Weighted Quasi-arithmetic Means and Conditional Expectations

Yuji Yoshida

Faculty of Economics and Business Administration, University of Kitakyushu
4-2-1 Kitagata, Kokuraminami, Kitakyushu 802-8577, Japan
yoshida@kitakyu-u.ac.jp

Abstract. In this paper, the weighted quasi-arithmetic means are discussed from the viewpoint of utility functions and background risks in economics, and they are represented by weighting functions and conditional expectations. Using these representations, an index for background risks in stochastic environments is derived through the weighted quasi-arithmetic means. The first-order stochastic dominance and the risk premium are demonstrated using the weighted quasi-arithmetic means and the aggregated mean ratios, and they are characterized by the background risk index. Finally, examples of the weighted quasi-arithmetic mean and the aggregated mean ratio for various typical utility functions are given.

1 Introduction

Weighted quasi-arithmetic means are important tools in the subjective estimation of data in decision making such as management, artificial intelligence and so on ([3,4,5]), and it is also strongly related to utility functions and background risks in economics ([6]). This paper analyzes quasi-arithmetic means of an interval through utility functions and weighting functions. Yoshida [12,13] has studied weighted quasi-arithmetic means of an interval by weighted aggregation operations from the viewpoint of subjective decision making where Kolmogorov [9] and Nagumo [10] studied the aggregation operators and Aczél [1] developed the theory regarding weighted aggregation. In this paper, we take a continuous strictly increasing function $f : [a, b] \mapsto (-\infty, \infty)$ as a decision maker's utility function, and we put a continuous function $w : [a, b] \mapsto (0, \infty)$ as a weighting function. Then we define a *weighted quasi-arithmetic mean* on a closed interval $[a, b]$ with the utility f in the background risk w by

$$f^{-1}\left(\int_a^b f(x)w(x)\,\mathrm{d}x \Big/ \int_a^b w(x)\,\mathrm{d}x\right).$$

Hence, it represents a *mean value* given by a real number $c(\in [a, b])$ satisfying

$$f(c)\int_a^b w(x)\,\mathrm{d}x = \int_a^b f(x)w(x)\,\mathrm{d}x$$

V. Torra, Y. Narukawa, and M. Daumas (Eds.): MDAI 2010, LNAI 6408, pp. 31–42, 2010.

in the *mean value theorem*. This paper discusses the weighted quasi-arithmetic means from the viewpoint of utility functions and background risks in economics. Representing the weighted quasi-arithmetic means by conditional expectations, we derive an index for risks in stochastic environments, and we also discuss the first-order stochastic dominance and the risk premium using the weighted quasi-arithmetic means and the aggregated mean ratios.

In Section 2, we give definitions of *the weighted quasi-arithmetic mean* and an *aggregated mean ratio* of the weighted quasi-arithmetic mean by an interior ratio on the interval, and we demonstrate the relation among the weighted quasi-arithmetic mean, the aggregated mean ratio and the decision maker's preference/attitude based on his utility. In economics, the decision maker's attitudes, for example neutral, risk averse and risk loving, are characterized to Arrow-Pratt index of the utility function([2,11,7,8]). In Section 3, this paper characterizes the weighted quasi-arithmetic means and the mean ratios by not only utility functions but also weighing functions as an index for risks in stochastic environments. Next we investigate the properties of the weighted quasi-arithmetic means and the aggregated mean ratios regarding combinations of utility functions and weighting functions. Representing the weighted quasi-arithmetic means by conditional expectations, we investigate the relation between the index for background risks and the risk premium in economics. We also discuss the first-order stochastic dominance through the weighted quasi-arithmetic means. Finally, in Section 4, we show a lot of examples of the weighted quasi-arithmetic means and the aggregated mean ratios with various typical utility functions, and we demonstrate their relations with the classical quasi-arithmetic means.

2 Weighted Quasi-arithmetic Means and Their Properties

In this section, we introduce weighted quasi-arithmetic means and aggregated mean ratios regarding with utility functions and weighting functions, and we discuss sufficient conditions on utility functions and weighting functions to characterize the decision maker's attitude based on the quasi-arithmetic mean and the aggregated mean ratio. Let D be a fixed interval which is not a singleton and we call it a domain. Let $\mathcal{C}(D)$ be the set of all nonempty bounded closed subintervals of D and let $\mathcal{C}(D)_< := \{[a,b] \in \mathcal{C}(D) | a < b\}$. Let $f : D \mapsto (-\infty, \infty)$ be a continuous strictly increasing function for utility, and let $w : D \mapsto (0, \infty)$ be a continuous function for weighting. For a closed interval $[a,b] \in \mathcal{C}(D)_<$, a mapping $M_w^f : \mathcal{C}(D) \mapsto D$ given by

$$M_w^f([a,b]) := f^{-1}\left(\int_a^b f(x)w(x)\,\mathrm{d}x \Big/ \int_a^b w(x)\,\mathrm{d}x \right) \tag{1}$$

is called the *weighted quasi-arithmetic mean* with a specified weighting w. Next for a closed interval $[a,b] \in \mathcal{C}(D)_<$ we define an interior ratio $\theta_w^f(a,b)$ from a position of the weighted quasi-arithmetic mean $M_w^f([a,b])$ on the interval $[a,b]$ by

$$\theta_w^f(a,b) := \frac{M_w^f([a,b]) - a}{b-a}. \tag{2}$$

Dujmović [3,4,5] studied a *conjunction/disjunction degree*, which is a similar type of ratio in the power case, for computer science. This paper discusses their characterizations from the viewpoint of economics by conditional expectations. Now we let $g : D \mapsto (-\infty, \infty)$ be another continuous strictly increasing function for utility. Let $M_w^g : \mathcal{C}(D) \mapsto D$ be the weighted quasi-arithmetic mean defined by g instead of f in the way of (1) and we put the aggregated mean ratio θ_w^g for M_w^g. Then we obtain the following results.

Lemma 1 ([13]). *Let f and g be C^2-class utility functions on D. Let $[a,b] \in \mathcal{C}(D)_<$. Then the following (a) – (c) are equivalent.*

(a) $f''/f' \le g''/g'$ *on* (a,b).
(b) $M_w^f([c,d]) \le M_w^g([c,d])$ *for all* $[c,d]$ *satisfying* $[c,d] \subset [a,b]$ *and* $c < d$.
(c) $\theta_w^f(c,d) \le \theta_w^g(c,d)$ *for all* $[c,d]$ *satisfying* $[c,d] \subset [a,b]$ *and* $c < d$.

When we may choose two utility functions f and g as decision maker's utilities, Lemma 1 implies that the utility f yields more risk averse results than g if $f''/f' \le g''/g'$ on (a,b). Thus, the inequality $\theta_w^f(a,b) \le \theta_w^g(a,b)$ implies that the aggregated mean ratio $\theta_w^f(a,b)$ is more risk averse than $\theta_w^g(a,b)$. The function $-f''/f'$ is called the *Arrow-Pratt index* and it implies the degree of absolute risk aversion in economics ([2,11]).

3 Weighted Quasi-arithmetic Means and Background Risks

In this paper, we focus on weighting functions w as risk factors of stochastic environments in the weighted quasi-arithmetic mean (1) and we characterize it in relation to the conditional expectation. Let D be a fixed domain and let $f : D \mapsto (-\infty, \infty)$ be a fixed continuous strictly increasing function for utility. The following theorem implies the properties of the weighted quasi-arithmetic mean M_w^f and the ratio θ_w^f concerning weighting w.

Theorem 1. *Let $w : D \mapsto (0, \infty)$ and $v : D \mapsto (0, \infty)$ be C^1-class weighting functions. Let $[a,b] \in \mathcal{C}(D)_<$. Then the following (i) and (ii) hold.*

(i) *If w and v satisfy $w'/w < v'/v$ on (a,b), it holds that $M_w^f([a,b]) < M_v^f([a,b])$ and $\theta_w^f([a,b]) < \theta_v^f([a,b])$.*
(ii) *If w and v satisfy $w'/w \le v'/v$ on (a,b), it holds that $M_w^f([a,b]) \le M_v^f([a,b])$ and $\theta_w^f([a,b]) \le \theta_v^f([a,b])$.*

In Theorem 1, we note that $w'/w \le v'/v$ on (a,b) is a sufficient condition so that the weighting w yields lower estimation than the weighting v. Further, the following Theorem 2 shows an equivalence regarding the assertion 'if - then' in Theorem 1(ii).

Theorem 2. *Let $w : D \mapsto (0, \infty)$ and $v : D \mapsto (0, \infty)$ be C^1-class weighting functions. Let $[a,b] \in \mathcal{C}(D)_<$. Then the following (a) – (c) are equivalent.*

(a) $w'/w \le v'/v$ *on* (a,b).
(b) $M_w^f([c,d]) \le M_v^f([c,d])$ *for all* $[c,d]$ *satisfying* $[c,d] \subset [a,b]$ *and* $c < d$.
(c) $\theta_w^f(c,d) \le \theta_v^f(c,d)$ *for all* $[c,d]$ *satisfying* $[c,d] \subset [a,b]$ *and* $c < d$.

In the following proposition, (i) implies that the estimation by a utility $h = (f+g)/2$ gives a *middle attitude* by the both utilities f and g and (ii) shows that a weighting function $u = (w+v)/2$ gives a *middle-level risk* of the both risks w and v in stochastic environments.

Proposition 1. *Let* $[a,b] \in \mathcal{C}(D)_<$. *Then the following (i) and (ii) holds.*

(i) *Let* f *and* g *be* C^2*-class utility functions on* D. *Let* $h := (f+g)/2$. *If* f *and* g *satisfy* $f''/f' \le g''/g'$ *on* (a,b), *then* $M_w^f([a,b]) \le M_w^h([a,b]) \le M_w^g([a,b])$ *and* $\theta_w^f(a,b) \le \theta_w^h(a,b) \le \theta_w^g(a,b)$.
(ii) *Let* $w : D \mapsto (0,\infty)$ *and* $v : D \mapsto (0,\infty)$ *be* C^1*-class weighting functions. Let* $u := (w+v)/2$. *If* w *and* v *satisfy* $w'/w \le v'/v$ *on* (a,b), *then* $M_w^f([a,b]) \le M_u^f([a,b]) \le M_v^f([a,b])$ *and* $\theta_w^f(a,b) \le \theta_u^f(a,b) \le \theta_v^f(a,b)$.

The Arrow-Pratt index $-f''/f'$ implies the degree of absolute risk aversion. On the other hand, the index $-w'/w$, which is introduced in this paper, is related to the *background risks* of stochastic environments in economics ([8]). In the rest of this section, using the representation of conditional expectations, we investigate the relation between the index $-w'/w$ and the background risks. Let (Ω, P) be a probability space, where P is a non-atomic probability measure on Ω.

Definition 1. For random variables X and Y on Ω, it is said that the random variable X is *dominated by* the random variable Y in the sense of *the first-order stochastic dominance* if

$$P(X < x) \ge P(Y < x) \text{ for any real number } x. \tag{3}$$

Then the following result is well-known for the first-order stochastic dominance in economics (Arrow [2], Gollier [7], Eeckhoudt et al. [8]).

Proposition 2. *Let* X *and* Y *be random variables on* Ω. *Then, the random variable* X *is dominated by the random variable* Y *in the sense of the first-order stochastic dominance if and only if it holds that*

$$E(f(X)) \le E(f(Y)) \tag{4}$$

for any increasing utility function $f : (-\infty,\infty) \mapsto (-\infty,\infty)$ *satisfying a tail condition* $\lim_{x\to\pm\infty} f(x)(P(X<x) - P(Y<x)) = 0$.

The first-order stochastic dominance (3) means that the stochastic environment X is risky than the stochastic environment Y, and it shows in (4) that all decision makers estimate the stochastic environment X lower than the stochastic environment Y. Then the decision makers prefer the stochastic environment Y to the stochastic environment X with their any increasing utility functions f.

Let X be a real random variable on Ω with a C^1-class density function w on $(-\infty, \infty)$. Since the conditional expectation of the utility $f(X)$ is

$$E(f(X) \mid a < X < b) = \frac{E(f(X)1_{\{a<X<b\}})}{P(a < X < b)} = \frac{\int_a^b f(x)w(x)\,\mathrm{d}x}{\int_a^b w(x)\,\mathrm{d}x}, \tag{5}$$

it holds that

$$M_w^f([a,b]) = f^{-1}\left(\frac{\int_a^b f(x)w(x)\,\mathrm{d}x}{\int_a^b w(x)\,\mathrm{d}x}\right) = f^{-1}(E(f(X) \mid a < X < b)) \tag{6}$$

for real numbers $a, b\,(a < b)$, where $1_{\{\cdot\}}$ implies the characteristic function of a set. From Theorem 2 and (6), we obtain the following result together with Proposition 2.

Corollary 1. *Let X and Y be random variables on Ω which have C^1-class density functions w and v on $(-\infty, \infty)$ respectively. If*

$$\frac{w'}{w} \leq \frac{v'}{v} \quad \textit{on } (-\infty, \infty), \tag{7}$$

then the random variable X is dominated by the random variable Y in the sense of the first-order stochastic dominance.

From this corollary, (7) is a sufficient condition for the first-order stochastic dominance (3) where the stochastic environment X is risky than the stochastic environment Y. Hence we find that (7) is useful to estimate the risk-level of stochastic environments and it is easy to check in actual problems (Example 3). In this paper, we call $-w'/w$ the *background risk index*. We note that the first-order stochastic dominance (3) is a risk criterion in a global area $D = (-\infty, \infty)$ for stochastic environments and it is represented by integrals in (4), however the background risk index $-w'/w$ can measure risks even in local areas since it is represented by differentials.

Next we discuss risk premiums regarding risk averse in financial management ([7,8]). Let $z \in D$, which implies an *initial wealth*, and let $[a, b] \in \mathcal{C}(D_z)_<$, where $D_z := \{x - z \mid x \in D\}$. Let X be a random variable on Ω, which implies a *stochastic environment with some risk*. A decision maker with a utility f is called *risk averse on* (a, b) if

$$E(f(z + X) \mid a < X < b) \leq f(E(z + X \mid a < X < b)). \tag{8}$$

A sufficient condition for the risk averse is that the utility function f is concave. Let w be a density function on D for the random variable X. Hence, in the following (9), a real number $\pi_w^f(a, b)$ is called *the risk premium on* (a, b) ([7,8]) if it satisfies

$$E(f(z + X) \mid a < X < b) = f(z - \pi_w^f(a, b)). \tag{9}$$

Eq.(9) means that the decision maker accepts the risk arising from the random variable X by paying the risk premium $\pi_w^f(a, b)$.

Theorem 3. *Let f be a continuous strictly increasing utility function on D. Let X be a random variable on Ω which has a C^1-class density function w on D. The risk premium in (9) is given by*

$$\pi_w^f(a,b) = -M_w^h([a,b]), \tag{10}$$

where $h(x) := f(z+x)$ for $x \in (a-z, b-z)$.

Then we obtain the following two theorems. Theorem 4 is from Lemma 1 and Theorem 3, and it gives the relation between the Arrow-Pratt index and the risk premium. On the other hand, Theorem 4 is from Theorems 2 and 3, and it gives the relation between the background risk index and the risk premium.

Theorem 4. *Let an initial wealth $z \in D$ and let $[a,b] \in \mathcal{C}(D_z)_<$. Let f and g be continuous strictly increasing utility functions on D. Let X be random variable on Ω which has a C^1-class density function w. Then the following (a) and (b) are equivalent.*

(a) $f''/f' \le g''/g'$ *on* $(z+a, z+b)$.
(b) $\pi_w^f(c,d) \ge \pi_w^g(c,d)$ *for all* $[c,d]$ *satisfying* $[c,d] \subset [a,b]$ *and* $c<d$.

Theorem 5. *Let f be a continuous strictly increasing utility function on D. Let X and Y be random variables on Ω which have C^1-class density functions w and v respectively. Then the following (a) and (b) are equivalent.*

(a) $w'/w \le v'/v$ *on* (a,b).
(b) $\pi_w^f(c,d) \ge \pi_v^f(c,d)$ *for all* $[c,d]$ *satisfying* $[c,d] \subset [a,b]$ *and* $c<d$.

4 Examples

In this section, we give examples for weighted quasi-arithmetic means $M_w^f([a,b])$ and the aggregated mean ratio $\theta_w^f(a,b)$ which are presented in the previous sections. First we investigate examples of weighting functions w, and next we discuss examples of utility functions f.

Example 1. We deal with a utility function $f(x) = x$ for $x \in (-\infty, \infty)$. Then $f''(x)/f'(x) = 0$. For a closed interval $[a,b] \in \mathcal{C}(D)_<$, we define the *neutral weighted mean* $N_w([a,b])$ and its aggregated mean ratio $\nu_w(a,b)$ by

$$N_w([a,b]) := \int_a^b x\, w(x)\, \mathrm{d}x \Big/ \int_a^b w(x)\, \mathrm{d}x \tag{11}$$

and

$$\nu_w(a,b) := \frac{N_w([a,b]) - a}{b-a} = \int_a^b (x-a) w(x)\, \mathrm{d}x \Big/ \int_a^b (b-a) w(x)\, \mathrm{d}x. \tag{12}$$

(i) Take a weighting function $w(x) = x^\alpha$ on $D = (0, \infty)$ with a constant α such that $\alpha \neq -2$ and $\alpha \neq -1$. Then $w'(x)/w(x) = \alpha/x$. Let $[a, b] \subset D = (0, \infty)$ such that $a < b$. Then, we have

$$N_w([a,b]) = \frac{(\alpha+1)(b^{\alpha+2} - a^{\alpha+2})}{(\alpha+2)(b^{\alpha+1} - a^{\alpha+1})}.$$

Further, it holds that $\lim_{b \downarrow a} \nu_w(a,b) = \lim_{a \uparrow b} \nu_w(a,b) = 1/2$ ([13, Theorem 5.9]) and $\lim_{a \downarrow 0} \nu_w(a,b) = \lim_{b \to \infty} \nu_w(a,b) = (\alpha+1)/(\alpha+2)$. Weighted quasi-arithmetic means $M_w^f([a,b])$ for other utility functions f are given by Table 1.

Table 1. Weighted quasi-arithmetic means for utility functions f $(w(x) = x^\alpha)$

f	f''/f'	$M_w^f([a,b])$
$rx + s$ $(r > 0)$	0	$\dfrac{(\alpha+1)(b^{\alpha+2} - a^{\alpha+2})}{(\alpha+2)(b^{\alpha+1} - a^{\alpha+1})}$
x^r $(r \neq 0)$	$\dfrac{r-1}{x}$	$\left(\dfrac{(\alpha+1)(b^{r+\alpha+1} - a^{r+\alpha+1})}{(r+\alpha+1)(b^{\alpha+1} - a^{\alpha+1})}\right)^{1/r}$
$r \log x$ $(r > 0)$	$-\dfrac{1}{x}$	$\exp\left(\dfrac{b^{\alpha+1}\log b - a^{\alpha+1}\log a}{b^{\alpha+1} - a^{\alpha+1}} - \dfrac{1}{\alpha+1}\right)$
e^{sx} $(s \neq 0)$	s	$\dfrac{1}{s}\log\left(\dfrac{(\alpha+1)(\Gamma(\alpha+1,-sb) - \Gamma(\alpha+1,-sa))}{s^{\alpha+1}(b^{\alpha+1} - a^{\alpha+1})}\right)$

Here in Table 1 we put

$$\Gamma(\alpha+1, c) := \int_c^\infty t^\alpha e^{-t} dt$$

for real numbers c.

(ii) Take a weighting function $w(x) = x^{-2}$ on $D = (0, \infty)$ with $\alpha = -2$. Then $w'(x)/w(x) = -2/x$. Let $[a, b] \subset D = (0, \infty)$ such that $a < b$. Then, we have

$$N_w([a,b]) = \frac{ab(\log b - \log a)}{b - a}.$$

Further, it holds that $\lim_{b \downarrow a} \nu_w(a,b) = \lim_{a \uparrow b} \nu_w(a,b) = 1/2$ ([13, Theorem 5.9]) and $\lim_{a \downarrow 0} \nu_w(a,b) = \lim_{b \to \infty} \nu_w(a,b) = 0$.

(iii) Take a weighting function $w(x) = x^{-1}$ on $D = (0, \infty)$ with $\alpha = -1$. Then $w'(x)/w(x) = -1/x$. Let $[a, b] \subset D = (0, \infty)$ such that $a < b$. Then, we have

$$N_w([a,b]) = \frac{b-a}{\log b - \log a}.$$

Further, it holds that $\lim_{b\downarrow a} \nu_w(a,b) = \lim_{a\uparrow b} \nu_w(a,b) = 1/2$ ([13, Theorem 5.9]) and $\lim_{a\downarrow 0} \nu_w(a,b) = \lim_{b\to\infty} \nu_w(a,b) = 0$.

(iv) Take a weighting function $w(x) = c_0 + c_1 x + c_2 x^2$ on $D = (0, \infty)$ with positive constants $c_0, c_1.c_2$. Then

$$\frac{w'(x)}{w(x)} = \frac{c_1 + 2c_2 x}{c_0 + c_1 x + c_2 x^2}.$$

Let $[a, b] \subset D = (0, \infty)$ such that $a < b$. Then, we have

$$N_w([a,b]) = \frac{\frac{1}{2}c_0(b^2-a^2) + \frac{1}{3}c_1(b^3-a^3) + \frac{1}{4}c_2(b^4-a^4)}{c_0(b-a) + \frac{1}{2}c_1(b^2-a^2) + \frac{1}{3}c_2(b^3-a^3)}.$$

Further, it holds that $\lim_{b\downarrow a} \nu_w(a,b) = \lim_{a\uparrow b} \nu_w(a,b) = 1/2$,

$$\lim_{a\downarrow 0} \nu_w(a,b) = \frac{6c_0 + 4c_1 b + 3c_2 b^2}{12c_0 + 6c_1 b + 4c_2 b^2} \quad \text{and} \quad \lim_{b\to\infty} \nu_w(a,b) = \frac{3}{4}.$$

(v) Take a weighting function $w(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$ on $D = (0, \infty)$ with positive constants $c_0, c_1.c_2, \cdots, c_n$. Then

$$\frac{w'(x)}{w(x)} = \frac{\sum_{k=0}^{n-1}(k+1)c_{k+1}x^k}{\sum_{k=0}^{n} c_k x^k}.$$

Let $[a, b] \subset D = (0, \infty)$ such that $a < b$. Then, we have

$$N_w([a,b]) = \frac{\sum_{k=0}^{n} \frac{1}{k+2} c_k (b^{k+2} - a^{k+2})}{\sum_{k=0}^{n} \frac{1}{k+1} c_k (b^{k+1} - a^{k+1})}.$$

Further, it holds that $\lim_{b\downarrow a} \nu_w(a,b) = \lim_{a\uparrow b} \nu_w(a,b) = 1/2$,

$$\lim_{a\downarrow 0} \nu_w(a,b) = \frac{\sum_{k=0}^{n} \frac{1}{k+2} c_k b^{k+2}}{\sum_{k=0}^{n} \frac{1}{k+1} c_k b^{k+1}} \quad \text{and} \quad \lim_{b\to\infty} \nu_w(a,b) = \frac{n+1}{n+2}.$$

(vi) Take a weighting function $w(x) = e^{-\beta x}$ on $D = (-\infty, \infty)$ with a non-zero constant β. Then $w'(x)/w(x) = -\beta$. Let $[a, b] \subset D = (-\infty, \infty)$ such that $a < b$. Then, we have

$$N_w([a,b]) = \frac{e^{-\beta b}(\beta b + 1) - e^{-\beta a}(\beta a + 1)}{\beta(e^{-\beta b} - e^{-\beta a})}.$$

Further, $\lim_{b\downarrow a} \nu_w(a,b) = \lim_{a\uparrow b} \nu_w(a,b) = 1/2$ and $\lim_{a\to-\infty} \nu_w(a,b) = \lim_{b\to\infty} \nu_w(a,b) = 1$.

(vii) Take a weighting function $w(x) = e^{-\gamma^2 x^2}$ on $D = (-\infty, \infty)$ with a positive constant γ. Then $w'(x)/w(x) = -2\gamma^2 x$. Let $[a, b] \subset D = (-\infty, \infty)$ such that $a < b$. Then, we have

$$N_w([a, b]) = \frac{e^{-\gamma^2 a^2} - e^{-\gamma^2 b^2}}{2\gamma \int_{\gamma a}^{\gamma b} e^{-x^2}\, dx}.$$

Further, $\lim_{b \downarrow a} \nu_w(a, b) = \lim_{a \uparrow b} \nu_w(a, b) = 1/2$, $\lim_{a \to -\infty} \nu_w(a, b) = 1$ and $\lim_{b \to \infty} \nu_w(a, b) = 0$.

Table 2. Neutral weighted means for weighting functions w ($f(x) = x$)

w	w'/w	$N_w([a, b])$
x^α ($\alpha \neq -2, -1$)	$\frac{\alpha}{x}$	$\frac{(\alpha+1)(b^{\alpha+2} - a^{\alpha+2})}{(\alpha+2)(b^{\alpha+1} - a^{\alpha+1})}$
$\sum_{k=0}^{n} c_k x^k$ ($c_0, c_1, \cdots, c_n > 0$)	$\frac{\sum_{k=0}^{n-1}(k+1)c_{k+1}x^k}{\sum_{k=0}^{n} c_k x^k}$	$\frac{\sum_{k=0}^{n} \frac{1}{k+2} c_k (b^{k+2} - a^{k+2})}{\sum_{k=0}^{n} \frac{1}{k+1} c_k (b^{k+1} - a^{k+1})}$
$e^{-\beta x}$ ($\beta \neq 0$)	$-\beta$	$\frac{e^{-\beta b}(\beta b + 1) - e^{-\beta a}(\beta a + 1)}{\beta(e^{-\beta b} - e^{-\beta a})}$
$e^{-\gamma^2 x^2}$ ($\gamma > 0$)	$-2\gamma^2 x$	$\frac{e^{-\gamma^2 a^2} - e^{-\gamma^2 b^2}}{2\gamma \int_{\gamma a}^{\gamma b} e^{-x^2}\, \mathrm{d}x}$

Some results in Example 1 are listed in Table 2. Next we show the relation between the weighted quasi-arithmetic means and the typical means.

Example 2. Let the domain $D = (0, \infty)$. Take a function $f(x) = x^r$ and $w(x) = x^\alpha$ on D with constants r, α satisfying $r \neq 0$. Then $f''(x)/f'(x) = (r-1)/x$ and $w'(x)/w(x) = \alpha/x$. Hence we can deal with not only $r > 0$ for increasing function $f = x^r$ but also $r < 0$ for decreasing function $f(x) = x^r$ ([13, Remark 3.2(1)]). Then, for $[a, b] \subset D$ such that $a < b$, the weighted quasi-arithmetic mean is given by the following $M_{(\alpha)}^{(r)}([a, b]) := M_w^f([a, b])$:

$$M_{(\alpha)}^{(r)}([a, b]) = \left(\frac{(\alpha+1)(b^{r+\alpha+1} - a^{r+\alpha+1})}{(r+\alpha+1)(b^{\alpha+1} - a^{\alpha+1})} \right)^{1/r}$$

if $r \neq 0, \alpha \neq -1, r + \alpha \neq -1$. The limiting values regrading r and α are

$$\lim_{\alpha \to -r-1} M_{(\alpha)}^{(r)}([a,b]) = ab \left(\frac{r(\log b - \log a)}{b^r - a^r} \right)^{1/r} \qquad \text{if } r \neq 0,$$

$$\lim_{\alpha \to -1} M_{(\alpha)}^{(r)}([a,b]) = \left(\frac{r(\log b - \log a)}{b^r - a^r} \right)^{-1/r} \qquad \text{if } r \neq 0,$$

$$\lim_{r \to 0} M_{(\alpha)}^{(r)}([a,b]) = \exp \left(\frac{b^{\alpha+1} \log b - a^{\alpha+1} \log a}{b^{\alpha+1} - a^{\alpha+1}} - \frac{1}{\alpha+1} \right) \quad \text{if } \alpha \neq -1,$$

$$\lim_{\alpha \to -1} \lim_{r \to 0} M_{(\alpha)}^{(r)}([a,b]) = \sqrt{ab},$$

$$\lim_{r \to -\infty} M_{(\alpha)}^{(r)}([a,b]) = a,$$

$$\lim_{r \to \infty} M_{(\alpha)}^{(r)}([a,b]) = b.$$

Finally we show the relation between the weighted quasi-arithmetic means and their application to economics.

Example 3. We give an example for Corollary 1 by normal distributions on stochastic environments. Let random variables X and Y have normal distributions on Ω with density functions w and v respectively as follows. Let μ_X and μ_X be the means and let σ_X and σ_Y be the standard deviations for w and v respectively, i.e.,

$$w(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(-\frac{(x-\mu_X)^2}{2\,\sigma_X^2} \right) \text{ and } v(x) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left(-\frac{(x-\mu_Y)^2}{2\,\sigma_Y^2} \right)$$

for real numbers x. Then we have

$$\begin{aligned} & \frac{w'(x)}{w(x)} \le \frac{v'(x)}{v(x)} \\ \iff & -\frac{x-\mu_X}{\sigma_X^2} \le -\frac{x-\mu_Y}{\sigma_Y^2} \\ \iff & \sigma_X^2 \mu_Y - \sigma_Y^2 \mu_X \ge (\sigma_X^2 - \sigma_Y^2) x \\ \iff & \begin{cases} x \ge \frac{\sigma_X^2 \mu_Y - \sigma_Y^2 \mu_X}{\sigma_X^2 - \sigma_Y^2} & \text{if } \sigma_X < \sigma_Y \\ x \le \frac{\sigma_X^2 \mu_Y - \sigma_Y^2 \mu_X}{\sigma_X^2 - \sigma_Y^2} & \text{if } \sigma_X > \sigma_Y \\ \text{all } x \in (-\infty, \infty) & \text{if } \sigma_X = \sigma_Y \text{ and } \mu_X \le \mu_Y \\ \text{no } x & \text{if } \sigma_X = \sigma_Y \text{ and } \mu_X > \mu_Y. \end{cases} \end{aligned}$$

Define a domain D by

$$D := \begin{cases} \left(\frac{\sigma_X^2 \mu_Y - \sigma_Y^2 \mu_X}{\sigma_X^2 - \sigma_Y^2}, \infty \right) & \text{if } \sigma_X < \sigma_Y \\ \left(-\infty, \frac{\sigma_X^2 \mu_Y - \sigma_Y^2 \mu_X}{\sigma_X^2 - \sigma_Y^2} \right) & \text{if } \sigma_X > \sigma_Y \\ (-\infty, \infty) & \text{if } \sigma_X = \sigma_Y \text{ and } \mu_X \le \mu_Y \\ \emptyset & \text{if } \sigma_X = \sigma_Y \text{ and } \mu_X > \mu_Y. \end{cases}$$

From Theorems 2 and 5, we get $M_w^f([a,b]) \leq M_v^f([a,b])$ and $\pi_w^f(a,b) = -M_w^f([a,b]) \geq -M_v^f([a,b]) = \pi_v^f(a,b)$ for subintervals $[a,b] \subset D$. By Theorem 3, we can also calculate the risk premium $\pi_w^f(a,b)$ for a classical utility function $f(x) = 1 - e^{-x}$ as follows:

$$\pi_w^f(a,b) = \frac{\mathrm{Erf}\left(\frac{a-\mu_X}{\sqrt{2}\sigma_X}\right) - \mathrm{Erf}\left(\frac{b-\mu_X}{\sqrt{2}\sigma_X}\right)}{2(-a+b+e^{a+z}-e^{b+z})} + \frac{e^{z+\mu_X+\frac{\sigma_X^2}{2}}\left(\mathrm{Erf}\left(\frac{-a+\mu_X+\sigma_X^2}{\sqrt{2}\sigma_X}\right) - \mathrm{Erf}\left(\frac{-b+\mu_X+\sigma_X^2}{\sqrt{2}\sigma_X}\right)\right)}{2(-a+b+e^{a+z}-e^{b+z})},$$

where

$$\mathrm{Erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

for real numbers x. Further, by Corollary 1, if $\sigma_X = \sigma_Y$ and $\mu_X \leq \mu_Y$, all decision makers prefers the stochastic environment Y to the stochastic environment X for his any increasing utility f, i.e. it holds that $E(f(X)) \leq E(f(Y))$ for any increasing utility function f, which is equivalent that X is dominated by Y in the sense of the first-order stochastic dominance (Proposition 1).

5 Conclusions

We have analyzed the weighted quasi-arithmetic means with utility functions and weighting for random factors in stochastic environments. The background risk index is first introduced through weighting functions as an index of risk-levels for stochastic environments, and its relations to the first-order stochastic dominance and the risk premium are demonstrated with conditional expectations. We have investigated a lot of examples of the weighted quasi-arithmetic mean and the aggregated mean ratio for various typical utility functions. The stochastic dominance is a risk criterion in a global area for stochastic environments however using the background risk index $-w'/w$ we can analyze risks even in local areas. The background risk index $-w'/w$ will be useful and easy to calculate in actual problems.

References

1. Aczél, J.: On weighted mean values. Bulletin of the American Math. Society 54, 392–400 (1948)
2. Arrow, K.J.: Essays in the Theory of Risk-Bearing, Markham, Chicago (1971)
3. Dujmović, J.J.: Weighted Conjunctive and disjunctive means and their application in system evaluation. Univ. Beograd. Publ. Elektotech. Fak. Ser. Mat. Fiz. 483, 147–158 (1974)
4. Dujmović, J.J., Larsen, H.L.: Generalized Conjunction/disjunction. International Journal of Approximate Reasoning 46, 423–446 (2007)

5. Dujmović, J.J., Nagashima, H.: LSP method and its use for evaluation of Java IDEs. International Journal of Approximate Reasoning 41, 3–22 (2006)
6. Fishburn, P.C.: Utility Theory for Decision Making. John Wiley and Sons, New York (1970)
7. Gollier, G.: The Economics of Risk and Time. MIT Publishers, Cambridge (2001)
8. Eeckhoudt, L., Gollier, G., Schkesinger, H.: Economic and Financial Decisions under Risk. Princeton University Press, Princeton (2005)
9. Kolmogoroff, A.N.: Sur la notion de la moyenne. Acad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. 12, 388–391 (1930)
10. Nagumo, K.: Über eine Klasse der Mittelwerte. Japanese Journal of Mathematics 6, 71–79 (1930)
11. Pratt, J.W.: Risk Aversion in the Small and the Large. Econometrica 32, 122–136 (1964)
12. Yoshida, Y.: Aggregated mean ratios of an interval induced from aggregation operations. In: Torra, V., Narukawa, Y. (eds.) MDAI 2008. LNCS (LNAI), vol. 5285, pp. 26–37. Springer, Heidelberg (2008)
13. Yoshida, Y.: Quasi-arithmetic means and ratios of an interval induced from weighted aggregation operations. Soft Computing 14, 473–485 (2010)