Sugeno Utility Functions I: Axiomatizations

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Abstract. In this paper we consider a multicriteria aggregation model where local utility functions of different sorts are aggregated using Sugeno integrals, and which we refer to as Sugeno utility functions. We propose a general approach to study such functions via the notion of pseudo-Sugeno integral (or, equivalently, pseudo-polynomial function), which naturally generalizes that of Sugeno integral, and provide several axiomatizations for this class of functions.

Keywords: Pseudo-Sugeno integral, pseudo-polynomial function, local utility function, overall utility function, Sugeno utility function, axiomatization.

1 Introduction

The importance of aggregation functions is made apparent by their wide use, not only in pure mathematics (e.g., in the theory of functional equations, measure and integration theory), but also in several applied fields such as operations research, computer and information sciences, economics and social sciences, as well as in other experimental areas of physics and natural sciences. For general background, see [1,14] and for a recent reference, see [13].

In many applications, the values to be aggregated are first to be transformed by mappings $\varphi_i \colon X_i \to Y$, $i = 1, \ldots, n$, so that the transformed values (which are usually real numbers) can be aggregated in a meaningful way by a function $M \colon Y^n \to Y$. The resulting composed function $U \colon X_1 \times \cdots \times X_n \to Y$ is then defined by

$$U(x_1, \ldots, x_n) = M(\varphi_1(x_1), \ldots, \varphi_n(x_n)). \tag{1}$$

Such an aggregation model is used for instance in multicriteria decision making where the criteria are not commensurate. Here each φ_i is a local utility function, i.e., order-preserving mapping, and the resulting function U is referred to as an overall utility function (also called global preference function). For general background see [2].

In this paper, we consider this aggregation model in a purely ordinal decision setting, where Y and each X_i are bounded chains L and L_i, respectively, and

V. Torra, Y. Narukawa, and M. Daumas (Eds.): MDAI 2010, LNAI 6408, pp. 79–90, 2010.

where $M\colon L^n \to L$ is a Sugeno integral (see [10,19,20]) or, more generally, a lattice polynomial function. We refer to the resulting compositions as pseudo-Sugeno integrals and pseudo-polynomial functions, respectively. The particular case when each L_i is the same chain L', and each φ_i is the same mapping $\varphi\colon L' \to L$, was studied in [8] where the corresponding compositions $U = M \circ \varphi$ were called quasi-Sugeno integrals and quasi-polynomial functions. Such mappings were characterized as solutions of certain functional equations and in terms of necessary and sufficient conditions which have natural interpretations in decision making and aggregation theory.

Here, we take a similar approach and study pseudo-Sugeno integrals from an axiomatic point of view, and seek necessary and sufficient conditions for a given function to be factorizable as a composition of a Sugeno integral with unary maps. The importance of such an axiomatization is attested by the fact that this framework subsumes the Sugeno utility model. Since overall utility functions (1) where M is a Sugeno integral, coincide exactly with order-preserving pseudo-Sugeno integrals (see Sect. 5 in the companion paper [9]), we are particular interested in the case when the inner mappings φ_i are local utility functions.

The paper is organized as follows. In Sect. 2 we recall the basic definitions and terminology, as well as the necessary results concerning polynomial functions (and, in particular, Sugeno integrals) used in the sequel. In Sect. 3, we focus on pseudo-Sugeno integrals as a tool to study certain overall utility functions. We introduce the notion of pseudo-polynomial function in Subsect. 3.1 and show that, even though seemingly more general, it can be equivalently defined in terms of Sugeno integrals. An axiomatization of this class of generalized polynomial functions is given in Subsect. 3.2. Sugeno utility functions are introduced in Subsect. 3.3, as certain order-preserving pseudo-Sugeno integrals, and then characterized in Subsect. 3.4 by means of necessary and sufficient conditions which extend well-known properties in aggregation function theory. Within this general setting for studying Sugeno utility functions, it is natural to consider the inverse problem which asks for factorizations of a Sugeno utility function as a composition of a Sugeno integral with local utility functions. This question is addressed in Sect. 4, and left as an open problem to be considered in the companion paper [9] submitted to this same volume.

2 Lattice Polynomial Functions and Sugeno Integrals

2.1 Preliminaries

Throughout this paper, let L denote an arbitrary bounded chain endowed with lattice operations $\wedge$ and $\vee$, and with least and greatest elements 0_L and 1_L, respectively, where the subscripts may be omitted when the underlying lattice is clear from the context. A subset S of a chain L is said to be *convex* if for every $a, b \in S$ and every $c \in L$ such that $a \le c \le b$, we have $c \in S$. For any subset $S \subseteq L$, we denote by $\mathrm{cl}(S)$ the convex hull of S, that is, the smallest convex subset of L containing S. For instance, if $a, b \in L$ such that $a \le b$, then $\mathrm{cl}(\{a, b\}) = [a, b] = \{c \in L : a \le c \le b\}$.

For an integer $n \geq 1$, we set $[n] = \{1, \ldots, n\}$. Let σ be a permutation on $[n]$. The *standard simplex* of L^n associated with σ is the subset $L^n_\sigma \subseteq L^n$ defined by

$$L^n_\sigma = \{\boldsymbol{x} \in L^n \colon x_{\sigma(1)} \leq x_{\sigma(2)} \leq \cdots \leq x_{\sigma(n)}\}.$$

Given arbitrary bounded chains L_i, $i \in [n]$, their Cartesian product $\prod_{i\in[n]} L_i$ constitutes a bounded distributive lattice by defining

$$\boldsymbol{a} \wedge \boldsymbol{b} = (a_1 \wedge b_1, \ldots, a_n \wedge b_n), \quad \text{and} \quad \boldsymbol{a} \vee \boldsymbol{b} = (a_1 \vee b_1, \ldots, a_n \vee b_n).$$

For $k = 1, \ldots, n$ and $c \in L_k$, we use $\boldsymbol{x}^c_k$ to denote the vector whose ith component is c, if $i = k$, and x_i, otherwise.

In the case when $L_i = L$, for all $i \in [n]$, we also use the following notation. For $c \in L$ and $\boldsymbol{x} \in L^n$, let $\boldsymbol{x} \wedge c = (x_1 \wedge c, \ldots, x_n \wedge c)$ and $\boldsymbol{x} \vee c = (x_1 \vee c, \ldots, x_n \vee c)$, and denote by $[\boldsymbol{x}]_c$ the n-tuple whose ith component is 0, if $x_i \leq c$, and x_i, otherwise, and by $[\boldsymbol{x}]^c$ the n-tuple whose ith component is 1, if $x_i \geq c$, and x_i, otherwise.

Let $f\colon \prod_{i\in[n]} L_i \to L$ be a function. The *range* of f is given by $\text{ran}(f) = \{f(\boldsymbol{x}) : \boldsymbol{x} \in \prod_{i\in[n]} L_i\}$. Also, f is said to be *order-preserving* if, for every $\boldsymbol{a}, \boldsymbol{b} \in \prod_{i\in[n]} L_i$ such that $\boldsymbol{a} \leq \boldsymbol{b}$, we have $f(\boldsymbol{a}) \leq f(\boldsymbol{b})$. A well-known example of an order-preserving function is the *median* function $\text{med}\colon L^3 \to L$ given by $\text{med}(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (x_2 \wedge x_3)$. Given a vector $\boldsymbol{x} \in L^m$, $m \geq 1$, set $\langle \boldsymbol{x} \rangle_f = \text{med}(f(\mathbf{0}), \boldsymbol{x}, f(\mathbf{1}))$.

2.2 Basic Background on Polynomial Functions and Sugeno Integrals

In this subsection we recall some well-known results concerning polynomial functions that will be needed hereinafter. For further background, we refer the reader to [4,5,6,7,11,12,18].

Recall that a (*lattice*) *polynomial function* on L is any map $p\colon L^n \to L$ which can be obtained as a composition of the lattice operations $\wedge$ and $\vee$, the projections $\boldsymbol{x} \mapsto x_i$ and the constant functions $\boldsymbol{x} \mapsto c$, $c \in L$.

Fact 1. *Every polynomial function $p : L^n \to L$ is order-preserving and range-idempotent, that is, $p(c, \ldots, c) = c$, for every $c \in \text{ran}(p)$.*

Polynomial functions are known to generalize certain prominent fuzzy integrals, namely, the so-called (*discrete*) *Sugeno integrals*. Indeed, as observed in [17], Sugeno integrals coincide exactly with those polynomial functions $q : L^n \to L$ which are *idempotent*, that is, satisfy $q(c, \ldots, c) = c$, for every $c \in L$, and in particular satisfy $\text{ran}(q) = L$. We shall take this as our working definition of the Sugeno integral; for the original definition (as an integral with respect to a fuzzy measure) see, e.g., [13,19,20].

As shown by Goodstein [11], polynomial functions over bounded distributive lattices (in particular, over bounded chains) have very neat normal form representations. For $I \subseteq [n]$, let $\boldsymbol{e}_I$ be the *characteristic vector* of I, i.e., the n-tuple in L^n whose i-th component is 1 if $i \in I$, and 0 otherwise.

Proposition 2 (Goodstein [11]). *A function $p\colon L^n \to L$ is a polynomial function if and only if*

$$p(x_1, \ldots, x_n) = \bigvee_{I \subseteq [n]} \big(p(\boldsymbol{e}_I) \wedge \bigwedge_{i \in I} x_i\big). \tag{2}$$

Furthermore, the function given by (2) is a Sugeno integral if and only if $p(\boldsymbol{e}_\emptyset) = 0$ and $p(\boldsymbol{e}_{[n]}) = 1$.

Remark 3. Observe that, by Proposition 2, every polynomial function $p\colon L^n \to L$ is uniquely determined by its restriction to $\{0,1\}^n$. Also, since every lattice polynomial function is order-preserving, we have that the coefficients in (2) are monotone increasing, i.e., $p(\boldsymbol{e}_I) \leq p(\boldsymbol{e}_J)$ whenever $I \subseteq J$. Moreover, a function $f\colon \{0,1\}^n \to L$ can be extended to a polynomial function over L if and only if it is order-preserving.

Remark 4. It follows from Goodstein's theorem that every unary polynomial function is of the form

$$p(x) = s \vee (x \wedge t) = \operatorname{med}(s, x, t) = \begin{cases} s, & \text{if } x < s, \\ x, & \text{if } x \in [s,t], \\ t, & \text{if } t < x, \end{cases} \tag{3}$$

where $s = p(0), t = p(1)$. In other words, $p(x)$ is a truncated identity. Figure 1 shows the graph of this function in the case when L is the real unit interval $[0,1]$.

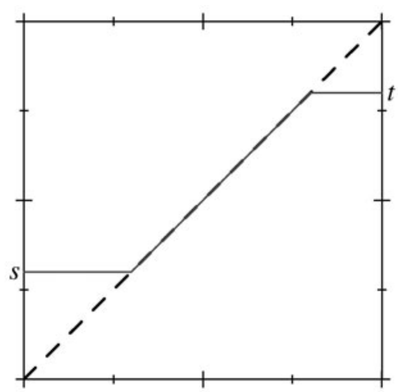

Fig. 1. A typical unary polynomial function

It is noteworthy that every polynomial function p as in (2) can be represented by $p = \langle q \rangle_p$ where q is the Sugeno integral given by

$$q(x_1, \ldots, x_n) = \bigvee_{\emptyset \subsetneq I \subsetneq [n]} \big(p(\boldsymbol{e}_I) \wedge \bigwedge_{i \in I} x_i\big) \vee \bigwedge_{i \in [n]} x_i.$$

2.3 Characterizations of Polynomial Functions

The following results reassemble the various characterizations of polynomial functions obtained in [5]. For further background see, e.g., [6,7,13].

Theorem 5. *Let $p\colon L^n \to L$ be a function on an arbitrary bounded chain L. The following conditions are equivalent:*

(i) p is a polynomial function.
(ii) p is median decomposable, that is, for every $\boldsymbol{x} \in L^n$,

$$p(\boldsymbol{x}) = \text{med}\big(p(\boldsymbol{x}_k^0), x_k, p(\boldsymbol{x}_k^1)\big) \qquad (k = 1, \ldots, n).$$

(iii) p is order-preserving, and $\text{cl}(\text{ran}(p))$-min and $\text{cl}(\text{ran}(p))$-max homogeneous, that is, for every $\boldsymbol{x} \in L^n$ and every $c \in \text{cl}(\text{ran}(p))$,

$$p(\boldsymbol{x} \wedge c) = p(\boldsymbol{x}) \wedge c \quad \textit{and} \quad p(\boldsymbol{x} \vee c) = p(\boldsymbol{x}) \vee c, \quad \textit{resp.}$$

(iv) p is order-preserving, range-idempotent, and horizontally minitive and maxitive, that is, for every $\boldsymbol{x} \in L^n$ and every $c \in L$,

$$p(\boldsymbol{x}) = p(\boldsymbol{x} \vee c) \wedge p([\boldsymbol{x}]^c) \quad \textit{and} \quad p(\boldsymbol{x}) = p(\boldsymbol{x} \wedge c) \vee p([\boldsymbol{x}]_c), \quad \textit{resp.}$$

Remark 6. Note that, by the equivalence $(i) \Leftrightarrow (iii)$, for every polynomial function $p\colon L^n \to L$, $p(\boldsymbol{x}) = \langle p(\boldsymbol{x})\rangle_p = p(\langle \boldsymbol{x}\rangle_p)$. Moreover, for every function $f\colon L^m \to L$ and every Sugeno integral $q\colon L^n \to L$, we have $\langle q(\boldsymbol{x})\rangle_f = q(\langle \boldsymbol{x}\rangle_f)$.

Theorem 5 is a refinement of the Main Theorem in [5] stated for functions over bounded distributive lattices. As shown in [7], in the case when L is a chain, Theorem 5 can be strengthened since the conditions need to be verified only on vectors of a certain prescribed type. Moreover, further characterizations are available and given in terms of conditions of somewhat different flavor, as the following theorem illustrates [7].

Theorem 7. *A function $p\colon L^n \to L$ is a polynomial function if and only if it is range-idempotent, and comonotonic minitive and maxitive, that is, for every permutation σ on $[n]$, and every $\boldsymbol{x}, \boldsymbol{x}' \in L^n_\sigma$,*

$$p(\boldsymbol{x} \wedge \boldsymbol{x}') = p(\boldsymbol{x}) \wedge p(\boldsymbol{x}') \quad \textit{and} \quad p(\boldsymbol{x} \vee \boldsymbol{x}') = p(\boldsymbol{x}) \vee p(\boldsymbol{x}'), \quad \textit{resp.}$$

3 Pseudo-Sugeno Integrals and Sugeno Utility Functions

In this section we study certain prominent function classes in the realm of multicriteria decision making. More precisely, we investigate overall utility functions $U\colon \prod_{i\in[n]} L_i \to L$ which can obtained by aggregating various local utility functions (i.e., order-preserving mappings) $\varphi_i\colon L_i \to L$, $i \in [n]$, using Sugeno integrals.

To this extent, in Subsect. 3.1 we introduce the wider class of pseudo-polynomial functions, and we present their axiomatization in Subsect. 3.2.

As we will see, pseudo-polynomial functions can be equivalently defined in terms of Sugeno integrals, and thus they model certain processes within multicriteria decision making. This is observed in Subsect. 3.3 where the notion of a Sugeno utility function $U\colon \prod_{i\in[n]} L_i \to L$ associated with given local utility functions $\varphi_i\colon L_i \to L$, $i \in [n]$, is discussed. Using the axiomatization of pseudo-polynomial functions, in Subsect. 3.4 we establish several characterizations of Sugeno utility functions based on Sugeno integrals given in terms of necessary and sufficient conditions which naturally extend those presented in Subsect. 2.3.

3.1 Pseudo-Sugeno Integrals and Pseudo-Polynomial Functions

Let L and $L_1,\dots,L_n$ be bounded chains. In the sequel, we shall denote the top and bottom elements of $L_1,\dots,L_n$ and L by 1 and 0, respectively. This convention will not give rise to ambiguities. We shall say that a mapping $\varphi_i\colon L_i \to L$, $i \in [n]$, satisfies the *boundary conditions* if for every $x \in L_i$,

$$\varphi(0) \le \varphi(x) \le \varphi(1) \quad \text{or} \quad \varphi(1) \le \varphi(x) \le \varphi(0).$$

Observe that if φ is order-preserving, then it satisfies the boundary conditions.

A function $f\colon \prod_{i\in[n]} L_i \to L$ is a *pseudo-polynomial function* if there is a polynomial function $p\colon L^n \to L$ and there are unary functions $\varphi_i\colon L_i \to L$, $i \in [n]$, satisfying the boundary conditions, such that

$$f(\boldsymbol{x}) = p(\varphi_1(x_1),\dots,\varphi_n(x_n)). \tag{4}$$

If p is a Sugeno integral, then we say that f is a *pseudo-Sugeno integral.* As the following result asserts, the notions of pseudo-polynomial function and pseudo-Sugeno integral turn out to be equivalent.

Proposition 8. *A function $f\colon \prod_{i\in[n]} L_i \to L$ is a pseudo-polynomial function if and only if it is a pseudo-Sugeno integral.*

Proof. Clearly, every pseudo-Sugeno integral is a pseudo-polynomial function. Conversely, if $f\colon \prod_{i\in[n]} L_i \to L$ of the form $f = p(\varphi_1(x_1),\dots,\varphi_n(x_n))$ for a lattice polynomial p, then by setting $\phi_i = \langle\varphi_i\rangle_p$ and taking q as a Sugeno integral such that $p = \langle q\rangle_p$, we have

$$\begin{aligned} f(\boldsymbol{x}) = \langle q(\varphi_1(x_1),\dots,\varphi_n(x_n))\rangle_p \;&=\; q(\langle\varphi_1(x_1)\rangle_p,\dots,\langle\varphi_n(x_n)\rangle_p) \\ = q(\phi_1(x_1),\dots,\phi_n(x_n)),& \end{aligned}$$

and thus f is a pseudo-Sugeno integral. □

Remark 9. Clearly, $f(\boldsymbol{x}_k^0) \le f(\boldsymbol{x}) \le f(\boldsymbol{x}_k^1)$ or $f(\boldsymbol{x}_k^1) \le f(\boldsymbol{x}) \le f(\boldsymbol{x}_k^0)$ depending on whether $\varphi_k(0) \le \varphi_k(x) \le \varphi_k(1)$ or $\varphi_k(1) \le \varphi_k(x) \le \varphi_k(0)$, respectively.

3.2 A Characterization of Pseudo-Sugeno Integrals

Throughout this subsection, we assume that the unary maps $\varphi_i \colon L_i \to L$ considered, satisfy the boundary conditions.

We say that $f \colon \prod_{i\in[n]} L_i \to L$ is *pseudo-median decomposable* if for each $k \in [n]$ there is a unary function $\varphi_k \colon L_k \to L$ such that

$$f(\boldsymbol{x}) = \text{med}\left(f(\boldsymbol{x}_k^0), \varphi_k(x_k), f(\boldsymbol{x}_k^1)\right) \tag{5}$$

for every $\boldsymbol{x} \in \prod_{i\in[n]} L_i$. Note that if f is pseudo-median decomposable w.r.t. unary functions $\varphi_i \colon L_i \to L$, $i \in [n]$, then for every $\boldsymbol{x} \in \prod_{i\in[n]} L_i$ and $k \in [n]$, we have $f(\boldsymbol{x}_k^0) \le f(\boldsymbol{x}) \le f(\boldsymbol{x}_k^1)$ or $f(\boldsymbol{x}_k^1) \le f(\boldsymbol{x}) \le f(\boldsymbol{x}_k^0)$.

Theorem 10. *Let $f \colon \prod_{i\in[n]} L_i \to L$ be a function. Then f is a pseudo-Sugeno integral if and only if f is pseudo-median decomposable.*

Proof. First we show that the condition is necessary. Suppose that $f \colon \prod_{i\in[n]} L_i \to L$ is of the form $f(\boldsymbol{x}) = q(\varphi_1(x_1), \ldots, \varphi_n(x_n))$ for some Sugeno integral q and unary functions φ_k satisfying the boundary conditions. Without loss of generality, assume that $k = 1$. So let us fix the values of $x_2, \ldots, x_n$, and let us consider the unary polynomial function $u\left(y\right) = q\left(y, \varphi_2\left(x_2\right), \ldots, \varphi_n\left(x_n\right)\right)$.

Setting $a = \varphi_1\left(0\right), b = \varphi_1\left(1\right), y_1 = \varphi_1\left(x_1\right)$, the equality to prove takes the form $u\left(y_1\right) = \text{med}\left(u\left(a\right), y_1, u\left(b\right)\right)$. This becomes clear if we take into account that u is of the form (3), and by the boundary conditions either $a \le y_1 \le b$ or $b \le y_1 \le a$ (see also Fig. 1).

To verify that the condition is sufficient, just observe that applying (5) repeatedly to each variable of f we can straightforwardly obtain a representation of f as $f(\boldsymbol{x}) = p(\varphi_1(x_1), \ldots, \varphi_n(x_n))$ for some polynomial function p. Thus, f is a pseudo-polynomial function and, by Proposition 8, it is a pseudo-Sugeno integral. □

3.3 Motivation: Overall Utility Functions

Despite the theoretical interest, the motivation for the study of pseudo-Sugeno integrals (or, equivalently, pseudo-polynomial functions) is deeply rooted in multicriteria decision making. Let $\varphi_i \colon L_i \to L$, $i \in [n]$, be local utility functions (i.e., order-preserving mappings) having a common range $\mathcal{R} \subseteq L$, and let $M \colon L^n \to L$ be an aggregation function. The *overall utility function* associated with φ_i, $i \in [n]$, and M is the mapping $U \colon \prod_{i\in[n]} L_i \to L$ defined by

$$U(\boldsymbol{x}) = M(\varphi_1(x_1), \ldots, \varphi_n(x_n)). \tag{6}$$

For background on overall utility functions, see e.g. [2].

Thus, pseudo-Sugeno integrals subsume those overall utility functions (6) where the aggregation function M is a Sugeno integral. In the sequel we shall refer to a mapping $f \colon \prod_{i\in[n]} L_i \to L$ for which there are local utility functions

φ_i, $i \in [n]$, and a Sugeno integral (or, equivalently, a polynomial function) q, such that

$$f(\boldsymbol{x}) = q(\varphi_1(x_1), \ldots, \varphi_n(x_n)), \tag{7}$$

as a *Sugeno utility function.* As it will become clear in [9], these Sugeno utility functions coincide exactly with those pseudo-Sugeno integrals (or equivalently, pseudo-polynomial functions) which are order-preserving. Also, by taking $L_1 = \cdots = L_n = L$ and $\varphi_1 = \cdots = \varphi_n = \varphi$, it follows that Sugeno utility functions subsume the notions of quasi-Sugeno integral and quasi-polynomial function in the terminology of [8].

Remark 11. Note that the condition that $\varphi_i\colon L_i \to L$, $i \in [n]$ have a common range $\mathcal{R}$ is not really restrictive, since each φ_i can be extended to a local utility function $\varphi_i'\colon L_i' \to L$, where $L_i \subseteq L_i'$, in such a way that each φ_i', $i \in [n]$, has the same range $\mathcal{R} \subseteq L$. In fact, if $\mathcal{R}_i$ is the range of φ_i, for each $i \in [n]$, then $\mathcal{R}$ can be chosen as the interval

$$\mathrm{cl}(\bigcup_{i\in[n]} \mathcal{R}_i) = [\bigwedge_{i\in[n]} \varphi_i(0), \bigvee_{i\in[n]} \varphi_i(1)].$$

In this way, if $f'\colon \prod_{i\in[n]} L_i' \to L$ is such that $f'(\boldsymbol{x}) = q(\varphi_1'(x_1), \ldots, \varphi_n'(x_n))$, then the restriction of f' to $\prod_{i\in[n]} L_i$ is of the form $f(\boldsymbol{x}) = q(\varphi_1(x_1), \ldots, \varphi_n(x_n))$.

3.4 Characterizations of Sugeno Utility Functions

In view of the remark above, in this subsection we will assume that the local utility functions $\varphi_i\colon L_i \to L$, $i \in [n]$, considered have the same range $\mathcal{R} \subseteq L$. Since local utility functions satisfy the boundary conditions, from Theorem 10 we get the following characterization of Sugeno utility functions.

Corollary 12. *A function $f\colon \prod_{i\in[n]} L_i \to L$ is a Sugeno utility function if and only if it is pseudo-median decomposable w.r.t. local utility functions.*

We will provide further axiomatizations of Sugeno utility functions extending those of polynomial functions given in Subsect. 2.3 as well as those of quasi-polynomial functions given in [8]. For the sake of simplicity, given $\varphi_i\colon L_i \to L$, $i \in [n]$, we make use of the shorthand notation $\overline{\varphi}(\boldsymbol{x}) = (\varphi_1(x_1), \ldots, \varphi_n(x_n))$ and $\overline{\varphi}^{-1}(c) = \{\boldsymbol{d} : \overline{\varphi}(\boldsymbol{d}) = c\}$, for every $c \in \mathcal{R}$.

We say that a function $f\colon \prod_{i\in[n]} L_i \to L$ is *pseudo-max homogeneous* (resp. *pseudo-min homogeneous*) if there are local utility functions $\varphi_i\colon L_i \to L$, $i \in [n]$, such that for every $\boldsymbol{x} \in \prod_{i\in[n]} L_i$ and $c \in \mathcal{R}$,

$$f(\boldsymbol{x} \vee \boldsymbol{d}) = f(\boldsymbol{x}) \vee c \quad \text{(resp. } f(\boldsymbol{x} \wedge \boldsymbol{d}) = f(\boldsymbol{x}) \wedge c), \quad \text{whenever } \boldsymbol{d} \in \overline{\varphi}^{-1}(c). \tag{8}$$

Fact 13. *Let $f\colon \prod_{i\in[n]} L_i \to L$ be a function, and let $\varphi_i\colon L_i \to L$, $i \in [n]$, be local utility functions. If f is pseudo-min homogeneous and pseudo-max homogeneous w.r.t. $\varphi_1, \ldots, \varphi_n$, then it satisfies the condition*

$$\text{for every } c \in \mathcal{R} \text{ and } \boldsymbol{d} \in \overline{\varphi}^{-1}(c),\ f(\boldsymbol{d}) = c. \tag{9}$$

Lemma 14. *If* $f(x_1,\ldots,x_n) = q(\varphi(x_1),\ldots,\varphi_n(x_n))$ *for some Sugeno integral* $q\colon L^n \to L$ *and local utility functions* $\varphi_1,\ldots,\varphi_n$*, then* f *is pseudo-min homogeneous and pseudo-max homogeneous w.r.t.* $\varphi_1,\ldots,\varphi_n$*.*

Proof. Let $\mathcal{R}$ be the common range of $\varphi_1,\ldots,\varphi_n$, let $c \in \mathcal{R}$ and $\boldsymbol{d} \in \overline{\varphi}^{-1}(c)$. By Theorem 5 and the fact that each φ_k is order-preserving, we have

$$\begin{aligned} f(\boldsymbol{x} \vee \boldsymbol{d}) &= q(\overline{\varphi}(\boldsymbol{x} \vee \boldsymbol{d})) = q(\overline{\varphi}(\boldsymbol{x}) \vee \overline{\varphi}(\boldsymbol{d})) \\ &= q(\overline{\varphi}(\boldsymbol{x}) \vee c) = q(\overline{\varphi}(\boldsymbol{x})) \vee c = f(\boldsymbol{x}) \vee c, \end{aligned}$$

and hence, f is pseudo-max homogeneous. The dual statement follows similarly. □

For $\boldsymbol{x}, \boldsymbol{d} \in \prod_{i\in[n]} L_i$, let $[\boldsymbol{x}]_{\boldsymbol{d}}$ be the n-tuple whose ith component is 0_{L_i}, if $x_i \leq d_i$, and x_i, otherwise, and dually let $[\boldsymbol{x}]^{\boldsymbol{d}}$ be the n-tuple whose ith component is 1_{L_i}, if $x_i \geq d_i$, and x_i, otherwise. We say that $f\colon \prod_{i\in[n]} L_i \to L$ is *pseudo-horizontally maxitive* (resp. *pseudo-horizontally minitive*) if there are local utility functions $\varphi_i\colon L_i \to L$, $i \in [n]$, such that for every $\boldsymbol{x} \in \prod_{i\in[n]} L_i$ and $c \in \mathcal{R}$, if $\boldsymbol{d} \in \overline{\varphi}^{-1}(c)$, then

$$f(\boldsymbol{x}) = f(\boldsymbol{x} \wedge \boldsymbol{d}) \vee f([\boldsymbol{x}]_{\boldsymbol{d}}) \quad \text{(resp. } f(\boldsymbol{x}) = f(\boldsymbol{x} \vee \boldsymbol{d}) \wedge f([\boldsymbol{x}]^{\boldsymbol{d}})). \tag{10}$$

Lemma 15. *If* $f\colon \prod_{i\in[n]} L_i \to L$ *is order-preserving, pseudo-horizontally minitive (resp. pseudo-horizontally maxitive) and satisfies* (9)*, then it is pseudo-min homogeneous (resp. pseudo-max homogeneous).*

Proof. If $f\colon \prod_{i\in[n]} L_i \to L$ is order-preserving, pseudo-horizontally minitive and satisfies (9) w.r.t. $\varphi_1,\ldots,\varphi_n$, then for every $\boldsymbol{x} \in \prod_{i\in[n]} L_i$, $c \in \mathcal{R}$, $\boldsymbol{d} \in \overline{\varphi}^{-1}(c)$

$$\begin{aligned} f(\boldsymbol{x}) \wedge c = f(\boldsymbol{x}) \wedge f(\boldsymbol{d}) \ &\geq \ f(\boldsymbol{x} \wedge \boldsymbol{d}) \ = \ f((\boldsymbol{x} \wedge \boldsymbol{d}) \vee \boldsymbol{d}) \wedge f([\boldsymbol{x} \wedge \boldsymbol{d}]^{\boldsymbol{d}}) \\ = f(\boldsymbol{d}) \wedge f([\boldsymbol{x}]^{\boldsymbol{d}}) \ &\geq \ f(\boldsymbol{d}) \wedge f(\boldsymbol{x}) \ = \ f(\boldsymbol{x}) \wedge c. \end{aligned}$$

Hence f is pseudo-min homogeneous w.r.t. $\varphi_1,\ldots,\varphi_n$. The dual statement can be proved similarly. □

Lemma 16. *Suppose that* $f\colon \prod_{i\in[n]} L_i \to L$ *is order-preserving and pseudo-min homogeneous (resp. pseudo-max homogeneous), and satisfies* (9)*. Then* f *is pseudo-max homogeneous (resp. pseudo-min homogeneous) if and only if it is pseudo-horizontally maxitive (resp. pseudo-horizontally minitive).*

Proof. Suppose that $f\colon \prod_{i\in[n]} L_i \to L$ is order-preserving and pseudo-min homogeneous and satisfies (9) w.r.t. $\varphi_1,\ldots,\varphi_n$. Assume first that f is pseudo-max homogeneous w.r.t. $\varphi_1,\ldots,\varphi_n$. For every $\boldsymbol{x} \in \prod_{i\in[n]} L_i$ and $\boldsymbol{d} \in \overline{\varphi}^{-1}(c)$, where $c \in \mathcal{R}$, we have

$$\begin{aligned} f(\boldsymbol{x} \wedge \boldsymbol{d}) \vee f([\boldsymbol{x}]_{\boldsymbol{d}}) &= \big(f(\boldsymbol{x}) \wedge c\big) \vee f([\boldsymbol{x}]_{\boldsymbol{d}}) = \big(f(\boldsymbol{x}) \vee f([\boldsymbol{x}]_{\boldsymbol{d}})\big) \wedge \big(c \vee f([\boldsymbol{x}]_{\boldsymbol{d}})\big) \\ &= f(\boldsymbol{x}) \wedge f(\boldsymbol{d} \vee [\boldsymbol{x}]_{\boldsymbol{d}}) = f(\boldsymbol{x}), \end{aligned}$$

and hence f is pseudo-horizontally maxitive w.r.t. $\varphi_1,\ldots,\varphi_n$.

Conversely, if f is pseudo-horizontally maxitive w.r.t. $\varphi_1, \ldots, \varphi_n$, then by Lemma 15 f is pseudo-max homogeneous w.r.t. $\varphi_1, \ldots, \varphi_n$. The dual statement can be proved similarly. □

Lemma 17. *If $f\colon \prod_{i\in[n]} L_i \to L$ is order-preserving, pseudo-min homogeneous and pseudo-horizontally maxitive, then it is pseudo-median decomposable w.r.t. local utility functions.*

Proof. Let $\boldsymbol{x} \in \prod_{i\in[n]} L_i$ and let $k \in [n]$. If f is pseudo-horizontally maxitive, say w.r.t. $\varphi_1, \ldots, \varphi_n$, then $f(\boldsymbol{x}) = f(\boldsymbol{x} \wedge \boldsymbol{d}) \vee f([\boldsymbol{x}]_{\boldsymbol{d}})$, where the kth component of $\boldsymbol{d} \in \overline{\varphi}^{-1}(\varphi_k(x_k))$ is x_k. Now if f is pseudo-min homogeneous, then $f(\boldsymbol{x} \wedge \boldsymbol{d}) = f(\boldsymbol{x}_k^1 \wedge \boldsymbol{d}) = f(\boldsymbol{x}_k^1) \wedge \varphi_k(x_k)$, and by the definition of $[\boldsymbol{x}]_{\boldsymbol{d}}$, we have $f([\boldsymbol{x}]_{\boldsymbol{d}}) \leq f(\boldsymbol{x}_k^0)$. Thus,

$$\begin{aligned} f(\boldsymbol{x}) &= \operatorname{med}\big(f(\boldsymbol{x}_k^0), f(\boldsymbol{x}), f(\boldsymbol{x}_k^1)\big) = \big(f(\boldsymbol{x}_k^0) \vee f(\boldsymbol{x})\big) \wedge f(\boldsymbol{x}_k^1) \\ &= \big(f(\boldsymbol{x}_k^0) \vee (f(\boldsymbol{x}_k^1) \wedge \varphi_k(x_k))\big) \wedge f(\boldsymbol{x}_k^1) = f(\boldsymbol{x}_k^0) \vee \big(f(\boldsymbol{x}_k^1) \wedge \varphi_k(x_k)\big) \\ &= \operatorname{med}\big(f(\boldsymbol{x}_k^0), \varphi_k(x_k), f(\boldsymbol{x}_k^1)\big). \end{aligned}$$

Since this holds for every $\boldsymbol{x} \in \prod_{i\in[n]} L_i$ and $k \in [n]$, f is pseudo-median decomposable. □

We can also extend the comonotonic properties as follows. We say that a function $f\colon \prod_{i\in[n]} L_i \to L$ is *pseudo-comonotonic minitive* (resp. *pseudo-comonotonic maxitive*) if there are local utility functions $\varphi_i\colon L_i \to L$, $i \in [n]$, such that for every permutation σ on $[n]$, and every $\boldsymbol{x}, \boldsymbol{x}'$ such that $\overline{\varphi}(\boldsymbol{x}), \overline{\varphi}(\boldsymbol{x}') \in L_\sigma^n$,

$$f(\boldsymbol{x} \wedge \boldsymbol{x}') = f(\boldsymbol{x}) \wedge f(\boldsymbol{x}') \quad (\text{resp. } f(\boldsymbol{x} \vee \boldsymbol{x}') = f(\boldsymbol{x}) \vee f(\boldsymbol{x}')).$$

The following fact is straightforward.

Fact 18. *Every Sugeno utility function of the form* (7) *is pseudo-comonotonic minitive and maxitive. Moreover, if a function is pseudo-comonotonic minitive (resp. pseudo-comonotonic maxitive) and satisfies* (9)*, then it is pseudo-min homogeneous (resp. pseudo-max homogeneous).*

Let $\mathbf{P}$ be the set comprising the properties of pseudo-min homogeneity, pseudo-horizontal minitivity and pseudo-comonotic minitivity, and let $\mathbf{P}^d$ be the set comprising the corresponding dual properties. The following result generalizes the various characterizations of polynomial functions given in Subsect. 2.3.

Theorem 19. *Let $f\colon \prod_{i\in[n]} L_i \to L$ be an order-preserving function. The following assertions are equivalent:*

(i) f is a Sugeno utility function.
(ii) f is pseudo-median decomposable w.r.t. local utility functions.
(iii) f is $P_1 \in \mathbf{P}$ and $P_2 \in \mathbf{P}^d$, and satisfies (9).

Proof. By Corollary 12, we have $(i) \Leftrightarrow (ii)$. By Lemma 14, we also have that if (i) holds, then f is pseudo-min homogeneous and pseudo-max homogeneous. Furthermore, by Fact 18 and Lemmas 15, 16 and 17, we have that any two formulations of (iii) are equivalent. By Lemma 17, $(iii) \Rightarrow (ii)$. □

Remark 20. By Fact 13, if P_1 and P_2 are the pseudo-homogeneity properties, then (9) becomes redundant in (iii). Similarly, by Lemma 17, Corollary 12, and $(i) \Rightarrow (iii)$ of Theorem 19, if P_1 is pseudo-min homogeneity (pseudo-horizontal minitivity) property, and P_2 is pseudo-horizontal maxitivity (pseudo-max homogeneity) property, then (9) is redundant in (iii).

4 Concluding Remarks

Theorem 19 provides necessary and sufficient conditions for an order-preserving function $f\colon \prod_{i\in[n]} L_i \to L$ to be a Sugeno utility function, that is, to be factorized into a composition

$$f(x_1,\ldots,x_n) = q(\varphi_1(x_1),\ldots,\varphi_n(x_n)), \tag{11}$$

where $\varphi_i\colon L_i \to L$, $i \in [n]$, are local utility functions and q is a Sugeno integral. However, knowing that f is a Sugeno utility function, no clues are given on how to derive such a factorization (11). Thus, we are left with the following problem:

Problem. Given a Sugeno utility function $f\colon \prod_{i\in[n]} L_i \to L$, construct local utility functions $\varphi_k\colon L_k \to L$, $k \in [n]$, and a Sugeno integral q such that f fulfills (11).

This problem is considered and solved in the companion paper [9] also submitted to MDAI2010.

Acknowledgments. We would like to thank Jean-Luc Marichal for useful discussions and for bringing this topic to our attention. The second named author acknowledges that the present project is supported by the National Research Fund, Luxembourg, and cofunded under the Marie Curie Actions of the European Commission (FP7-COFUND), and supported by the Hungarian National Foundation for Scientific Research under grant no. K77409.

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