi(d) \right\rangle + \left\langle \frac{\partial V(d,x)}{\partial x}, \nu \right\rangle - \mathbf{l}(d,x;\nu) \; \geq \; 0 \end{array} \right\}$$

Inclusion $AR_V(d,x) \subset R_V(d,Ax)$ *always holds true. If we assume that the solution* V *satisfies*

$$\begin{cases} (i) \ \ \forall (d,u) \in \mathrm{Dom}(\mathbf{l}), \forall \nu \in \mathbb{R}^n, \exists \mu \in \mathrm{Dom}(D_\uparrow \mathbf{l}(d,u)) \cap c\|\nu\| \text{ and } A\nu = \mu \\ (ii) \ \forall (d,x) \in \mathrm{Dom}(V), \forall \nu \in \mathbb{R}^n, \exists \mu \in \mathrm{Dom}(D_\uparrow V(d,x)) \cap c\|\nu\| \text{ and } A\nu = \mu \end{cases} \tag{13.41}$$

equality

$$\exists\, x \text{ satisfying } Ax = y \text{ and } R_W(d,y) \ = \ AR_V(d;x) \tag{13.42}$$

ensues.

Proof. 1. *Proof of inclusion* $AR_V(d,x) \subset R_W(d,Ax)$. Let us consider $\nu \in R_V(d,x)$. This means that

$$(-\varphi(d), -\nu, -\mathbf{l}(d;\nu)) \ \in \ T_{\mathcal{E}p(V)}(d,x,V(d,x))$$

Therefore,

$$\begin{cases} (-\varphi(d), -A\nu, -\mathbf{l}(d;\nu)) \ = \ (\mathbb{I} \times A \times \mathbb{I})(-\varphi(d), -\nu, -\mathbf{l}(d;\nu)) \\ \in \ (\mathbb{I} \times A \times \mathbb{I}) T_{\mathcal{E}p(V)}(d,x,V(d,x)) \ \subset \ T_{(\mathbb{I} \times A \times \mathbb{I})\mathcal{E}p(V)}(d,Ax,V(d,x)) \\ = \ T_{\mathcal{E}p(W)}(d,y,V(d,x)) \end{cases}$$

Since by Proposition 6.1.4, p. 226 of *Set-Valued Analysis*, [27, Aubin & Frankowska], $T_{\mathcal{E}p(W)}(d,y,V(d,x)) \ = \ \mathrm{Dom}(D_\uparrow W(d,y)) \times \mathbb{R}$ if $W(d,y) < V(d,x)$ and $T_{\mathcal{E}p(W)}(d,x,V(d,x)) \ = \ T_{\mathcal{E}p(W)}(d,y,W(d,y))$ if $W(d,y) = V(d,x)$, we infer that $A\nu$ belongs to $R_W(d,Ax)$.

2. *Proof of inclusion* $\forall\, y$, $\exists\, x$ such that $Ax - y$ and $R_W(d,y) \subset AR_V(d,x)$. Let us consider $\mu \in R_W(d,y)$, i.e., satisfying

$$(-\varphi(d), -\mu, -m(d;\mu)) \ \in \ T_{\mathcal{E}p(W)}(d,y,W(d,y))$$

Theorem 18.4.14, p. 734 and assumption (13.41)(ii), p. 550 imply that there exist x such that $Ax = y$ and $V(d,x) = W(d,y)$.
By applying Theorem 18.4.14, p. 734 for the linear operator $\mathbb{I} \times A \times \mathbb{I}$ and the subset $\mathcal{E}p(V)$, we infer that assumption (13.41)(ii), p. 550 implies equality

$$(\mathbb{I} \times A \times \mathbb{I}) T_{\mathcal{E}p(V)}(d,x,V(d,x)) \ = \ T_{(\mathbb{I} \times A \times \mathbb{I})\mathcal{E}p(V)}(d,Ax,V(d,x)) \tag{13.43}$$

holds true. Then, there exist ν such that $A\nu = \mu$ and $\mathbf{l}(d,\nu) = m(d,\mu)$. Consequently,

$$\begin{cases} (-\varphi(d), -\mu, -m(d;\mu)) \ = \ (\mathbb{I} \times A \times \mathbb{I})(-\varphi(d), -\nu, -\mathbf{l}(d;\nu)) \\ \in \ (\mathbb{I} \times A \times \mathbb{I}) T_{\mathcal{E}p(W)}(d,y,W(d,y)) \ = \ (\mathbb{I} \times A \times \mathbb{I}) T_{\mathcal{E}p(W)}(d,x,V(d,x)) \end{cases}$$

Therefore,

$$(-\varphi(d), -\nu, -\mathbf{l}(d;\nu)) \in T_{\mathcal{E}p(V)}(d, x, V(d,x))$$

and we infer that ν belongs to $R_V(d,x)$. □

13.8 Hamilton–Jacobi–Cournot Equations

The question whether we can obtain beforehand *the initial states* $x(0)$ *and the specific regulation maps* $\widetilde{R}(d, x(0), x)$ *(depending upon* $x(0)$*) driving optimal viable evolutions* $x(\cdot)$ arriving at terminal state x at optimal time $t^\sharp$ (see Definition 8.4.8, p. 288). This would avoid *computing initial states* $x(0)$ by solving the variational problem or, equivalently, avoid the regulation of *all* optimal evolutions from the terminal state through the regulation map R_V as in Theorem 13.6.3, p. 543.

For computing the formerly missing initial conditions, we just introduce an auxiliary parameter $\chi \in \mathbb{R}^n$ which plays the role of *candidate to be an initial condition*. We then extend internal and viability conditions $\widetilde{\mathbf{k}}$ and $\widetilde{\mathbf{c}}$ by setting

$$\begin{cases} (i) \ \widetilde{\mathbf{k}}(d,\chi,x) \ := \ \mathbf{k}(d,x) \\ (ii) \ \widetilde{\mathbf{c}}(d,x,x) \ := \ \mathbf{c}(d,x) \text{ and } \widetilde{\mathbf{c}}(d,\chi,x) := +\infty \ \ \chi \neq x \end{cases} \tag{13.44}$$

The answer is obtained by computing the viability solution $\widetilde{V}(d,\chi,x)$ to the Hamilton–Jacobi–Cournot partial differential equation

$$\left\langle \frac{\partial \widetilde{V}(d,\chi,x)}{\partial d}, \varphi(d) \right\rangle + \mathbf{l}^\star \left(d, x; \frac{\partial \widetilde{V}(d,\chi,x)}{\partial x} \right) \ = \ 0 \tag{13.45}$$

satisfying

$$\widetilde{\mathbf{k}}(d,\chi,x) \ \le \ \widetilde{V}(d,\chi,x) \ \le \ \widetilde{\mathbf{c}}(d,\chi,x) \tag{13.46}$$

We associate with the viability solution $\widetilde{V}$ of the Hamilton–Jacobi–Cournot equation:

1. the *Cournot regulation map* $R_{\widetilde{V}}$ defined by

$$\begin{cases} R_{\widetilde{V}}(d,\chi,x) \ = \\ \left\{ \upsilon \in F(d,x) \text{ such that } D_\uparrow^{\star\star}\widetilde{V}(d,\chi,x)(-\varphi(d), 0, -\upsilon) + \mathbf{l}(d,x) \ \le \ 0 \right\} \end{cases}$$

2. the *Cournot map* $\mathbb{C}_{\widetilde{V}}$ defined by

$$\mathbb{C}_{\widetilde{V}}(d,x) \ := \ \left\{ \chi \text{ such that } \widetilde{V}(d,\chi,x) < \infty \right\} \tag{13.47}$$

Theorem 13.8.1 ***[Regulation of Optimal Evolutions from Cournot Initial States]*** *We posit the assumptions of Theorem 13.10.2, p. 558. For any initial state* $\chi \in \mathbb{C}_{\widetilde{V}}(d,x)$ *provided by the Cournot map, there exists at least one optimal viable optimal evolution* $x(\cdot) \in \mathcal{O}_{\mathbf{k}}(t^\sharp; d, \chi, x)$ *where* $t^\sharp \in \mathbb{T}_{(\mathbf{k},\mathbf{c})}(d,x)$ *starting from initial conditions* $x(0) = \chi$*, arriving at* $x = x(t^\sharp)$ *at optimal* $t^\sharp$ *when* $d(t^\sharp) = d$*.*

It is regulated by differential inclusion

$$\forall\, t \in [0, t^\sharp[, \quad x'(t) \;\in\; R_{\widetilde{V}}(d(t), \chi, x(t))$$

Proof. Let us consider the viability solution $(d, \chi, x) \mapsto \widetilde{V}(d, \chi, x)$ and the associated regulation map $R_{\widetilde{V}}$.

Its epigraph is the viable-capture basin of $\mathcal{E}p(\widetilde{c})$ viable in $\mathcal{E}p(\widetilde{k})$ under the characteristic system

$$\begin{cases} (i) \quad \delta'(t) \;=\; -\varphi(\delta(t)) \\ (ii) \;\; \chi'(t) \;=\; 0 \\ (iii) \; (\xi'(t), \eta'(t)) \;\in\; -\mathcal{E}p(\mathbf{l}(\delta(t), \xi(t); \cdot)) \end{cases} \tag{13.48}$$

There exist at least some time $t^\sharp$ and one evolution $t \mapsto (\delta(t), \chi, \xi(t), \eta(t))$ starting from $(d, x, \chi, V(d,x)) \in \mathcal{E}p(\widetilde{V})$ viable in $\mathcal{E}p(\widetilde{\mathbf{k}})$ until it reaches $\mathcal{E}p(\widetilde{\mathbf{c}})$ at some time $t^\sharp$. The viability condition implies that $t^\sharp \le \tau^\sharp(d)$ and that

$$\sup_{s \in [0, t^\sharp]} \left(\mathbf{k}\,(\delta(s), \xi(s)) + \int_0^s \mathbf{l}(\delta(\tau), \xi(\tau), \upsilon(\tau)) d\tau \right) \;\le\; \widetilde{V}(d, \chi, x)$$

so that

$$\overleftarrow{\mathbf{I}}_{\mathbf{k}}(t^\sharp; \upsilon(\cdot))(d,x) \;\le\; \widetilde{V}(d, \chi, x)$$

and the target condition that

$$\widetilde{\mathbf{c}}\left(\delta(t^\sharp), \chi, \xi(t^\sharp)\right) + \int_0^{t^\sharp} \mathbf{l}(\delta(\tau), \xi(\tau), \upsilon(\tau)) d\tau \;\le\; \widetilde{V}(d, \chi, x) \;<\; +\infty$$

Since $\widetilde{\mathbf{c}}(d, \chi, x) := +\infty$ whenever $\chi \ne x$, we infer that $\chi = \xi(t^\sharp)$, that $\chi \in \mathbb{C}_{\widetilde{V}}(d,x)$ and that

$$\mathbf{c}(\delta(t^\sharp), \chi) + \int_0^{t^\sharp} \mathbf{l}(\delta(\tau), \xi(\tau), \upsilon(\tau)) d\tau \;\le\; \widetilde{V}(d, \chi, x) \;<\; +\infty$$

Setting

$$\begin{cases} \overleftarrow{\mathbf{J}}_{\mathbf{c}}(t^\sharp; \upsilon(\cdot))(d, \chi, x) \\ := \ \mathbf{c}(\delta(t^\sharp), \chi) + \displaystyle\int_0^{t^\sharp} \mathbf{l}(\delta(\tau), \xi(\tau), \upsilon(\tau)) d\tau \text{ if } \xi(t^\sharp) = \chi \\ \text{and } +\infty \text{ otherwise} \end{cases}$$

we infer that

$$\overleftarrow{\mathbf{J}}_{\mathbf{c}}(t^\sharp; \upsilon(\cdot))(d, \chi, x) \ \leq \ \widetilde{V}(d, \chi, x)$$

and thus that, setting

$$\begin{cases} \overleftarrow{\mathbf{L}}_{(\mathbf{k},\mathbf{c})}(t^\sharp; \upsilon(\cdot))(d, \chi, x) \\ := \ \max(\overleftarrow{\mathbf{I}}_{\mathbf{k}}(t^\sharp; x(\cdot), \upsilon(\cdot))(d, x), \overleftarrow{\mathbf{J}}_{\mathbf{c}}(t^\sharp; \upsilon(\cdot))(d, \chi, x)) \end{cases}$$

we have proved that

$$\overleftarrow{\mathbf{L}}_{(\mathbf{k},\mathbf{c})}(t^\sharp; \upsilon(\cdot))(d, \chi, x) \ \leq \ \widetilde{V}(d, \chi, x)$$

The proof of Theorem 13.4.3, p. 533 implies that

$$\widetilde{V}(d, \chi, x) \ = \ \inf_{(t^\sharp; \upsilon(\cdot))} \overleftarrow{\mathbf{L}}_{(\mathbf{k},\mathbf{c})}(t^\sharp; \upsilon(\cdot))(d, \chi, x)$$

and that the viable evolution is regulated by

$$\upsilon(t) \ \in \ R_{\widetilde{V}}(\delta(t), \chi, \xi(t))$$

where, under assumptions of Theorem 13.10.2, p. 558, the regulation map $R_{\widetilde{V}}$ is equal to

$$R_{\widetilde{V}}(d, \chi, x) = \left\{ \upsilon \in F(d, x) \text{such that } D_{\uparrow}^{\star\star}\widetilde{V}(d, \chi, x)(-\varphi(d), 0, -\upsilon) + \mathbf{l}(d, x) \leq 0 \right\}$$

By setting $d(t) := \delta(t^\sharp - t)$, $x(t) := \xi(t^\sharp - t)$ and $x'(t) := \upsilon(t^\sharp - t)$, this is equivalent to saying that an optimal viable evolution starting from $\chi \in \mathbb{C}_{\widetilde{V}}(d, x)$ is regulated by

$$\upsilon(t) \ \in \ R_{\widetilde{V}}(\delta(t), \chi, \xi(t))$$

and satisfy the terminal condition $d(t^\sharp) = d$ and $x(t^\sharp) = x$ and initial condition $x(0) = \chi$.

Hence, for each initial condition $\chi \in \mathbb{C}_{\widetilde{V}}(d, x)$, there exists an optimal viable evolution starting at χ and arriving at x in optimal time $t^\sharp$. □

Consequently, for any pair (d, x), if

1. $\mathbb{C}_{\widetilde{V}}(d, x) = \{\chi\}$ is a singleton, then there exists one optimal viable evolution starting at χ and regulated by $x'(t) \in R_{\widetilde{V}}(d(t), \chi, x(t))$ until it arrives at x at time $t^\sharp$,

2. $\mathbb{C}_{\widetilde{V}}(d,x) = \emptyset$, and no optimal evolution arrive at x,
3. $\mathbb{C}_{\widetilde{V}}(d,x)$ contains several initial states χ. From all $\chi \in \mathbb{C}_{\widetilde{V}}(d,x)$ start optimal viable evolutions regulated by $x'(t) \in R_{\widetilde{V}}(d(t),\chi,x(t))$ until they *collide* at x at time $t^\sharp$.

This is the property which motivates the terminology of Cournot map, since Antoine Cournot (1801–1877) suggested to capture one aspect of uncertainty or chance as the collision of several independent causal series.

13.9 Lax–Hopf Formula

Theorem 13.9.1 ***[Lax–Hopf Formula]*** *We assume that both φ and $\mathbf{l}$ do not depend upon d and x. Let us assume that the function $\mathbf{c}$ is lower semicontinuous and that the functions $\mathbf{k}$ and $\mathbf{l}$ are convex and lower semicontinuous. Then the viability solution V is equal to the* Lax–Hopf value function

$$V(d,x) \;=\; \max\left(\mathbf{k}(d,x), \inf_{t^\sharp \geq 0} \inf_{u \in \mathrm{Dom}(\mathbf{l})} \left(\mathbf{c}(d - t^\sharp\varphi, x - t^\sharp u) + t^\sharp \mathbf{l}(u)\right)\right) \tag{13.49}$$

which is the marginal function of a static minimization theorem.

The regulation map is the set of elements $u \in \mathrm{Dom}(\mathbf{l})$ minimizing this function:

$$\begin{cases} R_V(d,x) \;=\; \{u \in \mathrm{Dom}(\mathbf{l}) \text{ such that} \\ V(d,x) = \max\left(\mathbf{k}(d,x), \left(\mathbf{c}(d - \tau^\sharp(d)\varphi, x - \tau^\sharp(d)u) + \tau^\sharp(d)\mathbf{l}(u)\right)\right)\} \end{cases} \tag{13.50}$$

If the viability solution V and the Lagrangian $\mathbf{l}$ are differentiable and if $u := r(d,x) \in R_V(d,x)$ is the unique minimizer, Lax–Oleinik formula holds

$$\frac{\partial V}{\partial x} \;=\; \frac{d}{du}\mathbf{l}(d,x;r(d,x)) \tag{13.51}$$

For boundary value problems without constraints, the formula boils down to

$$V(d,x) \;=\; \inf_{u \in \mathrm{Dom}(\mathbf{l})} \left(\mathbf{c}(d - \tau^\sharp(d)\varphi, x - \tau^\sharp(d)u) + \tau^\sharp(d)\mathbf{l}(u)\right) \tag{13.52}$$

Proof. Since the epigraph of $\mathbf{k}$ is convex and since both the map φ and Lagrangian $\mathbf{l}$ do not depend on (d,x), differential inclusion

$$\begin{cases} (i) \ \ \delta'(t) \ = \ -\varphi \\ (ii) \ (\xi'(t), \eta'(t)) \ \in \ -\mathcal{E}p(\mathbf{l}(\cdot)) \end{cases} \tag{13.53}$$

can be written in the form

$$(\delta'(t), \xi'(t), \eta'(t)) \ \in \ \mathcal{G} \ := \ \{-\varphi\} \times -\mathcal{E}p(\mathbf{l}(\cdot))$$

where $\mathcal{G}$ is a constant closed convex subset. Lax–Hopf Formula (11.27), p. 469 of Theorem 11.5.4, p. 469 states that if the environment $\mathcal{E}p(\mathbf{k})$ is closed and convex, the target $C \subset K$ is closed and $\mathcal{G}$ is a closed convex subset, then the viable-capture basin enjoys the *Lax–Hopf formula*

$$\mathrm{Capt}_{\mathcal{G}}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})) \ = \ \mathcal{E}p(\mathbf{k}) \cap (\mathcal{E}p(\mathbf{c}) - \mathbb{R}_+\mathcal{G})$$

Hence, the epigraph of the valuation function V, defined as the viable-capture basin $\mathrm{Capt}_{(13.52)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))$ under the differential inclusion (13.53), is equal to

$$\begin{cases} \mathrm{Capt}_{(13.52)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})) \ = \ \mathrm{Capt}_{\mathcal{G}}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})) \\ = \ \mathcal{E}p(\mathbf{k}) \cap \left(\mathcal{E}p(\mathbf{c}) - \bigcup_{\lambda \geq 0} \lambda(\{\varphi\} \times \mathcal{E}p(\mathbf{l}))\right) \end{cases} \tag{13.54}$$

Therefore (d, x, y) belongs the viable-capture basin if $\mathbf{k}(d, x) \leq y$ and if there exist $(\delta^\star, \xi^\star, \eta^\star) \in \mathcal{E}p(\mathbf{c})$, $t^\sharp > 0$ and $u \in \mathrm{Dom}(\mathbf{l})$ such that

$$\mathbf{c}(d - t^\sharp, x - t^\sharp u) \ = \ \mathbf{c}(\delta^\star, \xi^\star) \ \leq \ \eta \ \leq \ y - t^\sharp \mathbf{l}(u)$$

This means that

$$\max\left(\mathbf{k}(d, x), \inf_{t^\sharp} \inf_{u} \left(\mathbf{c}(t - t^\sharp \varphi, x - t^\sharp u) + t^\sharp \mathbf{l}(u)\right)\right) \ = \ V(d, x)$$

which is the Lax–Hopf formula we were looking for. □

We associate with the *internal condition* function $(d, x) \mapsto \mathbf{c}(d, x) \in \mathbb{R} \cup \{+\infty\}$ the *domain map* $d \rightsquigarrow \mathbf{C}(d)$ defined by

$$\mathbf{C}(d) \ := \ \{x \text{ such that } \mathbf{c}(d, x) < +\infty\} \tag{13.55}$$

The question arises to know precisely the domains $\mathrm{Dom}(V(t, \cdot))$ of the traffic profiles, i.e., the set of states x such that $V(t, x) < +\infty$: we shall prove that it is couched in terms of the domain map $\mathbf{C}$:

Theorem 13.9.2 ***[Domain of the Viability Solution]*** *We posit the assumptions of Theorem 13.9.1, p. 554. For any $d \in \mathcal{D}$, the domains of the function $V(d, \cdot)$ associated with the internal condition $\mathbf{c}$ are equal to*

$$\mathrm{Dom}(V(d,\cdot)) \; = \; \bigcup_{s\in[0,\tau^\sharp(d)],\; u\in\mathrm{Dom}(\mathbf{l})} (\mathbf{C}(d-s\varphi)+su) \tag{13.56}$$

If we assume furthermore that $\mathbf{C}$ *satisfies*

$$\forall\, s\in[0,\tau^\sharp(d)],\;\; \mathbf{C}(d-s\varphi) \;\subset\; \mathbf{C}(d-\tau^\sharp(d)\varphi)+(\tau^\sharp(d)-s)\mathrm{Dom}(\mathbf{l}) \tag{13.57}$$

then

$$\mathrm{Dom}(V(d,\cdot)) \; = \; \mathbf{C}(d-\tau^\sharp(d)\varphi)+\tau^\sharp(d)\mathrm{Dom}(\mathbf{l}) \tag{13.58}$$

Proof. For any $x \in \mathrm{Dom}(V(d,\cdot))$ and any $\varepsilon > 0$, there exist $s \in [0,\tau^\sharp(d)]$, $u \in \mathrm{Dom}(\mathbf{l})$ such that

$$\mathbf{c}(d-s\varphi, x-su)+s\mathbf{l}(u) \;\leq\; V(d,x)+\varepsilon \;<\; +\infty$$

so that $x-su \in \mathbf{C}(d-s\varphi)$, and thus,

$$\mathrm{Dom}(V(d,\cdot)) \;\subset\; \mathbf{C}(d-s\varphi)+su \;\subset\; \bigcup_{s\in[0,\tau^\sharp(d)],\; u\in\mathrm{Dom}(\mathbf{l})} (\mathbf{C}(d-s\varphi)+su)$$

Conversely, let us take $x \in \bigcup_{s\in[0,\tau^\sharp(d)],\; u\in\mathrm{Dom}(\mathbf{l})} (\mathbf{C}(d-s\varphi)+su)$, and thus, take $s \in [0,\tau^\sharp(d)]$ and $u \in \mathrm{Dom}(\mathbf{l})$ such that $x \in \mathbf{C}(d-s\varphi)+su$. Therefore, $\mathbf{c}(d-s\varphi, x-su) < +\infty$ is finite and

$$V(d,x) \;\leq\; \mathbf{c}(d-s\varphi, x-su)+s\mathbf{l}(u) \;<\; +\infty$$

so that $x \in \mathrm{Dom}(V)(d,\cdot)$.

As a particular case, for $s := \tau^\sharp(d)$

$$\mathbf{C}(d-\tau^\sharp(d)\varphi)+\tau^\sharp(d)\mathrm{Dom}(\mathbf{l}) \;\subset\; \mathrm{Dom}(V(t,\cdot))$$

Assume now that the tube $\mathbf{C}$ satisfies (14.22), p. 584. Take any $x \in \mathrm{Dom}(V(d,\cdot))$, with which we associate $s \in [0,\tau^\sharp(d)]$, $\xi \in \mathbf{C}(d-s\varphi)$ and $u \in \mathrm{Dom}(\mathbf{l})$ such that $x = \xi + su$ thanks to (14.21), p. 584. Therefore

$$\xi \;\in\; \mathbf{C}(d-s\varphi) \;\subset\; \mathbf{C}(d-\tau^\sharp(d)\varphi)+(\tau^\sharp(d)-s)\mathrm{Dom}(\mathbf{l})$$

so that there exists $v \in \mathrm{Dom}(\mathbf{l})$ such that $\xi \in \mathbf{C}(d-s\varphi)+(\tau^\sharp(d)-s)v$. Hence $x \in \mathbf{C}(d-s\varphi)+su+(\tau^\sharp(d)-s)v$. Since $\mathrm{Dom}(\mathbf{l})$ is convex and since u and v belong to it and $\tau^\sharp(d)-s \geq 0$, then $su+(\tau^\sharp(d)-s)v = \tau^\sharp(d)w$ where $w \in \mathrm{Dom}(\mathbf{l})$. Hence x belongs to $\mathbf{C}(d-s\varphi)+\tau^\sharp(d)\mathrm{Dom}(\mathbf{l})$ and thus,

$$\mathrm{Dom}(V(d,\cdot)) \subset \mathbf{C}(d-\tau^{\sharp}(d)\varphi)+\tau^{\sharp}(d)\mathrm{Dom}(\mathbf{l})$$

This completes the characterization of the domain of the traffic solution. □

13.10 Barron–Jensen/Frankowska Viscosity Solution

This (technical) section is devoted to the proof that the viability solution defined through a viable-capture basin is actually the unique solution to the structured Hamilton-Jacobi equation. This is done by translating the Frankowska property characterizing a viable-capture basin of a target at the level of differential inclusions in terms of gradients (or subdifferential) of the viability solutions.

For simplicity of the exposition, we begin with the assumption that the viability solution is differentiable, first in the case classical case without viability constraint, easier to formulate, next, under viability constraint, before dropping the differentiability assumption and proving that the viability solution is the Barron–Jensen/Frankowska Viscosity Solution (Theorem 13.10.3, p. 560).

Proposition 13.10.1 *[**Hamilton–Jacobi Equation without Viability Constraints**] We assume, for simplicity, that the viability solution is differentiable. We posit the following assumptions:*

1. *the Lagrangian is Marchaud,*
2. *φ, $(d,x,\upsilon) \mapsto \mathbf{l}(d,x;\upsilon)$ and the set-valued map $(d,x) \rightsquigarrow F(d,x)$ are Lipschitz.*

Then the viability solution is the unique function V solution to the Hamilton–Jacobi equation (13.9), p. 530

$$\text{if } V(d,x)<\mathbf{c}(d,x), \text{ then } \left\langle \frac{\partial V(d,x)}{\partial d},\varphi(d)\right\rangle + \mathbf{l}^{\star}\left(d,x;\frac{\partial V(d,x)}{\partial x}\right) = 0$$

satisfying conditions $V(d,x) \le \mathbf{c}(d,x)$.
The regulation map satisfies

$$\text{if } V(d,x) < \mathbf{c}(d,x), \text{ then } R_V(d,x) \subset \partial_p\mathbf{l}^{\star}\left(d,x;\frac{\partial V(d,x)}{\partial x}\right)$$

This is a consequence of the following theorem with state constraints:

Theorem 13.10.2 *[**Hamilton–Jacobi Equation with Viability Constraints**] We assume, for simplicity, that the viability solution is differentiable. We posit the following assumptions:*

1. *the Lagrangian is Marchaud,*
2. *φ, $(d, x, \upsilon) \mapsto \mathbf{l}(d, x; \upsilon)$ and the set-valued map $(d, x) \rightsquigarrow F(d, x)$ are Lipschitz*
3. *the viability constraint function $\mathbf{k}$ is differentiable and satisfies*

$$\left\langle \frac{\partial \mathbf{k}(d,x)}{\partial d}, \varphi(d) \right\rangle + \mathbf{l}^{\star}\left(d, x; \frac{\partial \mathbf{k}(d,x)}{\partial x}\right) \leq 0$$

Then the viability solution is the unique function V satisfying conditions (13.10), p. 530

$$\mathbf{k}(d,x) \leq V(d,x) \leq \mathbf{c}(d,x)$$

and solution to the Hamilton–Jacobi equation (13.9), p. 530 in the sense that

$$\begin{cases} (i) \text{ if } \mathbf{k}(d,x) < V(d,x) < \mathbf{c}(d,x), \text{ then} \\ \quad \left\langle \frac{\partial V(d,x)}{\partial d}, \varphi(d) \right\rangle + \mathbf{l}^{\star}\left(d, x; \frac{\partial V(d,x)}{\partial x}\right) = 0 \\ (ii) \text{ if } \mathbf{k}(d,x) = V(d,x) \leq \mathbf{c}(d,x), \text{ then} \\ \quad \left\langle \frac{\partial V(d,x)}{\partial d}, \varphi(d) \right\rangle + \mathbf{l}^{\star}\left(d, x; \frac{\partial V(d,x)}{\partial x}\right) \leq 0 \end{cases} \tag{13.59}$$

When $\mathbf{k}(d,x) < V(d,x)$, the regulation map satisfies

$$\text{if } \mathbf{k}(d,x) < V(d,x) < \mathbf{c}(d,x), \text{ then } R_V(d,x) \subset \partial_p \mathbf{l}^{\star}\left(d, x; \frac{\partial V(d,x)}{\partial x}\right) \tag{13.60}$$

This formula can be written

$$\text{if } \mathbf{k}(d,x) < V(d,x) < \mathbf{c}(d,x), \text{ then } \frac{\partial V(d,x)}{\partial x} \in \bigcap_{u \in R_V(d,x)} \partial_u \mathbf{l}(d, x; u) \tag{13.61}$$

and regarded as an extension of the Lax–Oleinik formula *to this general case.*

Proof. Actually, we shall derive the equality in formula (13.59), p. 558 from two inequalities, the first one valid when the Lagrangian is Marchaud, the second one under the Lipschitz conditions.

1. Since the Lagrangian is Marchaud, so is the set-valued map $\mathcal{F}$ defined by (13.18), p. 534 and, by (13.22), p. 537, the viable-capture basin

$\mathcal{E}p(V) := \text{Capt}_{(13.10)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))$ is equal to $\text{Capt}_{(13.17)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})) = \text{Capt}_{(13.10)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))$.
It is then the largest closed subset between $\mathcal{E}p(\mathbf{c})$ and $\mathcal{E}p(\mathbf{k})$, locally viable in $\mathcal{E}p(V) \setminus \mathcal{E}p(\mathbf{c})$ thanks to the Viability Theorem 11.4.6, p. 463, which also states that, whenever $(d, x, V(d, x)) \in \mathcal{E}p(V) \setminus \mathcal{E}p(\mathbf{c})$, i.e., whenever $V(d, x) < \mathbf{c}(d, x)$, there exists some $v \in \text{Dom}(\mathbf{l})$ such that

$$(-\varphi(d), -v, -\mathbf{l}(d, x; v)) \in T_{\mathcal{E}p(V)}(d, x, V(d, x))$$

Recall that $N_{\mathcal{E}p(V)}(d, x, V(d, x)) := T^{\star}_{\mathcal{E}p(V)}(d, x, V(d, x))$ and that *the subdifferential* $\partial V(d, x)$ *is the set of* (p_d, p_x) *such that* $(p_d, p_x, -1) \in N_{\mathcal{E}p(V)}(d, x, V(d, x))$.
Therefore, we infer that

$$\forall\, v \in R_V(d, x),\ \forall\, (p_d, p_x) \in \partial V(d, x),\ \ 0 \le \langle p_d, \varphi(d)\rangle + \langle p_x, v\rangle - \mathbf{l}(d, x; v) \tag{13.62}$$

and thus, that, by taking the supremum over $F(d, x) := \text{Dom}(\mathbf{l}(d, x; \cdot))$,

$$\forall\, (p_d, p_x) \in \partial V(d, x),\ \ 0 \le \langle p_d, \varphi(d)\rangle + \mathbf{l}^{\star}(d, x; p_x) \tag{13.63}$$

2. Since $(d, x, v) \mapsto \mathbf{l}(d, x; v)$ and the set-valued map $(d, x) \rightsquigarrow F(d, x)$ are Lipschitz. it is easy to observe that the set-valued map $(d, x) \rightsquigarrow \mathcal{E}p(\mathbf{l}(d, x; \cdot))$ is Lipschitz. The viable-capture basin $\mathcal{E}p(V) := \text{Capt}_{(13.10)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))$ is the smallest closed subset between $\mathcal{E}p(\mathbf{c})$ and $\mathcal{E}p(\mathbf{k})$ backward invariant relatively to $\mathcal{E}p(\mathbf{k})$.
If we assume that $\mathcal{E}p(\mathbf{k})$ is itself backward invariant, then $\mathcal{E}p(V)$ is also backward invariant. It remains to translate properties of the Invariance Theorem 11.4.6, p. 463 in terms of subdifferentials.
Backward invariance of the epigraph $\mathcal{E}p(V)$ means that whenever $(d, x, V(d, x)) \in \mathcal{E}p(V)$, then

$$\forall v \in F(d, x), \forall (p_d, p_x) \in \partial V(d, x), (\varphi(d), v, \mathbf{l}(d, x; v)) \in T_{\mathcal{E}p(V)}(d, x, V(d, x))$$

This amounts to saying that

$$\forall\, v \in F(d, x),\ \forall\, (p_d, p_x) \in \partial V(d, x),\ \ \langle p_d, \varphi(d)\rangle + \langle p_x, u\rangle - \mathbf{l}(d, x; u) \le 0$$

and thus, that

$$\forall\, (p_d, p_x) \in \partial V(d, x),\ \ \langle p_d, \varphi(d)\rangle + \mathbf{l}^{\star}(d, x; p_x) \le 0 \tag{13.64}$$

In the same way, the epigraph of $\mathbf{k}$ is backward invariant if

$$\forall\, (q_d, q_x) \in \partial \mathbf{k}(d, x),\ \ \langle q_d, \varphi(d)\rangle + \mathbf{l}^{\star}(d, x; q_x) \le 0$$

If $\mathbf{k}(d,x) < V(d,x) < \mathbf{c}(d,x)$, inequality (13.63), p. 559 and (13.64), p. 559 imply that the viability solution satisfies

$$\forall\ (p_d, p_x) \in \partial V(d,x),\ \ \langle p_d, \varphi(d)\rangle + \mathbf{l}^\star(d,x;p_x)\ =\ 0 \tag{13.65}$$

Taking into account that $\langle p_d, \varphi(d)\rangle = -\mathbf{l}^\star(d,x;p_x)$ in inequality (13.62), p. 559, we infer that

$$\forall\ \upsilon \in R_V(d,x),\ \forall\ (p_d,p_x) \in \partial V(d,x),\ \ 0\ \le\ \langle p_x, \upsilon\rangle - \mathbf{l}^\star(d,x;p_x) - \mathbf{l}(d,x;\upsilon)$$

The Legendre property (13.7), p. 529 of the Fenchel transform implies that this is equivalent to saying that $\upsilon \in \partial_p \mathbf{l}^\star(d,x;p_x)$ or that $p_x \in \partial \mathbf{l}_u(d,x;u)$. Consequently:

$$R_V(d,x)\ \subset \bigcap_{(p_d,p_x)\in\partial V(d,x)} \partial_p \mathbf{l}^\star(d,x;p_x)$$

or, equivalently,

$$\bigcup_{(p_d,p_x)\in\partial V(d,x)} p_x\ \subset \bigcap_{u\in R_V(d,x)} \partial_u \mathbf{l}(d,x;u)$$

Assuming that V is differentiable, we have proved the formulas (13.59), p. 558 and (13.60), p. 558 and, consequently, that the viability solution is the unique solution to the structured Hamilton–Jacobi equation. □

Under the assumptions of Theorem 13.10.2, p. 558, when the viability solution V is not differentiable, it is still the unique solution satisfying (13.65), p. 560:

$$\forall\ (p_d, p_x) \in \partial V(d,x),\ \ \langle p_d, \varphi(d)\rangle + \mathbf{l}^\star(d,x;p_x)\ =\ 0$$

Theorem 13.10.3 ***[Barron–Jensen/Frankowska Viscosity Solution]*** *We posit the following assumptions:*

1. *the Lagrangian is Marchaud,*
2. φ, $(d,x,\upsilon) \mapsto \mathbf{l}(d,x;\upsilon)$ *and the set-valued map* $(d,x) \rightsquigarrow F(d,x)$ *are Lipschitz*
3. *The viability constraint function* $\mathbf{k}$ *satisfies*

$$\forall\ (q_d, q_x) \in \partial \mathbf{k}(d,x),\ \ \langle q_d, \varphi(d)\rangle + \mathbf{l}^\star(d,x;q_x)\ \le\ 0 \tag{13.66}$$

Then the viability solution is the unique function V satisfying conditions (13.10), p. 530

$$\mathbf{k}(d,x)\ \le\ V(d,x)\ \le\ \mathbf{c}(d,x)$$

and solution to the Hamilton–Jacobi equation (13.9), p. 530 in the sense that

$$\begin{cases} (i) \text{ if } \mathbf{k}(d,x) \ < \ V(d,x) \ < \ \mathbf{c}(d,x), \text{ then} \\ \quad \forall \ (p_d, p_x) \in \partial V(d,x), \ \ \langle p_d, \varphi(d)\rangle + \mathbf{l}^\star(d,x;p_x) \ = \ 0 \\ (ii) \text{ if } \mathbf{k}(d,x) \ = \ V(d,x) \ \leq \ \mathbf{c}(d,x), \text{ then} \\ \quad \forall \ (p_d, p_x) \in \partial V(d,x), \ \ \langle p_d, \varphi(d)\rangle + \mathbf{l}^\star(d,x;p_x) \ \leq \ 0 \end{cases} \tag{13.67}$$

which is the very definition of a Barron–Jensen/Frankowska viscosity solution for lower semicontinuous functions.
Furthermore, when $\mathbf{k}(d,x) < V(d,x)$, the regulation map and the subdifferential of the viability solution with respect to x are related by

$$\begin{cases} R_V(d,x) \ \subset \ \bigcap_{(p_d,p_x)\in\partial V(d,x)} \partial_p \mathbf{l}^\star(d,x;p_x) \\ \text{or, equivalently,} \\ \bigcup_{(p_d,p_x)\in\partial V(d,x)} \{p_x\} \ \subset \ \bigcap_{u\in R_V(d,x)} \partial_u \mathbf{l}(d,x;u) \end{cases} \tag{13.68}$$

Remark. The theorem holds true by dropping assumption (13.66), p. 560, but the conclusion translating the property that the epigraph of the viability solution is backward invariant relatively to the epigraph of $\mathbf{k}$ when $V(d,x) = \mathbf{k}(d,x)$ is quite technical and ugly:

$$\begin{cases} \text{if } \mathbf{k}(d,x) \ = \ V(d,x) \ \leq \ \mathbf{c}(d,x), \text{ then } \forall \ (p_d,p_x) \in \partial V(d,x), \\ \inf_{(q_d,q_x)\in\partial\mathbf{k}(d,x)} (\langle p_d - q_d, \varphi(d)\rangle + \mathbf{l}^\star(d,x;p_x - q_x)) \ \leq \ 0 \end{cases} \tag{13.69}$$

□