Sugeno Utility Functions II: Factorizations

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Abstract. In this paper we address and solve the problem posed in the companion paper [3] of factorizing an overall utility function as a composition $q(\varphi_1(x_1),\ldots,\varphi_n(x_n))$ of a Sugeno integral q with local utility functions φ_i, if such a factorization exists.

Keywords: Sugeno integral, local utility function, overall utility function, Sugeno utility function, factorization.

1 Introduction

In the companion paper [3], we considered a multicriteria aggregation model where local utility functions (i.e., order-preserving mappings) $\varphi_i\colon L_i \to L$, $i = 1,\ldots,n$, are aggregated using a (discrete) Sugeno integral $q\colon L^n \to L$, thus giving rise to an overall utility function $f\colon L_1 \times \cdots \times L_n \to L$ defined by

$$f(x_1,\ldots,x_n) = q(\varphi_1(x_1),\ldots,\varphi_n(x_n)). \tag{1}$$

Such functions were called Sugeno utility functions in [3]. More general classes of functions were also considered, where the inner functions φ_i are not necessarily order-preserving, and where the outer function q is either a Sugeno integral or a (lattice) polynomial function. The resulting functions were referred to as pseudo-Sugeno integrals and pseudo-polynomial functions, respectively.

This aggregation model is deeply rooted in multicriteria decision making, where the variables x_i represent different properties of the alternatives (e.g., price, speed, safety, comfort level of a car), and the overall utility function (also called global preference function) assigns a score to the alternatives that helps the decision maker to choose the best one (e.g., to choose the car to buy). A similar situation is that of subjective evaluation (see [1]): f outputs the overall rating of a certain product by customers, and the variables x_i represent the various properties of that product. The way in which these properties influence the overall rating can give information about the attitude of the customers. A factorization of the (empirically) given overall utility function f in the form (1) can be used for such an analysis; this is our main motivation for addressing this problem.

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In [3] we established necessary and sufficient conditions which guarantee the existence of factorizations of functions $f\colon L_1 \times \cdots \times L_n \to L$ as compositions (1). However, no hint was given on how to obtain such factorizations. In this paper, we address and solve this problem by providing a canonical construction of such a Sugeno integral q and utility functions φ_i so that their composition satisfies $f = q(\varphi_1, \ldots, \varphi_n)$.

The paper is organized as follows. In Sect. 2 we recall the background on lattice polynomial functions, Sugeno integrals and Sugeno utility functions needed throughout the paper (for further background and references see the companion paper [3]). In Sect. 3 we present a method to construct a factorization of a Sugeno utility function f, which is illustrated in Subsect. 3.3 by means of a concrete example. Finally, in Sect. 4 we prove the correctness of the procedure by showing that the Sugeno integral and the local utility functions constructed in Sect. 3 indeed give a factorization of f.

2 Preliminaries

2.1 Lattice Polynomials and Sugeno Integrals

Let L be a chain endowed with the lattice operations $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Clearly, L is a distributive lattice. A chain L is *complete* if every nonempty subset S of L has a greatest lower bound (infimum) denoted by $\bigwedge S$, and a least upper bound (supremum) denoted by $\bigvee S$. A chain is *bounded*, if it has least and greatest elements, usually denoted by 0_L and 1_L, respectively, or simply by 0 and 1, when there is no risk of ambiguity. Observe that if L is complete, then it is bounded. In most applications the chains considered are either closed real intervals or finite chains, and these are all complete. Hence throughout the paper, $L_1, \ldots, L_n$ and L will always denote complete chains.

An n-ary (*lattice*) *polynomial function* on L is a function $p\colon L^n \to L$ that can be built from projections $(x_1, \ldots, x_n) \mapsto x_i$ and constants by a finite number of applications of the lattice operations $\wedge, \vee$ (for a recent reference, see [2]). The notion of polynomial functions subsumes certain important fuzzy integrals, namely, Sugeno integrals. As it was observed in [8,9], (*discrete*) *Sugeno integrals* can be defined as certain polynomial functions, namely, those polynomial functions $q\colon L^n \to L$ satisfying $q(a, \ldots, a) = a$ for all $a \in L$. We will work with this definition of the Sugeno integral; for the original definition (as an integral with respect to a fuzzy measure) see, e.g., [7,10,11].

Polynomial functions have a neat disjunctive normal form representation, as shown by the following theorem of Goodstein [5]. Let $[n] = \{1, \ldots, n\}$, and for $I \subseteq [n]$ let $\mathbf{e}_I \in L^n$ be the characteristic vector of I, i.e., the vector whose ith component is 1 if $i \in I$ and 0 if $i \notin I$.

Theorem 1. *A function $p\colon L^n \to L$ is a polynomial function if and only if*

$$p(x_1, \ldots, x_n) = \bigvee_{I \subseteq [n]} \bigl(p(\mathbf{e}_I) \wedge \bigwedge_{i \in I} x_i\bigr).$$

Such a function is a Sugeno integral iff $p(\mathbf{e}_\emptyset) = 0$ and $p\left(\mathbf{e}_{[n]}\right) = 1$.

In the sequel we will make use of the following property of polynomial functions [2].

Proposition 2. *For every polynomial function* $p\colon L^n \to L$ *and* $k \in [n]$ *we have*

$$p\big(x_1, \ldots, x_{k-1}, p\,(x_1, \ldots, x_n)\,, x_{k+1}, \ldots, x_n\big) = p\,(x_1, \ldots, x_n)\,.$$

An important polynomial function is the *median function* $\mathrm{med}\colon L^3 \to L$ defined by $\mathrm{med}\,(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$. If a, b, c are pairwise different, then $\mathrm{med}\,(a, b, c)$ is the middle one of these three elements (w.r.t. the ordering of L), while if there is a repetition among a, b, c, then $\mathrm{med}\,(a, b, c)$ equals this repeated value.

2.2 Sugeno Utility Functions

By a *Sugeno utility function* we mean a function $f\colon L_1 \times \cdots \times L_n \to L$ of the form

$$f\,(x_1, \ldots, x_n) = q\,(\varphi_1\,(x_1)\,, \ldots, \varphi_n\,(x_n))\,, \tag{2}$$

where $q\colon L^n \to L$ is a Sugeno integral, and each $\varphi_i\colon L_i \to L$ is an order-preserving function, so-called *local utility function*. Such functions can model various situations where one needs to aggregate several inputs into a single output in a meaningful way. The local utility functions φ_i map the various inputs x_i (which are measured on possibly different scales L_i) into a single scale L, and then the aggregation function q, in this case a Sugeno integral, combines them into a single value. For general background see [1,4,6,7].

A function $\varphi_i\colon L_i \to L$ satisfies the *boundary conditions* if $\varphi_i\,(x_i)$ lies between $\varphi_i\,(0_{L_i})$ and $\varphi_i\,(1_{L_i})$ for all $x_i \in L_i$. *Pseudo-Sugeno integrals* were defined in [3] as functions f of the form (2), where q is a Sugeno integral, and the inner functions φ_i satisfy the boundary conditions. Order-preserving functions clearly satisfy the boundary conditions, hence the class of pseudo-Sugeno integrals subsumes the class of Sugeno utility functions. We will see in Sect. 5 that Sugeno utility functions coincide with order-preserving pseudo-Sugeno integrals.

A fundamental tool in our study of Sugeno utility functions is the following *pseudo-median decomposition* formula. This result was stated and proved in [3] in a stronger form, where the pseudo-median decomposition formula was shown to characterize the wider class of pseudo-Sugeno integrals.

Theorem 3. *If* $f\colon L_1 \times \cdots \times L_n \to L$ *is a Sugeno utility function as in* (2), *then for all* $k \in [n]$ *and* $\mathbf{x} \in L_1 \times \cdots \times L_n$ *we have*

$$f\,(\mathbf{x}) = \mathrm{med}\left(f\left(\mathbf{x}_k^0\right), \varphi_k\,(x_k)\,, f\left(\mathbf{x}_k^1\right)\right), \tag{3}$$

where $\mathbf{x}_k^0$ *(resp.* $\mathbf{x}_k^1$*) is the vector obtained from* $\mathbf{x}$ *by replacing its* k*th component by* 0_{L_k} *(resp.* 1_{L_k}*).*

Let us give a rough idea of how we can use the pseudo-median decomposition to extract from the global function f information about the local utility functions

φ_k. The key observation is that if $f\left(\mathbf{x}_k^0\right) < f(\mathbf{x}) < f\left(\mathbf{x}_k^1\right)$, then (3) implies that $\varphi_k(x_k) = f(\mathbf{x})$. So let us imagine that we fix all but the kth component of $\mathbf{x}$, and we continuously increase x_k from 0 to 1 in L_k. Let a (resp. b) be the first (resp. last) value of x_k where $f(\mathbf{x}) > f\left(\mathbf{x}_k^0\right)$ (resp. $f(\mathbf{x}) < f\left(\mathbf{x}_k^1\right)$). Then $f(\mathbf{x})$, viewed as a unary function of x_k, consists of three pieces: it is constant $s = f\left(\mathbf{x}_k^0\right)$ from 0 to a, coincides with φ_k from a to b, and is constant $t = f\left(\mathbf{x}_k^1\right)$ from b to 1 (see Fig. 1, where L_k and L are chosen to be the unit interval $[0,1] \subseteq \mathbb{R}$). Thus we can see some part of φ_k through the "window" $[a,b]$. Fixing the components of $\mathbf{x}$ (other than x_k) to some other values, we may open other windows, which may expose other parts of φ_k. If we could find sufficiently many windows, then we could recover φ_k but, unfortunately, this is not always the case. (In fact, as we shall see in the example of Subsect. 3.3, the local utility functions are not always uniquely determined by f.) In Sect. 3 we will develop this idea to find a candidate for φ_k, and we will show in Sect. 4 that the candidate that we construct is indeed appropriate in the sense that it can be used in factorizing a given Sugeno utility function.

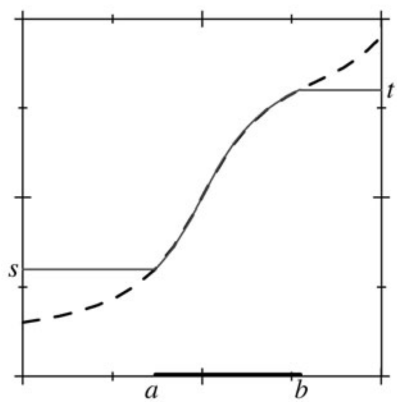

Fig. 1. The graph of φ_k as seen through a window

3 The Construction

Throughout this section let $f\colon L_1 \times \cdots \times L_n \to L$ be a Sugeno utiliy function. Knowing that f can be factorized as in (2), we will show how to construct in a canonical way a possibly different Sugeno integral q^f and local utility functions φ_i^f such that $f = q^f\left(\varphi_1^f, \ldots, \varphi_n^f\right)$. It is important that q^f and φ_i^f can be computed only from f, without having any information about q and φ_i (which are assumed to exist). In the first subsection we construct the Sugeno integral q^f, and then we describe the procedure to find suitable local utility functions φ_i^f. The latter is substantially more involved, therefore we conclude this section with a concrete example, and we defer the proof of correctness of the procedure to Sect. 4.

We will assume in the sequel that f depends on all of its variables. If this is not the case, e.g., f does not depend on its first variable, then there is a Sugeno utility function $g\colon L_2 \times \cdots \times L_n \to L$ such that $f(x_1, \ldots, x_n) = g(x_2, \ldots, x_n)$. In this case one could consider the function g instead of f, and find a factorization for this function. (If g still has inessential variables, then we can eliminate them in a similar way.)

3.1 Constructing the Sugeno Integral

The following result, which is essentially a generalization of Theorem 1, provides an appropriate Sugeno integral in order to factorize a Sugeno utility function.

Theorem 4. *If $f(x_1, \ldots, x_n) = q(\varphi_1(x_1), \ldots, \varphi_n(x_n))$ is a Sugeno utility function, then $f(x_1, \ldots, x_n) = q^f(\varphi_1(x_1), \ldots, \varphi_n(x_n))$, where $q^f : L^n \to L$ is the polynomial function given by*

$$q^f(y_1, \ldots, y_n) = \bigvee_{I \subseteq [n]} \Big(f(\mathbf{e}_I) \wedge \bigwedge_{i \in I} y_i \Big).$$

Proof. We need to prove that the following identity holds:

$$f(x_1, \ldots, x_n) = \bigvee_{I \subseteq [n]} \Big(f(\mathbf{e}_I) \wedge \bigwedge_{i \in I} \varphi_i(x_i) \Big). \tag{4}$$

We apply induction on n. If $n = 1$, then the right hand side of (4) takes the form $f(0) \vee (f(1) \wedge \varphi_1(x_1)) = \operatorname{med}(f(0), \varphi_1(x_1), f(1))$, which equals $f(x_1)$ by (3). Now suppose that the statement of the theorem is true for all Sugeno utility functions in $n-1$ variables. Applying the pseudo-median decomposition to f with $k = n$ we obtain

$$\begin{aligned} f(x_1, \ldots, x_n) &= \operatorname{med}(f_0(x_1, \ldots, x_{n-1}), \varphi_n(x_n), f_1(x_1, \ldots, x_{n-1})) \\ &= f_0(x_1, \ldots, x_{n-1}) \vee (f_1(x_1, \ldots, x_{n-1}) \wedge \varphi_n(x_n)), \end{aligned} \tag{5}$$

where f_0 and f_1 are the $(n-1)$-ary Sugeno utility functions defined by

$$\begin{aligned} f_0(x_1, \ldots, x_{n-1}) &= f(x_1, \ldots, x_{n-1}, 0), \\ f_1(x_1, \ldots, x_{n-1}) &= f(x_1, \ldots, x_{n-1}, 1). \end{aligned}$$

Let us apply the induction hypothesis for these functions:

$$f_0(x_1, \ldots, x_{n-1}) = \bigvee_{I \subseteq [n-1]} \Big(f_0(\mathbf{e}_I) \wedge \bigwedge_{i \in I} \varphi_i(x_i) \Big) = \bigvee_{I \subseteq [n-1]} \Big(f(\mathbf{e}_I) \wedge \bigwedge_{i \in I} \varphi_i(x_i) \Big),$$

$$f_1(x_1, \ldots, x_{n-1}) = \bigvee_{I \subseteq [n-1]} \Big(f_1(\mathbf{e}_I) \wedge \bigwedge_{i \in I} \varphi_i(x_i) \Big) = \bigvee_{I \subseteq [n-1]} \Big(f(\mathbf{e}_{I \cup \{n\}}) \wedge \bigwedge_{i \in I} \varphi_i(x_i) \Big).$$

Substituting back into (5) and using distributivity we obtain the desired equality (4). □

The polynomial q^f given in the above theorem is a Sugeno integral if and only if $f(0, \ldots, 0) = 0$ and $f(1, \ldots, 1) = 1$. It is natural to assume that the latter holds, since otherwise the parts of L that lie outside the interval $[f(0, \ldots, 0), f(1, \ldots, 1)]$ are "useless"; we may remove them without changing anything in the problem.

3.2 Constructing the Local Utility Functions

We only present the construction of φ_1^f; the other local utility functions can be constructed similarly. For any $x_1 \in L_1$ we partition $L_2 \times \cdots \times L_n$ into the following four disjoint sets:

$$\begin{aligned}
\mathcal{W}_{x_1} &= \{(x_2, \ldots, x_n) : f(0, x_2, \ldots, x_n) < f(x_1, x_2, \ldots, x_n) < f(1, x_2, \ldots, x_n)\}, \\
\mathcal{L}_{x_1} &= \{(x_2, \ldots, x_n) : f(0, x_2, \ldots, x_n) < f(x_1, x_2, \ldots, x_n) = f(1, x_2, \ldots, x_n)\}, \\
\mathcal{U}_{x_1} &= \{(x_2, \ldots, x_n) : f(0, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n) < f(1, x_2, \ldots, x_n)\}, \\
\mathcal{E}_{x_1} &= \{(x_2, \ldots, x_n) : f(0, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n) = f(1, x_2, \ldots, x_n)\}.
\end{aligned}$$

Observe that $\mathcal{E}_{x_1}$ bears no information on x_1; we only introduce it for notational convenience.

From the pseudo-median decomposition formula (3) we know that

$$f(x_1, x_2, \ldots, x_n) = \operatorname{med}(f(0, x_2, \ldots, x_n), \varphi_1(x_1), f(1, x_2, \ldots, x_n)).$$

Examining this formula, we immediately get the following implications for all $x_1 \in L_1$ and $(x_2, \ldots, x_n) \in L_2 \times \cdots \times L_n$:

$$\begin{aligned}
(x_2, \ldots, x_n) \in \mathcal{W}_{x_1} &\implies \varphi_1(x_1) = f(x_1, x_2, \ldots, x_n), \\
(x_2, \ldots, x_n) \in \mathcal{L}_{x_1} &\implies \varphi_1(x_1) \geq f(x_1, x_2, \ldots, x_n), \\
(x_2, \ldots, x_n) \in \mathcal{U}_{x_1} &\implies \varphi_1(x_1) \leq f(x_1, x_2, \ldots, x_n).
\end{aligned}$$

Thus, if $\mathcal{W}_{x_1}$ is not empty, then we can see $\varphi_1(x_1)$ through a window, and we can determine its exact value. Furthermore, $\mathcal{L}_{x_1}$ and $\mathcal{U}_{x_1}$ provide lower and upper bounds, respectively, whenever they are not empty. We introduce the following notation for these values:

$$\varphi_1(x_1) = w_{x_1} = f(x_1, x_2, \ldots, x_n) \text{ if } (x_2, \ldots, x_n) \in \mathcal{W}_{x_1}, \tag{6}$$

$$\varphi_1(x_1) \geq l_{x_1} = \bigvee_{(x_2, \ldots, x_n) \in \mathcal{L}_{x_1}} f(x_1, x_2, \ldots, x_n) \text{ if } \mathcal{L}_{x_1} \neq \emptyset, \tag{7}$$

$$\varphi_1(x_1) \leq u_{x_1} = \bigwedge_{(x_2, \ldots, x_n) \in \mathcal{U}_{x_1}} f(x_1, x_2, \ldots, x_n) \text{ if } \mathcal{U}_{x_1} \neq \emptyset. \tag{8}$$

If any of the sets $\mathcal{W}_{x_1}, \mathcal{L}_{x_1}, \mathcal{U}_{x_1}$ is empty, then the corresponding values $w_{x_1}, l_{x_1}, u_{x_1}$ are undefined.

Now we are able to define a function $\varphi_1^f : L_1 \to L$ that will serve as a replacement of φ_1:

(W) if $\mathcal{W}_{x_1} \neq \emptyset$ then let $\varphi_1^f(x_1) = w_{x_1}$;
(L) if $\mathcal{W}_{x_1} = \emptyset, \mathcal{L}_{x_1} \neq \emptyset, \mathcal{U}_{x_1} = \emptyset$ then let $\varphi_1^f(x_1) = l_{x_1}$;
(U) if $\mathcal{W}_{x_1} = \emptyset, \mathcal{L}_{x_1} = \emptyset, \mathcal{U}_{x_1} \neq \emptyset$ then let $\varphi_1^f(x_1) = u_{x_1}$;
(LU) if $\mathcal{W}_{x_1} = \emptyset, \mathcal{L}_{x_1} \neq \emptyset, \mathcal{U}_{x_1} \neq \emptyset$ then let $\varphi_1^f(x_1) = l_{x_1}$.

It is important to note that φ_1^f is computed only from f, without reference to φ_1. Let us also observe that the four cases above cover all possibilities since $\mathcal{W}_{x_1} = \mathcal{U}_{x_1} = \mathcal{L}_{x_1} = \emptyset$ is ruled out by the assumption that f depends on its first variable. In the case (LU) we could have chosen any element from the interval $[l_{x_1}, u_{x_1}]$ (see Remark 6); we chose l_{x_1} just to make the construction canonical. We will also prove in Lemma 8 that φ_1^f is indeed a good candidate in the sense that $f = q\big(\varphi_1^f, \varphi_2, \ldots, \varphi_n\big)$.

3.3 An Example

Let us illustrate our construction with a concrete (albeit fictitious) example. Customers evaluate hotels along three criteria, namely quality of services, price, and whether the hotel has a good location. Service is evaluated on a four-level scale L_1: `*<**<***<****`, price is evaluated on a three-level scale L_2: `-<0<+` (where "`-`"means expensive, thus less desirable, and "`+`"means cheap, thus more desirable), and the third scale is L_3: `n(o)<y(es)`. In addition, each hotel receives an overall rating on the scale $L : 1 < \cdots < 8$, which gives the overall utility function $f\colon L_1 \times L_2 \times L_3 \to L$ (see Table 1(a)). We will find a factorization of this function, and we will analyse its structure in order to draw conclusions about the nature of the "human aggregation" that the customers (unconsciously) perform when forming their opinions about hotels. First we apply Theorem 4 to find the underlying Sugeno integral:

$$\begin{aligned} q^f\,(y_1, y_2, y_3) = {} & 1 \vee (2 \wedge y_1) \vee (2 \wedge y_2) \vee (3 \wedge y_3) \\ & \vee (2 \wedge y_1 \wedge y_2) \vee (8 \wedge y_1 \wedge y_3) \vee (6 \wedge y_2 \wedge y_3) \vee (8 \wedge y_1 \wedge y_2 \wedge y_3)\,. \end{aligned}$$

Since 1 (resp. 8) is the least (resp. greatest) element of L, this polynomial function q^f is indeed a Sugeno integral. We can simplify q^f by cancelling those terms which are absorbed by some other terms in the disjunction:

$$q^f\,(y_1, y_2, y_3) = (2 \wedge y_1) \vee (2 \wedge y_2) \vee (3 \wedge y_3) \vee (y_1 \wedge y_3) \vee (6 \wedge y_2 \wedge y_3)\,.$$

We will be able to perform further simplifications after constructing the local utility functions. Table 1(b) shows the partitions of $L_2 \times L_3$ corresponding to the four possible elements $x_1 \in L_1$. The numbers in parentheses are the values of $f\,(x_1, x_2, x_3)$ (recall that we do not compute any values for the sets $\mathcal{E}_{x_1}$); these are used to compute the numbers $l_{x_1}, w_{x_1}, u_{x_1}$ shown in Table 1(c). This table contains these data for all $x_2 \in L_2$ and $x_3 \in L_3$ as well, together with the values of $\varphi_1^f\,(x_1)\,, \varphi_2^f\,(x_2)\,, \varphi_3^f\,(x_3)$.

Now that we know that the greatest value of φ_2^f is 6, we can simplify the Sugeno integral q^f by replacing $6 \wedge y_2 \wedge y_3$ with $y_2 \wedge y_3$, and "factoring out" $y_1 \vee y_2$:

$$(3 \wedge y_3) \vee ((y_1 \vee y_2) \wedge (2 \vee y_3)) = \operatorname{med}\,(3 \wedge y_3, y_1 \vee y_2, 2 \vee y_3)\,.$$

Note that this polynomial function is different from q^f, but it gives the same overall utility function f. This example shows that the Sugeno integral is not

Table 1. The hotel example

(a) The overall utility function

service	price	location	f
*	–	n	1
**	–	n	2
***	–	n	2
****	–	n	2
*	0	n	2
**	0	n	2
***	0	n	2
****	0	n	2
*	+	n	2
**	+	n	2
***	+	n	2
****	+	n	2
*	–	y	3
**	–	y	3
***	–	y	7
****	–	y	8
*	0	y	5
**	0	y	5
***	0	y	7
****	0	y	8
*	+	y	6
**	+	y	6
***	+	y	7
****	+	y	8

(b) The partitions of $L_2 \times L_3$

	*	**	***	****
(–,n)	$\mathcal{U}_{*}$ (1)	$\mathcal{L}_{**}$ (2)	$\mathcal{L}_{***}$ (2)	$\mathcal{L}_{****}$ (2)
(0,n)	$\mathcal{E}_{*}$	$\mathcal{E}_{**}$	$\mathcal{E}_{***}$	$\mathcal{E}_{****}$
(+,n)	$\mathcal{E}_{*}$	$\mathcal{E}_{**}$	$\mathcal{E}_{***}$	$\mathcal{E}_{****}$
(–,y)	$\mathcal{U}_{*}$ (3)	$\mathcal{U}_{**}$ (3)	$\mathcal{W}_{***}$ (7)	$\mathcal{L}_{****}$ (8)
(0,y)	$\mathcal{U}_{*}$ (5)	$\mathcal{U}_{**}$ (5)	$\mathcal{W}_{***}$ (7)	$\mathcal{L}_{****}$ (8)
(+,y)	$\mathcal{U}_{*}$ (6)	$\mathcal{U}_{**}$ (6)	$\mathcal{W}_{***}$ (7)	$\mathcal{L}_{****}$ (8)

(c) The local utility functions

	l	w	u	φ_1^f
*			1	1
**	2		3	2
***	2	7		7
****	8			8

	l	w	u	φ_2^f
–			1	1
0		5		5
+	6			6

	l	w	u	φ_3^f
n			1	1
y	8			8

uniquely determined by f, and neither are the local utility functions (e.g., we could have chosen $\varphi_1^f(\texttt{**}) = 3$ according to Remark 6).

To better understand the behaviour of f, let us separate two cases upon the location of the hotel:

$$f(x_1, x_2, x_3) = \operatorname{med}\big(3 \wedge \varphi_3^f(x_3), \varphi_1^f(x_1) \vee \varphi_2^f(x_2), 2 \vee \varphi_3^f(x_3)\big) \qquad (9)$$
$$= \begin{cases} \varphi_1^f(x_1) \vee \varphi_2^f(x_2) \vee 3, & \text{if } x_3 = \texttt{y}, \\ \big(\varphi_1^f(x_1) \vee \varphi_2^f(x_2)\big) \wedge 2, & \text{if } x_3 = \texttt{n}. \end{cases}$$

We can see from (9) that once x_3 is fixed, what matters is the higher one of $\varphi_1^f(x_1)$ and $\varphi_2^f(x_2)$. Thus, instead of aiming at an average level in both, a better strategy would be to maximize one of them. Moreover, φ_1^f either outputs very low or very high scores, whereas φ_2^f is almost maximized once the price is not very bad. Hence it seems more reasonable to focus on service rather than on price. The third variable can radically change the final outcome, but little can be done to improve the location of the hotel.

4 Proof of Correctness

In this section we show that the construction described in the previous section indeed provides a factorization of the Sugeno utility function f. First we prove that the functions φ_i^f are local utility functions, i.e., order-preserving functions. As before, we only consider the case $i = 1$; the other cases can be treated in an analogous way.

Theorem 5. *For any Sugeno utility function f, the function φ_1^f defined by the rules* (W),(L),(U),(LU) *in Sect. 3 is order-preserving.*

Proof. We fix $a \leq b \in L_1$ and show that $\varphi_1^f(a) \leq \varphi_1^f(b)$. First let us assume that $\mathcal{W}_a \neq \emptyset$, and let us fix an arbitrary $(x_2, \ldots, x_n) \in \mathcal{W}_a$. Then $\varphi_1^f(a) = w_a = f(a, x_2, \ldots, x_n)$, and since f is order-preserving, by the definition of $\mathcal{W}_a$ we get

$$f(0, x_2, \ldots, x_n) < f(a, x_2, \ldots, x_n) \leq f(b, x_2, \ldots, x_n) \leq f(1, x_2, \ldots, x_n).$$

If $f(b, x_2, \ldots, x_n) < f(1, x_2, \ldots, x_n)$ then $(x_2, \ldots, x_n) \in \mathcal{W}_b$, hence, by (6), $\varphi_1^f(b) = w_b = f(b, x_2, \ldots, x_n)$. If $f(b, x_2, \ldots, x_n) = f(1, x_2, \ldots, x_n)$, then $(x_2, \ldots, x_n) \in \mathcal{L}_b$, therefore $\varphi_1^f(b) \geq l_b \geq f(b, x_2, \ldots, x_n)$ by (7). In both cases we obtain that

$$\varphi_1^f(a) = w_a = f(a, x_2, \ldots, x_n) \leq f(b, x_2, \ldots, x_n) \leq \varphi_1^f(b),$$

since f is order-preserving.

The case $\mathcal{W}_b \neq \emptyset$ can be dealt with similarly. So let us consider the remaining case $\mathcal{W}_a = \mathcal{W}_b = \emptyset$. Then

$$\mathcal{L}_a \cup \mathcal{U}_a = L_2 \times \cdots \times L_n \setminus \mathcal{E}_a = L_2 \times \cdots \times L_n \setminus \mathcal{E}_b = \mathcal{L}_b \cup \mathcal{U}_b.$$

Futhermore, from $a \leq b$ we can conclude that $\mathcal{L}_a \subseteq \mathcal{L}_b$ and $\mathcal{U}_a \supseteq \mathcal{U}_b$ by making use of the fact that f is order-preserving. This implies that either $\mathcal{L}_a \subset \mathcal{L}_b$ and $\mathcal{U}_a \supset \mathcal{U}_b$, or $\mathcal{L}_a = \mathcal{L}_b$ and $\mathcal{U}_a = \mathcal{U}_b$. In the first case, choosing an arbitrary $(x_2, \ldots, x_n) \in \mathcal{L}_b \setminus \mathcal{L}_a = \mathcal{U}_a \setminus \mathcal{U}_b$ we obtain the desired inequality with the help of (7) and (8): $\varphi_1^f(a) \leq u_a \leq f(a, x_2, \ldots, x_n) \leq f(b, x_2, \ldots, x_n) \leq l_b \leq \varphi_1^f(b)$.

In the second case, we claim that $f(a, x_2, \ldots, x_n) = f(b, x_2, \ldots, x_n)$ for all $(x_2, \ldots, x_n) \in L_2 \times \cdots \times L_n$. This is clear if $(x_2, \ldots, x_n) \in \mathcal{E}_a = \mathcal{E}_b$. If $(x_2, \ldots, x_n) \in \mathcal{L}_a = \mathcal{L}_b$, then

$$f(a, x_2, \ldots, x_n) = f(1, x_2, \ldots, x_n) = f(b, x_2, \ldots, x_n).$$

If $(x_2, \ldots, x_n) \in \mathcal{U}_a = \mathcal{U}_b$, then

$$f(a, x_2, \ldots, x_n) = f(0, x_2, \ldots, x_n) = f(b, x_2, \ldots, x_n).$$

Thus, when determining l_a and l_b according to (7), we have to compute the join of exactly the same elements, hence $l_a = l_b$ (if they are defined). Similarly, we have $u_a = u_b$ whenever they are defined. Therefore $\varphi_1^f(a)$ and $\varphi_1^f(b)$ coincide, no matter which rule (L),(U) or (LU) was used to compute their values. □

Remark 6. We can see from the proof of the above theorem that (LU) could be relaxed: $\varphi_1^f(x_1)$ could be chosen to be any element of $[l_{x_1}, u_{x_1}]$ with the convention that whenever we encounter the same interval $[l_{x_1}, u_{x_1}]$ for different values of x_1, we always choose the same element of this interval. This guarantees that φ_1^f will be order-preserving. All of the proofs below work with this relaxed rule as well, since they rely only on the fact that $\varphi_1^f(x_1) \in [l_{x_1}, u_{x_1}]$ whenever $\varphi_1^f(x_1)$ is determined by rule (LU).

Next we prove that the function φ_1^f can be used in the factorization of the Sugeno utility function f. Let us recall that, since f is a Sugeno utility function, $f(x_1, x_2, \ldots, x_n) = q(\varphi_1(x_1), \varphi_2(x_2), \ldots, \varphi_n(x_n))$ for some Sugeno integral q and local utility functions φ_i. Let us denote $f'(x_1, x_2, \ldots, x_n) = q\big(\varphi_1^f(x_1), \varphi_2(x_2), \ldots, \varphi_n(x_n)\big)$. Observe that f' is also a Sugeno utility function.

Lemma 7. *For all $(x_2, \ldots, x_n) \in L_2 \times \cdots \times L_n$ we have*

$$f'(0, x_2, \ldots, x_n) = f(0, x_2, \ldots, x_n),$$
$$f'(1, x_2, \ldots, x_n) = f(1, x_2, \ldots, x_n).$$

Proof. We prove the first equality; the proof of the second equality is similar. Let us observe first that $\mathcal{W}_0 = \mathcal{L}_0 = \emptyset$, and $\mathcal{U}_0 \neq \emptyset$, since otherwise we had $L_2 \times \cdots \times L_n = \mathcal{E}_0$, contradicting our assumption that f depends on its first variable. Thus $\varphi_1^f(0)$ is determined by the rule (U), and $\varphi_1^f(0) = u_0 \geq \varphi_1(0)$ according to (8). Since q is order-preserving, this immediately implies that $f'(0, x_2, \ldots, x_n) \geq f(0, x_2, \ldots, x_n)$. For the other inequality we treat the two cases $(x_2, \ldots, x_n) \in \mathcal{U}_0$ and $(x_2, \ldots, x_n) \in \mathcal{E}_0$ separately.

If $(x_2, \ldots, x_n) \in \mathcal{U}_0$, then $f(0, x_2, \ldots, x_n)$ is one of the elements whose meet gives u_0 in (8), therefore $u_0 \leq f(0, x_2, \ldots, x_n)$. Thus we have

$$\begin{aligned} f'(0, x_2, \ldots, x_n) &= q\big(\varphi_1^f(0), \varphi_2(x_2), \ldots, \varphi_n(x_n)\big) \\ &= q(u_0, \varphi_2(x_2), \ldots, \varphi_n(x_n)) \\ &\leq q\big(f(0, x_2, \ldots, x_n), \varphi_2(x_2), \ldots, \varphi_n(x_n)\big) \\ &= q\big(q(\varphi_1(0), \varphi_2(x_2), \ldots, \varphi_n(x_n)), \varphi_2(x_2), \ldots, \varphi_n(x_n)\big). \end{aligned}$$

By Proposition 2, the right hand side equals

$$q(\varphi_1(0), \varphi_2(x_2), \ldots, \varphi_n(x_n)) = f(0, x_2, \ldots, x_n).$$

Hence we can conclude that $f'(0, x_2, \ldots, x_n) \leq f(0, x_2, \ldots, x_n)$.

Now let us assume that $(x_2, \ldots, x_n) \in \mathcal{E}_0$. We have observed at the beginning of the proof that $\varphi_1^f(0) = u_0 \geq \varphi_1(0)$. In a similar manner one can see that $\varphi_1^f(1) = l_1 \leq \varphi_1(1)$, and therefore $\varphi_1^f(0) \leq \varphi_1^f(1) \leq \varphi_1(1)$ since φ_1^f is order-preserving. This allows us to make the following estimate:

$$\begin{aligned} f'(0, x_2, \ldots, x_n) &= q\big(\varphi_1^f(0), \varphi_2(x_2), \ldots, \varphi_n(x_n)\big) \\ &\leq q(\varphi_1(1), \varphi_2(x_2), \ldots, \varphi_n(x_n)) \\ &= f(1, x_2, \ldots, x_n). \end{aligned}$$

However, $f(1, x_2, \ldots, x_n) = f(0, x_2, \ldots, x_n)$ as $(x_2, \ldots, x_n) \in \mathcal{E}_0$, so we can again conclude that $f'(0, x_2, \ldots, x_n) \leq f(0, x_2, \ldots, x_n)$. □

Lemma 8. *For all* $(x_1, \ldots, x_n) \in L_1 \times \cdots \times L_n$ *we have* $f'(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$.

Proof. Using the pseudo-median decomposition and Lemma 7 we obtain the following expression for f':

$$\begin{aligned} f'(x_1, x_2, \ldots, x_n) &= \operatorname{med}\big(f'(0, x_2, \ldots, x_n), \varphi_1^f(x_1), f'(1, x_2, \ldots, x_n)\big) \\ &= \operatorname{med}\big(f(0, x_2, \ldots, x_n), \varphi_1^f(x_1), f(1, x_2, \ldots, x_n)\big). \end{aligned}$$

Thus it suffices to show that

$$\operatorname{med}\big(f(0, x_2, \ldots, x_n), \varphi_1^f(x_1), f(1, x_2, \ldots, x_n)\big) = f(x_1, x_2, \ldots, x_n). \quad (10)$$

We separate four cases with respect to the partition of $L_2 \times \cdots \times L_n$.

If $(x_2, \ldots, x_n) \in \mathcal{W}_{x_1}$, then $\varphi_1^f(x_1) = w_{x_1} = \varphi_1(x_1)$ by (6), hence (10) is nothing else but the pseudo-median decomposition of f.

If $(x_2, \ldots, x_n) \in \mathcal{L}_{x_1}$, then $\varphi_1^f(x_1) \geq l_{x_1}$ no matter which one of the rules (W),(L),(U),(LU) was used to define $\varphi_1^f(x_1)$. Then by (7) and by the definition of $\mathcal{L}_{x_1}$ we get

$$\varphi_1^f(x_1) \geq l_{x_1} \geq f(x_1, x_2, \ldots, x_n) = f(1, x_2, \ldots, x_n).$$

Therefore, the left hand side of (10) equals $f(1, x_2, \ldots, x_n)$, and right hand side has the same value, since $(x_2, \ldots, x_n) \in \mathcal{L}_{x_1}$.

The case $(x_2, \ldots, x_n) \in \mathcal{U}_{x_1}$ follows similarly. Finally, if $(x_2, \ldots, x_n) \in \mathcal{E}_{x_1}$, then the left hand side of (10) is $f(0, x_2, \ldots, x_n) = f(1, x_2, \ldots, x_n)$ independently of the value of $\varphi_1^f(x_1)$, and right hand side has the same value, since $(x_2, \ldots, x_n) \in \mathcal{E}_{x_1}$. □

Now we are ready to prove the main result of this paper.

Theorem 9. *For any Sugeno utility function f, the Sugeno integral q^f and the local utility functions φ_i^f defined in Sect. 3 give a factorization of f:*

$$f(x_1, \ldots, x_n) = q^f\left(\varphi_1^f(x_1), \ldots, \varphi_n^f(x_n)\right).$$

Proof. Lemma 8 shows that $f(x_1, \ldots, x_n) = q\left(\varphi_1^f(x_1), \varphi_2(x_2), \ldots, \varphi_n(x_n)\right)$. In a similar way one can show that φ_2 can be replaced by φ_2^f: $f(x_1, \ldots, x_n) = q\left(\varphi_1^f(x_1), \varphi_2^f(x_2), \ldots, \varphi_n(x_n)\right)$. By recursive reasoning, we can replace the local utility functions one by one, and we get

$$f(x_1, x_2, \ldots, x_n) = q\left(\varphi_1^f(x_1), \varphi_2^f(x_2), \ldots, \varphi_n^f(x_n)\right).$$

Now applying Theorem 4 to this latter factorization of f we obtain $f(x_1, \ldots, x_n) = q^f\left(\varphi_1^f(x_1), \varphi_2^f(x_2), \ldots, \varphi_n^f(x_n)\right)$. □

5 Concluding Remarks

We have given a method to factorize any Sugeno utility function f into a composition $f = q^f\left(\varphi_1^f, \ldots, \varphi_n^f\right)$ of a Sugeno integral q^f with local utility functions φ_i^f. Such a factorization can be applied to analyse the behaviour of f, which can be useful in many problems in decision making. However, in many situations, we do not know whether our overall utility function f is a Sugeno utility function. This can be decided by making use of the various characterizations given in [3]. Alternatively, one can apply the construction of Sect. 3 directly to f. If at some point the construction fails (e.g, there are several values for w_{x_1} or $l_{x_1} > w_{x_1}$, etc.), then f does not have such a factorization. If the construction works, then we obtain a function $q^f\left(\varphi_1^f(x_1), \varphi_2^f(x_2), \ldots, \varphi_n^f(x_n)\right)$. If this function coincides with f, then we have obtained the desired factorization of f, otherwise f is not a Sugeno utility function.

As we have mentioned, the pseudo-median decomposition formula (3) is valid for the wider class of pseudo-Sugeno integrals [3]. Let us observe that in our proofs we never made use of the fact that the functions φ_i are order-preserving, only the order-preservation of f (and the pseudo-median decomposition) was used. Thus Theorems 5 and 9 hold in this more general setting, and this implies that a pseudo-Sugeno integral is order-preserving if and only if it is a Sugeno utility function.

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