Chapter 10
Viability and Capturability Properties of Evolutionary Systems

10.1 Introduction

This chapter presents properties proved at the level of evolutionary systems, whereas Chap. 11, p. 437 focuses on specific results on evolutionary systems generated by control systems based on the Viability Theorem 11.3.4, p. 455 and Invariance Theorem 11.3.7, p. 457 involving tangential conditions. Specific results of the same nature are presented in Sect. 12.3, p. 503 for *impulse* systems, in Chap. 11 of the first edition of [18, Aubin] for evolutionary systems generated by *history dependent* (or path dependent) systems and in [23, Aubin] for *mutational* and *morphological* systems, which will inherit properties uncovered in this chapter.

This chapter is mainly devoted to the first and second fundamental viability characterizations of kernels and basins. The first one, in Sect. 10.2, p. 377, characterizes them as bilateral fixed points. The second one, in Sect. 10.5, p. 399, translates these fixed point theorems in terms of viability properties which will be exploited in Chap. 11, p. 437. The first one states that *the viability kernel is the largest subset viable outside the target* and the second one that *it is the smallest isolated subset*, and thus, the *unique* one satisfying both. This uniqueness theorem plays an important role, in particular for deriving the uniqueness property of viability episolutions of Hamilton–Jacobi–Bellman partial differential equations in Chap. 17, p. 681.

For that purpose, we uncover the topological properties of evolutionary systems in Sect. 10.3, p. 382 for the purpose of proving that under these topological properties, kernels and basins are closed. We need to define in Sect. 10.3.1, p. 382 two concepts of semicontinuity of evolutionary systems: "upper semicompact" evolutionary systems, under which viability properties hold true, and "lower semicontinuous" evolutionary systems, under which invariance (or tychastic) properties are satisfied. These assumptions are realistic, because they are respectively satisfied for systems generated by the

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"Marchaud control systems" for the class of "upper semicompact" evolutionary systems and "Lipschitz" ones for "lower semicontinuous" evolutionary systems.

This allows us to prove that exit and minimal time functions are semicontinuous in Sect. 10.4, p. 392 and that the optimization problems defining them are achieved by "persistent and minimal time evolutions." Some of these properties are needed for proving the viability characterization in Sect. 10.5, p. 399, the most useful one of this chapter. It characterizes:

1. subsets viable outside a target by showing that the complement of the target in the environment is *locally viable*, a viability concept which can be characterized further for specific evolutionary systems, or, equivalently, that its exit set is contained in the target,
2. isolated subsets as subsets *backward invariant* relatively to the environment, again a viability concept which can be exploited further. Therefore, viability kernels being the unique isolated subset viable outside a target, they are the unique ones satisfying such local viability and backward invariance properties.

Section 10.6, p. 411 presents such characterizations for invariance kernels and connection basins.

We pursue by studying in Sect. 10.7, p. 416 under which conditions the capture basin of a (Painlevé–Kuratowski) limit of targets is the limit of the capture basins of those targets. This is quite an important property which is studied in Sect. 10.7.1, p. 416.

The concepts of viability and invariance kernels of environments are defined as the *largest subsets* of the environments satisfying either one of these properties. The question arises whether it is possible to define the concepts of viability and invariance envelopes of given subsets, which are the *minimal subsets* containing an environment which are viable and invariant respectively. This issue is dealt with in Sect. 10.7.2, p. 420. In the case of invariance kernels, this envelope is unique: it is the intersection of invariant subsets containing it. In the case of viability envelopes, we obtain, under adequate assumptions, the existence of nonempty viability envelopes. Equilibria are viable singletons which are necessarily minimal. They do not necessarily exist, except in the case of compact convex environments. For plain compact environments, minimal viability envelopes are not empty, and they enjoy a singular property, a weaker property than equilibria, which are the asymptotic limits of evolutions. *Minimal viability envelopes are the subsets coinciding with the limit sets of their elements*, i.e., are made of limit sets of evolutions instead of limit points (equilibria) which may not exist.

We briefly uncover without proofs the links between invariance of an environment under a tychastic system and stochastic viability in Sect. 10.10, p. 433. They share the same underlying philosophy: the viability property is satisfied *by all* evolutions of a tychastic system (tychastic viability), *by almost all* evolutions under a stochastic system. This is made much more precise by using the Strook–Varadhan Theorem, implying, so to speak, that stochastic viability is a very particular case in comparison to tychastic viability (or invariance). Hence the results dealing with invariant subsets can bring another point of view on the mathematical translation of this type uncertainty: *either by stochastic systems, or by tychastic systems*.

Exit sets also play a crucial role for regulating viable evolutions with a finite number of non viable feedbacks (instead of a viable feedback), but which are, in some sense made precise, "collectively viable": this is developed in Sect. 10.8, p. 422 for regulating viable punctuated evolutions satisfying the "hard" version of the inertia principle.

Section 10.9, p. 427 is devoted to *inverse problems* of the following type: assuming that both the dynamics $F(\lambda, (\cdot))$, the environment $K(\lambda)$ and the target $C(\lambda)$ depend upon a parameter λ, and given any state x, what is the set of parameters λ for which x lies in the viability kernel $\mathrm{Viab}_{F(\lambda,(\cdot))}(K(\lambda), C(\lambda))$?

This is a prototype of a *parameter identification problem*. It amounts to inverting the viability kernel map $\lambda \rightsquigarrow \mathrm{Viab}_{F(\lambda,(\cdot))}(K(\lambda), C(\lambda))$. For that purpose, we need to know the graph of this map, since the set-valued map and its inverse share the same "graphical properties." It turns out that the graph of the viability kernel map is itself the viability kernel of an auxiliary map, implying that both the viability kernel map and its inverse inherits the properties of viability kernels. When the parameters $\lambda \in \mathbb{R}$ are scalar, under some monotonicity condition, the inverse of this viability kernel map is strongly related to an extended function associating with any state x the best parameter λ, as we saw in many examples of Chaps. 4, p. 125 and 6, p. 199.

10.2 Bilateral Fixed Point Characterization of Kernels and Basins

We begin our investigation of viability kernels and capture basins by emphasizing simple algebraic properties of utmost importance, due to the collaboration with Francine Catté, which will be implicitly used all along the book. We begin by reviewing simple algebraic properties of viability invariance kernels as maps depending on the evolutionary system, environment and the target.

Lemma 10.2.1 ***[Monotonicity Properties of Viability and Invariance Kernels]*** *Let us consider the maps* $(\mathcal{S}, K, C) \mapsto \mathrm{Viab}_{\mathcal{S}}(K, C)$ *and* $(\mathcal{S}, K, C) \mapsto \mathrm{Inv}_{\mathcal{S}}(K, C)$*. Assume that* $\mathcal{S}_1 \subset \mathcal{S}_2$, $K_1 \subset K_2$, $C_1 \subset C_2$*. Then*

$$\begin{cases} (i) \ \ \mathrm{Viab}_{\mathcal{S}_1}(K_1, C_1) \subset \mathrm{Viab}_{\mathcal{S}_2}(K_2, C_2) \\ (ii) \ \mathrm{Inv}_{\mathcal{S}_2}(K_1, C_1) \subset \mathrm{Inv}_{\mathcal{S}_1}(K_2, C_2). \end{cases} \tag{10.1}$$

The consequences of these simple observations are important:

Lemma 10.2.2 ***[Union of Targets and Intersection of Environments]***

$$\begin{cases} (i) \ \ \mathrm{Viab}_{\mathcal{S}}\left(K, \bigcup_{i\in I} C_i\right) = \bigcup_{i\in I} \mathrm{Viab}_{\mathcal{S}}(K, C_i) \\ (ii) \ \mathrm{Inv}_{\mathcal{S}}\left(\bigcap_{i\in I} K_i, C\right) = \bigcap_{i\in I} \mathrm{Inv}_{\mathcal{S}}(K_i, C) \end{cases} \tag{10.2}$$

Evolutionary systems $\mathcal{R} \subset \mathcal{S}$ satisfying equality $\mathrm{Inv}_{\mathcal{R}}(K, C) = \mathrm{Viab}_{\mathcal{S}}(K, C)$ enjoy the following monotonicity property:

Lemma 10.2.3 ***[Comparison between Invariance Kernels under Smaller Evolutionary Systems and Viability Kernels of a Larger System]*** *Let us assume that there exists an evolutionary system* $\mathcal{R}$ *contained in* $\mathcal{S}$ *such that* $\mathrm{Inv}_{\mathcal{R}}(K, C) = \mathrm{Viab}_{\mathcal{S}}(K, C)$. *Then, for all evolutionary systems* $\mathcal{Q} \subset \mathcal{R}$, $\mathrm{Inv}_{\mathcal{Q}}(K, C) = \mathrm{Viab}_{\mathcal{S}}(K, C)$.

Proof. Indeed, by the monotonicity property with respect to the evolutionary system, we infer that

$$\mathrm{Inv}_{\mathcal{R}}(K, C) \subset \mathrm{Inv}_{\mathcal{Q}}(K, C) \subset \mathrm{Viab}_{\mathcal{Q}}(K, C) \subset \mathrm{Viab}_{\mathcal{S}}(K, C) = \mathrm{Inv}_{\mathcal{R}}(K, C)$$

Hence equality $\mathrm{Inv}_{\mathcal{Q}}(K, C) = \mathrm{Viab}_{\mathcal{S}}(K, C)$ ensues. □

Next, we need the following properties.

Lemma 10.2.4 ***[Fundamental Properties of Viable and Capturing Evolutions]*** *Let* $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ *be an evolutionary system,* $K \subset X$ *be an environment and* $C \subset K$ *be a target.*

1. *Every evolution* $x(\cdot) \in \mathcal{S}(x)$ *viable in* K*, forever or until it reaches* C *in finite time, is actually viable in the viability kernel* $\mathrm{Viab}_{\mathcal{S}}(K, C)$,
2. *Every evolution* $x(\cdot) \in \mathcal{S}(x)$ *viable in* K *forever or which captures the viability kernel* $\mathrm{Viab}_{\mathcal{S}}(K, C)$ *in finite time, remains viable in* $\mathrm{Viab}_{\mathcal{S}}(K, C)$ *until it captures also the target* C *in finite time.*

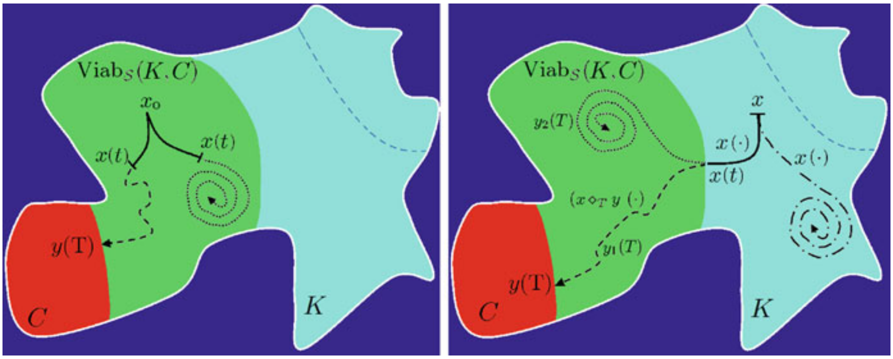


Fig. 10.1 Illustration of the proof of Lemma 10.2.4, p. 378.
Left*: an evolution $x(\cdot)$ viable in K is viable in Viab$_{\mathcal{S}}(K,C)$ forever (*spiral*) or reaches C at time T.* Right*: an evolution $x(\cdot)$ viable in K forever or which captures Viab$_{\mathcal{S}}(K,C)$ in finite time remains viable in Viab$_{\mathcal{S}}(K,C)$ until it captures the target at T (*dotted trajectory*).*

Proof. The first statement follows from the translation property. Let us consider an evolution $x(\cdot) \in \mathcal{S}(x)$ viable in K, forever or until it reaches C in finite time T. Therefore, for all $t \in [0, T[$, the translation $y(\cdot) := \kappa(-t)x(\cdot)$ of $x(\cdot)$ defined by $y(\tau) := x(t+\tau)$ is an evolution $y(\cdot) \in \mathcal{S}(x(t))$ starting at $x(t)$ and viable in K until it reaches C at time $T-t$. Hence $x(t)$ does belong to $\text{Viab}_{\mathcal{S}}(K,C)$ for every $t \in [0, T[$.

The second statement follows from the concatenation property because it can be concatenated with an evolution either remaining in $\text{Viab}_{\mathcal{S}}(K,C) \subset K$ or reaching the target C in finite time. □

10.2.1 Bilateral Fixed Point Characterization of Viability Kernels

We shall start our presentation of kernels and basins properties by a simple and important algebraic property:

Theorem 10.2.5 ***[The Fundamental Characterization of Viability Kernels]*** *Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,+\infty; X)$ be an evolutionary system, $K \subset X$ be an environment and $C \subset K$ be a target. The viability kernel* $\text{Viab}_{\mathcal{S}}(K,C)$ *of K with target C (see Definition 2.10.2, p. 86) is the* ***unique*** *subset between C and K that is both:*

1. ***viable outside*** C *(and is the largest subset* $D \subset K$ *viable outside* C*),*
2. ***isolated in*** K *(and is the smallest subset* $D \supset C$ *isolated in* K*):*

$$\text{Viab}_{\mathcal{S}}(K, \text{Viab}_{\mathcal{S}}(K, C)) = \text{Viab}_{\mathcal{S}}(K, C) \; = \; \text{Viab}_{\mathcal{S}}(\text{Viab}_{\mathcal{S}}(K, C), C) \tag{10.3}$$

The viability kernel satisfies the properties of both the subsets viable outside a target and of isolated subsets in a environment, and is the unique one to do so.

This statement is at the root of uniqueness properties of solutions to some Hamilton–Jacobi–Bellman partial differential equations whenever the epigraph of a solution is a viability kernel of the epigraph of a function outside the epigraph of another function.

Proof. We begin by proving the two following statements:

1. The translation property implies that the viability kernel $\text{Viab}_{\mathcal{S}}(K, C)$ is viable outside C:

$$\text{Viab}_{\mathcal{S}}(K, C) \; \subset \; \text{Viab}_{\mathcal{S}}(K, \text{Viab}_{\mathcal{S}}(K, C)) \; \subset \; \text{Viab}_{\mathcal{S}}(K, C)$$

   Take $x_0 \in \text{Viab}_{\mathcal{S}}(K, C)$ and prove that there exists an evolution $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 viable in $\text{Viab}_{\mathcal{S}}(K, C)$ until it possibly reaches C. Indeed, there exists an evolution $x(\cdot) \in \mathcal{S}(x_0)$ viable in K until some time $T \geq 0$ either finite when it reaches C or infinite. Then the first statement of Lemma 10.2.4, p. 378 implies that x_0 belongs to the viability kernel $\text{Viab}_{\mathcal{S}}(\text{Viab}_{\mathcal{S}}(K, C), C)$ of the viability kernel $\text{Viab}_{\mathcal{S}}(K, C)$ of K with target C.
2. The concatenation property implies that the viability kernel $\text{Viab}_{\mathcal{S}}(K, C)$ is isolated in K:

$$\text{Viab}_{\mathcal{S}}(K, \text{Viab}_{\mathcal{S}}(K, C)) \; \subset \; \text{Viab}_{\mathcal{S}}(K, C)$$

   Let x_0 belong to $\text{Viab}_{\mathcal{S}}(K, \text{Viab}_{\mathcal{S}}(K, C))$. There exists at least one evolution $x(\cdot) \in \mathcal{S}(x_0)$ that would either remain in K or reach the viability kernel $\text{Viab}_{\mathcal{S}}(K, C)$ in finite time. Lemma 10.2.4, p. 378 implies that $x_0 \in \text{Viab}_{\mathcal{S}}(K, C)$. □

We now observe that the map $(K, C) \mapsto \text{Viab}_{\mathcal{S}}(K, C)$ satisfies

$$\begin{cases} (i) \;\; C \; \subset \; \text{Viab}_{\mathcal{S}}(K, C) \; \subset \; K \\ (ii) \; (K, C) \mapsto \; \text{Viab}_{\mathcal{S}}(K, C) \text{ is increasing} \end{cases} \tag{10.4}$$

in the sense that if $K_1 \subset K_2$ and $C_1 \subset C_2$, then $\text{Viab}_{\mathcal{S}}(K_1, C_1) \subset \text{Viab}_{\mathcal{S}}(K_2, C_2)$.

Setting $\mathcal{A}(K,C) := \text{Viab}_{\mathcal{S}}(K,C)$, the statements below follow from general algebraic Lemma 10.2.6 below.

Lemma 10.2.6 ***[Uniqueness of Bilateral Fixed Points]*** *Let us consider a map $\mathcal{A} : (K,C) \mapsto \mathcal{A}(K,C)$ satisfying*

$$\begin{cases} (i)\ \ C \ \subset \ \mathcal{A}(K,C) \ \subset K \\ (ii)\ (K,C) \mapsto \mathcal{A}(K,C) \text{ is increasing} \end{cases} \tag{10.5}$$

1. *If $\mathcal{A}(K,C) = \mathcal{A}(\mathcal{A}(K,C),C)$, it is the largest fixed point of the map $D \mapsto \mathcal{A}(D,C)$ between C and K,*
2. *If $\mathcal{A}(K,C) = \mathcal{A}(K,\mathcal{A}(K,C))$, it is the smallest fixed point of the map $E \mapsto \mathcal{A}(K,E)$ between C and K.*

Then, any subset D between C and K satisfying

$$D = \mathcal{A}(D,C) \text{ and } \mathcal{A}(K,D) = D$$

is the unique bilateral fixed point *D between C and K of the map $\mathcal{A}$ in the sense that:*

$$\mathcal{A}(K,D) = D = \mathcal{A}(D,C)$$

and is equal to $\mathcal{A}(K,C)$.

Proof. If $D = \mathcal{A}(D,C)$ is a fixed point of $D \mapsto \mathcal{A}(D,C)$, we then deduce that $D = \mathcal{A}(D,C) \subset \mathcal{A}(K,C)$, so that *whenever $\mathcal{A}(K,C) = \mathcal{A}(\mathcal{A}(K,C),C)$, we deduce that $\mathcal{A}(K,C)$ is the largest fixed point of $D \mapsto \mathcal{A}(D,C)$* contained in K. In the same way, *if $\mathcal{A}(K,\mathcal{A}(K,C)) = \mathcal{A}(K,C)$, then $\mathcal{A}(K,C)$ is the smallest fixed points of $E \mapsto \mathcal{A}(K,E)$* containing C. Furthermore, equalities

$$\mathcal{A}(K,D) \ = \ D \ = \ \mathcal{A}(D,C)$$

imply that $D = \mathcal{A}(K,C)$ because the monotonicity property implies that

$$\mathcal{A}(K,C) \ \subset \ \mathcal{A}(K,D) \ \subset \ D \ \subset \ \mathcal{A}(D,C) \ \subset \ \mathcal{A}(K,C) \ \square$$

10.2.2 Bilateral Fixed Point Characterization of Invariance Kernels

This existence and uniqueness of a "bilateral fixed point" is shared by the invariance kernel with target, the capture basin and the absorption basin of a target that satisfy property (10.5), and thus, the conclusions of Lemma 10.2.6:

Theorem 10.2.7 ***[Characterization of Kernels and Basins as Unique Bilateral Fixed Point]*** *Let* $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,+\infty;X)$ *be an evolutionary system,* $K \subset X$ *be a environment and* $C \subset K$ *be a target.*

1. *The viability kernel* $\mathrm{Viab}_{\mathcal{S}}(K,C)$ *of a subset* K *with target* $C \subset K$ *is the* unique *bilateral fixed point* D *between* C *and* K *of the map* $(K,C) \mapsto \mathrm{Viab}_{\mathcal{S}}(K,C)$ *in the sense that*

$$D = \mathrm{Viab}_{\mathcal{S}}(K,D) = \mathrm{Viab}_{\mathcal{S}}(D,C)$$

2. *The invariance kernel* $\mathrm{Inv}_{\mathcal{S}}(K,C)$ *of a subset* K *with target* $C \subset K$ *is the* unique *bilateral fixed point* D *between* C *and* K *of the map* $(K,C) \mapsto \mathrm{Inv}_{\mathcal{S}}(K,C)$ *in the sense that*

$$D = \mathrm{Inv}_{\mathcal{S}}(K,D) = \mathrm{Inv}_{\mathcal{S}}(D,C)$$

The same properties are shared by the maps $(K,C) \mapsto \mathrm{Capt}_{\mathcal{S}}(K,C)$ *and* $(K,C) \mapsto \mathrm{Abs}_{\mathcal{S}}(K,C)$*.*

10.3 Topological Properties

We begin this section by introducing adequate semicontinuity concepts for evolutionary systems in Sect. 10.3.1, p. 382 for uncovering topological properties of kernels and basins in Sect. 10.3.2, p. 387.

10.3.1 Continuity Properties of Evolutionary Systems

In order to go further in the characterization of viability and invariance kernels with targets in terms of properties easier to check, we need to bring in the forefront some continuity requirements on the evolutionary system $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,+\infty;X)$. First, both the state space X and the evolutionary $\mathcal{C}(0,+\infty;X)$ have to be complete topological spaces.

32 ***[The Evolutionary Space]*** *Assume that the state space X is a complete metric space. We supply the space $\mathcal{C}(0,+\infty;X)$ of continuous evolutions with the "compact topology": A sequence of continuous evolutions $x_n(\cdot) \in \mathcal{C}(0,+\infty;X)$ converges to the continuous evolution $x(\cdot)$ as $n \to +\infty$ if for every $T > 0$, the sequence $\sup_{t\in[0,T]} d(x_n(t),x(t))$ converges to 0. It is a complete metrizable space. The Ascoli Theorem states that a subset $\mathcal{H}$ is compact if and only if it is closed, equicontinuous and for any $t \in \mathbb{R}_+$, the subset $\mathcal{H}(t) := \{x(t)\}_{x(\cdot)\in\mathcal{H}}$ is compact in X.*

Stability, a polysemous word, means formally that the solution of a problem depends "continuously" upon its data. Here, for evolutionary systems, the data are principally the initial states: In this case, stability means that the set of solutions depends "continuously" on the initial state. We recall that a deterministic system $\mathcal{S} := \{\mathbf{s}\} : X \mapsto \mathcal{C}(0,+\infty;X)$ is continuous at some $x \in X$ if it maps any sequence $x_n \in X$ converging to x to a sequence $\mathbf{s}(x_n)$ converging to $\mathbf{s}(x)$.

However, when the evolutionary system $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,+\infty;X)$ is no longer single-valued, there are several ways of describing the convergence of the set $\mathcal{S}(x_n)$ to the set $\mathcal{S}(x)$. We shall use in this book only two of them, that we present in the context of evolutionary systems (see Definition 18.4.3, p. 729 and other comments in the Appendix 18, p. 713). We begin with the notion of upper semicompactness:

Definition 10.3.1 ***[Upper Semicompactness]*** *Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,+\infty;X)$ be an evolutionary system, where both the state space X and the evolutionary space $\mathcal{C}(0,+\infty;X)$ are topological spaces. The evolutionary system is said to be* upper semicompact *at x if for every sequence $x_n \in X$ converging to x and for every sequence $x_n(\cdot) \in \mathcal{S}(x_n)$, there exists a subsequence $x_{n_p}(\cdot)$ converging to some $x(\cdot) \in \mathcal{S}(x)$. It is said to be* upper semicompact *if it is upper semicompact at every point $x \in X$ where $\mathcal{S}(x)$ is not empty.*

Before using this property, we need to provide examples of evolutionary system exhibiting it: this is the case for Marchaud differential inclusions:

Definition 10.3.2 ***[Marchaud Set-Valued Maps]*** *We say that F is a* Marchaud map *if*

$$\begin{cases} (i) & \text{the graph and the domain of } F \text{ are nonempty and closed} \\ (ii) & \text{the values } F(x) \text{ of } F \text{ are convex} \\ (iii) & \exists\, c > 0 \text{ such that } \forall x \in X,\ \|F(x)\| := \sup_{v \in F(x)} \|v\| \le c(\|x\| + 1) \end{cases} \tag{10.6}$$

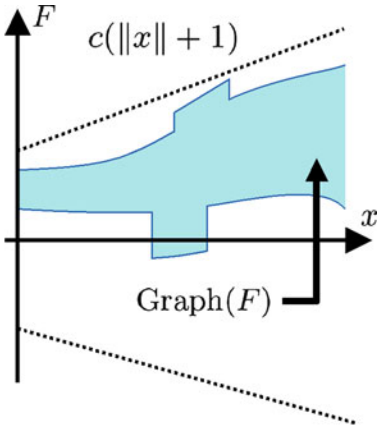
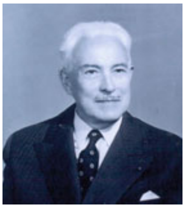


Fig. 10.2 Marchaud map.
Illustration of a Marchaud map, with convex images, closed graph and linear growth.

André Marchaud was with *Stanislas Zaremba* among the firsts to study what did become known 50 years later differential inclusions:



André Marchaud [1887–1973]. *After entering École Normale Supérieure in 1909, he fought First World War, worked in ministry of armement and industrial reconstruction and became professor and Dean at Faculté des Sciences de Marseille from 1927 to 1938 before being Recteur of several French Universities. He was a student of Paul Montel, was in close relations with Georges Bouligand and Stanislas Zaremba, and was a mentor of André Lichnerowicz [1915–1998]. His papers dealt with analysis and differentiability.*

The (difficult) Stability Theorem states that the set of solutions depends continuously upon the initial states in the upper semicompact sense:

Theorem 10.3.3 *[**Upper Semicompactness of Marchaud Evolutionary Systems**] If $F : X \rightsquigarrow X$ is Marchaud, the solution map $\mathcal{S}$ is an upper semicompact evolutionary system.*

Proof. This a consequence of the Convergence Theorem 19.2.3, p. 771. □

The other way to take into account the idea of continuity in the case of evolutionary systems is by introducing the following concept:

Definition 10.3.4 *[**Lower Semicontinuity of Evolutionary Systems**] Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,+\infty;X)$ be an evolutionary system, where both the state space X and the evolutionary space $\mathcal{C}(0,+\infty;X)$ are topological spaces. The evolutionary system is said to be* lower semicontinuous *at x if for every sequence $x_n \in X$ converging to x and for every sequence $x(\cdot) \in \mathcal{S}(x)$ (thus assumed to be nonempty), there exists a sequence $x_n(\cdot) \in \mathcal{S}(x_n)$ converging to $x(\cdot) \in \mathcal{S}(x)$. It is said to be* lower semicontinuous *if it is lower semicontinuous at every point $x \in X$ where $\mathcal{S}(x)$ is not empty.*

Warning: An evolutionary system can be upper semicompact at x without being lower semicontinuous and lower semicontinuous at x without being upper semicompact. If the evolutionary system is deterministic, lower semicontinuity coincides with continuity and upper semicompactness coincides with "properness" of single-valued maps (in the sense of Bourbaki). Note also the unfortunate confusions between the semicontinuity of numerical and extended functions (Definition 18.6.3, p. 744) and the semicontinuity of set-valued maps (Definition 18.4.3, p. 729).

Recall that a single-valued map $f : X \mapsto Y$ is said to be λ-*Lipschitz* if for any $x_1,\ x_2 \in X$, $d(f(x_1), f(x_2)) \leq \lambda d(x_1, x_2)$. In the case of normed vector spaces, denoting by B the unit ball of the vector space, this inequality can be translated in the form $f(x_1) \in f(x_2) + \lambda\|x_1 - x_2\|B$. This is this formulation which is the easiest to adapt to set-valued maps in the case of (finite) dimensional vector spaces:

Definition 10.3.5 *[**Lipschitz Maps**] A set-valued map $F : X \rightsquigarrow Y$ is said to be λ-*Lipschitz *(or Lipschitz for the constant $\lambda > 0$) if*

$$\forall x_1,\ x_2,\ F(x_1) \subset F(x_2) + \lambda\|x_1 - x_2\|B$$

The Lipschitz norm $\|F\|_\Lambda$ *of a map* $F : x \rightsquigarrow Y$ *is the smallest Lipschitz constants of* F*. The evolutionary system* $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ *associated with a Lipschitz set-valued map is called a* Lipschitz evolutionary system*.*

The Filippov Theorem 11.3.9, p. 459 implies that Lipschitz systems are lower semicontinuous:

Theorem 10.3.6 ***[Lower Semicontinuity of Lipschitz Evolutionary Systems]*** *If* $F : X \rightsquigarrow X$ *is Lipschitz, the associated evolutionary system* $\mathcal{S}$ *is lower semicontinuous.*

Under appropriate topological assumptions, we can prove that inverse images and cores of closed subsets of evolutions are closed.

Definition 10.3.7 ***[Closedness of Inverse Images]*** *Let* $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ *be an upper semicompact evolutionary system. Then for any subset* $\mathcal{H} \subset \mathcal{C}(0, +\infty; X)$*,*

$$\overline{\mathcal{S}^{-1}(\mathcal{H})} \subset \mathcal{S}^{-1}(\overline{\mathcal{H}})$$

Consequently, the inverse images $\mathcal{S}^{-1}(\mathcal{H})$ *under* $\mathcal{S}$ *of any closed subset* $\mathcal{H} \subset \mathcal{C}(0, +\infty; X)$ *are closed.*

Furthermore, the evolutionary system $\mathcal{S}$ *maps compact sets* $K \subset X$ *to compact sets* $\mathcal{H} \subset \mathcal{C}(0, +\infty; X)$*.*

Proof. Let us consider a subset $\mathcal{H} \subset \mathcal{C}(0, +\infty; X)$, a sequence of elements $x_n \in \mathcal{S}^{-1}(\mathcal{H})$ converging to some x and prove that x belongs to $\mathcal{S}^{-1}(\mathcal{H})$. Hence there exist elements $x_n(\cdot) \in \mathcal{S}(x_n) \cap \mathcal{H}$. Since $\mathcal{S}$ is upper semicompact, there exists a subsequence $x_{n_p}(\cdot) \in \mathcal{S}(x_{n_p})$ converging to some $x(\cdot) \in \mathcal{S}(x)$. It belongs also to the closure of $\mathcal{H}$, so that $x \in \mathcal{S}^{-1}(\overline{\mathcal{H}})$.

Take now any compact subset $K \subset X$. For proving that $\mathcal{S}(K)$ is compact, take any sequence $x_n(\cdot) \in \mathcal{S}(x_n)$ where $x_n \in K$. Since K is compact, a subsequence $x_{n'}$ converges to some $x \in K$ and since $\mathcal{S}$ is upper semicompact, a subsequence $x_{n''}(\cdot) \in \mathcal{S}(x_{n''})$ converges to some $x(\cdot) \in \mathcal{S}(x)$. □

Example: Let us consider an upper semicompact evolutionary system $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$.
If $K \subset X$ is a closed subset, then the set of equilibria of the evolutionary system that belong to K is closed: Indeed, it is the inverse images $\mathcal{S}^{-1}(\mathcal{K})$ of the set $\mathcal{K}$ of stationary evolutions in K, which is closed whenever K is closed.
In the same way, the set of points through which passes at least one T-periodic evolution of an upper semicompact evolutionary system is closed, since it is the inverse images $\mathcal{S}^{-1}(\mathcal{P}_T(X))$ of the set $\mathcal{P}_T(X)$ of T-periodic evolutions, which is closed.
If a function $\mathbf{v} : X \mapsto \mathbb{R}$ is continuous, the set of initial states from which starts at least one evolution of the evolutionary system monotone along the function $\mathbf{v}$ is closed, since it is the inverse images of the set $\mathcal{M}_{\mathbf{v}}$ of monotone evolutions, which is closed.

For cores, we obtain

Theorem 10.3.8 *[**Closedness of Cores**] Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ be a lower semicontinous evolutionary system. Then for any subset $\mathcal{H} \subset \mathcal{C}(0, +\infty; X)$,*

$$\overline{\mathcal{S}^{\ominus 1}(\mathcal{H})} \subset \mathcal{S}^{\ominus 1}(\overline{\mathcal{H}})$$

Consequently, the core $\mathcal{S}^{\ominus 1}(\mathcal{H})$ under $\mathcal{S}$ of any closed subset $\mathcal{H} \subset \mathcal{C}(0, +\infty; X)$ is closed.

Proof. Let us consider a closed subset $\mathcal{H} \subset \mathcal{C}(0, +\infty; X)$, a sequence of elements $x_n \in \mathcal{S}^{\ominus 1}(\mathcal{H})$ converging to some x and prove that x belongs to $\mathcal{S}^{\ominus 1}(\mathcal{H})$. We have to prove that any $x(\cdot) \in \mathcal{S}(x)$ belongs to $\mathcal{H}$. But since $\mathcal{S}$ is lower semicontinuous, there exists a sequence of elements $x_n(\cdot) \in \mathcal{S}(x_n) \subset \mathcal{H}$ converging to $x(\cdot) \in \overline{\mathcal{H}}$. Therefore $\mathcal{S}(x) \subset \overline{\mathcal{H}}$, i.e., $x \in \mathcal{S}^{\ominus 1}(\overline{\mathcal{H}})$. □

10.3.2 Topological Properties of Viability Kernels and Capture Basins

Recall that the set $\mathcal{V}(K, C)$ of evolutions *viable in K outside C* is defined by (2.5), p. 49:

$$\begin{cases} \mathcal{V}(K, C) := \{x(\cdot) \text{ such that } \forall t \geq 0, \ \ x(t) \in K \\ \text{or } \exists\, T \geq 0 \text{ such that } x(T) \in C \ \& \ \forall t \in [0, T], \ x(t) \ \in \ K\} \end{cases}$$

Lemma 10.3.9 ***[Closedness of the Subset of Viable Evolutions]*** *Let us consider a environment $K \subset X$ and a (possibly empty) target $C \subset K$. Then*

$$\overline{\mathcal{V}(K,C)} \subset \mathcal{V}(\overline{K},\overline{C})$$

and consequently, if C and K are closed, the set $\mathcal{V}(K,C)$ of evolutions that are viable in K forever or until they reach the target C in finite time is closed.

Proof. Let us consider a sequence of evolutions $x_n(\cdot) \in \mathcal{V}(K,C)$ converging to some evolution $x(\cdot)$. We have to prove that $x(\cdot)$ belongs to $\mathcal{V}(\overline{K},\overline{C})$, i.e., that it is viable in $\overline{K}$ forever or until it reaches the target $\overline{C}$ in finite time. Indeed:

1. either for any $T > 0$ and any $N > 0$, there exist $n \geq N$, $t_n \geq T$ and an evolution $x_n(\cdot)$ for which $x_n(t) \in K$ for every $t \in [0, t_n]$,
2. or there exist $T > 0$ and $N > 0$ such that for any $t \geq T$ and $n \geq N$ and any evolution $x_n(\cdot)$, there exists $t_n \leq t$ such that $x_n(t_n) \notin K$.

In the first case, we deduce that for any $T > 0$, $x(T) \in \overline{K}$, so that the limit $x(\cdot)$ is viable in $\overline{K}$ forever.

In the second case, all the solutions $x_n(\cdot)$ leave K before T. This is impossible if evolutions $x_n(\cdot)$ are viable in K forever. Therefore, since $x_n(\cdot) \in \mathcal{V}(K,C)$, they have to reach C before leaving K: There exist $s_n \leq T$ such that

$$x_n(s_n) \in C \ \& \ \forall t \in [0, s_n], \ x_n(t) \in K$$

Then some subsequence $s_{n'}$ converges to some $S \in [0,T]$. Therefore, for any $s < S$, then $s < s_{n'}$ for n' large enough, so that $x_{n'}(s) \in K$. By taking the limit, we infer that for every $s < S$, $x(s) \in \overline{K}$. Furthermore, since $x_n(\cdot)$ converges to $x(\cdot)$ uniformly on the compact interval $[0,T]$, then $x_n(s_n)$ converges to $x(S)$, that belongs to $\overline{C}$.

This shows that the limit $x(\cdot)$ belongs to $\mathcal{V}(\overline{K},\overline{C})$. □

Consequently, the viability kernel of a closed subset with a closed target under an upper semicompact evolutionary subset is closed:

Theorem 10.3.10 ***[Closedness of the Viability Kernel]*** *Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,+\infty;X)$ be an upper semicompact evolutionary system. Then for any environment $K \subset X$ and any target $C \subset K$,*

$$\overline{\mathrm{Viab}_{\mathcal{S}}(K,C)} \subset \mathrm{Viab}_{\mathcal{S}}(\overline{K},\overline{C})$$

Consequently, if $C \subset K$ *and* K *are closed, so is the viability kernel* $\mathrm{Viab}_{\mathcal{S}}(K,C)$ *of* K *with target* C*. Furthermore, if* $K\backslash C$ *is a repeller, the capture basin* $\mathrm{Capt}_{\mathcal{S}}(K,C)$ *of* C *viable in* K *under* $\mathcal{S}$ *is closed.*

Proof. Since the viability kernel $\mathrm{Viab}_{\mathcal{S}}(K,C) := \mathcal{S}^{-1}(\mathcal{V}(K,C))$ is the inverse image of the subset $\mathcal{V}(K,C)$ by Definition 2.10.2, the closedness of the viability kernel follows from Theorem 10.3.7 and Lemma 10.3.9. □

Theorem 10.3.8 implies the closedness of the invariance kernels:

Theorem 10.3.11 *[**Closedness of Invariance Kernels**] Let* $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,+\infty;X)$ *be a lower semicontinuous evolutionary system. Then for any environment* $K \subset X$ *and any target* $C \subset K$*,*

$$\overline{\mathrm{Inv}_{\mathcal{S}}(K,C)} \subset \mathrm{Inv}_{\mathcal{S}}(\overline{K},\overline{C})$$

Consequently, if $C \subset K$ *and* K *are closed, so is the invariance kernel* $\mathrm{Inv}_{\mathcal{S}}(K,C)$ *of* K *with target* C*.*

Therefore, if $K\backslash C$ *is a repeller, the absorption basin* $\mathrm{Abs}_{\mathcal{S}}(K,C)$ *of* C *invariant in* K *under* $\mathcal{S}$ *is closed.*

As for interiors of capture and absorption basins, we obtain the following statements:

Theorem 10.3.12 *[**Interiors of Capture and Absorption Basins**] For any environment* $K \subset X$ *and any target* $C \subset K$*:*

- *if* $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,+\infty;X)$ *is lower semicontinuous, then*

$$\mathrm{Capt}_{\mathcal{S}}(\mathrm{Int}(K),\mathrm{Int}(C)) \subset \mathrm{Int}(\mathrm{Capt}_{\mathcal{S}}(K,C))$$

- *if* $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,+\infty;X)$ *is upper semicontinuous, then*

$$\mathrm{Abs}_{\mathcal{S}}(\mathrm{Int}(K),\mathrm{Int}(C)) \subset \mathrm{Int}(\mathrm{Abs}_{\mathcal{S}}(K,C))$$

Consequently, if $C \subset K$ *and* K *are open, so are the capture basin* $\mathrm{Capt}_{\mathcal{S}}(K,C)$ *and the absorption basin* $\mathrm{Capt}_{\mathcal{S}}(K,C)$ *whenever the evolutionary system is respectively lower semicontinuous and upper semicompact.*

Proof. Observe that, taking the complements, Lemma 2.12.2 implies that if $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ is lower semicontinuous, then Theorem 10.7.8, p. 420 implies that

$$\mathrm{Capt}_{\mathcal{S}}(\mathrm{Int}(K), \mathrm{Int}(C)) \subset \mathrm{Int}(\mathrm{Capt}_{\mathcal{S}}(K, C))$$

since the complement of an invariance kernel is the capture basin of the complements and since the complement of a closure is the interior of the complement, and Theorem 10.3.10, p. 388 imply the similar statement for absorption basins. □

For capture basins, we obtain another closedness property based on backward invariance (see Definition 8.2.4, p. 278):

Proposition 10.3.13 *[**Closedness of Capture Basins**] If the set-valued map $\overleftarrow{\mathcal{S}}$ is lower semicontinuous and if K is backward invariant, then for any closed subset $C \subset K$,*

$$\mathrm{Capt}_{\mathcal{S}}(\overline{K}, \overline{C}) \subset \overline{\mathrm{Capt}_{\mathcal{S}}(K, C)} \tag{10.7}$$

Proof. Let us take $x \in \mathrm{Capt}_{\mathcal{S}}(\overline{K}, \overline{C})$ and an evolution $x(\cdot) \in \mathcal{S}(x)$ viable in $\overline{K}$ until it reaches the target $\overline{C}$ at time $T < +\infty$ at $c := x(T) \in \overline{C}$. Hence the function $t \mapsto y(t) := x(T - t)$ is an evolution $y(\cdot) \in \overleftarrow{\mathcal{S}}(c)$.

Let us consider a sequence of elements $c_n \in C$ converging to c. Since $\overleftarrow{\mathcal{S}}$ is lower semicontinuous, there exist evolutions $y_n(\cdot) \in \overleftarrow{\mathcal{S}}(c_n)$ converging uniformly over compact intervals to $y(\cdot)$. These evolutions $y_n(\cdot)$ are viable in K, since K is assumed to be backward invariant. The evolutions $x_n(\cdot)$ defined by $x_n(t) := y_n(T - t)$ satisfy $x_n(0) = y_n(T) \in K$, $x_n(T) = c_n$ and, for all $t \in [0, T]$, $x_n(t) \in K$. Therefore $x_n(0) := y_n(T)$ belongs to $\mathrm{Capt}_{\mathcal{S}}(K, C)$ and converges to $x := x(0)$, so that $x \in \overline{\mathrm{Capt}_{\mathcal{S}}(K, C)}$. □

As a consequence, we obtain the following topological regularity property (see Definition 18.2.2, p. 714) of capture basins:

Proposition 10.3.14 *[**Topological Regularity of Capture Basins**] If the set-valued map $\mathcal{S}$ is upper semicompact and the set-valued map $\overleftarrow{\mathcal{S}}$ is lower semicontinuous, if $K = \overline{\mathrm{Int}(K)}$ and $C = \overline{\mathrm{Int}(C)}$, if $K \setminus C$ is a repeller and if $\mathrm{Int}(K)$ is backward invariant, then*

$$\mathrm{Capt}_{\mathcal{S}}(K, C) = \overline{\mathrm{Capt}_{\mathcal{S}}(\mathrm{Int}(K), \mathrm{Int}(C))} = \overline{\mathrm{Int}(\mathrm{Capt}_{\mathcal{S}}(K, C))} \tag{10.8}$$

Proof. Since $K = \overline{\mathrm{Int}(K)}$ and $C = \overline{\mathrm{Int}(C)}$, since $\overleftarrow{\mathcal{S}}$ is lower semicontinuous and since $\mathrm{Int}(K)$ is backward invariant, Proposition 10.3.14, p. 390 implies that

$$\mathrm{Capt}_{\mathcal{S}}(K,C) \;=\; \mathrm{Capt}_{\mathcal{S}}(\overline{\mathrm{Int}(K)},\overline{\mathrm{Int}(C)}) \;\subset\; \overline{\mathrm{Capt}_{\mathcal{S}}(\mathrm{Int}(K),\mathrm{Int}(C))}$$

Inclusion

$$\mathrm{Capt}_{\mathcal{S}}(\mathrm{Int}(K),\mathrm{Int}(C)) \;\subset\; \mathrm{Int}(\mathrm{Capt}_{\mathcal{S}}(K,C))$$

follows from Theorem 10.3.12, p. 389. On the other hand, since $\mathcal{S}$ is upper semicompact and $K \setminus C$ is a repeller, Theorem 10.3.10, p. 388 implies that

$$\overline{\mathrm{Capt}_{\mathcal{S}}(\mathrm{Int}(K),\mathrm{Int}(C))} \;\subset\; \overline{\mathrm{Int}(\mathrm{Capt}_{\mathcal{S}}(K,C))} \;\subset\; \mathrm{Capt}_{\mathcal{S}}(K,C)$$

so that $\mathrm{Capt}_{\mathcal{S}}(K,C) = \overline{\mathrm{Int}(\mathrm{Capt}_{\mathcal{S}}(K,C))}$. □

We turn now our attention to connectedness properties of viability kernels:

Lemma 10.3.15 *[**The connectedness Lemma**] Assume that the evolutionary system $\mathcal{S}$ is upper semicompact. Let K be a closed environment and $C_1 \subset K$ and $C_2 \subset K$ be nonempty closed disjoint targets. If the viability kernel $\mathrm{Viab}_{\mathcal{S}}(K, C_1 \cup C_2)$ is connected, then the intersection*

$$\mathrm{Viab}_{\mathcal{S}}(K,C_1) \cap \mathrm{Viab}_{\mathcal{S}}(K,C_2)$$

is closed and not empty. Consequently, if we assume further that $K \setminus C_1$ and $K \setminus C_2$ are repellers, we infer that

$$\mathrm{Capt}_{\mathcal{S}}(K,C_1) \cap \mathrm{Capt}_{\mathcal{S}}(K,C_2) \;\neq\; \emptyset$$

Proof. This follows from the definition of connectedness since $\mathcal{S}$ being upper semicompact, the viability kernels $\mathrm{Viab}_{\mathcal{S}}(K,C_1)$ and $\mathrm{Viab}_{\mathcal{S}}(K,C_2)$ are closed, nonempty (they contain their nonempty targets) and cover the viability kernel with the union of targets :

$$\mathrm{Viab}_{\mathcal{S}}(K,C_1 \cup C_2) \;=\; \mathrm{Viab}_{\mathcal{S}}(K,C_1) \cup \mathrm{Viab}_{\mathcal{S}}(K,C_2)$$

Since this union is assumed connected, the intersection $\mathrm{Viab}_{\mathcal{S}}(K,C_1) \cap \mathrm{Viab}_{\mathcal{S}}(K,C_2)$ must be empty (and is closed). □

Motivating Remark. The intersection

$$\mathrm{Capt}_{\mathcal{S}}(K,C_1) \cap \mathrm{Capt}_{\mathcal{S}}(K,C_2) \;\neq\; \emptyset$$

being the subset of states from which at least one volution reaches one target in finite time and another one reaches the other target also in finite time, could be use as a proto-concept of a *"watershed"*. It is closed when the evolutionary system is upper semicompact and when the environment and the targets are closed (see *Morphologie Mathématique*, [187, Schmitt M. & Mattioli], and *Mathematical Morphology*, [166, Najman]).

10.4 Persistent Evolutions and Exit Sets

This section applies the above topological results to the study of exit and minimal time functions initiated in Sect. 4.3, p. 132. We shall begin by proving that these functions are respectively upper and lower semicontinuous and that persistent and minimal time evolutions exist under upper semicompact evolutionary systems. We next study the exit sets, the subset of states at the boundary of the environment from which all evolutions leave the environment immediately. They play an important role in the characterization of local viability, of *transversality.*

10.4.1 Persistent and Minimal Time Evolutions

Let us recall the definition of the *exit function* of K defined by

$$\tau_K(x(\cdot)) := \inf\{t \in [0,\infty[\mid x(t) \notin K\} \text{ and } \tau_K^\sharp(x) := \sup_{x(\cdot)\in\mathcal{S}(x)} \tau_K(x(\cdot))$$

and of *minimal time function* $\varpi_{(K,C)}$ defined by

$$\varpi_{(K,C)}(x(\cdot)) := \inf\{t \geq 0 \mid x(t) \in C \ \& \ \forall s \in [0,t],\ x(s) \in K\ \}$$

and

$$\varpi^\flat_{(K,C)}(x) := \inf_{x(\cdot)\in\mathcal{S}(x)} \varpi_{(K,C)}(x(\cdot))$$

We summarize the semi-continuity properties of the exit and minimal time functions in the following statement:

Theorem 10.4.1 ***[Semi-Continuity Properties of Exit and Minimal Time Functions]*** *Let us assume that the evolutionary system is upper semicompact and that the subsets K and $C \subset K$ are closed. Then:*

1. the hypograph of the exit function $\tau_K^\sharp(\cdot)$ is closed,

2. *the epigraph of the minimal time function* $\varpi^{\flat}_{(K,C)}(\cdot)$ *is closed*
This can be translated by saying that the exit function is upper semicontinuous and the minimal time function is lower semicontinuous.

Proof. The first statements follow from Theorems 4.3.6 and 10.3.10. □

Actually, in several applications, we would like to maximize the exit functional and minimize the minimal time or minimal time functional. Indeed, when an initial state $x \in K$ does not belong to the viability kernel, all evolutions $x(\cdot) \in \mathcal{S}(x)$ leave K in finite time. The question arises to select the "persistent evolutions" in K which persist to remain in K as long as possible:

Definition 10.4.2 *[**Persistent Evolutions**] Let us consider an evolutionary system* $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,+\infty;X)$ *and a subset* $K \subset X$*.*
The solutions $x^{\sharp}(\cdot) \in \mathcal{S}(x)$ *which maximize the exit time function*

$$\forall x \in K,\ \tau_K(x^{\sharp}(\cdot)) = \tau^{\sharp}_K(x) := \max_{x(\cdot)\in\mathcal{S}(x)} \tau_K(x(\cdot)) \tag{10.9}$$

are called persistent evolutions *in* K *(Naturally, when* $x \in \text{Viab}_{\mathcal{S}}(K)$*, persistent evolutions starting at* x *are the viable ones).*
We denote by $\mathcal{S}^{K^{\sharp}} : K \rightsquigarrow \mathcal{C}(0,+\infty;X)$ *the evolutionary system* $\mathcal{S}^{K^{\sharp}} \subset \mathcal{S}$ *associating with any* $x \in K$ *the set of persistent evolutions in* K*.*

In a symmetric way, we single out the evolutions which minimize the minimal time to a target:

Definition 10.4.3 *[**Minimal Time Evolutions**] Let us consider an evolutionary system* $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,+\infty;X)$ *and subsets* $K \subset X$ *and* $C \subset K$*.*
The evolutions $x^{\flat}(\cdot) \in \mathcal{S}(x)$ *which minimize the minimal time function*

$$\forall x \in K,\ \varpi_{(K,C)}(x^{\flat}(\cdot)) = \varpi^{\flat}_{(K,C)}(x) := \min_{x(\cdot)\in\mathcal{S}(x)} \varpi_{(K,C)}(x(\cdot)) \tag{10.10}$$

are called minimal time evolutions *in* K*.*

Persistent evolutions and minimal time evolutions exist when the evolutionary system is upper semicompact:

Theorem 10.4.4 ***[Existence of Persistent and Minimal Time Evolutions]*** *Let $K \subset X$ be a closed subset and $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ be an upper semicompact evolutionary system. Then:*

1. *For any $x \notin \mathrm{Viab}_{\mathcal{S}}(K)$, there exists at least one persistent evolution $x^\sharp(\cdot) \in \mathcal{S}^{K^\sharp}(x) \subset \mathcal{S}(x)$ viable in K on the interval $[0, \tau_K^\sharp(x)]$.*
2. *For any $x \in \mathrm{Capt}_{\mathcal{S}}(K, C)$, there exists at least one evolution $x^\flat(\cdot) \in \mathcal{S}(x)$ reaching C in minimal time while being viable in K.*

Proof. Let $t < \tau_K^\sharp(x)$ and $n > 0$ such that $t < \tau_K^\sharp(x) - \frac{1}{n}$. Hence, by definition of the supremum, there exists an evolution $x_n(\cdot) \in \mathcal{S}(x)$ such that $\tau_K(x_n(\cdot)) \geq \tau_K^\sharp(x) - \frac{1}{n}$, and thus, such that $x_n(t) \in K$. Since the evolutionary system $\mathcal{S}$ is upper semicompact, we can extract a subsequence of evolutions $x_{n'}(\cdot) \in \mathcal{S}(x)$ converging to some evolution $x_\star(\cdot) \in \mathcal{S}(x)$. Therefore, we infer that $x_\star(t)$ belongs to K because K is closed. Since this is true for any $t < \tau_K^\sharp(x)$ and since the evolution $x_\star(\cdot)$ is continuous, we infer that $\tau_K^\sharp(x) \leq \tau_K(x_\star(\cdot))$. We deduce that such an evolution $x_\star(\cdot) \in \mathcal{S}(x)$ is persistent in K because $\tau_K(x_\star(\cdot)) \leq \tau_K^\sharp(x)$ by definition.

By definition of $T := \varpi^\flat_{(K,C)}(x)$, for every $\varepsilon > 0$, there exists N such that for $n \geq N$, there exists an evolution $x_n(\cdot) \in \mathcal{S}(x_n)$ and $t_n \leq T + \varepsilon$ such that $x_n(t_n) \in C$ and for every $s < t_n$, $x_n(s) \in K$. Since $\mathcal{S}$ is upper semicompact, a subsequence (again denoted by) $x_n(\cdot)$ converges uniformly on compact intervals to some evolution $x(\cdot) \in \mathcal{S}(x)$. Let us also consider a subsequence (again denoted by) t_n converging to some $T^\star \leq T + \varepsilon$. By taking the limit, we infer that $x(T^\star)$ belongs to C and that, for any $s < T^\star$, $x(s)$ belongs to K. This implies that

$$\varpi^\flat_{(K,C)}(x) \leq \varpi_{(K,C)}(x(\cdot)) \leq T^\star \leq T + \varepsilon$$

We conclude by letting ε converge to 0: The evolution $x(\cdot)$ obtained above achieves the infimum. □

We deduce the following characterization of viability kernels and viable-capture basins:

Proposition 10.4.5 ***[Sections of Exit and Minimal Time Functions]*** *Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ be a strict upper semicompact evolutionary system and C and K be two closed subsets such that $C \subset K$. Then the* viability kernel *is characterized by*

$$\mathrm{Viab}_{\mathcal{S}}(K) = \{x \in K \mid \tau_K^\sharp(x) = +\infty\}$$

and the viable-capture basin

$$\text{Capt}_{\mathcal{S}}(K,C) = \{x \in K \mid \varpi^{\flat}_{(K,C)}(x) \ < \ +\infty\}$$

is the domain of the (constrained) minimal time function $\varpi^{\flat}_{(K,C)}$.

Furthermore, for any $T \geq 0$, *the viability kernel and capture basin tubes defined in Definition 4.3.1, p. 133 can be characterized by exit and minimal time functions:*

$$\begin{cases} \text{Viab}_{\mathcal{S}}(K)(T) \quad := \left\{x \in K \mid \tau^{\sharp}_K(x) \ \geq \ T\right\} \\ \\ \text{Capt}_{\mathcal{S}}(K,C)(T) := \ \left\{x \in X \mid \varpi^{\flat}_{(K,C)}(x) \leq T\right\} \end{cases} \tag{10.11}$$

Proof. Inclusions

$$\begin{cases} \text{Viab}_{\mathcal{S}}(K) \ \subset \ \{x \in K \mid \tau^{\sharp}_K(x) = +\infty\} \\ \text{Capt}_{\mathcal{S}}(K,C) \ \subset \ \{x \in K \mid \varpi^{\flat}_{(K,C)}(x) \ < \ +\infty\} \end{cases}$$

as well as

$$\begin{cases} \text{Viab}_{\mathcal{S}}(K)(T) \quad \subset \left\{x \in K \mid \tau^{\sharp}_K(x) \ \geq \ T\right\} \\ \\ \text{Capt}_{\mathcal{S}}(K,C)(T) \subset \left\{x \in X \mid \varpi^{\flat}_{(K,C)}(x) \leq T\right\} \end{cases}$$

are obviously always true.

Equalities follow from Theorem 10.4.4 by taking one persistent evolution $x^{\sharp}(\cdot) \in \mathcal{S}(x)$ when $T \leq \tau^{\sharp}_K(x) \leq +\infty$, since we deduce that $T \leq \tau^{\sharp}_K(x) = \tau_K(x^{\sharp}(\cdot))$, so that $x(\cdot)$ is viable in K on the interval $[0,T]$. In the same way, taking one minimal time evolution $x^{\flat}(\cdot) \in \mathcal{S}(x)$ when $\varpi^{\flat}_{(K,C)}(x) \leq T < +\infty$, we deduce that $\varpi_{(K,C)}(x^{\flat}(\cdot)) = \varpi^{\flat}_{(K,C)}(x) \leq T$, so that $x(\cdot)$ is viable in K before it reaches C at T. $\square$

The viability kernel is in some sense the paradise for viability, which is lost whenever an environment K is a repeller. Even though there is no viable evolutions in K, one can however look for an ersatz of viability kernel, which is the subset of evolutions which survive with the highest life expectancy:

Proposition 10.4.6 ***[Persistent Kernel]*** *If K is a compact repeller under a upper semicompact evolutionary system $\mathcal{S}$, then there exists a nonempty compact subset of initial states which maximize their exit time. This set can be regarded as a* persistent kernel.

Proof. By Theorem 10.4.1, p. 392, the exit functional $\tau_K^\sharp$ is upper semicontinuous. Therefore, it achieves its maximum whenever the environment K is compact. □

10.4.2 Temporal Window

Let $x \in K$ and $x(\cdot) \in \mathcal{S}(x)$ be a (full) evolution passing through x (see Sect. 8.2, p. 275). The sum of the exit times $\tau_K(\overrightarrow{x}(\cdot))$ of the forward part of the evolution and of the exit time $\tau_K(\overleftarrow{x}(\cdot))$ of its backward part can be regarded as the "*temporal window*" of the evolution in K. One can observe that the *maximal temporal window* is the sum of the exit time function of its backward time and of its forward part since

$$\sup_{x(\cdot)\in\mathcal{S}(x)} (\tau_K(\overleftarrow{x}(\cdot))+\tau_K(\overrightarrow{x}(\cdot))) = \sup_{\overleftarrow{x}(\cdot)\in\overleftarrow{\mathcal{S}}(x)} (\tau_K(\overleftarrow{x}(\cdot)))+\sup_{\overrightarrow{x}(\cdot)\in\mathcal{S}(x)} (\tau_K(\overrightarrow{x}(\cdot)))$$

Any (full) evolution $x^\sharp(\cdot) \in \mathcal{S}(x)$ passing through $x \in K$ maximizing the temporal window is still called *persistent*. It is the concatenation of the persistent forward part $\tau_K(\overrightarrow{x}^\sharp(\cdot))$ and of its backward part $\tau_K(\overleftarrow{x}^\sharp(\cdot))$. The maximal temporal window of a (full) evolution viable in K is infinite, and the converse is true whenever the evolutionary system is upper semicompact (see Proposition 10.4.6, p. 395). If the subset $K \setminus B$ is a backward repeller and the subset $K \setminus C$ is a forward repeller, the bilateral viability kernel is empty, but the subset of states $x \in K$ maximizing their temporal window function is not empty and can be called the persistent kernel of K.

10.4.3 Exit Sets and Local Viability

We continue the study of local viability initiated in Sect. 2.13, p. 94 by characterizing it in terms of exit sets:

Definition 10.4.7 *[**Exit Subsets**] Let us consider an evolutionary system $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,+\infty;X)$ and a subset $K \subset X$. The* exit subset $\mathrm{Exit}_{\mathcal{S}}(K)$ *is the (possibly empty) subset of elements $x \in \partial K$ which leave K immediately:*

$$\mathrm{Exit}_{\mathcal{S}}(K) := \left\{x \in K \text{ such that } \tau_K^\sharp(x) = 0\right\}$$

Exit sets characterize viability and local viability of environments. Recall that Definition 2.13.1, p. 94 states that a subset D is said *locally viable* under $\mathcal{S}$ if from any initial state $x \in D$, *there exists at least one evolution* $x(\cdot) \in \mathcal{S}(x)$ and a strictly positive $T_{x(\cdot)} > 0$ such that $x(\cdot)$ is viable in D on the nonempty interval $[0, T_{x(\cdot)}[$.

Proposition 10.4.8 ***[Local Viability Kernel]*** *The subset* $K \setminus \text{Exit}_{\mathcal{S}}(K)$ *is the largest locally viable subset of* K *(and thus, can be regarded as the* "local viability kernel *of* K"*).*

Proof. Let $D \subset K$ be locally viable. If an evolution $x(\cdot) \in \mathcal{S}(x)$ starting from $x \in D$ is locally viable in D, it is clear that $\tau_K^\sharp(x) \geq \tau_D^\sharp(x) \geq \tau_D(x(\cdot)) > 0$, so that $x \in K \setminus \text{Exit}_{\mathcal{S}}(K)$. Furthermore, the subset $K \setminus \text{Exit}_{\mathcal{S}}(K)$ itself is locally viable because to say that $x \in K \setminus \text{Exit}_{\mathcal{S}}(K)$ means that $\tau_K^\sharp(x) > 0$. Hence for any $0 < \lambda < \tau_K^\sharp(x)$, there exists $x(\cdot) \in \mathcal{S}(x)$ such that $0 < \lambda \leq \tau_K(x(\cdot))$, i.e., such that $x(\cdot)$ is viable in K on the nonempty interval $[0, \tau_K(x(\cdot))]$. □

If an environment K is not viable, the subset K can be covered in the following way:

$$K \;=\; \text{Viab}_{\mathcal{S}}(K) \cup \text{Abs}_{\mathcal{S}}(K, \text{Exit}_{\mathcal{S}}(K))$$

because, starting outside the viability kernel of K, all solutions leave K in finite time through the exit set.

We also observe that $K \setminus (\text{Viab}_{\mathcal{S}}(K) \cup \text{Exit}_{\mathcal{S}}(K))$ is the set of initial states from which starts at least one evolution locally viable in K, but not viable in K.

Proposition 10.4.8, p. 397 implies

Proposition 10.4.9 ***[Locally Viable Subsets]*** *The following statements are equivalent:*

1. *the complement* $K \setminus C$ *of a target* $C \subset K$ *in the environment* K *is locally viable*
2. $\text{Exit}_{\mathcal{S}}(K) \subset C$,
3. $C \cap \text{Exit}_{\mathcal{S}}(K) \subset \text{Exit}_{\mathcal{S}}(C)$.

In particular, K *is locally viable if and only if its exit set* $\text{Exit}_{\mathcal{S}}(K) = \emptyset$ *is empty.*

Proof. Indeed, $K \setminus C$ is locally viable if and only if $K \setminus C \subset K \setminus \text{Exit}_{\mathcal{S}}(K)$ is contained in the local viability kernel $K \setminus \text{Exit}_{\mathcal{S}}(K)$, i.e., if and only if

$\text{Exit}_{\mathcal{S}}(K) \subset C$. On the other hand, since $C \cap \text{Exit}_{\mathcal{S}}(K) \subset \text{Exit}_{\mathcal{S}}(C)$. Hence the three statements are equivalent. □

There is a close link between the closedness of exit sets and the continuity of the exit function:

Theorem 10.4.10 ***[Closedness of Exit Sets and Continuity of Exit Functions]*** *Let us assume that the evolutionary system is upper semicompact and that the subset K is closed. Then the epigraph of the exit function $\tau_K^\sharp(\cdot)$ is closed if and only if the exit subset $\text{Exit}_{\mathcal{S}}(K)$ is closed.*

Proof. Since the exit function is upper semicontinuous, its continuity is equivalent to its lower semicontinuity, i.e., to the closedness of its epigraph. The lower semicontinuity of the exit function implies the closedness of the exit subset

$$\text{Exit}_{\mathcal{S}}(K) := \left\{ x \in K \text{ such that } \tau_K^\sharp(x) = 0 \right\}$$

because the lower sections of a lower semicontinuous function are closed. Let us prove the converse statement. Consider a sequence (x_n, y_n) of the epigraph of the exit function converging to some (x, y) and prove that the limit belongs to its epigraph, i.e., that $\tau_K^\sharp(x) \le y$.

Indeed, since $t_n := \tau_K^\sharp(x_n) \le y_n \le y + 1$ when n is large enough, there exists a subsequence (again denoted by) t_n converging to $t_\star \le y + 1$. Since the evolutionary system is assumed to be upper semicompact, there exists a persistent evolution $x_n^\sharp(\cdot) \in \mathcal{S}(x_n)$ such that $t_n := \tau_K(x_n^\sharp(\cdot))$. Furthermore, a subsequence (again denoted by) $x_n^\sharp(\cdot)$ converges to some evolution $x_\star(\cdot) \in \mathcal{S}(x)$ uniformly on the interval $[0, y+1]$. By definition of the persistent evolution, for all $t \in [0, t_n]$, $x_n^\sharp(t) \in K$ and $x_n(t_n) \in \text{Exit}_{\mathcal{S}}(K)$, which is closed by assumption. We thus infer that for all $t \in [0, t_\star]$, $x_\star(t) \in K$ and $x_\star(t_\star) \in \text{Exit}_{\mathcal{S}}(K)$. This means that $t_\star = \tau_K(x_\star(\cdot))$ and consequently, that $\tau_K(x_\star(\cdot)) \le y$. This completes the proof. □

We single out the important case in which the evolutions leaving K cross the boundary at a single point:

Definition 10.4.11 ***[Transverse Sets]*** *Let $\mathcal{S}$ be an evolutionary system and K be a closed subset. We shall say that K is transverse to $\mathcal{S}$ if for every $x \in K$ and for every evolution $x(\cdot) \in \mathcal{S}(x)$ leaving K in finite time, $\tau_K(x(\cdot)) = \varpi_{\partial K}(x(\cdot))$.*

Transversality of an environment means that all evolutions governed by an evolutionary system cross the boundary as soon as they reach it to leave the environment immediately.

We deduce the following consequence:

Proposition 10.4.12 *[**Continuity of the Exit Function of a Transverse Set**] Assume that the evolutionary system $\mathcal{S}$ is upper semicompact and that the subset K is closed and transverse to $\mathcal{S}$. Then the exit function $\tau_K^\sharp$ is continuous and the exit set $\mathrm{Exit}_{\mathcal{S}}(K)$ of K is closed.*

10.5 Viability Characterizations of Kernels and Basins

We shall review successively the viability characterizations of viable subsets outside a target, introduce the concept of relative backward invariance for characterizing isolated systems before proving the second viability characterizations of viability kernels and capture basins. They enjoy semi-permeable barrier properties investigated at the end of this section.

10.5.1 Subsets Viable Outside a Target

We now provide a characterization of a subset D viable outside a target C in terms of local viability of $D \backslash C$:

Proposition 10.5.1 *[**Characterization of Viable Subsets Outside a Target**] Assume that $\mathcal{S}$ is upper semicompact. Let $C \subset D$ and D be closed subsets. The following conditions are equivalent:*

1. *D is viable outside C under $\mathcal{S}$ ($\mathrm{Viab}_{\mathcal{S}}(D, C) = D$ by Definition 2.2.3, p. 49),*
2. *$D \setminus C$ is locally viable under $\mathcal{S}$,*
3. *The exit set of D is contained in the exit set of C: $C \cap \mathrm{Exit}_{\mathcal{S}}(D) \subset \mathrm{Exit}_{\mathcal{S}}(C)$*

In particular, a closed subset D is viable under $\mathcal{S}$ if and only if its exit set is empty:

$$\mathrm{Viab}_{\mathcal{S}}(D) = D \text{ if and only if } \mathrm{Exit}_{\mathcal{S}}(D) = \emptyset$$

Proof.

1. First, assume that $\text{Viab}_{\mathcal{S}}(D, C) = D$ and derive that $D \backslash C$ is locally viable. Take $x_0 \in D \backslash C$ and prove that there exists an evolution $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 viable in $D \backslash C$ on a nonempty interval. Indeed, since C is closed, there exists $\eta > 0$ such that $B(x_0, \eta) \cap C = \emptyset$, so that $x(t) \in B(x_0, \eta) \cap D \subset D \backslash C$ on some nonempty interval. This means that $\text{Viab}_{\mathcal{S}}(D, C) \backslash C$ is locally viable.
2. Assume that $D \setminus C$ is locally viable and derive that $\text{Viab}_{\mathcal{S}}(D, C) = D$. Take any $x \in D \setminus C$ and, since the evolutionary system is assumed to be semicompact, at least one persistent evolution $x^\sharp(\cdot) \in \mathcal{S}(x)$, thanks to Theorem 10.4.4. Either this persistent evolution is viable forever, and thus $x \in \text{Viab}_{\mathcal{S}}(D) \subset \text{Viab}_{\mathcal{S}}(D, C)$, or else, it leaves D in finite time $\tau_D^\sharp(x)$ at $x^\Rightarrow := x^\sharp(\tau_D^\sharp(x)) \in \partial D$.
   Such an element $x^\Rightarrow$ belongs to C because, otherwise, since $D \backslash C$ is locally viable and C is closed, one could associate with $x^\Rightarrow \in D \backslash C$ another evolution $y(\cdot) \in \mathcal{S}(x^\Rightarrow)$ and $T > 0$ such that $y(\tau) \in D \backslash C$ for all $\tau \in [0, T]$, so that $\tau_D^\sharp(x^\Rightarrow) = T > 0$, contradicting the fact that $x^\sharp(\cdot)$ is a persistent evolution.
3. The equivalence between the second and third statement follows from Propositions 10.4.9, p. 397 on exit sets. □

As a consequence, Proposition 10.5.2, p. 400 and Theorem 10.3.10, p. 388 (guaranteeing that the viability kernels $\text{Viab}_{\mathcal{S}}(D, C)$ are closed) Theorem 2.15.2 imply the following:

Theorem 10.5.2 ***[Characterization of Viable Subsets Outside a Target]*** *Assume that $\mathcal{S}$ is upper semicompact. Let $C \subset K$ and K be closed subsets.*

Then the viability kernel $\text{Viab}_{\mathcal{S}}(K, C)$ *of K with target C under $\mathcal{S}$ is:*

- *either the* largest closed subset $D \subset K$ *containing C such that $D \backslash C$ is locally viable,*
- *or, equivalently, the* largest closed subset *satisfying*

$$C \cap \text{Exit}_{\mathcal{S}}(D) \subset \text{Exit}_{\mathcal{S}}(C) \subset C \subset D \subset K \tag{10.12}$$

Therefore, under these equivalent assumptions (10.12), p. 400, inclusion

$$D \cap \text{Exit}_{\mathcal{S}}(K) \subset C \subset D \tag{10.13}$$

holds true. In particular, the viability kernel $\text{Viab}_{\mathcal{S}}(K)$ *of K is the largest closed viable subset contained in K.*

Remark. We shall see that inclusion (10.13), p. 400 is the "mother of boundary conditions" when the subsets C, K and D are graphs of set-valued maps or epigraphs of extended functions. □

10.5.2 Relative Invariance

We characterize further isolated subsets in terms of backward invariance properties – discovered by Hélène Frankowska in her investigations of Hamilton-Jacobi equations associated with value functions of optimal control problems under state constraints. They play a crucial role for enriching the Characterization Theorem 10.2.5 stating that the viability kernel of an environment with a target is the smallest subset containing the target and isolated in this environment. We already introduced the concept of backward relative invariance (see Definition 2.15.3, p. 100):

Definition 10.5.3 *[**Relative Invariance**] We shall say that a subset $C \subset K$ is* (backward) invariant relatively to K *under $\mathcal{S}$ if for every $x \in C$, all (backward) evolutions starting from x and viable in K on an interval $[0, T[$ are viable in C on the same interval $[0, T[$.*

If K is itself (backward) invariant, any subset (backward) invariant relatively to K is (backward) invariant.

If $C \subset K$ is (backward) invariant relatively to K, then $C \cap \text{Int}(K)$ is (backward) invariant.

Proposition 10.5.4 *[**Capture Basins of Relatively Invariant Targets**] Let $C \subset D \subset K$ three subsets of X.*

1. *If D is backward invariant relatively to K, then $\text{Capt}_{\mathcal{S}}(K, C) = \text{Capt}_{\mathcal{S}}(D, C)$,*
2. *If C is backward invariant relatively to K, then $\text{Capt}_{\mathcal{S}}(K, C) = C$.*

Proof. Since $\text{Capt}_{\mathcal{S}}(D, C) \subset \text{Capt}_{\mathcal{S}}(K, C)$, let us consider an element $x \in \text{Capt}_{\mathcal{S}}(K, C)$, an evolution $x(\cdot)$ viable in K until it reaches C in finite time $T \geq 0$ at $z := x(T) \in C$. Setting $\overleftarrow{y}(t) := x(T - t)$, we observe that $\overleftarrow{y}(\cdot) \in \overleftarrow{S}(x(T))$, satisfies $\overleftarrow{y}(T) = x \in K$ and is viable in K on the interval $[0, T]$. Since D is backward invariant relatively to K, we infer that this evolution $\overleftarrow{y}(\cdot)$ is viable in D on the interval $[0, T]$, so that $x(t) = \overleftarrow{y}(T - t)$ belongs to D for all $t \in [0, T]$. This implies that x belongs to $\text{Capt}_{\mathcal{S}}(D, C)$.

Taking $C := D$, then $\text{Capt}_{\mathcal{S}}(D,C) = C$, so that $\text{Capt}_{\mathcal{S}}(K,C) = \text{Capt}_{\mathcal{S}}(D,C) = C$. $\square$

Capture basins of targets viable in environments are backward invariants relatively to this environment:

Proposition 10.5.5 *[**Relative Backward Invariance of Capture Basins**] The capture basin $\text{Capt}_{\mathcal{S}}(K,C)$ of a target C viable in the environment K is backward invariant relatively to K.*

Proof. We have to prove that for every $x \in \text{Capt}_{\mathcal{S}}(K,C)$, every backward evolution $\overleftarrow{y}(\cdot) \in \overleftarrow{\mathcal{S}}(x)$ viable in K on some interval $[0,T]$ is actually viable in $\text{Capt}_{\mathcal{S}}(K,C)$ on the same interval.

Since x belongs to $\text{Capt}_{\mathcal{S}}(K,C)$, there exists an evolution $z(\cdot) \in \mathcal{S}(x)$ and $S \geq 0$ such that $z(S) \in C$ and, for all $t \in [0,S]$, $z(t) \in K$. We associate with it the evolution $\overrightarrow{x}_T(\cdot) \in \mathcal{S}(\overleftarrow{y}(T))$ defined by

$$\overrightarrow{x}_T(t) := \begin{cases} \overleftarrow{y}(T-t) \text{ if } t \in [0,T] \\ \overrightarrow{z}(t-T) \text{ if } t \in [T, T+S] \end{cases}$$

starting at $y(T) \in K$. It is viable in K until it reaches C at time $T+S$. This means that $y(T)$ belongs to $\text{Capt}_{\mathcal{S}}(K,C)$ and this implies that for every $t \in [0, T+S]$, $\overrightarrow{x}_T(t)$ belongs to the capture basin $\text{Capt}_{\mathcal{S}}(K,C)$. This is in particular the case when $t \in [0,T]$: then $\overleftarrow{y}(t) = \overrightarrow{x}_T(T-t)$ belongs to the capture basin. Therefore, the backward evolution $\overleftarrow{y}(\cdot) \in \overleftarrow{\mathcal{S}}(x)$ is viable in $\text{Capt}_{\mathcal{S}}(K,C)$ on the interval $[0,T]$. $\square$

We deduce that a subset $C \subset K$ is backward invariant relatively to K if and only if K is the capture basin of C:

Theorem 10.5.6 [***Characterization of Relative Invariance***] *A subset $C \subset K$ is backward invariant relatively to K if and only if $\text{Capt}_{\mathcal{S}}(K,C) = C$.*

Proof. First, Proposition 10.5.5, p. 402 implies that whenever $\text{Capt}_{\mathcal{S}}(K,C) = C$, C is backward invariant relatively to K. Conversely, assume that C is backward invariant relatively to K and we shall derive a contradiction by assuming that there exists $x \in \text{Capt}_{\mathcal{S}}(K,C) \backslash C$: in this case, there would exist a forward evolution denoted $\overrightarrow{x}(\cdot) \in \mathcal{S}(x)$ starting at x and viable in

K until it reaches C at time $T > 0$ at $c = x(T)$. Let $\overleftarrow{z}(\cdot) \in \overleftarrow{\mathcal{S}}(x)$ be any backward evolution starting at x and viable in K on some interval $[0, T]$. We associate with it the function $\overleftarrow{y}(\cdot)$ defined by

$$\overleftarrow{y}(t) \ := \begin{cases} \overrightarrow{x}(T-t) \text{ if } t \in [0,T] \\ \overleftarrow{z}(t-T) \text{ if } t \geq T \end{cases}$$

Then $\overleftarrow{y}(\cdot) \in \overleftarrow{\mathcal{S}}(c)$ and is viable in K on the interval $[0, T]$. Since C is assumed to be backward invariant relatively to K, then $\overleftarrow{y}(t) \in C$ for all $t \in [0, T]$, and in particular $\overleftarrow{y}(T) = x$ belongs to C. We have obtained a contradiction since we assumed that $x \notin C$. Therefore $\text{Capt}_{\mathcal{S}}(K, C) \backslash C = \emptyset$, i.e., $\text{Capt}_{\mathcal{S}}(K, C) = C$. □

As a consequence of Proposition 10.5.6, we obtain:

Proposition 10.5.7 ***[Backward Invariance of the Complement of an Invariant Set]*** *A subset C is backward invariant under an evolutionary system $\mathcal{S}$ if and only if its complement $\complement C$ is invariant under $\mathcal{S}$.*

Proof. Applying Proposition 10.5.6 with $K := X$, we infer that C is backward invariant if and only if $C = \text{Capt}_{\mathcal{S}}(X, C)$, which is equivalent, by Lemma 2.12.2, to the statement that $\complement C = \text{Inv}_{\mathcal{S}}(\complement C, \emptyset) =: \text{Inv}_{\mathcal{S}}(\complement C)$ is invariant. □

10.5.3 Isolated Subsets

The following Lemma is useful because it allows isolated subsets to be also characterized by viability properties:

Lemma 10.5.8 ***[Isolated Subsets]*** *Let D and K be two subsets such that $D \subset K$. Then the following properties are equivalent:*

1. *D is isolated in K under $\mathcal{S}$:* $\text{Viab}_{\mathcal{S}}(K, D) = D$,
2. $\text{Viab}_{\mathcal{S}}(K) = \text{Viab}_{\mathcal{S}}(D)$ *and* $\text{Capt}_{\mathcal{S}}(K, D) = D$,
3. *$K \backslash D$ is a repeller and* $\text{Capt}_{\mathcal{S}}(K, D) = D$.

Proof. Assume that D is isolated in K. This amounts to writing that,

1. by definition,

$$D = \mathrm{Viab}_{\mathcal{S}}(K, D) = \mathrm{Viab}_{\mathcal{S}}(K) \cup \mathrm{Capt}_{\mathcal{S}}(K, D)$$

   and thus, equivalently, that $\mathrm{Capt}_{\mathcal{S}}(K, D) = D$ and $\mathrm{Viab}_{\mathcal{S}}(K) \subset D$. Since $D \subset K$, inclusion $\mathrm{Viab}_{\mathcal{S}}(K) \subset D$ is equivalent to $\mathrm{Viab}_{\mathcal{S}}(K) = \mathrm{Viab}_{\mathcal{S}}(D)$.
2. by formula (2.26),

$$D = \mathrm{Viab}_{\mathcal{S}}(K, D) = \mathrm{Viab}_{\mathcal{S}}(K \backslash D) \cup \mathrm{Capt}_{\mathcal{S}}(K, D)$$

   and thus, equivalently, that $\mathrm{Capt}_{\mathcal{S}}(K, D) = D$ and that $\mathrm{Viab}_{\mathcal{S}}(K \backslash D) \subset D$. Since $D \cap \mathrm{Viab}_{\mathcal{S}}(K \backslash D) = \emptyset$, this implies that $\mathrm{Viab}_{\mathcal{S}}(K \backslash D) = \emptyset$. $\square$

We derive the following characterization:

Theorem 10.5.9 ***[Characterization of Isolated Subsets]*** *Let us consider a closed subset $D \subset K$. Then D is isolated in K by $\mathcal{S}$ if and only if:*

1. *D is backward invariant relatively to K,*
2. *either $K \backslash D$ is a repeller or* $\mathrm{Viab}_{\mathcal{S}}(K) = \mathrm{Viab}_{\mathcal{S}}(D)$.

We provide now another characterization of isolated subsets involving complements:

Proposition 10.5.10 ***[Complement of an Isolated Subset]*** *Let us assume that K and $D \subset K$ are closed.*

1. *If the evolutionary system $\mathcal{S}$ is lower semicontinuous and if* $D = \mathrm{Capt}_{\mathcal{S}}(K, D)$, *then either one of the following equivalent properties:*

$$\begin{cases} (i) \;\; \overline{\complement D} \;=\; \mathrm{Inv}_{\mathcal{S}}(\overline{\complement D}, \overline{\complement K}) \; (\overline{\complement D} \text{ is invariant outside } \overline{\complement K}) \\ (ii) \;\; \mathrm{Int}(D) \;=\; \mathrm{Capt}_{\mathcal{S}}(\mathrm{Int}(K), \mathrm{Int}(D)) \\ (iii) \; \mathrm{Int}(D) \text{ is backward invariant relatively to } \mathrm{Int}(K) \end{cases} \quad (10.14)$$

   hold true.
2. *Conversely, if* $\mathrm{Int}(K)$ *is backward invariant and if the set-valued map* $\overleftarrow{\mathcal{S}}$ *is lower semicontinuous, then any of the equivalent properties (10.14), p. 404 implies that* $\overline{\mathrm{Int}(D)} = \mathrm{Capt}_{\mathcal{S}}(\overline{\mathrm{Int}(K)}, \overline{\mathrm{Int}(D)})$.

Proof. Lemma 2.12.2 implies that $\mathrm{Capt}_{\mathcal{S}}(K, D) = D$ if and only if

$$\complement D = \mathrm{Inv}_{\mathcal{S}}(\complement D, \complement K)$$

Since $\mathcal{S}$ is assumed to be lower semicontinuous, we deduce from Theorem 10.3.7 that

$$\begin{cases} \complement(\mathrm{Int}(D)) \;=\; \overline{\complement D} \;=\; \overline{\mathrm{Inv}_{\mathcal{S}}(\complement D, \complement K)} \\ \subset\; \mathrm{Inv}_{\mathcal{S}}(\overline{\complement D}, \overline{\complement K}) \;=\; \mathrm{Inv}_{\mathcal{S}}(\complement(\mathrm{Int}(D)), \complement(\mathrm{Int}(K))) \;\subset\; \complement(\mathrm{Int}(D)) \end{cases}$$

so that the closure of the complement of D is invariant outside the closure of the complement of K. Observe that, taking the complements, Lemma 2.12.2 states that this is equivalent to property $\mathrm{Int}(D) = \mathrm{Capt}_{\mathcal{S}}(\mathrm{Int}(K), \mathrm{Int}(D))$, which, by Theorem 10.5.6, p. 402, amounts to saying that the interior of D is relatively backward invariant relatively to the interior of K.

For proving the converse statement, Proposition 10.3.14, p. 390 states that under the assumptions of the theorem, condition $\mathrm{Int}(D) = \mathrm{Capt}_{\mathcal{S}}(\mathrm{Int}(K), \mathrm{Int}(D))$ implies that

$$\overline{\mathrm{Int}(D)} \;\subset\; \mathrm{Capt}_{\mathcal{S}}(\overline{\mathrm{Int}(K)}, \overline{\mathrm{Int}(D)}) \;\subset\; \overline{\mathrm{Capt}_{\mathcal{S}}(\mathrm{Int}(K), \mathrm{Int}(D))} \;=\; \overline{\mathrm{Int}(D)} \quad \square$$

10.5.4 The Second Fundamental Characterization Theorem

Putting together the characterizations of viable subsets and isolated subsets, we reformulate Theorem 10.2.5 characterizing viability kernels with targets in the following way:

Theorem 10.5.11 ***[Viability Characterization of Viability Kernels]*** *Let us assume that $\mathcal{S}$ is upper semicompact and that the subsets $C \subset K$ and K are closed. The viability kernel* $\mathrm{Viab}_{\mathcal{S}}(K, C)$ *of a subset K with target C under $\mathcal{S}$ is the* ***unique*** *closed subset satisfying $C \subset D \subset K$ and*

$$\begin{cases} (i) \quad D\backslash C \text{ is } \textit{locally viable} \text{ under } \mathcal{S}, \text{ i.e., } D = \mathrm{Viab}_{\mathcal{S}}(D, C) \\ (ii) \quad D \textit{ is backward invariant relatively to } K \text{ under } \mathcal{S}, \\ (iii)\; K\backslash D \text{ is a } \textit{repeller} \text{ under } \mathcal{S}, \text{ i.e., } \mathrm{Viab}_{\mathcal{S}}(K \setminus D) = \emptyset. \end{cases} \tag{10.15}$$

We mentioned in Sect. 2.15, p. 98 the specific versions for viability kernels (Theorem 2.15.4, p. 101) and capture basin (Theorem 2.15.5, p. 101). However, Theorem 10.5.11 implies that when $K\backslash C$ is a repeller, the above theorem implies a characterization of the viable-capture basins in a more general context:

Theorem 10.5.12 *[**Characterization of Capture Basins**] Let us assume that $\mathcal{S}$ is upper semicompact and that a closed subset $C \subset K$ satisfies property*

$$\text{Viab}_{\mathcal{S}}(K \backslash C) = \emptyset \tag{10.16}$$

Then the viable-capture basin $\text{Capt}_{\mathcal{S}}(K, C)$ *is the* ***unique*** *closed subset D satisfying $C \subset D \subset K$ and*

$$\begin{cases} (i) \;\; D \backslash C \text{ is } \textit{locally viable} \text{ under } \mathcal{S} \\ (ii) \; D \textit{ is backward invariant relatively to } K \text{ under } \mathcal{S} \end{cases} \tag{10.17}$$

We deduce from Proposition 10.5.10, p. 404 another characterization of capture basins that provide existence and uniqueness of viscosity solutions to some Hamilton–Jacobi–Bellman equations:

Theorem 10.5.13 *[**"Viscosity" Characterization of Capture Basins**] Assume that the evolutionary system $\mathcal{S}$ is both upper semicompact and lower semicontinuous, that K is closed, that* $\text{Int}(K) \neq \emptyset$ *is backward invariant, that* $\text{Viab}_{\mathcal{S}}(K \backslash C) = \emptyset$*, that* $\overline{\text{Int}(K)} = K$ *and that* $\overline{\text{Int}(C)} = C$*.*

Then the capture basin $\text{Capt}_{\mathcal{S}}(K, C)$ *is the* ***unique*** *subset topologically regular subset D between C and K satisfying*

$$\begin{cases} (i) \;\; D \backslash C \text{ is } \textit{locally viable} \text{ under } \mathcal{S}, \\ (ii) \; \overline{\complement D} \text{ is } \textit{invariant} \text{ outside } \overline{\complement K} \text{ under } \mathcal{S}. \end{cases} \tag{10.18}$$

Proof. Since $\text{Viab}_{\mathcal{S}}(K \backslash C) = \emptyset$, the viability kernel and the capture basin are equal. By Theorem 10.2.5, p. 379, the capture basin is the unique subset D between C and K such that:

1. the largest subset $D \subset K$ such that $\text{Capt}_{\mathcal{S}}(D, C) = D$,
2. the smallest subset $D \supset C$ such that $\text{Capt}_{\mathcal{S}}(K, D) = D$.

The evolutionary system being upper semicompact, the first condition amounts to saying that $D \backslash C$ is locally viable.

By Proposition 10.5.10, p. 404, property $\text{Capt}_{\mathcal{S}}(K, D) = D$ implies that $\overline{\complement D}$ is invariant outside $\overline{\complement K}$, as well as the other properties (10.14), p. 404.

Conversely, let D satisfy those properties (10.14). Proposition 10.7.6, p. 419 implies that, under the assumptions of the theorem, the capture basin $\text{Capt}_{\mathcal{S}}(K, C)$ is topologically regular whenever K and C are topologically regular. Let D satisfy properties (10.18), p. 406. By Proposition 10.5.10, p. 404, property $\text{Capt}_{\mathcal{S}}(K,D) = D$ implies that $\text{Capt}_{\mathcal{S}}(K,D) = D$. Therefore, Theorem 10.2.5, p. 379 implies that $D = \text{Capt}_{\mathcal{S}}(K,C)$. □

Remark. We shall see that whenever the environment $K := \mathcal{E}p(\mathbf{k})$ and the target $C := \mathcal{E}p(\mathbf{c})$ are epigraphs of functions $\mathbf{k} \leq \mathbf{c}$, the capture basin under adequate dynamical system is itself the epigraph of a function $\mathbf{v}$. Theorem 10.5.13, p. 406 implies that $\mathbf{v}$ is a viscosity solution to a Hamilton–Jacobi–Bellman equation. □

10.5.5 The Barrier Property

Roughly speaking, an environment exhibits the barrier property if all viable evolutions starting from its boundary are viable on its boundary, so that no evolution can enter the interior of this environment: this is a *semi-permeability property* of the boundary.

For that purpose, we need to define the concept of boundary:

Definition 10.5.14 *[**Boundaries**] Let $C \subset K \subset X$ be two subsets of X. The subsets*

$$\partial_K C := \overline{C} \cap \overline{K \backslash C} \ \& \ \overset{\circ}{\partial}_K C := C \cap \overline{K \backslash C}$$

are called respectively the boundary and the pre-boundary of the subset C relatively to K. When $K := X$, we set

$$\partial C := \overline{C} \cap \overline{\complement C} \ \& \ \overset{\circ}{\partial} C := C \cap \overline{\complement C}$$

In other words, the interior of a set D and its pre-boundary form a partition of $D = \text{Int}(D) \cup \overset{\circ}{\partial} D$. Pre-boundaries are useful because of the following property:

Lemma 10.5.15 *[**Pre-boundary of an intersection with an open set**] Let $\Omega \subset X$ be an open subset and $D \subset X$ be a subset. Then*

$$\overset{\circ}{\partial} (\Omega \cap D) = \Omega \cap \overset{\circ}{\partial} D$$

In particular, if $C \subset D$ is closed, then

$$\text{Int}(D) \setminus C = \text{Int}(D \setminus C) \text{ and } \overset{\circ}{\partial} (D \setminus C) = \overset{\circ}{\partial} (D) \setminus C$$

Proof. Indeed, $D = \text{Int}(D) \cup \overset{\circ}{\partial} D$ being a partition of D, we infer that $D \cap \Omega = \text{Int}(D \cap \Omega) \cup \overset{\circ}{\partial} D \cap \Omega$ is still a partition. By definition, $D \cap \Omega = \text{Int}(D \cap \Omega) \cup \overset{\circ}{\partial} (D \cap \Omega)$ is another partition of $D \cap \Omega$. Since Ω is open, $\text{Int}(D \cap \Omega) = \text{Int}(D) \cap \text{Int}(\Omega) = \text{Int}(D) \cap \Omega$, so that $\overset{\circ}{\partial} (D \cap \Omega) = \Omega \cap \overset{\circ}{\partial} D$. □

Definition 10.5.16 *[**Barrier Property**] Let $D \subset X$ be a subset and $\mathcal{S}$ be an evolutionary system. We shall say that D exhibits the* barrier property *if its pre-boundary $\overset{\circ}{\partial} D$ is relatively invariant with respect to D itself. In other words, starting from any $x \in \overset{\circ}{\partial} D$, all evolutions viable in D on some time interval $[0, T[$ are actually viable in $\overset{\circ}{\partial} D$ on $[0, T[$.*

Remark. The barrier property of an environment is a *semi-permeability property* of D, since no evolution can enter the interior of D from the boundary (whereas evolutions may leave D). This is very important in terms of interpretation. Viability of a subset D having the barrier property is indeed a *very fragile property*, which cannot be restored from the outside, or equivalently, no solution starting from outside the viability kernel can cross its boundary from outside. In other words, starting from the pre-boundary of the environment, *love it or leave it...* The "barrier property" played an important role in control theory and the theory of differential games, because their boundaries could be characterized as solutions of first-order partial differential equations under (severe) regularity assumptions. *Marc Quincampoix* made the link at the end of the 1980s between this property and the boundary of the viability kernel: every solution starting from the boundary of the viability kernel can either remain in the boundary or leave the viability kernel, or equivalently, no solution starting from outside the viability kernel can cross its boundary. □

We deduce from Theorem 10.5.6, p. 402 that a subset D exhibits the barrier property if and only if its interior is backward invariant:

Proposition 10.5.17 *[**Backward Invariance of the interior and Barrier Property**] A subset D exhibits the barrier property if and only if its interior* $\mathrm{Int}(D)$ *is backward invariant.*

Proof. Theorem 10.5.6, p. 402 states that the pre-boundary $\overset{\circ}{\partial} D \subset D$ is invariant relatively to D if and only if $\mathrm{Capt}_{\overleftarrow{\mathcal{S}}}(D, \overset{\circ}{\partial} D) = \overset{\circ}{\partial} D$. Therefore, from every $x \in \mathrm{Int}(D) = D \setminus \overset{\circ}{\partial} D = D \setminus \mathrm{Capt}_{\overleftarrow{\mathcal{S}}}(D, \overset{\circ}{\partial} D)$, all backward evolutions are viable in $\mathrm{Int}(D) = D \setminus \overset{\circ}{\partial} D$ as long as they are viable in D. Such evolutions always remain in $\mathrm{Int}(D)$ because they can never reach $x(t) \in \overset{\circ}{\partial} D$ in some finite time t. □

Viability kernels exhibit the barrier property whenever the evolutionary system is both upper and lower semicontinuous:

Theorem 10.5.18 *[**Barrier Property of Boundaries of Viability Kernels**] Assume that K is closed and that the evolutionary system $\mathcal{S}$ is lower semicontinuous. Then the intersection* $\mathrm{Viab}_{\mathcal{S}}(K, C) \cap \mathrm{Int}(K)$ *of the viability kernel of K with the interior of K exhibits the* barrier property *and the interior* $\mathrm{Int}(\mathrm{Viab}_{\mathcal{S}}(K))$ *of the viability kernel of K is backward invariant.*

If $\mathrm{Viab}_{\mathcal{S}}(K) \subset \mathrm{Int}(K)$*, then* $\mathrm{Viab}_{\mathcal{S}}(K)$ *exhibits the barrier property, and thus, its interior is backward invariant.*

In some occasions, the boundary of the viability kernel can be characterized as the viability kernel of the complement of a target, and in this case, exhibits the properties of viability kernels, in particular, can be computed by the Viability Kernel Algorithm: see Theorem 9.2.18, p. 339.

Actually, Theorem 10.5.18, p. 409 is a consequence of the Barrier Theorem 10.5.19, p. 409 of viability kernels with nonempty targets:

Theorem 10.5.19 *[**Barrier Property of Viability Kernels with Targets**] Assume that K and $C \subset K$ are closed and that the evolutionary system $\mathcal{S}$ is lower semicontinuous. Then the intersection* $\mathrm{Viab}_{\mathcal{S}}(K, C) \cap \mathrm{Int}(K \setminus C)$ *of the viability kernel of K with target $C \subset K$ under $\mathcal{S}$ with the interior of $K \setminus C$ exhibits the* barrier property.

Furthermore, $\mathrm{Int}(\mathrm{Viab}_{\mathcal{S}}(K,C)) \setminus C$ *is backward invariant.*
In particular, if $\mathrm{Viab}_{\mathcal{S}}(K,C) \subset \mathrm{Int}(K \setminus C)$*, then* $\mathrm{Int}(\mathrm{Viab}_{\mathcal{S}}(K,C)) \setminus C$ *exhibits the barrier property.*

Proof. Let us set $D := \mathrm{Viab}_{\mathcal{S}}(K,C)$. Theorem 10.5.11, p. 405 implies that D satisfies (10.5.5), p. 410:

$$\begin{cases} (i) & D\backslash C \text{ is } \textit{locally viable} \text{ under } \mathcal{S} \\ (ii) & D \textit{ is backward invariant relatively to } K \text{ under } \mathcal{S} \\ (iii) & K\backslash D \text{ is a } \textit{repeller} \text{ under } \mathcal{S} \text{ or } \mathrm{Viab}_{\mathcal{S}}(K) = \mathrm{Viab}_{\mathcal{S}}(D). \end{cases}$$

and Proposition 10.5.10, p. 404 states that if the evolutionary system $\mathcal{S}$ is lower semicontinuous, then condition $D = \mathrm{Capt}_{\mathcal{S}}(K,D)$ implies that

$$\overline{\complement D} \;=\; \mathrm{Inv}_{\mathcal{S}}(\overline{\complement D}, \overline{\complement K}) \;(\overline{\complement D} \text{ is invariant outside } \overline{\complement K})$$

Lemma 10.5.15, p. 407 states that, since the target C is assumed to be closed,

$$\overset{\circ}{\partial}(\mathrm{Int}(K \setminus C) \cap D) \;=\; \mathrm{Int}(K \setminus C) \cap \overset{\circ}{\partial} D \;=\; (\overset{\circ}{\partial} D \cap \mathrm{Int}(K)) \setminus C$$

because the interior of a finite intersection of subsets is the intersection of their interiors.

Let x belong to $\mathrm{Int}(K \setminus C) \cap \overset{\circ}{\partial}(\mathrm{Viab}_{\mathcal{S}}(K,C))$. Since $x \in D := \mathrm{Viab}_{\mathcal{S}}(K,C)$, there exists at least one evolution belonging to $\mathcal{S}(x)$ viable in K forever or until it reaches C in finite time. Take any such evolution $x(\cdot) \in \mathcal{S}(x)$. Since $x \in \overline{\complement D} := \mathrm{Inv}_{\mathcal{S}}(\overline{\complement D}, \overline{\complement K})$, this evolution $x(\cdot)$, as well as every evolution starting from x, remains viable in $\overline{\complement D}$ as long as $x(t) \in \mathrm{Int}(K)$. Therefore, it remains viable in $\mathrm{Int}(K \setminus C) \cap \overset{\circ}{\partial}(D)$ as long as $x(t) \in \mathrm{Int}(K) \setminus C = \mathrm{Int}(K \setminus C)$ (since C is assumed to be closed, thanks to the second statement of Lemma 10.5.15, p. 407).

Proposition 10.5.17, p. 409 implies that the interior $\mathrm{Int}(D \cap (K \setminus C)) = \mathrm{Int}(D) \setminus C$ is backward invariant. □

Remark. If we assume furthermore that $\mathcal{S}$ is upper semicompact, then the viability kernel with target is closed, so that its pre-boundary coincides with its boundary. □

10.6 Other Viability Characterizations

10.6.1 Characterization of Invariance Kernels

We now investigate the viability property of invariance kernels.

Proposition 10.6.1 *[**Characterization of Invariant Subsets Outside a Target**] Assume that $\mathcal{S}$ is upper lower semicontinuous. Let $C \subset K$ be closed subsets.*

Then the invariance kernel $\mathrm{Inv}_{\mathcal{S}}(K, C)$ *of K with target C under $\mathcal{S}$ is the* largest closed subset *$D \subset K$ containing C such that $D \backslash C$ is locally invariant.*

In particular, K is invariant outside C if and only if $K \backslash C$ is locally invariant.

Proof. First, we have to check that if $D \supset C$ is invariant outside C, then $D \backslash C$ is locally invariant: take $x_0 \in D \backslash C$ and prove that all evolutions $x(\cdot) \in \mathcal{S}$ starting at x_0 are viable in $D \backslash C$ on a nonempty interval. Indeed, since C is closed, there exists $\eta > 0$ such that $B(x_0, \eta) \cap C = \emptyset$, so that $x(t) \in B(x_0, \eta) \cap D \subset D \backslash C$ on some nonempty interval.

In particular, $\mathrm{Inv}_{\mathcal{S}}(K, C) \backslash C$ is locally invariant and the invariance kernel $\mathrm{Inv}_{\mathcal{S}}(K, C)$ of K with target C under $\mathcal{S}$ is closed by Theorem 10.7.8.

Let us prove now that any subset D between C and K such that $D \backslash C$ is locally invariant is contained in the invariance kernel $\mathrm{Inv}_{\mathcal{S}}(K, C)$ of K with target C under $\mathcal{S}$.

Since $C \subset \mathrm{Inv}_{\mathcal{S}}(K, C)$, let us pick any x in $D \backslash C$ and show that it belongs to $\mathrm{Inv}_{\mathcal{S}}(K, C)$. Let us take any evolution $x(\cdot) \in \mathcal{S}(x)$. Either it is viable in D forever or, if not, leaves D in finite time $\tau_D(x(\cdot))$ at $\overline{x} := x(\tau_D(x(\cdot)))$: there exists a sequence $t_n \geq \tau_D(x(\cdot))$ converging to $\tau_D(x(\cdot))$ such that $x(t_n) \notin D$. Actually, this element $\overline{x}$ belongs to C. Otherwise, since $D \backslash C$ is locally invariant, this evolution remains in D in some nonempty interval $[\tau_D(x(\cdot)), T]$, a contradiction. □

Further characterizations require properties of the invariance kernels in terms of closed viable or invariant subsets. For instance:

Proposition 10.6.2 *[**Invariance Kernels**] Let us assume that $C \subset K$ and K are closed, that $K \backslash C$ is a repeller and that the evolutionary system $\mathcal{S}$ is both upper semicompact and lower semicontinuous. Then the invariance kernel* $\mathrm{Inv}_{\mathcal{S}}(K, C)$ *is a closed subset D between C and K satisfying*

$$\begin{cases} (i) \ \ D \ = \ \mathrm{Inv}_{\mathcal{S}}(D,C) \\ (ii) \ \overline{\complement D} \ = \ \mathrm{Capt}_{\mathcal{S}}(\overline{\complement D}, \overline{\complement K}) \end{cases} \tag{10.19}$$

Furthermore, condition (10.19)(ii), p. 412 is equivalent to

$$\mathrm{Int}(D) \ = \ \mathrm{Inv}_{\mathcal{S}}(\mathrm{Int}(K), \mathrm{Int}(D)) \text{ is invariant in } \mathrm{Int}(K) \text{ outside } \mathrm{Int}(D).$$

Proof. Let us consider the invariance kernel $D := \mathrm{Inv}_{\mathcal{S}}(K,C)$. By Theorem 10.2.7, p. 382, it is the unique subset between C and K such that $D = \mathrm{Inv}_{\mathcal{S}}(D,C)$ and $D = \mathrm{Inv}_{\mathcal{S}}(K,D)$. Thanks to Lemma 2.12.2, the latter condition is equivalent to

$$\complement \mathrm{Inv}_{\mathcal{S}}(K,D) = \mathrm{Capt}_{\mathcal{S}}(\complement D, \complement K)$$

Since $\mathcal{S}$ is upper semicompact and since $\complement C \backslash \complement K = K \backslash C$ is a repeller, we deduce from Theorem 10.3.10 that

$$\overline{\complement D} \ = \ \overline{\mathrm{Capt}_{\mathcal{S}}(\complement D, \complement K)} \ \subset \ \mathrm{Capt}_{\mathcal{S}}(\overline{\complement D}, \overline{\complement K}) \ \subset \ \overline{\complement D}$$

and thus, that $\complement \overset{\circ}{D} = \mathrm{Capt}_{\mathcal{S}}(\complement \overset{\circ}{D}, \complement \overset{\circ}{K})$. By Lemma 2.12.2, this amounts to saying that $\mathrm{Int}(D) = \mathrm{Inv}_{\mathcal{S}}(\mathrm{Int}(K), \mathrm{Int}(D))$. □

Lemma 10.6.3 *[**Complement of a Separated Subset**] Let us assume that the evolutionary system $\mathcal{S}$ is upper semicompact and that a closed subset $D \subset K$ is separated from K. Then $\mathrm{Int}(K \setminus D) \backslash \mathrm{Int}(D)$ is locally viable under $\mathcal{S}$. In particular, if $C \subset K$ is closed, $\mathrm{Int}(K) \backslash \mathrm{Int}(\mathrm{Inv}_{\mathcal{S}}(K,C))$ is locally viable.*

Proof. Let $x \in \mathrm{Int}(K) \backslash \mathrm{Int}(D)$ be given and $x_n \in \mathrm{Int}(K) \backslash D$ converge to x. Since $D = \mathrm{Inv}_{\mathcal{S}}(K,D)$ is separated by assumption, for any n, there exists $x_n(\cdot) \in \mathcal{S}(x_n)$ such that

$$T_n := \varpi_{\partial K}(x_n(\cdot)) \ \le \ \tau_K(x_n(\cdot)) \le \varpi_D(x_n(\cdot))$$

because $x_n \in K \backslash D$ and $\varpi_{\partial K}(x_n(\cdot)) \le \tau_K(x_n(\cdot)) < +\infty$. Therefore, for any $t < \varpi_{\partial K}(x_n(\cdot))$, $x_n(t) \in \mathrm{Int}(K) \setminus D$.

Since $\mathcal{S}$ is upper semicompact, a subsequence (again denoted by) $x_n(\cdot)$ converges to some $x(\cdot) \in \mathcal{S}(x)$. Since the functional $\varpi_{\partial K}$ is lower semicontinuous, we know that for any $t < \varpi_{\partial K}(x(\cdot))$, we have $t < \varpi_{\partial K}(x_n(\cdot))$ for n large enough. Consequently, $x_n(t) \in \complement D$, and, passing to the limit, we infer that for any $t < \varpi_{\partial K}(x(\cdot))$, $x(t) \in \overline{\complement D}$. This solution is thus locally viable in $\mathrm{Int}(K) \backslash \mathrm{Int}(D)$. □

The boundary of the invariance kernel is locally viable:

Theorem 10.6.4 *[**Local Viability of the Boundary of an Invariance Kernel**] If $C \subset K$ and K are closed and if $\mathcal{S}$ is upper semicompact, then, for every $x \in (\overset{\circ}{\partial}(\mathrm{Inv}_{\mathcal{S}}(K,C)) \cap \mathrm{Int}(K)) \setminus C$, there exists at least one solution $x(\cdot) \in \mathcal{S}(x)$ locally viable in*

$$(\overset{\circ}{\partial}(\mathrm{Inv}_{\mathcal{S}}(K,C)) \cap \mathrm{Int}(K)) \setminus C$$

Proof. Let x belong to $\overset{\circ}{\partial}\, \mathrm{Inv}_{\mathcal{S}}(K,C) \cap \mathrm{Int}(K \setminus C)$. Lemma 10.6.3, p. 412 states there exists an evolution $x(\cdot)$ viable in $\mathrm{Int}(K) \setminus (\mathrm{Inv}_{\mathcal{S}}(K,C))$ because the invariance kernel is separated from K. Since x belongs to the invariance kernel, it is viable in $\mathrm{Inv}_{\mathcal{S}}(K,C)$ until it reaches the target C, and thus viable in $\overset{\circ}{\partial}\, \mathrm{Inv}_{\mathcal{S}}(K,C)$ as long as it is viable in the interior of $K \setminus C$. □

10.6.2 Characterization of Connection Basins

The *connection basin* $\mathrm{Conn}_{\mathcal{S}}(K,(B,C))$ of K between B and C (see Definition 8.5.1, p. 291) can be written

$$\mathrm{Conn}_{\mathcal{S}}(K,(B,C)) = \mathrm{Det}_{\mathcal{S}}(K,B) \cap \mathrm{Capt}_{\mathcal{S}}(K,C) = \mathrm{Capt}_{\overleftarrow{\mathcal{S}}}(K,B) \cap \mathrm{Capt}_{\mathcal{S}}(K,C)$$

because $\mathrm{Det}_{\mathcal{S}}(K,B) := \mathrm{Capt}_{\overleftarrow{\mathcal{S}}}(K,B)$ thanks to Lemma 8.4.5, p. 287.

We begin by proving a statement analogous to Theorem 10.2.5, p. 379 for viability kernels:

Theorem 10.6.5 *[**Characterization of Connection Basins**] Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(-\infty,\infty;X)$ be an evolutionary system, $K \subset X$ be a environment, and $B \subset K$ be a source and $C \subset K$ be a target. The connection basin $\mathrm{Conn}_{\mathcal{S}}(K,(B,C))$ is the intersection of the detection and capture basin*

$$\mathrm{Conn}_{\mathcal{S}}(K,(B,C)) \;=\; \mathrm{Det}_{\mathcal{S}}(K,B) \cap \mathrm{Capt}_{\mathcal{S}}(K,C)$$

*The connection basin is the **largest** subset $D \subset K$ of K that is connecting B to C* viable in *D, i.e., the **largest** fixed point of the map $D \mapsto \mathrm{Conn}_{\mathcal{S}}(D,(B \cap D, C \cap D))$ contained in K.*

Furthermore, all evolutions connecting B to C viable in K are actually viable in $\mathrm{Conn}_{\mathcal{S}}(D,(B \cap D, C \cap D))$.

Proof. Let us set $\overline{D} := \mathrm{Conn}_{\mathcal{S}}(K,(B,C))$.

If $\overline{D} = \emptyset$, and since $\emptyset = \mathrm{Conn}_{\mathcal{S}}(\emptyset,(B \cap \emptyset, C \cap \emptyset))$, the empty set is a fixed point of $D \mapsto \mathrm{Conn}_{\mathcal{S}}(D,(B \cap D, C \cap D))$.

Otherwise, we shall prove that

$$\overline{D} \subset \mathrm{Det}_{\mathcal{S}}(\overline{D}, \overline{D} \cap B) \cap \mathrm{Capt}_{\mathcal{S}}(\overline{D}, \overline{D} \cap C)$$

and thus, since $\overline{D} \subset \mathrm{Det}_{\mathcal{S}}(\overline{D}, \overline{D} \cap B) \cap \mathrm{Capt}_{\mathcal{S}}(\overline{D}, \overline{D} \cap C) \subset \mathrm{Conn}_{\mathcal{S}}(K,(B,C)) =: \overline{D}$, that $\overline{D}$ is a fixed point of the map $D \mapsto \mathrm{Conn}_{\mathcal{S}}(D,(B \cap D, C \cap D))$.

Indeed, let x belong to the connection basin $\overline{D}$. By Definition 8.5.1, p. 291, there exist an evolution $x(\cdot) \in \mathcal{S}(x)$ passing through x and times $\overleftarrow{T} \geq 0$ and $\overrightarrow{T} \geq 0$ such that

$$\forall\, t \in [-\overleftarrow{T}, +\overrightarrow{T}],\;\; x(t) \in K,\;\; x(\overleftarrow{T}) \in B \text{ and } x(\overrightarrow{T}) \in C$$

Now, let us consider any such evolution $x(\cdot) \in \mathcal{S}(x)$ connecting B to C and viable in K and prove that it is viable in $\mathrm{Conn}_{\mathcal{S}}(\overline{D},(B \cap \overline{D}, C \cap \overline{D}))$.

Let us consider the evolution $y(\cdot) := (\kappa(\overleftarrow{T})x(\cdot))(\cdot) \in \mathcal{S}(x(\overleftarrow{T}))$ defined by $y(t) := x(t - \overleftarrow{T})$, viable in K until it reaches the target C in finite time $\overleftarrow{T} + \overrightarrow{T}$ at $y(\overleftarrow{T} + \overrightarrow{T}) = x(\overrightarrow{T}) \in C$. This implies that $x(\overleftarrow{T}) \in \mathrm{Capt}_{\mathcal{S}}(K,C)$. Since all evolutions capturing C viable in K are actually viable in $\mathrm{Capt}_{\mathcal{S}}(K,C)$ by Lemma 10.2.4, p. 378, this implies that $y(\cdot)$ is viable in $\mathrm{Capt}_{\mathcal{S}}(K,C)$ on the interval $[0, \overleftarrow{T} + \overrightarrow{T}]$. Hence the evolution $x(\cdot) = (\kappa(-\overleftarrow{T})y(\cdot))(\cdot) \in \mathcal{S}(x)$ is viable in $\mathrm{Capt}_{\mathcal{S}}(K,C)$ on the interval $[-\overleftarrow{T}, +\overrightarrow{T}]$. We prove in the same way that the evolution $x(\cdot)$ is viable in $\mathrm{Det}_{\mathcal{S}}(K,B)$ on the interval $[-\overleftarrow{T}, +\overrightarrow{T}]$.

Therefore, this evolution $x(\cdot)$ is connecting B to C in the connection basin $\mathrm{Conn}_{\mathcal{S}}(\overline{D},(B \cap \overline{D}, C \cap \overline{D}))$ itself. Therefore, we deduce that $\overline{D}$ is a fixed point $\overline{D} = \mathrm{Conn}_{\mathcal{S}}(\overline{D},(B \cap \overline{D}, C \cap \overline{D}))$ and the largest one, obviously. □

Proposition 10.6.6 ***[Relative Bilateral Invariance of Connection Basins]*** *The connection basin* $\mathrm{Conn}_{\mathcal{S}}(K,(B,C))$ *between a source* B *and a target* C *viable in the environment* K *is both forward and backward invariant relatively to* K.

Proof. We have to prove that for every $x \in \mathrm{Conn}_{\mathcal{S}}(K,(B,C))$, every evolution $x(\cdot) \in \mathcal{S}(x)$ connecting B to C, viable in K on some time interval $[S,T]$, is actually viable in $\mathrm{Conn}_{\mathcal{S}}(K,(B,C))$ on the same interval.

Since $x(S)$ belongs to $\mathrm{Det}_{\mathcal{S}}(K,B)$, there exist an element $b \in B$, a time $T_b \geq 0$ and an evolution $z_b(\cdot) \in \mathcal{S}(b)$ viable in K until it reaches $x(S) = z_b(T_b)$ at time T_b. Since $x(T)$ belongs to $\mathrm{Capt}_{\mathcal{S}}(K,C)$, there exist $T_c \geq 0$, an element

$c \in C$ and an evolution $z_c(\cdot) \in \mathcal{S}(x(T))$ such that $z_c(T_c) = c \in C$ and, for all $t \in [0, S]$, $z_c(t) \in K$. We associate with these evolutions their concatenation $y(\cdot) \in \mathcal{S}(b)$ defined by

$$y(t) \ := \begin{cases} z_b(t) & \text{if } t \in [0, T_b] \\ x(t + S - T_b) & \text{if } t \in [T_b, T_b + T - S] \\ z_c(t - T_b + S - T) & \text{if } t \in [T_b + T - S, T_b + T - S + T_c] \end{cases}$$

starting at $b \in B$ is viable in K until it reaches C at time $T_b + S + T + T_c$. This means that b belongs to $\text{Conn}_{\mathcal{S}}(K, (B, C))$ and this implies that for every $t \in [0, T_b+S+T+T_c]$, $y(t)$ belongs to the connection basin $\text{Conn}_{\mathcal{S}}(K, (B, C))$. This is in particular the case when $t \in [S, T]$: then $x(t) = y(t + T_b - S)$ belongs to the capture basin. Therefore, the evolution $x(\cdot)$ is viable in $\text{Conn}_{\mathcal{S}}(K, (B, C))$ on the same interval $[S, T]$. □

Theorem 10.6.7 ***[Characterization of Bilateral Relative Invariance]*** *A subset $D \subset K$ is bilaterally invariant relatively to K if and only if $\text{Conn}_{\mathcal{S}}(K, (D, D)) = D$.*

Proof. First, Proposition 10.6.6, p. 414 implies that whenever $\text{Conn}_{\mathcal{S}}(K, (D, D)) = D$, D is bilaterally invariant relatively to K.

Conversely, assume that D is bilaterally invariant relatively to K and we shall derive a contradiction by assuming that there exists $x \in \text{Conn}_{\mathcal{S}}(K, (D, D)) \setminus D$. Indeed, there would exist an evolution $x(\cdot) \in \mathcal{S}(x)$ through x, times $T_b \geq 0$ and $T_c \geq 0$ and elements $b \in D$ and $c \in D$ such that $x(-T_b) = b$, $x(T_c) = c$ and viable in K on the interval $[-T_b, +T_c]$. Since D is bilaterally viable and since $x(\cdot)$ is bilaterally viable in K, it is bilaterally viable in D by assumption. Therefore, for all $t \in [-T_b, +T_c]$, $x(t)$ belongs to D, and in particular, for $t = 0$: then $x(0) = x$ belongs to D, the contradiction we were looking for. □

Theorem 10.6.8 ***[Characterization of Connection Basins as Unique Bilateral Fixed Point]*** *Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ be an evolutionary system, $K \subset X$ be a environment and $C \subset K$ be a target. The connection basin $\text{Conn}_{\mathcal{S}}(K, (D, D))$ between subset C and itself is the* unique *bilateral fixed point between C and K of the map $(L, D) \mapsto \text{Conn}_{\mathcal{S}}(L, (D, D))$ in the sense that*

$$D \ = \ \text{Conn}_{\mathcal{S}}(D, (C, C)) \ = \ \text{Conn}_{\mathcal{S}}(K, (D, D))$$

Proof. Let us consider the map $(K,C) \mapsto \mathcal{A}(K,C) := \mathrm{Conn}_{\mathcal{S}}(K,(C,C))$. It satisfies properties (10.5), p. 381:

$$\begin{cases} (i) & C \subset \mathcal{A}(K,C) \subset K \\ (ii) & (K,C) \text{ is increasing} \end{cases}$$

Theorem 10.6.5, p. 413 states that $\mathcal{A}(K,C) := \mathrm{Conn}_{\mathcal{S}}(K,(C,C))$ is a fixed point of $L \mapsto \mathcal{A}(L,C)$ and Theorem 10.6.7, p. 415 that $\mathcal{A}(K,C)$ is a fixed point of $D \mapsto \mathcal{A}(K,D)$. Then $\mathcal{A}(K,C)$ is the unique bilateral fixed point of the map D between C and K of the map $\mathcal{A}$: $D = \mathcal{A}(D,C) = \mathcal{A}(K,D)$ thanks to the Uniqueness Lemma 10.2.6, p. 381. □

10.7 Stability of Viability and Invariance Kernels

In this section we study conditions under which kernels and basins of limit of a sequence of environments and/or of targets is the limit of these kernels and basins, and apply these results to the existence of viability envelopes in Sect. 10.7.2, p. 420.

10.7.1 Kernels and Basins of Limits of Environments

Let us consider a sequence of environments $K_n \subset X$, of targets $C_n \subset K_n$, and of viability kernels $\mathrm{Viab}_{\mathcal{S}}(K_n, C_n)$ of K_n with targets C_n under a given evolutionary system $\mathcal{S}$.

A natural and important question arises whether we can "take the limit" and *compare the limit of the viability kernels and the viability kernels of the limits.*

Answers to such questions require first an adequate concept of limit. Here, dealing with subsets, the natural concept of limit is the one of the Painlevé–Kuratowski upper limit of subsets. We recall the Definition 18.4.1, p. 728:

Definition 10.7.1 *[**Upper Limit of Sets**] Let $(K_n)_{n\in\mathbf{N}}$ be a sequence of subsets of a metric space E. We say that the subset*

$$\mathrm{Limsup}_{n\to\infty} K_n := \left\{ x \in E \;\middle|\; \liminf_{n\to\infty} d(x, K_n) = 0 \right\}$$

is the upper limit *of the sequence K_n.*

We would like to derive formulas of the type

$$\text{Limsup}_{n\to+\infty}\text{Viab}_{\mathcal{S}}(K_n, C_n) \subset \text{Viab}_{\mathcal{S}}(\text{Limsup}_{n\to+\infty}K_n, \text{Limsup}_{n\to+\infty}C_n)$$

and analogous formulas for invariance kernels.

10.7.1.1 Upper Limit of Subsets of Viable Evolutions

It is worth recalling that the viability kernel

$$\text{Viab}_{\mathcal{S}}(K, C) = \mathcal{S}^{-1}(\mathcal{V}(K, C))$$

is the inverse image of the subset $\mathcal{V}(K, C) \subset \mathcal{C}(0, +\infty; X)$ of evolutions *viable in* K *outside* C defined by (2.5):

$$\begin{cases} \mathcal{V}(K, C) := \{x(\cdot) \text{ such that } \forall t \geq 0, \;\; x(t) \in K \\ \text{or } \exists\, T \geq 0 \text{ such that } x(T) \in C \;\&\; \forall t \in [0, T], \; x(t) \;\in\; K\} \end{cases}$$

and that the invariance kernel

$$\text{Inv}_{\mathcal{S}}(K, C) = \mathcal{S}^{\ominus 1}(\mathcal{V}(K, C))$$

is the core of the subset $\mathcal{V}(K, C) \subset \mathcal{C}(0, +\infty; X)$.

Hence, we begin by studying the upper limits of subsets $\mathcal{V}(K_n, C_n)$:

Lemma 10.7.2 ***[Upper Limit of Subsets of Viable Evolutions]*** *For any sequence of environments* $K_n \subset X$ *and any target* $C_n \subset K_n$*,*

$$\text{Limsup}_{n\to+\infty}\mathcal{V}(K_n, C_n) \subset \mathcal{V}(\text{Limsup}_{n\to+\infty}K_n, \text{Limsup}_{n\to+\infty}C_n)$$

Proof. The proof is a slight generalization of the proof of Lemma 10.3.9, p. 388. Let us consider a sequence of evolutions $x_n(\cdot) \in \mathcal{V}(K_n, C_n)$ converging to some evolution $x(\cdot)$. We have to prove that $x(\cdot)$ belongs to $\mathcal{V}(\text{Limsup}_{n\to+\infty}K_n, \text{Limsup}_{n\to+\infty}C_n)$, i.e., that is viable in $\text{Limsup}_{n\to+\infty}K_n$ forever or until it reaches the target $\text{Limsup}_{n\to+\infty}C_n$ in finite time.

Indeed:

1. either for any $T > 0$ and any $N > 0$, there exist $n \geq N$, $t_n \geq T$ and an evolution $x_n(\cdot)$ for which $x_n(t) \in K_n$ for every $t \in [0, t_n]$,
2. Or there exist $T > 0$ and $N > 0$ such that for any $n \geq N$ and any evolution $x_n(\cdot)$, there exists $t_n \leq T$ such that $x_n(t_n) \notin K_n$.

In the first case, we deduce that for any $T > 0$, $x(T) \in \text{Limsup}_{n\to+\infty}K_n$, so that the limit $x(\cdot)$ is viable in $\text{Limsup}_{n\to+\infty}K_n$ forever. In the second case, all the solutions $x_n(\cdot)$ leave K_n before T. This is impossible if evolutions $x_n(\cdot)$ are viable in K_n forever. Therefore, since $x_n(\cdot) \in \mathcal{V}(K_n, C_n)$, they have to reach C_n before leaving K_n: there exists $s_n \leq T$ such that

$$x_n(s_n) \in C_n \ \& \ \forall t \in [0, s_n], \ x_n(t) \ \in \ K_n$$

Then a subsequence $s_{n'}$ converges to some $S \in [0,T]$. Therefore, for any $s < S$, then $s < s_{n'}$ for n' large enough, so that $x_{n'}(s) \ \in \ K_n$. By taking the limit, we infer that for every $s < S$, $x(s) \in \text{Limsup}_{n \to +\infty} K_n$. Furthermore, since $x_n(\cdot)$ converges to $x(\cdot)$ uniformly on the compact interval $[0,T]$, then $x_n(s_n) \in C_n$ converges to $x(S)$, which therefore belongs to $\text{Limsup}_{n \to +\infty} C_n$.

This shows that the limit $x(\cdot)$ belongs to $\mathcal{V}(\text{Limsup}_{n \to +\infty} K_n, \text{Limsup}_{n \to +\infty} C_n)$. $\square$

10.7.1.2 Upper Limits of Inverse Images and Cores

Stability problems amount to study the upper limits of inverse images and cores of subsets $\mathcal{H}_n \subset \mathcal{C}(0, +\infty; X)$ of evolutions, such as the subsets $\mathcal{V}(K_n, C_n)$ defined by (2.5), p. 49.

Theorem 10.7.3 ***[Upper Limit of Inverse Images]*** *Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ be an upper semicompact evolutionary system. Then for any sequence of subsets $\mathcal{H}_n \subset \mathcal{C}(0, +\infty; X)$,*

$$\text{Limsup}_{n \to +\infty} \mathcal{S}^{-1}(\mathcal{H}_n) \ \subset \ \mathcal{S}^{-1}(\text{Limsup}_{n \to +\infty} \mathcal{H}_n)$$

Proof. Let $x \in \text{Limsup}_{n \to +\infty} \mathcal{S}^{-1}(\mathcal{H}_n)$ be the limit of a sequence of elements $x_n \ \in \ \mathcal{S}^{-1}(\mathcal{H}_n)$. Hence there exist evolutions $x_n(\cdot) \ \in \ \mathcal{S}(x_n) \ \in \ \mathcal{H}_n$. Since $\mathcal{S}$ is upper semicompact, there exists a subsequence of evolutions $x_{n'}(\cdot) \ \in \ \mathcal{S}(x_{n'})$ starting at $x_{n'}$ and converging to some $x(\cdot) \ \in \ \mathcal{S}(x)$. It also belongs to the upper limit $\text{Limsup}_{n \to +\infty} \mathcal{H}_n$ of the subsets $\mathcal{H}_n$, so that $x \in \mathcal{S}^{-1}(\text{Limsup}_{n \to +\infty} \mathcal{H}_n)$. $\square$

For cores, we obtain

Theorem 10.7.4 ***[Upper Limit of Cores]*** *Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ be a lower semicontinous evolutionary system. Then for any sequence of subsets $\mathcal{H}_n \subset \mathcal{C}(0, +\infty; X)$,*

$$\text{Limsup}_{n \to +\infty} \mathcal{S}^{\ominus 1}(\mathcal{H}_n) \ \subset \ \mathcal{S}^{\ominus 1}(\text{Limsup}_{n \to +\infty} \mathcal{H}_n)$$

Proof. Let us consider a sequence of subsets $\mathcal{H}_n \subset \mathcal{C}(0,+\infty;X)$ and a sequence of elements $x_n \in \mathcal{S}^{\ominus 1}(\mathcal{H}_n)$ converging to some $x \in \mathrm{Limsup}_{n\to+\infty}\mathcal{S}^{\ominus 1}(\mathcal{H}_n)$. We have to prove that any $x(\cdot) \in \mathcal{S}(x)$ belongs to $\mathrm{Limsup}_{n\to+\infty}\mathcal{H}_n$. Indeed, since $\mathcal{S}$ is lower semicontinuous, there exists a sequence of elements $x_n(\cdot) \in \mathcal{S}(x_n) \subset \mathcal{H}_n$ converging to $x(\cdot)$. Therefore the evolution $x(\cdot)$ belongs to the upper limit $\mathrm{Limsup}_{n\to+\infty}\mathcal{H}_n$ of the subsets $\mathcal{H}_n$. Since the evolution $x(\cdot)$ was chosen arbitrarily in $\mathcal{S}(x)$, we infer that $x \in \mathcal{S}^{\ominus 1}(\mathrm{Limsup}_{n\to+\infty}\mathcal{H}_n)$. □

10.7.1.3 Upper Limits of Viability and Invariance Kernels

Theorem 10.7.3 and Lemma 10.7.2 imply

Theorem 10.7.5 ***[Upper Limit of Viability Kernels]*** *Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,+\infty;X)$ be an upper semicompact evolutionary system. Then for any sequence of environments $K_n \subset X$ and of targets $C_n \subset K_n$,*

$$\mathrm{Limsup}_{n\to+\infty}\mathrm{Viab}_{\mathcal{S}}(K_n,C_n) \subset \mathrm{Viab}_{\mathcal{S}}(\mathrm{Limsup}_{n\to+\infty}K_n, \mathrm{Limsup}_{n\to+\infty}C_n)$$

For capture basins, we obtain another property:

Lemma 10.7.6 ***[Upper Limit of Capture Basins]*** *If the set-valued map $\overleftarrow{\mathcal{S}}$ is lower semicontinuous and if K is backward invariant, then for any closed subset $C \subset K$,*

$$\mathrm{Capt}_{\mathcal{S}}(\mathrm{Limsup}_{n\to+\infty}K_n, \mathrm{Limsup}_{n\to+\infty}C_n) \subset \mathrm{Limsup}_{n\to+\infty}\mathrm{Capt}_{\mathcal{S}}(K_n,C_n) \tag{10.20}$$

Proof. Let us take $x \in \mathrm{Capt}_{\mathcal{S}}(\mathrm{Limsup}_{n\to+\infty}K_n, \mathrm{Limsup}_{n\to+\infty}C_n)$ and an evolution $x(\cdot) \in \mathcal{S}(x)$ viable in $\mathrm{Limsup}_{n\to+\infty}K_n$ until it reaches the target $\mathrm{Limsup}_{n\to+\infty}C_n$ at time $T < +\infty$ at $c := x(T) \in \mathrm{Limsup}_{n\to+\infty}C_n$. Hence the function $t \mapsto y(t) := x(T-t)$ is an evolution $y(\cdot) \in \overleftarrow{\mathcal{S}}(c)$. Let us consider a sequence of elements $c_n \in C_n$ converging to c. Since $\overleftarrow{\mathcal{S}}$ is lower semicontinuous, there exist evolutions $y_n(\cdot) \in \overleftarrow{\mathcal{S}}(c_n)$ converging uniformly over compact intervals to $y(\cdot)$. These evolutions $y_n(\cdot)$ are viable in K_n, since K_n is assumed to be backward invariant, so that $x_n(0)$ belongs to $\mathrm{Capt}_{\mathcal{S}}(K_n,C_n)$. Therefore $x_n(0) := y_n(T)$ converges to $x := x(0)$. □

Putting together these results, we obtain the following useful theorem on the stability of capture basins:

Theorem 10.7.7 ***[Stability Properties of Capture Basins]*** *Let us consider a sequence of closed subsets* C_n *satisfying* $\mathrm{Viab}_{\mathcal{S}}(K) \subset C_n \subset K$ *and*

$$\mathrm{Lim}_{n\to+\infty} C_n := \mathrm{Limsup}_{n\to+\infty} C_n = \mathrm{Liminf}_{n\to+\infty} C_n$$

If the evolutionary system $\mathcal{S}$ *is upper semicompact and lower semicontinuous and if* K *is closed and backward invariant under* $\mathcal{S}$*, then*

$$\mathrm{Lim}_{n\to+\infty}\mathrm{Capt}_{\mathcal{S}}(K, C_n) = \mathrm{Capt}_{\mathcal{S}}(K, \mathrm{Lim}_{n\to+\infty} C_n) \tag{10.21}$$

For invariance kernels, we deduce from Theorem 10.7.4 and Lemma 10.7.2 the stability theorem:

Lemma 10.7.8 ***[Upper Limit of Invariance Kernels]*** *Let* $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ *be a lower semicontinuous evolutionary system. Then for any sequence of environments* $K_n \subset X$ *and any target* $C_n \subset K_n$*,*

$$\mathrm{Limsup}_{n\to+\infty}\mathrm{Inv}_{\mathcal{S}}(K_n, C_n) \subset \mathrm{Inv}_{\mathcal{S}}(\mathrm{Limsup}_{n\to+\infty} K_n, \mathrm{Limsup}_{n\to+\infty} C_n)$$

10.7.2 Invariance and Viability Envelopes

Since the intersection of sets that are invariant under an evolutionary system is still invariant, it is natural to introduce the smallest invariant subset containing a given set:

Definition 10.7.9 *[**Invariance Envelope**] We shall say that the smallest invariant subset containing C is the* invariance envelope of C *and that the smallest subset of K containing C invariant outside C is the* invariance envelope of K outside C.

However, an intersection of subsets viable under an evolutionary system is not necessarily viable. Nevertheless, we may introduce the concept of minimal subsets viable outside a target:

Definition 10.7.10 *[**Viability Envelope**] Let L be any subset satisfying $C \subset L \subset \text{Viab}_{\mathcal{S}}(K, C)$. A (resp. closed)* viability envelope *of K with target C is any (resp. closed) set $L^\star \supset L$ viable outside C such that there is no strictly smaller subset $M \supset L$ viable outside C.*

We prove the existence of viability envelopes:

Proposition 10.7.11 *[**Existence of Viability Envelopes**] Let K be a closed subset viable under an upper semicompact evolutionary system $\mathcal{S}$. Then any closed subset $L \subset K$ is contained into a* viability envelopes of L *under $\mathcal{S}$.*

Proof. We apply Zorn's lemma for the inclusion order on the family of nonempty closed subsets viable under $\mathcal{S}$ between L and K. For that purpose, consider any decreasing family of closed subsets M_i, $i \in I$, viable under $\mathcal{S}$ and their intersection $M_\star := \bigcap_{i \in I} M_i$. It is a closed subset viable under $\mathcal{S}$ thanks to the Stability Theorem 10.7.7. Therefore every subset $L \subset K$ is contained in a minimal element for this preorder. □

When $L = \emptyset$, we have to assume that K is compact to guarantee that the intersection of any decreasing family of nonempty closed subset viable under $\mathcal{S}$ is not empty. In this case, we obtain the following

Proposition 10.7.12 *[**Non emptiness of Viability Envelopes**] Let K be a nonempty compact subset viable under an upper semicompact evolutionary system $\mathcal{S}$. Then nonempty minimal closed subsets M viable under $\mathcal{S}$ exist and are made of limit sets of viable evolutions. Actually, they exhibit the following property:*

$$\forall x \in M, \ \exists \, x(\cdot) \in \mathcal{S}(x) \mid x \in M = \omega(x(\cdot))$$

where, by Definition 9.3.1, p. 344, $\omega(x(\cdot)) := \bigcap_{T>0} \overline{(x([T,\infty[))}$ *is the* ω-limit set *of* $x(\cdot)$.

Proof. Let $M \subset K$ be a minimal closed subset viable under $\mathcal{S}$. We can associate with any $x \in M$ a viable evolution $x(\cdot) \in \mathcal{S}(x)$ starting at x. Hence its limit set $\omega(x(\cdot))$ is contained in M. But limit sets being closed subsets viable under $\mathcal{S}$ by Theorem 9.3.11 and M being minimal, it is equal to $\omega(x(\cdot))$, so that $x \in \omega(x(\cdot))$. □

10.8 The Hard Version of the Inertia Principle

Exit sets also play a crucial role for regulating viable evolutions with a finite number of feedbacks instead of the unique feedback, which, whenever it exists, regulates viable evolutions. However, even when its existence is guaranteed and when the Viability Kernel Algorithm allows us to compute it, it is often preferable to use available and well known feedbacks derived from a long history than computing a new one. Hence, arises the question of "quantized retroactions" using a subset of the set of all available retroactions (see Sect. 6.4, p. 207 for fixed degree open loop controls). In this section, we are investigating under which conditions a given finite subset of available feedbacks suffices to govern viable evolutions. We have to give a precise definition of the concept of "amalgams" of feedbacks for producing other ones, in the same way that a finite number of monomials generates the class of fixed degree polynomials. Once this operation which governs *concatenations* of evolutions defined, we can easily characterize a condition involving the exit set of the system under each of the finite class of systems. They govern specific evolutions satisfying the *hard version* of the *inertia principle*.

33 *[**Quantized Controls.**] Recent important issues in control theory are known under the name of "*quantized controls*", where, instead of finding adequate retroactions for governing evolutions satisfying such and such properties (viability, capturability, optimality, etc.), we are restricting the regulation of these evolutions by a smaller class of retroactions generated in some way by a* finite number *of feedbacks. Indeed, the* regulation map *(see Definition 2.14.3, p. 98) using the entire family of controls* $u \in U(x)$ *may be too difficult to construct.* Quantized control *combines only a finite number of retroactions to regulate viable, capturing or optimal evolutions.*

Chapter 6, p. 199 provided examples of such quantized systems where the retroactions are open loops controls made of polynomials of fixed degree m. The regulation by the *amalgam* of a finite number of given feedbacks provides another answer to the issue of quantization. Indeed, let us consider control system

$$x'(t) \ = \ f(x(t), u(t)) \text{ where } u(t) \ \in \ U(x(t))$$

We introduce a *finite family* of closed loop feedbacks $\widetilde{u}_i : x \rightsquigarrow \widetilde{u}_i(x) \ \in U(x)$ and $i \in I$ where I is a finite number of indices. They define a finite number of evolutionary systems $\mathcal{S}_i$ associated with differential equations

$$x'(t) \ = \ f(x(t), \widetilde{u}_i(x(t)))$$

Each of these a priori feedbacks is not especially designed to regulate viable evolutions in an arbitrary set, for instance. A compromise is obtained by "amalgamating" those closed loop feedbacks for obtaining the following class of retroactions (see Definition 2.7.2, p. 65):

Recall (Definition 18.3.12, p. 724) that the *mark* $\Xi_{[s,t]\times A} := \Xi^{\mathcal{U}}_{[s,t]\times A} : \mathbb{R} \times X \rightsquigarrow \mathcal{U}$ of a subset $[s,t] \times A$ is defined by

$$\Xi_{[s,t]\times A}(\tau, x) \ := \begin{cases} \mathcal{U} \text{ if } (\tau, x) \in [s,t] \times A \\ \emptyset \ \text{ if } (\tau, x) \notin [s,t] \times A \end{cases} \tag{10.22}$$

and plays the role of a "characteristic set-valued map of a subset". Therefore, for any $u \in \mathcal{U}$

$$u \cap \Xi_{[s,t]\times A}(\tau, x) \ := \begin{cases} \{u\} \text{ if } (\tau, x) \in [s,t] \times A \\ \emptyset \quad \text{ if } (\tau, x) \notin [s,t] \times A \end{cases}$$

Definition 10.8.1 *[**Amalgam of Feedbacks**] Let us consider a family of feedbacks $\widetilde{u}_i : X \rightsquigarrow \mathcal{U}$, a covering $X = \bigcup_{i \in I} A_i$ of X and an increasing sequence of instants t_i, $i = 0, \ldots, n$. The associated* amalgam *of these feedbacks is the retroaction*

$$\widetilde{u} := \bigcup_{i \geq 0} \widetilde{u}_i \cap \Xi_{[t_i, t_{i+1}[\times A_i}$$

defined by

$$\widetilde{u}(t, x) := \widetilde{u}_i(x) \text{ if } t \in [t_i, t_{i+1}[\text{ and } x \in A_i$$

Amalgams of feedbacks play an important role in the regulation of control and regulated systems.

Proposition 10.5.1, p. 399 characterizes environments viable outside a target C under an evolutionary system $\mathcal{S}$ if and only if $\text{Exit}_{\mathcal{S}}(K) \subset C$.

What happens if a given environment is not viable under any of the evolutionary systems $\mathcal{S}_i$ of a finite family $i \in I$? Is it possible to restore the viability by letting these evolutionary system cooperate?

To say that K is not viable outside C under $\mathcal{S}_i$ means that $\text{Exit}_{\mathcal{S}_i}(K) \cap \complement C \neq \emptyset$. However, even though K is not viable under each of the system $\mathcal{S}_i$, it may be possible that amalgated together, the collective condition

$$\bigcap_{i \in I} \text{Exit}_{\mathcal{S}_i}(K) \subset C$$

weaker than the individual condition $\text{Exit}_{\mathcal{S}}(K) \subset C$ may be enough to regulate a control system. This happens to be the case, if, for that purpose, we define the "cooperation" between evolutionary systems $\mathcal{S}_i$ by "amalgamating" them. For control systems, amalgamating feedbacks amounts to amalgamating the associated evolutionary system.

The examination of the exit sets of each of the evolutionary systems allows us to answer this important practical question by using the notion of the *amalgam* $\mathcal{S}^{\ddagger}$ of the evolutionary systems $\mathcal{S}_i$:

Definition 10.8.2 *[**Amalgam of a Family of Evolutionary Systems**] The* amalgamated system $\mathcal{S}^{\ddagger}$ *of the evolutionary systems $\mathcal{S}_i$ associates with any $x \in K$ concatenated evolutions $x(\cdot)$ associated with sequences of indices i_p, $p \in \mathbb{N}$, of times $\tau_{i_p} > 0$ and of evolutions $x_{i_p}(\cdot) \in \mathcal{S}_{i_p}(x_p)$ such that, defining*

$$t_0 = 0, \quad t_{p+1} := t_p + \tau_{i_p}$$

the evolution $x(\cdot)$ is defined by

$$\forall\, p \geq 0, \;\; \forall\, t \in [t_p, t_{p+1}], \;\; x(t) := x_{i_p}(t - t_p) \text{ and } x_{i_p}(t_{p+1}) = x_{p+1}$$

where $x_0 := x$ and $x_p := x_{i_{p-1}}(t_p)$, $p \geq 1$.

We derive a viability criterion allowing us to check whether a target $C \subset K$ can be captured under the amalgated system:

Theorem 10.8.3 *[**Viability Under Amalgams of Evolutionary Systems**] Let us consider a finite set I of indices and a family of upper semicompact evolutionary systems $\mathcal{S}_i$. Assume that K is a repeller under each evolutionary system $\mathcal{S}_i$ and that*

$$\bigcap_{i \in I} \text{Exit}_{\mathcal{S}_i}(K) \ \subset \ C \tag{10.23}$$

then K is viable outside C under the amalgam $\mathcal{S}^{\ddagger}$ of the evolutionary systems $\mathcal{S}_i$

$$\text{Viab}_{\mathcal{S}^{\ddagger}}(K, C) \ = \ K$$

Proof. For simplicity, we set $\mathcal{S}^{\sharp} := \mathcal{S}^{K^{\sharp}} \subset \mathcal{S}$ the sub-evolutionary system generating persistent evolutions (see Definition 10.4.2, p. 393).

Let us set $E_i := \text{Exit}_{\mathcal{S}_i}(K)$. Assumption $\bigcap_{i \in I} E_i \subset C$ amounts to saying that

$$K \setminus C \ = \ \bigcup_{i \in I} (K \setminus E_i)$$

Therefore, we associate with any $x \in K \setminus C$ the set $I(x) \subset I$ of indices i such that

$$\tau_K^{\mathcal{S}_i^{\sharp}}(x) \ := \ \max_{k \in I} \tau_K^{\mathcal{S}_k^{\sharp}}(x)$$

achieving the maximum of the exit times for each evolutionary system $\mathcal{S}_k^{\sharp}$. For each index $i \in I$, and $e_i \in E_i$, one can observe that $\max_{j \in I} \tau_K^{\mathcal{S}_j^{\sharp}}(e_i) = \max_{j \in I(e_i)} \tau_K^{\mathcal{S}_j^{\sharp}}(e_i)$. We next define the smallest of the largest exit times of states ranging the exit sets of each evolutionary system $\mathcal{S}_j^{\sharp}$:

$$\overline{\tau} \ := \ \min_{i \in I} \ \sup_{e_i \in \text{Exit}_{\mathcal{S}_i}(K)} \ \sup_{j \in I(e_i)} \tau_K^{\mathcal{S}_j^{\sharp}}(e_i)$$

Since the set I of indices is finite, assumption $\bigcap_{i \in I} E_i \ \subset \ C$ implies that $0 < \overline{\tau} < +\infty$.

This being said, we can build a concatenated evolution $x(\cdot)$ made of an increasing sequence of times t_p and of "pieces" of *persistent evolutions* $x_{i_p}^{\sharp}(\cdot) \in \mathcal{S}_{i_p}(x_{i_p})$ defined by

$$\forall \, p \ \geq \ 0, \ \ \forall \, t \in [t_p, t_{p+1}], \ \ x(t) \ := \ x_{i_p}^{\sharp}(t - t_p) \text{ and } x(t_{p+1}) \ = \ x_{p+1}$$

which is viable in K forever or until a finite time when it reaches C.

Indeed, we associate with any evolutionary system $\mathcal{S}_i$ and any $x_i \in K$ a *persistent evolution* $x_i^{\sharp}(\cdot) \in \mathcal{S}_i(x_i)$, its exit time $\tau_i^{\sharp} > 0$ (since we assumed that K is a repeller under each evolutionary system $\mathcal{S}_i$) and an exit state $e_i^{\sharp} \in E_i := \text{Exit}_{\mathcal{S}_i}(K)$.

To say that $\bigcap_{i \in I} E_i \subset C$ amounts to saying that

$$K \setminus C \;=\; \bigcup_{i \in I} (K \setminus E_i)$$

Therefore, starting with any initial state $x_0 \in K \setminus C$, we infer from the assumption $\bigcap_{i\in I} E_i \subset C$ that there exists $i_0 \in I$ such that $x \in K \setminus E_{i_0}$. Hence, we associate $x^\sharp_{i_0}(\cdot) \in \mathcal{S}_{i_0}(x_{i_0})$, its exit time $\tau^\sharp_{i_0} > 0$ and an exit state $e^\sharp_{i_0} \in E_{i_0}$. Setting $x_1 := e^\sharp_{i_0}$, either $x_1 \in C$, and the evolution $x(\cdot) := x^\sharp_{i_0}(\cdot)$ reaches C in finite time 0, or $x_1 \in K \setminus C$, and our assumption implies the existence of $i_1 \in I$ such that $x \in K \setminus E_{i_1}$, so that we can find a $x^\sharp_{i_1}(\cdot) \in \mathcal{S}_{i_1}(x_1)$, its exit time $\tau^\sharp_{i_1} > 0$ and an exit state $e^\sharp_{i_1} \in E_{i_1}$. And so on, knowing that $e^\sharp_{i_{p-1}} \in E_{i_{p-1}} \in K \setminus C$, we choose an index $i_p \in I$ such that $x_{i_p} \in K \setminus E_{i_p}$ and built recursively evolutions $x^\sharp_{i_p}(\cdot) \in \mathcal{S}_{i_p}(x_{i_p})$, its exit time $\tau^\sharp_{i_p} > 0$ and an exit state $e^\sharp_{i_p} \in E_{i_p}$.

We associate with this sequence of evolutions the sequence of times defined by

$$t_0 = 0,\;\; t_{p+1} \;:=\; t_p + \tau^\sharp_{i_p}$$

and evolutions

$$x_p(t) := (\kappa(\tau^\sharp_{i_p}) x^\sharp_{i_p}(\cdot))(t) = x^\sharp_{i_p}(t - \tau^\sharp_{i_p}) \text{ where } t \in [t_p, t_{p+1}] \text{ and } x_p(t_{p+1}) = x_{p+1}$$

and their concatenation $x(\cdot)$ defined by

$$\forall\, p \;\geq\; 0,\;\; \forall\, t \in [t_p, t_{p+1}],\;\; x(t) \;:=\; x_p(t)$$

Since the set I of indices is assumed to be finite, then $\overline{\tau} > 0$, so that the concatenated evolution is defined on $\mathbb{R}_+$ because $\sum_{p=0}^{+\infty} \tau^\sharp_{i_p} = +\infty$. Hence the concatenated evolution of persistent evolutions is viable forever or until it reaches the target C in finite time. □

We mentioned in Sect. 6.4, p. 207 the concept of the "soft" version of the inertia principle. Persistent evolutions and Theorem 10.8.3, p. 424 provide the "hard version" of this principle:

34 *[**The Hard Inertia Principle**] Theorem 10.8.3, p. 424 provides another answer to the* inertia principle *(see Sect. 6.4.4, p. 217) without inertia threshold:* When, where and how *change the available feedbacks (among them, constant controls) to maintain the viability of a system. Starting with a finite set of regulons, the system uses them successively as long as possible (*persistent evolutions*), up to the exit time (warning signal) and its exit set, which is its* critical zone *(see Definition 6.4.9, p. 216).*

In summary, when the viability is at stakes:

1. The *hard* version of the inertia principle requires that whenever the evolution reaches the boundary, then, and not before, the state has to switch instantaneously to a *new initial state* and a *new feedback* has to be chosen,
2. The *soft* version of the inertia principle involves an inertia threshold determining when, at the right time, the kairos, where, in the critical zone, the regulon only has to *evolve and how*.

10.9 Parameter Identification: Inverse Viability and Invariance Maps

When the differential inclusion (or parameterized system) $F(\lambda, \cdot)$, the environment $K(\lambda)$ and the target $C(\lambda)$ depend upon a parameter $\lambda \in Y$ ranging over a finite dimensional vector space Y, a typical example of *inverse problem* (see Comment 2, p. 5) is to associate with any state x the subset of parameters λ such that we know, for instance, that x belongs to the viability kernel $\mathbb{V}(\lambda) := \text{Viab}_{F(\lambda,\cdot)}(K(\lambda), C(\lambda))$.

The set of such parameters λ is equal to $\mathbb{V}^{-1}(x)$, where $\mathbb{V}^{-1} : X \rightsquigarrow Y$ is the inverse of the set-valued map $\mathbb{V} : Y \mapsto X$ associating with λ the viability kernel $\mathbb{V}(\lambda) := \text{Viab}_{F(\lambda,\cdot)}(K(\lambda), C(\lambda))$.

In control terminology, the search of those parameters λ such that a given x belongs to $\mathbb{V}(\lambda) := \text{Viab}_{F(\lambda,\cdot)}(K(\lambda), C(\lambda))$ is called a *parameter identification problem* formulated for viability problems. This covers as many examples as problems which can be formulated in terms of kernels and basins, as the ones covered in this book. As we shall see, most of the examples covered in Chaps. 4, p. 125 and 6, p. 199 are examples of inverse viability problems.

10.9.1 Inverse Viability and Invariance

It turns out that for these types of problems, the solution can be obtained by viability techniques. Whenever we know the graph of a set-valued map, we know both this map and its inverse (see Definition 18.3.1, p. 719). The graphs of such maps associating kernels and basins with those parameters are also kernels and basins of auxiliary environments and targets under auxiliary systems. Therefore, *they inherit their properties, which are then shared by both the set-valued map and its inverse.* This simple remark is quite useful.

Let us consider the parameterized differential inclusion

$$x'(t) \in F(\lambda, x(t)) \tag{10.24}$$

when environments $K(\lambda)$ and targets $C(\lambda)$ depend upon a parameter $\lambda \in Y$ ranging over a finite dimensional vector space Y. We set $F(\lambda, \cdot) : x \rightsquigarrow F(\lambda, x)$.

The problem is to *invert* the set-valued maps

$$\mathbb{V} : \lambda \rightsquigarrow \text{Viab}_{F(\lambda,\cdot)}(K(\lambda), C(\lambda)) \text{ and } \mathbb{I} : \lambda \rightsquigarrow \text{Inv}_{F(\lambda,\cdot)}(K(\lambda), C(\lambda))$$

For that purpose, we shall characterize the graphs of these maps:

Proposition 10.9.1 ***[Graph of the Viability Map]*** *The graph of the map* $\mathbb{V} : \lambda \rightsquigarrow \text{Viab}_{F(\lambda,\cdot)}(K(\lambda), C(\lambda))$ *is equal to the viability kernel*

$$\text{Graph}(\mathbb{V}) \;=\; \text{Viab}_{(10.25)}(\mathcal{K}, \mathcal{C})$$

of the graph $\mathcal{K} := \text{Graph}(\lambda \rightsquigarrow K(\lambda))$ *with target* $\mathcal{C} := \text{Graph}(\lambda \rightsquigarrow C(\lambda))$ *under the auxiliary system*

$$\begin{cases} (i) \;\; \lambda'(t) \;=\; 0 \\ (ii) \; x'(t) \;\in\; F(\lambda(t), x(t)) \end{cases} \tag{10.25}$$

Proof. The proof is easy: to say that (λ, x) belongs to the viability kernel $\text{Viab}_{(10.25)}(\mathcal{K}, \mathcal{C})$ amounts to saying that there exists a solution $t \mapsto (\lambda(t), x(t))$ viable in $\mathcal{K} := \text{Graph}(K(\cdot))$ of (10.25) until it possibly reaches $\mathcal{C} := \text{Graph}(C(\cdot))$, i.e., since $\lambda(t) = \lambda$ and $(\lambda(\cdot), x(\cdot)) \in \mathcal{S}_{\{0\}\times F}(\lambda, x)$ such that $x(t) \in K(\lambda)$ forever or until it reaches $C(\lambda)$. This means that (λ, x) belongs to the graph of the viability map $\mathbb{V}$. □

In the same way, one can prove the analogous statement for the invariance map:

Proposition 10.9.2 ***[Graph of the Invariance Map]*** *The graph of the map* $\mathbb{I} : \lambda \rightsquigarrow \text{Inv}_{F(\lambda,\cdot)}(K(\lambda), C(\lambda))$ *is equal to the invariance kernel*

$$\text{Graph}(\mathbb{I}) \;=\; \text{Inv}_{(10.25)}(\mathcal{K}, \mathcal{C})$$

of the graph $\mathcal{K} := \text{Graph}(\lambda \rightsquigarrow K(\lambda))$ *with target* $\mathcal{C} := \text{Graph}(\lambda \rightsquigarrow C(\lambda))$ *under the auxiliary system (10.25), p. 428.*

Consequently, the inverses $\mathbb{V}^{-1}$ and $\mathbb{I}^{-1}$ of the set-valued maps $\mathbb{V}$ and $\mathbb{I}$ associate with any $x \in X$ the subsets of parameters $\lambda \in Y$ such that the pairs (λ, x) belong to the viability and invariance kernels of the graph

$\mathcal{K} := \text{Graph}(\lambda \rightsquigarrow K(\lambda))$ with target $\mathcal{C} := \text{Graph}(\lambda \rightsquigarrow C(\lambda))$ under the auxiliary system (10.25) respectively.

10.9.2 Level Tubes of Extended Functions

When the parameters $\lambda \in \mathbb{R}$ are scalars, the set-valued maps $\lambda \rightsquigarrow \text{Graph}(F(\lambda, \cdot))$, $\lambda \rightsquigarrow \text{Graph}(K(\lambda))$ and $\lambda \rightsquigarrow \text{Graph}(C(\lambda))$, $\mathbb{V} : \lambda \rightsquigarrow \mathbb{V}(\lambda)$ and the viability and invariance maps $\mathbb{I} : \lambda \rightsquigarrow \mathbb{I}(\lambda)$ are tubes (see Fig. 4.3, p. 132).

We shall study the monotonicity properties of tubes:

Definition 10.9.3 ***[Monotone Tubes]*** *A tube is* increasing *(resp.* decreasing*) if whenever $\mu \leq \nu$, then $K(\mu) \subset K(\nu)$ (resp. $K(\nu) \subset K(\mu)$). A* monotone tube *is a tube which is either increasing or decreasing.*

The monotonicity properties of the tubes $\lambda \rightsquigarrow \mathbb{V}(\lambda)$ and $\lambda \rightsquigarrow \mathbb{I}(\lambda)$ depend upon the monotonicity properties of the tubes $\lambda \rightsquigarrow \text{Graph}(F(\lambda, \cdot))$, $\lambda \rightsquigarrow \text{Graph}(K(\lambda))$ and $\lambda \rightsquigarrow \text{Graph}(C(\lambda))$:

Lemma 10.9.4 ***[Monotonicity of the Viability and Invariance Maps]*** *The map $(F, K, C) \rightsquigarrow \text{Viab}_F(K, C)$ is increasing, the map $(K, C) \rightsquigarrow \text{Inv}_F(K, C)$ is increasing and the map $F \rightsquigarrow \text{Inv}_F(K, C)$ is decreasing.*

Recall, a tube is characterized by its *graph* (see Definition 18.3.1, p. 719): The graph of the tube $K : \mathbb{R} \rightsquigarrow X$ is the set of pairs (λ, x) such that x belongs to $K(\lambda)$:

$$\mathcal{K} \;:=\; \text{Graph}(K) \;=\; \{(\lambda, x) \in \mathbb{R} \times X \text{ such that } x \in K(\lambda)\}$$

Monotonic tubes can be characterized by their epilevel and hypolevel functions whenever the tubes $\lambda \rightsquigarrow \mathbb{V}(\lambda)$ and $\lambda \rightsquigarrow \mathbb{I}(\lambda)$ are monotone:

We then introduce the concepts of lower and upper level sets or sections of an extended function:

Definition 10.9.5 ***[Levels Sets or Sections of a Function]*** *Let $\mathbf{v} : X \mapsto \overline{\mathbb{R}}$ be an extended function. The* lower level map $\mathbf{L}^{\leq}_{\mathbf{v}}$ *associates with any $\lambda \in \mathbb{R}$ the* λ-lower section *or* λ-lower level set

$$\mathbf{L}_{\mathbf{v}}^{\leq}(\lambda) := \{x \in K \textit{ such that } \mathbf{v}(x) \leq \lambda\}$$

We define in the same way the strictly lower, exact, upper and strictly upper level maps $\mathbf{L}_{\mathbf{v}}^{\bigstar}$ *which associate with any* λ *the* λ*-level sets*

$$\mathbf{L}_{\mathbf{v}}^{\bigstar}(\lambda) := \{x \in K \textit{ such that } \mathbf{v}(x) \bigstar \lambda\}$$

where $\bigstar$ *denotes respectively the signs* $<$, $=$, $\geq$ *and* $>$.

We next introduce the concept of level function of a tube:

Definition 10.9.6 *[**Level Function of a Tube**] Let us consider a tube* $K : \mathbb{R} \rightsquigarrow X$. *The* epilevel function $\Lambda_K^{\uparrow}$ *of the tube* K *is the extended function defined by*

$$\Lambda_K^{\uparrow}(x) := \inf\{\lambda \textit{ such that } x \in K(\lambda)\} = \inf_{(\lambda,x)\in \mathrm{Graph}(K)} \lambda \tag{10.26}$$

and its hypolevel function $\Lambda_K^{\downarrow}$ *is the extended function defined by*

$$\Lambda_K^{\downarrow}(x) := \sup\{\lambda \textit{ such that } x \in K(\lambda)\} = \sup_{(\lambda,x)\in \mathrm{Graph}(K)} \lambda \tag{10.27}$$

We observe that level set map $\lambda \rightsquigarrow \mathbf{L}_{\mathbf{v}}^{\bigstar}(\lambda)$ is a tube from $\mathbb{R}$ to the vector space X. For instance, *the lower level map* $\lambda \rightsquigarrow \mathbf{L}_{\mathbf{v}}^{\leq}(\lambda)$ *is an increasing tube*:

$$\text{If } \lambda_1 \leq \lambda_2, \quad \text{then } \mathbf{L}_{\mathbf{v}}^{\leq}(\lambda_1) \subset \mathbf{L}_{\mathbf{v}}^{\leq}(\lambda_2)$$

and that the upper level map $\lambda \rightsquigarrow \mathbf{L}_{\mathbf{v}}^{\geq}(\lambda)$ is a decreasing tube. Lemma 18.6.3, p. 744 implies that *the level set map* $\mathbf{L}_{\mathbf{v}}^{\leq}$ *of a lower semicontinuous function is a closed tube.*

We observe at once that the images $K(\lambda)$ are contained in the λ-lower level sets:

$$\forall \lambda \in \mathbb{R}, \quad K(\lambda) \subset \mathbf{L}_{\Lambda_K^{\uparrow}}^{\leq}(\lambda)$$

The question arises whether the converse is true: is an increasing tube the lower level map of an extended function $\Lambda_K^{\uparrow}$, called the epilevel function of the tube? This means that we can represent the images $K(\lambda)$ of the tube in the form

$$\forall \lambda, \quad K(\lambda) = \left\{x \text{ such that } \Lambda_K^{\uparrow}(x) \leq \lambda\right\}$$

This property can be reformulated as $K(\lambda) = \mathbf{L}^{\leq}_{\Lambda^{\uparrow}_K}(\lambda)$, stating that the inverse of the set-valued map $x \rightsquigarrow K^{-1}(x)$ of the tube is the map $x \rightsquigarrow \Lambda^{\uparrow}_K(x) + \mathbb{R}_+$.

The answer is positive for closed monotonic tubes.

The equality between these two subsets is (almost) true for increasing tubes (a necessary condition) and really true when, furthermore, the tube is closed:

Proposition 10.9.7 *[**Inverses of Monotone Tubes and their Level Functions**] Let us assume that the tube K is increasing. Then it is related to its epilevel function by the relation*

$$\forall \lambda \in \mathbb{R}, \;\; \mathbf{L}^{<}_{\Lambda^{\uparrow}_K}(\lambda) \; \subset \; K(\lambda) \; \subset \; \mathbf{L}^{\leq}_{\Lambda^{\uparrow}_K}(\lambda) \tag{10.28}$$

Furthermore, if the graph of the tube is closed, then $x \in K(\lambda)$ if and only if $\Lambda^{\uparrow}_K(x) \leq \lambda$, i.e.,

$$\forall \lambda \in \mathbb{R}, \;\; K(\lambda) \; = \; \mathbf{L}^{\leq}_{\Lambda^{\uparrow}_K}(\lambda) \; =: \; \{x \mid \Lambda^{\uparrow}_K(x) \; \leq \; \lambda\} \tag{10.29}$$

Proof. By the very definition of the infimum, to say that $\Lambda^{\uparrow}_K(x) = \inf_{(\lambda,x)\in \text{Graph}(K)} \lambda$ amounts to saying that for any $\lambda > \Lambda^{\uparrow}_K(x)$, there exists $(\mu, x) \in \text{Graph}(K)$ such that $\mu \leq \lambda$. To say that $x \in \mathbf{L}^{<}_{\Lambda^{\uparrow}_K}(\lambda)$ means $\lambda > \Lambda^{\uparrow}_K(x)$. Hence there exists $(\mu, x) \in \text{Graph}(K)$, and there exist $\mu \leq \lambda$ and $x \in K(\mu)$. Since the tube K is decreasing, we deduce that $x \in K(\lambda)$. The first inclusion is thus proved, the other one being always obviously true.

If the graph of K is closed, then letting $\lambda > \Lambda^{\uparrow}_K(x)$ converge to $\Lambda^{\uparrow}_K(x)$ and knowing that (λ, x) belongs to Graph(K), we deduce $(\Lambda^{\uparrow}_K(x), x)$ belongs to the graph of K, and thus, that $x \in K(\Lambda^{\uparrow}_K(x))$. $\square$

The counterpart statement holds true for decreasing tubes and their hypolevel functions: If a tube is decreasing, then

$$\forall \lambda \in \mathbb{R}, \;\; \mathbf{L}^{>}_{\lambda^{\downarrow}_K}(\lambda) \; \subset \; K(\lambda) \; \subset \; \mathbf{L}^{\geq}_{\Lambda^{\downarrow}_K}(\lambda) \tag{10.30}$$

and if it is closed

$$\forall \lambda \in \mathbb{R}, \;\; K(\lambda) \; = \; \mathbf{L}^{\geq}_{\Lambda^{\downarrow}_K}(\lambda) \tag{10.31}$$

In this case, when the graph of the tube is closed, (10.29) and (10.31) can be written in terms of epigraphs and hypographs (see Definition 4.2.2, p. 131) in the form

$$\text{Graph}(K^{-1}) \;=\; \mathcal{E}p(\Lambda_K^{\uparrow}) \tag{10.32}$$

and

$$\text{Graph}(K^{-1}) \;=\; \mathcal{H}yp(\Lambda_K^{\downarrow}) \tag{10.33}$$

respectively.

In the scalar case, Theorem 10.9.7, p. 431 implies that these tubes are characterized by their epilevel or hypolevel functions (see Definition 10.9.6, p. 430). For instance, the epilevel function of the viability tube is defined by $\Lambda_{\mathbb{V}}^{\uparrow}(x) := \inf_{(\lambda,x)\in\text{Graph}(\mathbb{V})} \lambda$ whenever this map is increasing. In this case, if the graph of the viability tube is closed,

$$\mathbb{V}(\lambda) \;=\; \left\{ x \text{ such that } \Lambda_{\mathbb{V}}^{\uparrow}(x) \leq \lambda \right\}$$

If the tube is decreasing, the hypolevel function defined by $\Lambda_{\mathbb{V}}^{\downarrow}(x) := \sup_{(\lambda,x)\in\text{Graph}(\mathbb{V})} \lambda$ characterizes the tube in the sense that

$$\mathbb{V}(\lambda) \;=\; \left\{ x \text{ such that } \Lambda_{\mathbb{V}}^{\downarrow}(x) \geq \lambda \right\}$$

whenever the tube is closed.

For instance, for the viability map, we derive the following statement from Proposition 10.9.1, p. 428:

Proposition 10.9.8 ***[Level Functions of the Viability Tube]*** *Let us assume that the tubes* $\lambda \mapsto \text{Graph}(F(\lambda,\cdot))$, $\lambda \mapsto K(\lambda)$ *and* $\lambda \mapsto C(\lambda)$ *are increasing. Then the tube* $\mathbb{V}$ *is characterized by its epilevel function*

$$\Lambda_{\mathbb{V}}^{\uparrow}(x) \;:=\; \inf_{(\lambda,x)\in\text{Graph}(\mathbb{V})} \lambda \;:=\; \inf_{(\lambda,x)\in\text{Viab}_{(10.25)}(\mathcal{K},\mathcal{C})} \lambda \tag{10.34}$$

If $\lambda \mapsto \text{Graph}(F(\lambda,\cdot))$, $\lambda \mapsto K(\lambda)$ *and* $\lambda \mapsto C(\lambda)$ *are increasing, the tube* $\mathbb{V}$ *is characterized by its* hypolevel function

$$\Lambda_{\mathbb{V}}^{\downarrow}(x) \;:=\; \sup_{(\lambda,x)\in\text{Graph}(\mathbb{V})} \lambda \;=\; \sup_{(\lambda,x)\in\text{Viab}_{(10.25)}(\mathcal{K},\mathcal{C})} \lambda \tag{10.35}$$

The counterpart statement holds true for the invariance tubes.

10.10 Stochastic and Tychastic Viability

The invariance kernel is an example of the core $\mathcal{S}^{\ominus 1}(\mathcal{H})$ of a subset $\mathcal{H} \subset \mathcal{C}(0,\infty;\mathbb{R}^d)$ for $\mathcal{H} = \mathcal{K}_{(K,C)}$ being the set of evolutions viable in K reaching the target C in finite time.

Let us consider *random events* $\omega \in \Omega$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, instead of tyches $v(\cdot)$ ranging over the values $V(x(\cdot))$ of a tychastic map V (see (2.23), p. 89).

A stochastic system is a specific parameterized evolutionary system described by a map $\mathbb{X} : (x,\omega) \in \mathbb{R}^d \times \Omega \mapsto \mathbb{X}(x,\omega) \in \mathcal{C}(0,\infty;\mathbb{R}^d)$ (usually denoted by $t \mapsto \mathbb{X}_\omega^x$ in the stochastic literature) where $\mathcal{C}(0,\infty;\mathbb{R}^d)$ is the space of continuous evolutions. In other words, it defines evolutions $t \mapsto \mathbb{X}(x,\omega)(t) := \mathbb{X}_\omega^x(t) \in \mathbb{R}^d$ starting at x at time 0 and parameterized by random events $\omega \in \Omega$ satisfying technical requirements (measurability, filtration, etc.) that are not relevant to involve at this stage of the exposition. The initial state x being fixed, the random variable $\omega \mapsto \mathbb{X}(x,\omega) := \mathbb{X}_\omega^x(\cdot) \in \mathcal{C}(0,+\infty;\mathbb{R}^d)$ is called a stochastic process. A subset $\mathcal{H} \subset \mathcal{C}(0,\infty;\mathbb{R}^d)$ of evolutions sharing a given property being chosen, it is natural, as we did for tychastic systems, to introduce the *stochastic core* of $\mathcal{H}$ under the stochastic system: it is the subset of initial states x from which starts a stochastic process $\omega \mapsto \mathbb{X}(x,\omega)$ such that for almost all $\omega \in \Omega$, $\mathbb{X}(x,\omega) \in \mathcal{H}$:

$$\mathrm{Stoc}_{\mathbb{X}}(\mathcal{H}) := \{x \in \mathbb{R}^d \mid \text{for almost all } \omega \in \Omega,\ \ \mathbb{X}(x,\omega) := \mathbb{X}_\omega^x(\cdot) \ \in\ \mathcal{H}\} \tag{10.36}$$

Regarding a stochastic process as a set-valued map $\mathbb{X}$ associating with any state x the family $\mathbb{X}(x) := \{\mathbb{X}(x,\omega)\}_{\omega\in\Omega}$, the definitions of stochastic cores (10.36) of subsets of evolution properties are similar in spirit to definition:

$$\mathcal{S}^{\ominus 1}(\mathcal{H}) := \{x \in \mathbb{R}^d \mid \ \text{for all } \ v(\cdot) \in Q(x(\cdot)),\ \ x_{v(\cdot)}(\cdot) \ \in\ \mathcal{H}\}$$

under a tychastic system

$$x'(t) \ = \ f(x(t), v(t)) \text{ where } v(t) \ \in \ Q(x(t))$$

Furthermore, *the parameters ω are constant in the stochastic case, whereas the tychastic uncertainty $v(\cdot)$ is dynamic* in nature and involves a state dependence, two more realistic assumptions in the domain of life sciences.

There is however a deeper similarity that we mention briefly. When the stochastic system $(x,\omega) \mapsto \mathbb{X}(x,\omega)$ is derived from a stochastic differential equation, the Strook-Varadhan Support Theorem (see [201, Stroock &Varadhan]) states that there exists a tychastic system $(x,v) \mapsto \mathcal{S}(x,v)$ such that, whenever $\mathcal{H}$ is closed, the stochastic core of $\mathcal{H}$ under the stochastic system $\mathbb{X}$ and its tychastic core under the associated tychastic system $\mathcal{S}$ coincide:

$$\mathrm{Stoc}_{\mathbb{X}}(\mathcal{H}) = \mathcal{S}^{\ominus 1}(\mathcal{H})$$

To be more specific, let $\mathbb{X}(x,\omega)$ denote the solution to the stochastic differential equation

$$dx = \gamma(x)dt + \sigma(x)dW(t)$$

starting at x, where $W(t)$ ranges over $\mathbb{R}^c$, the drift $\gamma : \mathbb{R}^d \mapsto \mathbb{R}^d$ and the diffusion $\sigma : \mathbb{R}^d \mapsto \mathcal{L}(\mathbb{R}^c, \mathbb{R}^d)$ are smooth and bounded maps. Let us associate with them the Stratonovitch drift $\widehat{\gamma}$ defined by $\widehat{\gamma}(x) := \gamma(x) - \frac{1}{2}\sigma'(x)\sigma(x)$. The Stratonovitch stochastic integral is an alternative to the Ito integral, and easier to manipulate. Unlike the Ito calculus, Stratonovich integrals are defined such that the chain rule of ordinary calculus holds. It is possible to convert Ito stochastic differential equations to Stratonovich ones.

Then, the associated tychastic system is

$$x'(t) = \widehat{\gamma}(x(t)) + \sigma(x(t))v(t) \text{ where } v(t) \in \mathbb{R}^c \tag{10.37}$$

where *the tychastic map is constant and equal to* $\mathbb{R}^c$.

Consequently, *the tychastic system associated with a stochastic one by the Strook–Varadhan Support Theorem is very restricted: there are no bounds at all on the tyches, whereas general tychastic systems allow the tyches to range over subsets* $Q(x)$ *depending upon the state* x*, describing so to speak a state-dependent uncertainty*:

$$x'(t) = \widehat{\gamma}(x(t)) + \sigma(x(t))v(t) \text{ where } v(t) \in Q(x(t))$$

This state-dependent uncertainty, unfortunately absent in the mathematical representation of uncertainty in the framework of stochastic processes, is of utmost importance for describing uncertainty in problems dealing with living beings.

When $\mathcal{H}$ is a Borelian of $\mathcal{C}(0,\infty;\mathbb{R}^d)$, we denote by $\mathbb{P}_{\mathbb{X}(x,\cdot)}$ the *law* of the random variable $\mathbb{X}(x,\cdot)$ defined by

$$\mathbb{P}_{\mathbb{X}(x,\cdot)}(\mathcal{H}) \ := \ \mathbb{P}(\{\omega \mid \mathbb{X}(x,\omega) \in \mathcal{H}\}) \tag{10.38}$$

Therefore, we can reformulate the definition of the stochastic core of a set $\mathcal{H}$ of evolutions in the form

$$\text{Stoc}_{\mathbb{X}}(\mathcal{H}) = \{x \in \mathbb{R}^d \mid \mathbb{P}_{\mathbb{X}(x,\cdot)}(\mathcal{H}) = 1\} \tag{10.39}$$

In other words, the stochastic core of $\mathcal{H}$ is the set of initial states x such that the subset $\mathcal{H}$ has probability one under the law of the stochastic process $\omega \mapsto \mathbb{X}(x,\omega) \in \mathcal{C}(0,+\infty;\mathbb{R}^d)$ (if $\mathcal{H}$ is closed, $\mathcal{H}$ is called the *support* of the law $\mathbb{P}_{\mathbb{X}(x,\cdot)}$). The Strook–Varadhan Support Theorem states that under regularity assumptions, this support is the core of $\mathcal{H}$ under the tychastic system (10.37). It furthermore provides a characterization of stochastic viability in terms of tangent cones and general curvatures of the environments (see the contributions of Giuseppe da Prato, Halim Doss, Hélène Frankowska and Jerzy Zabczyk among many other).

These remarks further justify our choice of privileging tychastic systems because, as far as the properties of initial states of evolution are concerned, stochastic systems are just (very) particular cases of tychastic systems.