Aggregation of Bounded Fuzzy Natural Number-Valued Multisets

Jaume Casasnovas and J. Vicente Riera

University of Balearic Islands, Palma de Mallorca E 07122, Spain
jaume.casasnovas@uib.es, jvicente.riera@uib.es

Abstract. Multisets (also called bags) are like-structures where an element can appear more than once. Recently, several generalizations of this concept have been studied. In this article we deal with a new extension of this concept, the bounded fuzzy natural number-valued multisets. On this kind of bags, a bounded distributive lattice structure is presented and a partial order is defined. Moreover, we study operations of aggregations (t-norms and t-conorms) and we provide two methods for their construction.

1 Introduction

Multisets (also called bags in the literature [25]) are like-structures where an element can appear more than once. Formally, a multiset over a set of types X is a mapping M defined from X to the set $\mathbb{N} = \{1, 2, \cdots, \}$ of natural numbers. A survey of the mathematics of multisets, including their axiomatic foundation, can be found in [2]. The multisets have been studied by several researchers in computer science from different points of view. For example, their applications to data analysis and decision making [19], their applications to flexible querying [22] or the monograph on multiset processing [13].

According to the interpretation of a multiset $M : X \to \mathbb{N}$, it describes a set or *universe*, Ω, which consists of $M(x)$ "exact" copies of each type $x \in X$. Specifically, for each $x \in X$, $M(x)$ is the account of elements or cardinal of the subset $\Omega_x \subset \Omega$. The number $M(x)$ is usually called the *multiplicity* of x in the multiset M. One of the most natural and simple example is the multiset of prime factors of a natural number n. Thus, the number 504 has the factorization $504 = 2^3 \cdot 3^2 \cdot 7^1$ which gives the multiset $\{2, 2, 2, 3, 3, 7\}$.

Notice that all properties (inclusion, equality, etc.) and operations (addition, union, intersection, etc.) between multisets stem from similar properties and operations of the set of natural numbers. So, a deep study of the valuation set of multisets over a universe X allow us to obtain new properties. And a change of this valuation set allows us to get new extensions.

In [3], the authors introduced a more general definition of "extended multiset" as mappings $M : X \to L$, where L is a finite or infinite chain of natural numbers, or, even, it can be $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, with the usual operations and order. This definition allows to extend several aggregation operators defined in L, such as t-norms or t-conorms, to multisets.

V. Torra, Y. Narukawa, and M. Daumas (Eds.): MDAI 2010, LNAI 6408, pp. 67–78, 2010.

Another natural generalization of this interpretation of multisets leads to the notions *Real-Valued Bags and bag relations* [20] or *multisets with fuzzy values* [16,20] over a set of *types* X. Such a multiset describes for each $x \in X$, a set Ω_x consisting this time of "possibly inexact" copies of x with a degree of similarity valued in [0,1]. In this way, in [18] an immediate generalization of crisp multisets using fuzzy numbers instead of natural numbers is proposed. So, provided a suitable definition of fuzzy number (triangular, trapezoidal, Gauss-shaped, etc [15]), it is possible to consider fuzzy Number-Valued multisets defined over X.

Analogously to the crisp case and in order to define a "multiplicity" or *fuzzy multiplicity* of each type for a *fuzzy multiset* over X, we need to associate to each $x \in X$ the *cardinality* of the fuzzy set Ω_x. The problem of "counting" fuzzy sets has generated a lot of literature since Zadeh's first definition of the cardinality of fuzzy sets [14,15]. In particular, the scalar cardinalities of fuzzy sets, which associate to each fuzzy set a positive real number, have been studied from the axiomatic point of view [12] with the aim of capturing different ways of counting additive aspects of fuzzy sets like the cardinalities of supports, of levels, of cores, etc. In a similar way, the fuzzy cardinalities of fuzzy sets [11,14], which associate to any fuzzy set a fuzzy natural number, have also been studied from the axiomatic point of view.

Taking into account that the fuzzy cardinality of a fuzzy set is a fuzzy natural number, i.e., a discrete fuzzy number whose support is a subset of consecutive natural numbers, in [10] the authors defined *Fuzzy Natural Number-Valued multiset* as mappings $M : X \to FNN$ where FNN is the set of fuzzy natural numbers. On this type of multisets, monoidal and lattice structures were studied.

On the other hand, in [3] the authors deal with multisets whose multiplicities are possibly bounded due to circumstances of the framework where they are defined. As a consequence, in this paper we propose a new extension of the concept of multiset, the bounded fuzzy natural number-valued multisets. On this new set of multisets we define a structure of bounded distributive lattice. And, on this bounded partially ordered set we define triangular norms and conorms. Moreover, we propose two methods to get t-norms and t-conorms. The first method uses t-norms(t-conorms) on $\mathcal{A}_1^L = \{u \in FNN \mid supp(u) \subseteq L = \{0, 1, \cdots, m\}\}$. And the second one uses divisible t-norms(t-conorms) on the finite chain $L = \{0, 1, \cdots, m\}$ of natural numbers.

2 Preliminaries

2.1 Multisets

Let X be a crisp set. A (*crisp*) *multiset* over X is a mapping $M : X \to \mathbb{N}$, where $\mathbb{N}$ stands for the set of natural numbers including the 0. A multiset M over X is *finite* if its *support*

$$supp(M) = \{x \in X | M(x) > 0\}$$

is a finite subset of X. We shall denote the sets of all multisets over a set X by $MS(X)$, and by $\perp$ the *null multiset*, defined by $\perp(x) = 0$ for each $x \in X$.

For every $A, B \in MS(X)$, their *sum* [21] $A+B$ is the multiset defined pointwise by

$$(A+B)(x) = A(x) + B(x), \quad x \in X.$$

Let us mention here that it has been argued that this sum $+$, also called *additive union*, is the right notion of union of multisets. According to the interpretation of multisets as sets of copies of types explained in the introduction, this sum corresponds to the disjoint union of sets, as it interprets that all copies of each x in the set represented by A are different from all copies of it in the set represented by B. This additive sum has quite different properties from the ordinary union of sets. For instance, the collection of submultisets of a given multiset is not closed under this operation and consequently no sensible notion of complement within this collection exists.

For every $A, B \in MS(X)$, their *join* $A \vee B$ and *meet* $A \wedge B$ are respectively the multisets over X defined pointwise by $(A \vee B)(x) = \max(A(x), B(x))$ and $(A \wedge B)(x) = \min(A(x), B(x))$, $x \in X$. If A and B are finite, then $A+B$, $A \vee B$ and $A \wedge B$ are also finite. A partial order $\leq$ on $MS(X)$ is defined by $A \leq B$ if and only if $A(x) \leq B(x)$ for every $x \in X$. If $A \leq B$, then their *difference* $B - A$ is the multiset defined pointwise by

$$(B-A)(x) = B(x) - A(x).$$

2.2 Triangular Norms and Conorms on Partially Ordered Sets

Let $(P; \leq)$ be a non-trivial bounded partially ordered set (poset) with "e" and "m" as minimum and maximum elements respectively.

Definition 2.1. *[1] A* triangular norm *(briefly t-norm) on P is a binary operation $T : P \times P \to P$ such that for all $x, y, z \in P$ the following axioms are satisfied:*

1. $T(x, y) = T(y, x)$ *(commutativity)*
2. $T(T(x, y), z) = T(x, T(y, z))$ *(associativity)*
3. $T(x, y) \leq T(x', y')$ *whenever* $x \leq x', y \leq y'$ *(monotonicity)*
4. $T(x, m) = x$ *(boundary condition)*

Definition 2.2. *A* triangular conorm *(t-conorm for short) on P is a binary operation $S : P \times P \to P$ which, for all $x, y, z \in P$ satisfies (1), (2), (3) and (4'): $S(x, e) = x$, as boundary condition.*

2.3 Triangular Norms and Conorms on Discrete Settings

Let L be the totally ordered set $L = \{0, 1, \ldots, m\} \subset \mathbb{N}$. A t-norm(t-conorm) defined on L will be called a discrete t-norm(t-conorm).

Definition 2.3. *[17] A t-norm(t-conorm) $T(S) : L \times L \to L$ is said to be* smooth *if it satisfies $T(S)(x+1, y) - T(S)(x, y) \leq 1$ and $T(S)(x, y+1) - T(S)(x, y) \leq 1$.*

Definition 2.4. *[17] A t-norm(t-conorm) $T : L \times L \to L$ is said to be* divisible *if it satisfies: For all $x, y \in L$ with $x \leq y$, there is $z \in L$ such that $x = T(y, z)(y = S(x, z))$.*

2.4 Discrete Fuzzy Numbers

By a fuzzy subset of the set of real numbers, we mean a function $u : \mathbb{R} \rightarrow [0, 1]$. For each fuzzy subset u, let $u^\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}$ for any $\alpha \in (0, 1]$ be its α-level set (or α-cut). By $supp(u)$, we mean the support of u, i.e. the set $\{x \in \mathbb{R} : u(x) > 0\}$. By u^0, we mean the closure of $supp(u)$.

Definition 2.5. *[23] A fuzzy subset u of the set of real numbers* $\mathbb{R}$ *with membership mapping* $u : \mathbb{R} \rightarrow [0, 1]$ *is called* discrete fuzzy number *if its support is finite, i.e., there are* $x_1, ..., x_n \in \mathbb{R}$ *with* $x_1 < x_2 < ... < x_n$ *such that* $supp(u) = \{x_1, ..., x_n\}$*, and there are natural numbers* s, t *with* $1 \leq s \leq t \leq n$ *such that:*

1. *$u(x_i)$=1 for any natural number i with $s \leq i \leq t$ (*core*)*
2. *$u(x_i) \leq u(x_j)$ for each natural number i, j with $1 \leq i \leq j \leq s$*
3. *$u(x_i) \geq u(x_j)$ for each natural number i, j with $t \leq i \leq j \leq n$*

Remark 2.1. If the fuzzy subset u is a discrete fuzzy number then the support of u coincides with its closure, i.e. $supp(u) = u^0$.

From now on, the notation DFN stands for the set of discrete fuzzy numbers.

The operations addition, maximum and minimum between discrete fuzzy numbers defined through Extension principle [15] can yield fuzzy subsets that do not satisfy the conditions to be discrete fuzzy numbers [4,24]. In [4,5,6,24], this drawback is studied and a new method to define these operations is proposed. So, the next result holds [24]:

Theorem 2.1. *Let* $u, v \in DFN$*, the fuzzy subset denoted by* $u \underset{W}{\oplus} v$*, such that it has as r-cuts the sets* $[u \underset{W}{\oplus} v]^r = \{x \in supp(u) + supp(v) : \min([u]^r + [v]^r) \leq x \leq \max([u]^r + [v]^r)\}$ *for each* $r \in [0, 1]$ *where* $\min([u]^r + [v]^r) = \min\{x : x \in [u]^r + [v]^r\}$, $\max([u]^r + [v]^r) = \max\{x : x \in [u]^r + [v]^r\}$ *and* $(u \underset{W}{\oplus} v)(x) = \sup\{r \in [0, 1]$ *such that* $x \in [u \underset{W}{\oplus} v]^r\}$ *is a discrete fuzzy number.*

On the other hand, in [6], the following result is obtained:

Proposition 2.1. *For each* $u, v \in DFN$*, there exist two unique discrete fuzzy numbers, which we will denote by* $MIN_w(u, v)$ *and* $MAX_w(u, v)$*, such that they have the sets* $MIN_w(u, v)^\alpha$ *and* $MAX_w(u, v)^\alpha$ *as α-cuts respectively, where*

$$MIN_w(u, v)^\alpha = \{z \in supp(u) \bigwedge supp(v) | \min(x_1^\alpha, y_1^\alpha) \leq z \leq \min(x_p^\alpha, y_k^\alpha)\}$$

$$MAX_w(u, v)^\alpha = \{z \in supp(u) \bigvee supp(v) | \max(x_1^\alpha, y_1^\alpha) \leq z \leq \max(x_p^\alpha, y_k^\alpha)\}$$

for each $\alpha \in [0, 1]$*, being* $u^\alpha = \{x_1^\alpha, \cdots, x_p^\alpha\}$, $v^\alpha = \{y_1^\alpha, \cdots, y_k^\alpha\}$ *the α-cuts of u and v respectively. And,* $supp(u) \bigwedge supp(v) = \{z = min(x, y) | x \in supp(u), y \in supp(v)\}$ *and* $supp(u) \bigvee supp(v) = \{z = max(x, y) | x \in supp(u), y \in supp(v)\}$

3 Operations on Fuzzy Natural Numbers

From now on, the notation fnn stands for a fuzzy natural number (i.e. discrete fuzzy numbers whose support only includes consecutive natural numbers) and FNN stands for the set of fuzzy natural numbers.

3.1 Addition of Fuzzy Natural Numbers

It is well known [15] that, in the case of continuous fuzzy numbers the addition obtained by extending the usual addition of real numbers through the extension principle is associative and commutative. But the fuzzy natural numbers are not continuous on $\mathbb{R}$.

In [5], the authors proved that in the case in which the discrete fuzzy numbers have as support an arithmetic sequence or a subset of consecutive natural numbers it is possible to use the Zadeh's extension principle to obtain its addition. Moreover, we know [24] the next result:

Proposition 3.1. *Let us consider $u, v \in DFN$. If $u \oplus v \in DFN$ where $u \oplus v$ denotes the addition of u and v using the Zadeh's extension principle, then $u \oplus v$ and $u \underset{W}{\oplus} v$ are identical, where $u \underset{W}{\oplus} v$ is the discrete fuzzy number obtained from u and v according to Theorem 2.1.*

Remark 3.1. A consequence of the previous proposition is that if we prove a property for the operation $\underset{W}{\oplus}$ in the set of fuzzy natural numbers, we will obtain the same property for the operation $\oplus$ in this set.

Theorem 3.1. *[10] The set FNN of the fuzzy natural numbers is a commutative monoid with the Zadeh's addition as a monoidal operation.*

3.2 Maximum and Minimum of Fuzzy Natural Numbers

With respect to the maximum and the minimum of two fuzzy natural numbers, the authors have proved in [6] the following proposition:

Proposition 3.2. *[6] Let u, v be two fuzzy natural numbers. Then $MAX(u, v)$, defined through the extension principle, coincides with $MAX_w(u, v)$. So, if $u, v \in FNN$, $MAX(u, v)$ is a fuzzy natural number and $MAX(u, v) \in FNN$. Analogously, $MIN(u, v)$, defined through the extension principle, coincides with the fnn $MIN_w(u, v)$. So, if $u, v \in FNN$, then $MIN(u, v)$ is a fuzzy natural number and $MIN(u, v) \in FNN$.*

But we have studied in [7] the associativity, commutativity, idempotence, absorption and distributivity for the operations MIN_w and MAX_w between discrete fuzzy numbers in general and between fuzzy natural numbers in particular and we obtained the following proposition:

Proposition 3.3. *[7] The set of discrete fuzzy numbers whose support is a sequence of consecutive natural numbers* (FNN,MIN_w,MAX_w) *is a distributive lattice.*

If we gather the previous Propositions 3.2 and 3.3, then we obtain the following consequence:

Proposition 3.4. *[7] The set of discrete fuzzy numbers whose support is a sequence of consecutive natural numbers* (FNN,MIN,MAX) *is a distributive lattice.*

With the aim of studying the monotony for the addition of two fuzzy natural numbers, we need a definition of order:

Definition 3.1. *[7] From the operations* MIN_w *and* MAX_w*, we can define a partial order on* FNN *on the following way:*
$u \preceq v$ *if and only if* $MIN_w(u,v) = u$*, or equivalently,* $u \preceq v$ *if and only if* $MAX_w(u,v) = v$ *for any* $u, v \in FNN$*. Equivalently, we can also define the partial ordering in terms of* α*-cuts:*
$u \preceq v$ *if and only if* $\min(u^\alpha, v^\alpha) = u^\alpha$
$u \preceq v$ *if and only if* $\max(u^\alpha, v^\alpha) = v^\alpha$

Proposition 3.5. *[10] Let* $u, v, w, t \in FNN$*. If* $u \preceq v$ *and* $w \preceq t$ *where* $\preceq$ *denotes the partial order in* FNN *defined in Definition 3.1 then* $u \oplus w \preceq v \oplus t$*, where* $\oplus$ *denotes the Zadeh's addition.*

3.3 Discrete Fuzzy Numbers Obtained by Extending Discrete t-norms(t-conorms) Defined on a Finite Chain

Let us consider a discrete t-norm(t-conorm) $T(S)$ on the finite chain $L = \{0, 1, \cdots, m\} \subset \mathbb{N}$. If X and Y are subsets of L, then the subset $\{T(x,y) | x \in \mathsf{X}, y \in \mathsf{Y}\} \subseteq L$ will be denoted by $T(\mathsf{X}, \mathsf{Y})$. Analogously, $S(\mathsf{X}, \mathsf{Y}) = \{S(x,y) | x \in \mathsf{X}, y \in \mathsf{Y}\}$.

So, if we consider the α-cut sets, $u^\alpha = \{x_1^\alpha, ..., x_p^\alpha\}$, $v^\alpha = \{y_1^\alpha, ..., y_k^\alpha\}$, for u and v respectively then $T(u^\alpha, v^\alpha) = \{T(x,y) | x \in u^\alpha, y \in v^\alpha\}$ and $S(u^\alpha, v^\alpha) = \{S(x,y) | x \in u^\alpha, y \in v^\alpha\}$ for each $\alpha \in [0,1]$, where u^0 and v^0 denote $supp(u)$ and $supp(v)$ respectively.

Definition 3.2. *[8] For each* $\alpha \in [0,1]$*, let us consider the sets*

$$C^\alpha = \{z \in T(supp(u), supp(v)) | \min T(u^\alpha, v^\alpha) \leq z \leq \max T(u^\alpha, v^\alpha)\} \text{ and}$$

$$D^\alpha = \{z \in S(supp(u), supp(v)) | \min S(u^\alpha, v^\alpha) \leq z \leq \max S(u^\alpha, v^\alpha)\}$$

Theorem 3.2. *[8] There exists a unique discrete fuzzy number that will be denoted by* $\mathcal{T}(u,v)(\mathcal{S}(u,v))$ *such that* $\mathcal{T}(u,v)^\alpha = C^\alpha(\mathcal{S}(u,v)^\alpha = D^\alpha)$ *for each* $\alpha \in [0,1]$ *and* $\mathcal{T}(u,v)(z) = sup\{\alpha \in [0,1] : z \in C^\alpha\}(\mathcal{S}(u,v)(z) = sup\{\alpha \in [0,1] : z \in D^\alpha\})$

From now on the set $\mathcal{A}_1^L = \{u \in FNN \mid supp(u) \subseteq L = \{0, 1, \cdots, m\}\}$ will be called the set of bounded fuzzy natural numbers and each element of this set will be called a bounded fuzzy natural number (in short bfnn). In [9] the authors showed the next result

Theorem 3.3. *The triplet* $(\mathcal{A}_1^L, MIN_w, MAX_w)$ *is a bounded distributive lattice and the fnn* $\widehat{0}$ *and* $\widehat{m}$*, defined by* $\widehat{0}(i) = 1$ *if* $i = 0$ *and* $\widehat{0}(i) = 0$*, otherwise,* $\widehat{m}(i) = 1$ *if* $i = m$ *and* $\widehat{m}(i) = 0$*, otherwise, are the lower and the upper bound, respectively.*

On the other hand, it is well known [1] that it is possible to generalize the concept of t-norm (t-conorm) using any bounded partially ordered set instead of the unit interval. Using this idea and Theorem 3.3 we can build t-norms and t-conorms on the bounded distributive lattice $\mathcal{A}_1^L$.

Theorem 3.4. *[9] Let* $T(S)$ *be a divisible t-norm(t-conorm) on L and let*

$$\begin{aligned} \mathcal{T}(\mathcal{S}) : \mathcal{A}_1^L \times \mathcal{A}_1^L &\to \mathcal{A}_1^L \\ (A, B) &\longmapsto \mathcal{T}(u, v)(\mathcal{S}(u, v)) \end{aligned}$$

be the binary operation (which will be called the extension of the t-norm(t-conorm) $T(S)$ *to* $\mathcal{A}_1^L$*), where* $\mathcal{T}(u, v)(\mathcal{S}(u, v))$ *are defined according to Theorem 3.2. Then,* $\mathcal{T}(\mathcal{S})$ *is a t-norm(t-conorm) on the bounded set* $\mathcal{A}_1^L$*.*

4 Operations on Fuzzy Natural Number-Valued Multisets

4.1 FNN-Valued Multisets[10]

Definition 4.1. *A Fuzzy Natural Number-valued multiset defined over an universe* X *is a mapping* $M : X \to FNN$ *i.e. for all* $x \in X$*,* $M(x)$ *is a fuzzy natural number.*

Remark 4.1. We will denote the set of Fuzzy Natural Number-valued multisets defined over an universe X by $FNNM(X)$. Finally, the abbreviation fnnm will denote a Fuzzy Natural Number-valued multiset.

The properties of the addition of fuzzy natural numbers studied in the previous Section 3, will allow us to define the addition of fuzzy natural number-valued multisets and to study the monoidal structure of this set.

Definition 4.2. *Let* $A, B : X \to FNN$ *be two Fuzzy Natural Number-valued multisets. The* sum *of* A *and* B *will be the Fuzzy Natural Number-valued Multiset pointwise defined for all* $x \in X$ *by*

$$(A + B)(x) = A(x) \oplus B(x)$$

where the fnn $A(x) \oplus B(x)$ *is obtained following the Zadeh's extension principle or equivalently using the method considered in Theorem 2.1.*

Proposition 4.1. *The set $FNNM(X)$ of the fuzzy natural number-valued multisets over X is a commutative monoid with the addition as a monoidal operation.*

Analogously to the addition, the properties of the maximum and minimum of fnn studied in the previous section will allow us to define the maximum and minimum of fuzzy natural number-valued multisets and to study the order and the lattice structure of this set.

Definition 4.3. *Let $A, B : X \to FNN$ be two Fuzzy Natural Number-valued Multisets. The* join *and the* meet *of A and B will be the Fuzzy Natural Number-valued Multiset, pointwise defined for all $x \in X$ as*

$$(A \vee B)(x) = MAX\{A(x), B(x)\} \text{ and } (A \wedge B)(x) = MIN\{A(x), B(x)\}$$

respectively, where the fnn $MAX\{A(x), B(x)\}$ and $MIN\{A(x), B(x)\}$ are obtained according to the method presented in Proposition 2.1.

Proposition 4.2. *As long as, for all $x \in X$, $A(x) \in FNN$ and $B(x) \in FNN$, then $MAX\{A(x), B(x)\}$ and $MIN\{A(x), B(x)\}$ can be obtained by means of the extension principle.*

Proposition 4.3. *Let $A, B : X \to FNN$ be two Fuzzy Natural Number-valued Multisets. The binary relationship:*
$A \leq B$ if and only if $A \vee B = B$ and/or $A \wedge B = A$ i.e. $MAX\{A(x), B(x)\} = B(x), \forall x \in X$ (or $MIN\{A(x), B(x)\} = A(x), \forall x \in X$) is a partial order on the set $FNNM(X)$.

Proposition 4.4. *The set $FNNM(X)$ of the fuzzy natural number-valued multisets over X is a lattice with the partial order defined in Proposition 4.3 and the meet and join operations proposed in Definition 4.3.*

Proposition 4.5. *Let $A, B, C, D \in FNNM(X)$. If $A \leq B$ and $C \leq D$ where $\leq$ denotes the partial order in $FNNM(X)$ defined in Proposition 4.3 then $A + C \leq B + D$, where $+$ denotes the addition considered in Definition 4.2.*

5 Bounded Fuzzy Natural Numbers-Valued Multisets

Let us consider the finite chain $L = \{0, 1, \cdots, m\}$ of natural numbers and the set $\mathcal{A}_1^L = \{A \in FNN \mid supp(A) \subseteq L\}$ of bounded fuzzy natural numbers.

Definition 5.1. *Let X be a finite set or univers. A bounded fuzzy natural number-valued multiset is a function*

$$\begin{aligned} M : X &\longrightarrow \mathcal{A}_1^L \\ x &\longmapsto M(x) \end{aligned}$$

where $M(x)$ is a bounded fuzzy natural number. Usually, the function $\mathcal{M}(\cdot)$ is called count or multiplicity of M.

Remark 5.1. We will denote the set of Bounded Fuzzy Natural Number-valued multisets defined over an universe X by $BFNNM(X)$. Finally, the abbreviation bfnnm will denote a Bounded Fuzzy Natural Number-valued multiset.

The lattice structure on the set of fuzzy natural numbers considered in the previous section, will allow us to define similar algebraic structures and lattice operations (meet and join) on the set $BFNNM(X)$.

5.1 Distributive Bounded Sublattices of FNNM(X)

According to Proposition 4.4, we know that $FNNM(X)$ constitutes a partially ordered set which is a lattice. Now, using this fact, we want to see that the set $BFNNM(X)$ is a bounded distributive sublattice of the lattice $FNNM(X)$.

Definition 5.2. *Let $A, B : X \to BFNN$ be two Bounded Fuzzy Natural Number-valued Multisets. The* join *and the* meet *of A and B will be the Bounded Fuzzy Natural Number-valued Multiset, pointwise defined for all $x \in X$ as*

$$(A \vee B)(x) = MAX\{A(x), B(x)\} \text{ and } (A \wedge B)(x) = MIN\{A(x), B(x)\}$$

respectively, where the bfnn $MAX\{A(x), B(x)\}$ and $MIN\{A(x), B(x)\}$ are obtained according to the method presented in Proposition 2.1.

Remark 5.2. It is straightforward to see that if $A(x), B(x) \in \mathcal{A}_1^L$ for all $x \in X$ then the fnn $(A \vee B)(x) = MAX\{A(x), B(x)\}$ and $(A \wedge B)(x) = MIN\{A(x), B(x)\}$ belong to $BFNN$. So, the above operations $(A \vee B)$ and $(A \wedge B)$ are well defined. Moreover from proposition 3.2, $MAX\{A(x), B(x)\}$ and $MIN\{A(x), B(x)\}$ can be obtained by means of the extension principle as well.

Similarly to Proposition 4.3, it is possible to build a partial order on the set $BFNNM(X)$ using the operations join and meet considered in Definition 5.2:

Proposition 5.1. *Let $A, B : X \to BFNN$ be two Bounded Fuzzy Natural Number-valued Multisets. The binary relationship:*
$A \leq B$ if and only if $A \vee B = B$ and/or $A \wedge B = A$ i.e. $MAX\{A(x), B(x)\} = B(x), \forall x \in X$ (or $MIN\{A(x), B(x)\} = A(x), \forall x \in X$) is a partial order on the set $BFNNM(X)$.

Now, we will use this partial order to show that the set $BFNNM(X)$ has a structure of bounded distributive lattice.

Proposition 5.2. *The set $BFNNM(X)$ of the bounded fuzzy natural number-valued multisets over X is a bounded distributive lattice with the partial order defined in Proposition 5.1 and the meet and join operations proposed in Definition 5.2.*

Proof. The distributive lattice structure follows because $(\mathcal{A}_1^L, MIN, MAX)$ is a bounded distributive lattice (see Proposition 3.3). Moreover it is easy to see that the bfnnm M_0 such that $M_0(x) = \widehat{0}$ for all $x \in X$ (being $\widehat{0}$ the minimum of lattice of bounded fuzzy natural numbers $(\mathcal{A}_1^L, MIN, MAX)$) is the minimum of $BFNNM(X)$. And, M_m such that $M_m(x) = \widehat{m}$ for all $x \in X$ (being $\widehat{m}$ the maximum of the lattice of bounded fuzzy natural numbers $(\mathcal{A}_1^L, MIN, MAX)$) is the maximum of $BFNNM(X)$.

5.2 Triangular Norms and Triangular Conorms on $BFNNM(X)$

As we have discussed in the previous Section 3.3, we know [1] that it is possible to consider t-norms(t-conorms) on any bounded partially ordered set. For this reason, we can define t-norms(t-conorms) on the bounded distributive lattice $(BFNNM(X), MIN, MAX, M_0, M_m)$.

Definition 5.3. *A t-norm(t-conorm)* $\mathbb{T}(\mathbb{S})$ *on the bounded partially ordered set* $BFNNM(X)$ *is a function*

$$\begin{array}{rcl}\mathbb{T}(\mathbb{S}) : BFNNM(X) \times BFNNM(X) & \to & BFNNM(X) \\ (A, B) & \longmapsto & \mathbb{T}(A, B)(\mathbb{S}(A, B))\end{array}$$

such that fulfills the following properties:

i) Commutativity: For all $M, N \in BFNNM(X)$

$$\mathbb{T}(A, B) = \mathbb{T}(A, B) \text{ and } \mathbb{S}(A, B) = \mathbb{S}(A, B)$$

ii) Monotonicity: For $A \leq B$, $C \leq D$

$$\mathbb{T}(A, C) \leq \mathbb{T}(B, D) \text{ and } \mathbb{S}(A, C) \leq \mathbb{S}(B, D)$$

iii) Associativity: For all $A, B, C \in BFNNM(X)$

$$\mathbb{T}(\mathbb{T}(A, B), C) = \mathbb{T}(A, \mathbb{T}(B, C)) \text{ and } \mathbb{S}(\mathbb{S}(A, B), C) = \mathbb{S}(A, \mathbb{S}(B, C))$$

iv) Boundary condition: For all $A \in BFNNM(X)$

$$\mathbb{T}(A, M_m) = A \text{ and } \mathbb{S}(A, M_0) = A$$

In the next proposition we will see that it is possible to construct a t-norm on the bounded distributive lattice $BFNNM(X)$ from a t-norm defined on the bounded distributive lattice $\mathcal{A}_1^L$.

Proposition 5.3. *For each t-norm* $\mathcal{T}$ *defined on* $\mathcal{A}_1^L$ *it is possible to build a t-norm* $\mathbb{T}$ *on the bounded distributive lattice* $BFNNM(X)$ *on the following way:* $\mathbb{T}(A, B)$ *is the bfnnm such that for each* $x \in X$

$$\mathbb{T}(A, B)(x) = \mathcal{T}(A(x), B(x))$$

Proof. It is straightforward because $\mathcal{T}$ is a t-norm.

Analogously,

Proposition 5.4. *For each t-conorm $\mathcal{S}$ defined on $\mathcal{A}_1^L$ it is possible to give a t-norm $\mathbb{S}$ on the bounded distributive lattice $BFNNM(X)$ on the following way: $\mathbb{S}(A, B)$ is the bfnnm such that for each $x \in X$*

$$\mathbb{S}(A, B)(x) = \mathcal{S}(A(x), B(x))$$

Proof. It is straightforward because $\mathcal{S}$ is a t-conorm.

From Theorem 3.4, we know that if $T(S)$ are divisible t-norm(t-conorm) on $L = \{0, \cdots, m\}$ it is possible to construct a t-norm(t-conorm) on the bounded distributive lattice of bounded fuzzy natural numbers $\mathcal{A}_1^L$. Using this fact we will see that for each divisible t-norm(t-conorm) on L it is possible to obtain a t-norm(t-conorm) on bounded partially ordered set $BFNNM(X)$.

Proposition 5.5. *For each divisible t-norm T defined on the finite chain L it is possible to build a t-norm $\mathbb{T}$ on the bounded distributive lattice $BFNNM(X)$.*

Proof. From Theorem 3.4 for each divisible t-norm T on L it is possible to obtain a t-norm $\mathcal{T}$ on $\mathcal{A}_1^L$. Now from Proposition 5.3 the proof is straightforward.

Similarly,

Proposition 5.6. *For each divisible t-conorm S defined on the finite chain L it is possible to build a t-conorm $\mathbb{S}$ on the bounded distributive lattice $BFNNM(X)$.*

6 Conclusion

We have introduced a possible extension of the concept of multiset, the bounded fuzzy natural number-valued multisets. On these bags, a bounded distributive lattice structure is presented and triangular operations have been defined.

Future studies aim to investigate the properties of these triangular operations and their application to build negation function and implication function on this bounded lattice.

Acknowledgments. We would like to express our thanks to anonymous reviewers who have contributed to improve this article. This work has been partially supported by the MTM2009-10962 project grant.

References

1. De Baets, B., Mesiar, R.: Triangular norms on product lattices. Fuzzy Sets and Systems 104, 61–75 (1999)
2. Blizard, W.D.: The development of multiset theory. Modern Logic, 319–352 (1991)
3. Casasnovas, J., Mayor, G.: Discrete t-norms and operations on extended multisets. Fuzzy sets and Systems 159, 1165–1177 (2008)
4. Casasnovas, J., Riera, J.V.: On the addition of discrete fuzzy numbers. WSEAS Transactions on Mathematics, 549–554 (2006).

5. Casasnovas, J., Riera, J.V.: Discrete fuzzy numbers defined on a subset of natural numbers. Theoretical Advances and Applications of Fuzzy Logic and Soft Computing: Advances in Soft Computing 42, 573–582 (2007)
6. Casasnovas, J., Riera, J.V.: Maximum and minimum of discrete fuzzy numbers. Frontiers in Artificial Intelligence and Applications: Artificial Intelligence Research and Development 163, 273–280 (2007)
7. Casasnovas, J., Riera, J.V.: Lattice properties of discrete fuzzy numbers under extended min and max. In: Proceedings IFSA-EUSFLAT, pp. 647–652 (2009)
8. Casasnovas, J., Riera, J.V.: Extension of discrete t-norms and t-conorms to discrete fuzzy numbers. In: Proceedings of the Fifth International Summer School on Aggregation Operators (AGOP 2009), pp. 77–82 (2009)
9. Casasnovas, J., Riera, J.V.: Triangular norms and conorms on the set of discrete fuzzy numbers. In: IPMU (accepted 2010)
10. Casasnovas, J., Riera, J.V.: Aggregation and arithmetic operations on fuzzy natural number-valued multisets. In: Proceedings ESTYLF 2010, pp. 309–314 (2010)
11. Casasnovas, J., Torrens, J.: An Axiomatic Approach to the fuzzy cardinality of finite fuzzy sets. Fuzzy Sets and Systems 133, 193–209 (2003)
12. Casasnovas, J., Torrens, J.: Scalar cardinalities of finite fuzzy sets for t-norms and t-conorms. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 11, 599–615 (2003)
13. Calude, C.S., Păun, G., et al. (eds.): Multiset Processing, Mathematical, Computer Science and Molecular Computing Points of View. LNCS, vol. 2235. Springer, Heidelberg (2001)
14. Dubois, D.: A new definition of the fuzzy cardinality of finite sets preserving the classical additivity property. Bull. Stud. Ecxch. Fuzziness Appl. BUSEFAL (5), 11–12 (1981)
15. Klir, G., Bo, Y.: Fuzzy sets and fuzzy logic. Theory and Applications. Prentice Hall, Englewood Cliffs (1995)
16. Li, B.: Fuzzy Bags and Applications. Fuzzy Sets and Systems 34, 61–71 (1990)
17. Mayor, G., Torrens, J.: Triangular norms on discrete settings. In: Klement, E.P., Mesiar, R. (eds.) Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms, pp. 189–230. Elsevier, Amsterdam (2005)
18. Miyamoto, S.: Generalizations of Multisets and Rough Approximations. International Journal of Intel. Systems 19, 639–652 (2004)
19. Miyamoto, S.: Generalized Bags, Bag Relations and Applications to Data Analysis and Decision Making. In: Torra, V., Narukawa, Y., Inuiguchi, M. (eds.) MDAI 2009. LNCS (LNAI), vol. 5861, pp. 37–54. Springer, Heidelberg (2009)
20. Miyamoto, S.: Operations for Real-Valued Bags and Bag Relations. In: Proceedings IFSA-EUSFLAT 2009, pp. 612–617 (2009)
21. Syropulos, A.: Mathematics of Multisets. In: Calude, C.S., Pun, G., Rozenberg, G., Salomaa, A. (eds.) Multiset Processing. LNCS, vol. 2235, pp. 154–160. Springer, Heidelberg (2001)
22. Rocacher, D.: On fuzzy bags and their application to flexible quering. Fuzzy Sets and Systems 140, 93–110 (2003)
23. Voxman, W.: Canonical representations of discrete fuzzy numbers. Fuzzy Sets and Systems 54, 457–466 (2001)
24. Wang, G., Wu, C., Zhao, C.: Representation and Operations of discrete fuzzy numbers. Southeast Asian Bulletin of Mathematics 28, 1003–1010 (2005)
25. Yager, R.: On the theory of bags. Inter. J. General Systems 13, 23–37 (1986)