Chapter 7
Ultimate Boundedness and Invariant Measure

We introduce in this chapter the concept of ultimate boundedness in the mean square sense (m.s.s.) and relate it to the problem of the existence and uniqueness of invariant measure. We consider semilinear stochastic differential equations in a Hilbert space and their mild solutions under the usual linear growth and Lipschitz conditions on the coefficients. We also study stochastic differential equations in the variational case, assuming that the coefficients satisfy the coercivity condition, and study their strong solutions which are exponentially ultimately bounded in the m.s.s.

7.1 Exponential Ultimate Boundedness in the m.s.s.

Definition 7.1 We say that the mild solution of (6.10) is exponentially ultimately bounded in the mean square sense (m.s.s.) if there exist positive constants c, β, M such that

$$E\left\|X^x(t)\right\|_H^2 \le c e^{-\beta t}\|x\|_H^2 + M \quad \text{for all } x \in H. \tag{7.1}$$

Here is an analogue of Theorem 6.4.

Theorem 7.1 *The mild solution* $\{X^x(t),\ t \ge 0\}$ *of* (6.10) *is exponentially ultimately bounded in the m.s.s. if there exists a function* $\Psi \in C_{2p}^2(H)$ *satisfying the following conditions*:

(1) $c_1\|x\|_H^2 - k_1 \le \Psi(x) \le c_2\|x\|_H^2 - k_2$,
(2) $\mathscr{L}\Psi(x) \le -c_3\Psi(x) + k_3$,

for $x \in H$, *where* c_1, c_2, c_3 *are positive constants*, *and* $k_1, k_2, k_3 \in \mathbb{R}$.

Proof Similarly as in the proof of Theorem 6.4, using Itô's formula for the solutions of the approximating equations (6.17) and utilizing condition (2), we arrive at

$$E\Psi\big(X^x(t)\big) - E\Psi\big(X^x(0)\big) \le \int_0^t \big(-c_3\, E\Psi\big(X^x(s)\big) + k_3\big)\, ds.$$

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Hence, $\Phi(t) = E\Psi(X^x(t))$ satisfies

$$\Phi'(t) \le -c_3\Phi(t) + k_3.$$

By Gronwall lemma,

$$\Phi(t) \le \frac{k_3}{c_3} + \left(\Phi(0) - \frac{k_3}{c_3}\right)\mathrm{e}^{-c_3 t}.$$

Using condition (1), we have, for all $x \in H$,

$$c_1 E\left\|X^x(t)\right\|_H^2 - k_1 \le E\Psi\left(X^x(t)\right) \le \frac{k_3}{c_3} + \left(c_2\|x\|_H^2 - k_2 - \frac{k_3}{c_3}\right)\mathrm{e}^{-c_3 t},$$

and (7.1) follows. □

Theorem 7.2 *Assume that A generates a pseudo-contraction semigroup of operators $\{S(t),\ t \ge 0\}$ on H. If the mild solution $\{X_0^x(t),\ t \ge 0\}$ of the linear equation* (6.22) *is exponentially ultimately bounded in the m.s.s., then there exists a function $\Psi_0 \in C_{2p}^2(H)$ satisfying conditions* (1) *and* (2) *of Theorem* 7.1, *with the operator $\mathscr{L}_0$ replacing $\mathscr{L}$ in condition* (2).

Proof Since the mild solution $X_0^x(t)$ is exponentially ultimately bounded in the m.s.s., we have

$$E\left\|X_0^x(t)\right\|_H^2 \le c\mathrm{e}^{-\beta t}\|x\|_H^2 + M \quad \text{for all } x \in H.$$

Let

$$\Psi_0(x) = \int_0^T E\left\|X_0^x(s)\right\|_H^2 ds + \alpha\|x\|_H^2,$$

where T and α are constants to be determined later.

First, let us show that $\Psi_0 \in C_{2p}^2(H)$. It suffices to show that

$$\varphi_0(x) = \int_0^T E\left\|X_0^x(s)\right\|_H^2 ds \in C_{2p}^2(H).$$

Now,

$$\varphi_0(x) \le \frac{c}{\beta}\left(1 - \mathrm{e}^{-\beta T}\right)\|x\|_H^2 + MT \le \frac{c}{\beta}\|x\|_H^2 + MT.$$

If $\|x\|_H^2 = 1$, then $\varphi_0(x) \le c/\beta + MT$.

Since $X_0^x(t)$ is linear in x, we have that, for any positive constant k, $X_0^{kx}(t) = kX_0^x(t)$. Hence, $\varphi_0(kx) = k^2\varphi_0(x)$, and for any $x \in H$,

$$\varphi_0(x) = \|x\|_H^2\varphi\left(\frac{x}{\|x\|_H}\right) \le \left(\frac{c}{\beta} + MT\right)\|x\|_H^2.$$

Let $c' = c/\beta + MT$. Then $\phi_0(x) \le c' \|x\|_H^2$ for all $x \in H$. For $x, y \in H$, define

$$\tau(x, y) = \int_0^T E\big\langle X_0^x(s), X_0^y(s)\big\rangle_H \, ds.$$

Then $\tau(x, y)$ is a nonnegative definite bounded bilinear form on $H \times H$ since $\varphi_0(x) \le c' \|x\|_H^2$. Hence, $\tau(x, y) = \langle Cx, y\rangle_H$, where C is a nonnegative definite bounded linear operator on H with $\|C\|_{\mathscr{L}(H)} \le c'$. Therefore, $\varphi_0 = \langle Cx, x\rangle_H \in C_{2p}^2(H)$, and $\Psi_0 \in C_{2p}^2(H)$. Clearly Ψ_0 satisfies condition (1) of Theorem 7.1. To prove (2), observe that by the continuity of the function $t \to E\|X_0^x(t)\|_H^2$ and because

$$E\varphi_0\big(X_0^x(r)\big) = \int_0^T E\big\|X_0^x(r+s)\big\|_H^2 \, ds = \int_r^{T+r} E\big\|X_0^x(s)\big\|_H^2 \, ds,$$

we have

$$\begin{aligned}
\mathscr{L}_0\varphi_0(x) &= \frac{\mathrm{d}}{\mathrm{d}r}\big(E\varphi_0\big(X_0^x(r)\big)\big)\Big|_{r=0} \\
&= \lim_{r\to 0} \frac{E\varphi_0(X_0^x(r)) - E\varphi_0(x)}{r} \\
&= \lim_{r\to 0}\left(-\frac{1}{r}\int_0^r E\big\|X_0^x(s)\big\|_H^2 \, ds + \frac{1}{r}\int_T^{r+T} E\big\|X_0^x(s)\big\|_H^2 \, ds\right) \\
&= -\|x\|_H^2 + E\big\|X_0^x(T)\big\|_H^2 \\
&\le -\|x\|_H^2 + c\mathrm{e}^{-\beta T}\|x\|_H^2 + M \\
&\le \big(-1 + c\mathrm{e}^{-\beta T}\big)\|x\|_H^2 + M.
\end{aligned}$$

Therefore, since by (6.32), $\mathscr{L}_0\|x\|_H^2 \le (2\lambda + d^2\,\mathrm{tr}(Q))\|x\|_H^2$, we have

$$\begin{aligned}
\mathscr{L}_0\Psi_0(x) &= \mathscr{L}_0\varphi_0(x) + \mathscr{L}_0\|x\|_H^2 \\
&\le \big(-1 + c\mathrm{e}^{-\beta T}\big)\|x\|_H^2 + \alpha\big(2\lambda + d^2\,\mathrm{tr}(Q)\big)\|x\|_H^2 + M. \qquad (7.2)
\end{aligned}$$

If $T > \ln(c/\beta)$, then one can choose α small enough such that $\Psi_0(x)$ satisfies condition (2) with $\mathscr{L}$ replaced by $\mathscr{L}_0$. □

The following theorem is a counterpart of Remark 6.1 in the framework of exponential ultimate boundedness.

Theorem 7.3 *If the mild solution of* (6.10) *is exponentially ultimately bounded in the m.s.s. and, for some* $T > 0$,

$$\varphi(x) = \int_0^T E\big\|X^x(t)\big\|_H^2 \, dt \in C_{2p}^2(H),$$

then there exists a (Lyapunov) function $\Psi \in C^2_{2p}(H)$ *satisfying conditions* (1) *and* (2) *of Theorem* 7.1.

Theorem 7.4 *Suppose that the mild solution* $X_0^x(t)$ *of the linear equation* (6.22) *satisfies condition* (7.1). *Then the mild solution* $X^x(t)$ *of* (6.10) *is exponentially ultimately bounded in the m.s.s. if*

$$2\|x\|_H \big\|F(x)\big\|_H + \tau\big(B(x)QB^*(x) - B_0xQ(B_0x)^*\big) < \tilde{\omega}\|x\|_H^2 + M_1, \tag{7.3}$$

where $\tilde{\omega} < \max_{s>\ln(c/\beta)}(1 - c\mathrm{e}^{-\beta s})/(c/\beta + Ms)$.

Proof Let $\Psi_0(x)$ be the Lyapunov function as defined in Theorem 7.2, with $T > \ln(c/\beta)$, such that the maximum in the definition of $\tilde{\omega}$ is achieved. It remains to show that

$$\mathscr{L}\Psi_0(x) \le -c_3\Psi_0(x) + k_3.$$

Since $\Psi_0(x) = \langle Cx, x\rangle_H + \alpha\|x\|_H^2$ for some $C \in \mathscr{L}(H)$ with $\|C\|_{\mathscr{L}(H)} \le c/\beta + MT$ and α sufficiently small, we have

$$\begin{aligned}
&\mathscr{L}\Psi_0(x) - \mathscr{L}_0\Psi_0(x)\\
&\quad\le \big(\|C\|_{\mathscr{L}(H)} + \alpha\big)\big(2\|x\|_H\big\|F(x)\big\|_H + \tau\big(B(x)QB^*(x) - B_0xQ(B_0x)^*\big)\big)\\
&\quad\le (c/\beta + MT + \alpha)\big(\tilde{\omega}\|x\|_H^2 + M_1\big).
\end{aligned}$$

Using (7.2), we have

$$\begin{aligned}
\mathscr{L}\Psi_0(x) &\le \big(-1 + c\mathrm{e}^{-\beta T}\big)\|x\|_H^2 + \alpha\big(2\lambda + d^2\,\mathrm{tr}(Q)\big)\|x\|_H^2 + M\\
&\qquad + (c/\beta + MT + \alpha)\big(\tilde{\omega}\|x\|_H^2 + M_1\big)\\
&\le \big(-1 + c\mathrm{e}^{-\beta T} + \tilde{\omega}(c/\beta + MT)\big)\|x\|_H^2\\
&\qquad + \alpha\big(2\lambda + d^2\,\mathrm{tr}(Q) + \tilde{\omega}\big)\|x\|_H^2 + M + (c/\beta + MT + \alpha).
\end{aligned}$$

Using the bound for $\tilde{\omega}$, we have $-1 + c\mathrm{e}^{-\beta T} + \tilde{\omega}(c/\beta + MT) < 0$, so that we can choose α small enough to obtain condition (2) of Theorem 7.1. □

Corollary 7.1 *Suppose that the mild solution* $X_0^x(t)$ *of the linear equation* (6.22) *is exponentially ultimately bounded in the m.s.s. If, as* $\|x\|_H \to \infty$,

$$\big\|F(x)\big\|_H = o\big(\|x\|_H\big) \quad \textit{and} \quad \tau\big(B(x)QB^*(x) - B_0xQ(B_0x)^*\big) = o\big(\|x\|_H\big),$$

then the mild solution $X^x(t)$ *of* (6.10) *is exponentially ultimately bounded in the m.s.s.*

Proof We fix $\tilde{\omega} < \max_{s>\ln(c/\beta)}(1 - c\mathrm{e}^{-\beta t}/(c/\beta + Ms)$, and using the assumptions, we choose a constant K such that for $\|x\|_H \ge K$, condition (7.3) holds. But for

$\|x\|_H \le K$, by appealing to the growth conditions on F and B,

$$\begin{aligned}
&2\|x\|_H \big\|F(x)\big\|_H + \tau\big(B(x)QB^*(x) - B_0 x Q(B_0 x)^*\big) \\
&\quad\le \|x\|_H^2 + \big\|F(x)\big\|_H^2 + \tau\big(B(x)QB^*(x) - B_0 x Q(B_0 x)^*\big) \\
&\quad\le \|x\|_H^2 + \ell\big(1 + \|x\|_H^2\big) + \big(\|B_0 x\|_{\mathscr{L}(H)}^2\big)\,\mathrm{tr}(Q) \\
&\quad\le \|x\|_H^2 + \ell\big(1 + \|x\|_H^2\big) + d^2\|x\|_H^2\,\mathrm{tr}(Q) \\
&\quad\le K^2 + \ell\big(1 + K^2\big) + M'.
\end{aligned}$$

Hence, condition (7.3) holds with the constant $M_1 = K^2 + \ell(1 + K^2) + M'$, and the result follows from Theorem 7.4. □

Example 7.1 (Dissipative Systems) Consider SSDE (6.10) and, in addition to assumptions (1)–(3) in Sect. 6.2, impose the following *dissipativity condition*:

(D) (Dissipativity) There exists a constant $\omega > 0$ such that for all $x, y \in H$ and $n = 1, 2, \ldots,$

$$\begin{aligned}
&2\big\langle A_n(x-y), x-y\big\rangle_H + 2\big\langle F(x) - F(y), x-y\big\rangle_H + \big\|B(x) - B(y)\big\|_{\mathscr{L}_2(K_Q,H)} \\
&\quad\le -\omega\|x-y\|_H^2, \qquad (7.4)
\end{aligned}$$

where $A_n x = AR_n x,\ \ x \in H$, are the Yosida approximations of A defined in (1.22).

Then the mild solution to (6.10) is ultimately exponentially bounded in the m.s.s. (Exercise 7.1).

Exercise 7.1 (a) Show that condition (D) implies that for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that for any $x \in H$ and $n = 1, 2, \ldots,$

$$2\langle A_n x, x\rangle_H + 2\big\langle F(x), x\big\rangle_H + \big\|B(x)\big\|_{\mathscr{L}_2(K_Q,H)} \le -(\omega - \varepsilon)\|x\|_H^2 + C_\varepsilon$$

with A_n, the Yosida approximations of A. Use this fact to prove that the strong solutions $X_n^x(t)$ of the approximating SDEs (6.12) are ultimately exponentially bounded in the m.s.s. Conclude that the mild solution $X^x(t)$ of (6.10) is ultimately exponentially bounded in the m.s.s.

(b) Prove that if zero is a solution of (6.10), then the mild solution $X^x(t)$ of (6.10) is exponentially stable in the m.s.s.

7.2 Exponential Ultimate Boundedness in Variational Method

We study in this section strong solutions to (6.37) whose coefficients satisfy linear growth, coercivity, and monotonicity assumptions (6.38)–(6.40).

Definition 7.2 We extend Definition 7.1 of exponential ultimate boundedness in the m.s.s. to the strong solution $\{X^x(t),\ t \geq 0\}$ of (6.37) and say that $X^x(t)$ is exponentially ultimately bounded in the m.s.s. if it satisfies condition (7.1).

Let us begin by noting that the proof of Theorem 7.1 can be carried out in this case if we assume that the function Ψ satisfies conditions (1)–(5) of Theorem 6.10 and that the operator $\mathscr{L}$ is defined by

$$\mathscr{L}\Psi(u) = \big\langle \Psi'(u), A(u)\big\rangle + \operatorname{tr}\big(\Psi''(u)B(u)QB^*(u)\big). \tag{7.5}$$

Hence, we have the following theorem.

Theorem 7.5 *The strong solution* $\{X^x(t),\ t \geq 0\}$ *of* (6.37) *is exponentially ultimately bounded in the m.s.s. if there exists a function* $\Psi : H \to \mathbb{R}$ *satisfying conditions* (1)–(5) *of Theorem* 6.10 *and*, *in addition*, *such that*

(1) $c_1\|x\|_H^2 - k_1 \leq \Psi(x) \leq c_2\|x\|_H^2 + k_2$ *for some positive constants* c_1, c_2, k_1, k_2 *and for all* $x \in H$,
(2) $\mathscr{L}\Psi(x) \leq -c_3\Psi(x) + k_3$ *for some positive constants* c_3, k_3 *and for all* $x \in V$.

In the linear case, we have both, sufficiency and necessity, and the Lyapunov function has an explicit form under the general coercivity condition (C).

Theorem 7.6 *A solution* $\{X_0^x(t),\ t \geq 0\}$ *of the linear equation* (6.42) *whose coefficients satisfy coercivity condition* (6.39) *is exponentially ultimately bounded in the m.s.s. if and only if there exists a function* $\Psi_0 : H \to \mathbb{R}$ *satisfying conditions* (1)–(5) *of Theorem* 6.10 *and*, *in addition*, *such that*

(1) $c_1\|x\|_H^2 - k_1 \leq \Psi_0(x) \leq c_2\|x\|_H^2 + k_2$ *for some positive constants* c_1, c_2, k_1, k_2 *and for all* $x \in H$,
(2) $\mathscr{L}_0\Psi_0(x) \leq -c_3\Psi_0(x) + k_3$ *for some positive constants* c_3, k_3 *and for all* $x \in V$.

This function can be written in the explicit form

$$\Psi_0(x) = \int_0^T\int_0^t E\big\|X_0^x(s)\big\|_V^2\,ds\,dt \tag{7.6}$$

with $T > \alpha_0(c|\lambda|/(\alpha\beta) + 1/\alpha)$, *where* α_0 *is such that* $\|v\|_H^2 \leq \alpha_0\|v\|_V^2$, $v \in V$.

Proof Assume that the solution $\{X_0^x(t),\ t \geq 0\}$ of the linear equation (6.42) is exponentially ultimately bounded in the m.s.s., so that

$$E\big\|X_0^x(t)\big\|_H^2 \leq c\mathrm{e}^{-\beta t}\|x\|_H^2 + M \quad \text{for all } x \in H.$$

Applying Itô's formula to the function $\|x\|_H^2$, taking the expectations, and using the coercivity condition (6.39), we obtain

$$E\left\|X_0^x(t)\right\|_H^2 - \|x\|_H^2 = \int_0^t E\mathscr{L}_0\left\|X_0^x(s)\right\|_H^2 ds$$
$$\le \lambda \int_0^t E\left\|X_0^x(s)\right\|_H^2 ds - \alpha \int_0^t E\left\|X_0^x(s)\right\|_V^2 ds + \gamma t. \quad (7.7)$$

Hence,

$$\int_0^t E\left\|X_0^x(s)\right\|_V^2 \le \frac{1}{\alpha}\left(\lambda \int_0^t E\left\|X_0^x(s)\right\|_H^2 ds + \|x\|_H^2 + \gamma t\right).$$

Applying condition (7.1), we have

$$\int_0^t E\left\|X_0^x(s)\right\|_V^2 \le \frac{1}{\alpha}\left(\frac{c|\lambda|}{\beta}\left(1 - \mathrm{e}^{-\beta t}\right)\|x\|_H^2 + \|x\|_H^2 + \left(|\lambda|M + \gamma\right)t\right)$$
$$\le \left(\frac{c|\lambda|}{\alpha\beta} + \frac{1}{\alpha}\right)\|x\|_H^2 + \frac{|\lambda|M + \gamma}{\alpha}t.$$

Therefore, with Ψ_0 defined in (7.6),

$$\Psi_0(x) = \int_0^T\int_0^t E\left\|X_0^x(s)\right\|_V^2 ds\,dt \le \left(\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta}\right)T\|x\|_H^2 + \frac{|\lambda|M + \gamma}{2\alpha}T^2. \quad (7.8)$$

Now

$$\left|\mathscr{L}_0\|v\|_H^2\right| \le 2a_1\|v\|_V^2 + b_1^2\,\mathrm{tr}(Q)\|v\|_V^2 \le c'\|v\|_V^2$$

for some positive constant c'. Therefore, we conclude that

$$\mathscr{L}_0\|v\|_H^2 \ge -c'\|v\|_V^2.$$

From (7.7) we get

$$E\left\|X_0^x(t)\right\|_H^2 - \|x\|_H^2 \ge -c'\int_0^t E\left\|X_0^x(s)\right\|_V^2 ds.$$

Using (7.1), we have

$$c'\int_0^t E\left\|X_0^x(t)\right\|_V^2 ds \ge \left(1 - \mathrm{e}^{-\beta t}\right)\|x\|_H^2 - M.$$

Hence,

$$\Psi_0(x) \ge \frac{1}{c'}\int_0^T \|x\|_H^2\left(1 - \mathrm{e}^{-\beta t}\right)dt - MT \ge \frac{1}{c'}\left(T - \frac{c}{\beta}\right)\|x\|_H^2 - \frac{MT}{c'}.$$

Choose $T > c/\beta$ to obtain condition (1).

To prove that condition (2) holds, consider

$$E\Psi_0\big(X_0^x(r)\big)=\int_0^T\int_0^t E\big\|X_0^{X_0^x(r)}(s)\big\|_V^2\,ds\,dt.$$

By the Markov property of the solution and the uniqueness of the solution,

$$E\Psi_0\big(X_0^x(r)\big)=\int_0^T\int_0^t E\big\|X_0^x(s+r)\big\|_V^2\,ds\,dt=\int_0^T\int_r^{t+r} E\big\|X_0^x(s)\big\|_V^2\,ds\,dt.$$

We now need the following technical lemma that will be proved later.

Lemma 7.1 *If* $f\in L^1([0,T])$, $T>0$, *is a nonnegative real-valued function*, *then*

$$\lim_{\Delta t\to 0}\int_0^T\frac{\int_t^{t+\Delta t} f(s)\,ds}{\Delta t}\,dt=\int_0^T\lim_{\Delta t\to 0}\frac{\int_t^{t+\Delta t} f(s)\,ds}{\Delta t}\,dt=\int_0^T f(t)\,dt.$$

Assuming momentarily that Ψ_0 satisfies conditions (1)–(5) of Theorem 6.10, we have

$$\begin{aligned}\mathscr{L}_0\Psi_0(x)&=\frac{\mathrm{d}}{\mathrm{d}r}\big(E\Psi_0\big(X_0^x(r)\big)\big)\Big|_{r=0}\\&=\lim_{r\to 0}\int_0^T\frac{\int_t^{t+r}E\|X_0^x(s)\|_V^2\,ds}{r}\,dt-\lim_{r\to 0}\frac{T}{r}\int_0^r E\big\|X_0^x(s)\big\|_V^2\,ds\\&\le\int_0^T E\big\|X_0^x(t)\big\|_V^2\,dt-\lim_{r\to 0}\frac{T}{\alpha_0}\frac{1}{r}\int_0^r E\big\|X_0^x(s)\big\|_H^2\,ds\end{aligned}$$

for α_0 such that $\|v\|_H^2\le\alpha_0\|v\|_V^2$. This gives

$$\mathscr{L}_0\Psi_0(x)\le\left(\frac{c|\lambda|}{\alpha\beta}+\frac{1}{\alpha}-\frac{T}{\alpha_0}\right)\|x\|_H^2+\frac{|\lambda|M+\gamma}{\alpha}T.\tag{7.9}$$

With $T>\alpha_0(\frac{c|\lambda|}{\alpha\beta}+\frac{1}{\alpha})$, condition (2) holds.

It remains to prove that Ψ_0 satisfies conditions (1)–(5) of Theorem 6.10. We use linearity of (6.42) to obtain, for any positive constant k,

$$X_0^{kx}(t)=kX_0^x(t).$$

Then $\Psi_0(kx)=k^2\Psi_0(x)$, and by (7.8), for $\|x\|_H=1$,

$$\Psi_0(x)\le\left(\frac{1}{\alpha}+\frac{c|\lambda|}{\alpha\beta}\right)T+\frac{|\lambda|M+\gamma}{2\alpha}T^2.$$

Hence, for $x\in H$,

$$\Psi_0(x)\le\|x\|_H^2\Psi_0\left(\frac{x}{\|x\|_H}\right)\le\left[\left(\frac{1}{\alpha}+\frac{c|\lambda|}{\alpha\beta}\right)T+\frac{|\lambda|M+\gamma}{2\alpha}T^2\right]\|x\|_H^2,\tag{7.10}$$

which implies that $\Psi_0(x) \le c''\|x\|_H^2$ for all $x \in H$. For $x,\ y \in H$, denote

$$\tau(x, y) = \int_0^T \int_0^t E\langle X_0^x(s), X_0^y(s)\rangle_H \, ds\, dt \le \Psi_0^{\frac{1}{2}}(x)\Psi_0^{\frac{1}{2}}(y) \le c''\|x\|_H\|y\|_H.$$

Then τ is a continuous bilinear form on $H \times H$, and there exists $C \in \mathscr{L}(H)$, with $\|C\|_{\mathscr{L}(H)} \le c''$, such that

$$\tau(x, y) = \langle Cx, y\rangle_H. \tag{7.11}$$

Using the continuity of the embedding $V \hookrightarrow H$, we conclude that $\tau(x, y)$ is a continuous bilinear form on $V \times V$, and hence,

$$\tau(x, y) = \langle \tilde{C}x, y\rangle_V \quad \text{for } x,\ y \in V, \tag{7.12}$$

with $\tilde{C} \in \mathscr{L}(V)$. Now it is easy to verify that Ψ_0 satisfies conditions (1)–(5) of Theorem 6.10. □

Proof of Lemma 7.1 We are going to use the Fubini theorem to change the order of integrals,

$$\begin{aligned}
\int_0^T \frac{\int_t^{t+\Delta t} f(s)\, ds}{\Delta t}\, dt &= \frac{1}{\Delta t}\int_0^T \left(\int_0^{t+\Delta t} f(s)\, ds\right) dt \\
&= \frac{1}{\Delta t}\left[\int_0^{\Delta t}\left(\int_0^s f(s)\, dt\right) ds + \int_{\Delta t}^T \left(\int_{s-\Delta t}^T f(s)\, dt\right) ds \right.\\
&\quad \left. + \int_T^{T+\Delta t}\left(\int_{s-\Delta t}^T f(s)\, dt\right) ds\right] \\
&= \frac{1}{\Delta t}\left[\int_0^{\Delta t} s f(s)\, ds + \int_{\Delta t}^T f(s)\Delta t\, ds + \int_T^{T+\Delta t} f(s)(T + \Delta t - s)\, ds\right] \\
&\le \frac{1}{\Delta t}\left[\Delta t \int_0^{\Delta t} f(s)\, ds + \Delta t \int_{\Delta t}^T f(s)\, ds + \Delta t \int_T^{T+\Delta t} f(s)\, ds\right] \\
&= \int_0^{\Delta t} f(s)\, ds + \int_{\Delta t}^T f(s)\, ds + \int_T^{T+\Delta t} f(s)\, ds.
\end{aligned}$$

The first and third terms converge to zero as $\Delta t \to 0$, so that

$$\lim_{\Delta t \to 0} \int_0^T \frac{\int_t^{t+\Delta t} f(s)\, ds}{\Delta t}\, dt \le \int_0^T f(t)\, dt.$$

The opposite inequality follows directly from Fatou's lemma. □

By repeating the proof of Theorem 7.6, we obtain a partial converse of Theorem 7.5.

Theorem 7.7 *Let the strong solution* $\{X^x(t),\ t \geq 0\}$ *of* (6.37) *be exponentially ultimately bounded in the m.s.s. Let*

$$\Psi(x) = \int_0^T \int_0^t E\left\|X^x(s)\right\|_V^2\, ds\, dt \tag{7.13}$$

with $T > \alpha_0(c|\lambda|/(\alpha\beta) + 1/\alpha)$, *where* α_0 *is such that* $\|v\|_H^2 \leq \alpha_0\|v\|_V^2$, $v \in V$. *Suppose that* $\Psi(x)$ *satisfies conditions* (1)–(5) *of Theorem* 6.10. *Then* $\Psi(x)$ *satisfies conditions* (1) *and* (2) *of Theorem* 7.5.

To study exponential ultimate boundedness, i.e., condition (7.1), for the strong solution of (6.37), we use linear approximation and the function Ψ_0 of the corresponding linear equation (6.42) as the Lyapunov function. We will prove the following result.

Theorem 7.8 *Suppose that the coefficients of the linear equation* (6.42) *satisfy the coercivity condition* (6.39) *and its solution* $\{X_0^x(t),\ t \geq 0\}$ *is exponentially ultimately bounded in the m.s.s. Let* $\{X^x(t),\ t \geq 0\}$ *be the solution of the nonlinear equation* (6.37). *Furthermore, we suppose that*

$$A(v) - A_0 v \in H \quad \text{for all } v \in V$$

and that, for $v \in V$,

$$2\|v\|_H \left\|A(v) - A_0 v\right\|_H + \tau\left(B(v)QB^*(v) - B_0 v Q(B_0 v)^*\right) \leq \tilde{\omega}\|v\|_H^2 + k,$$

where $\tilde{\omega}$ *and* k *are constants, and*

$$\tilde{\omega} < \frac{c}{\alpha_0\beta\left[\left(\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta}\right) + \left(\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta} + \frac{c}{\beta}\right) + \frac{|\lambda|M}{2\alpha}\left(\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta} + \frac{c}{\beta}\right)^2\right]}.$$

Then $X^x(t)$ *is exponentially ultimately bounded in the m.s.s.*

Proof Let

$$\Psi_0(x) = \int_0^{T_0} \int_0^t E\left\|X_0^x(t)\right\|_V^2\, ds\, dt$$

with $T_0 = \alpha_0(c|\lambda|/(\alpha\beta) + 1/\alpha) + c/\beta$. Then $\Psi_0(s)$ satisfies conditions (1)–(5) of Theorem 6.10, and for all $x \in H$,

$$c_1\|x\|_H^2 - k_1 \leq \Psi_0(x) \leq c_2\|x\|_H^2 + k_2.$$

It remains to prove that, for all $x \in V$,

$$\mathscr{L}\Psi_0(x) \leq -c_3\Psi_0(s) + k_3.$$

Then we can conclude the result by Theorem 7.5. Now, for $x \in V$,

$$\begin{aligned}
&\mathscr{L}\Psi_0(x) - \mathscr{L}_0\Psi_0(x) \\
&\quad = \langle \Psi_0'(x), A(x) - A_0x \rangle + \frac{1}{2}\operatorname{tr}\big(\Psi_0''(x)\big(B(x)QB^*(x) - B_0xQ(B_0x)^*\big)\big) \\
&\quad = \langle \Psi_0'(x), A(x) - A_0x \rangle_H + \frac{1}{2}\operatorname{tr}\big(\Psi_0''(x)\big(B(x)QB^*(x) - B_0xQ(B_0x)^*\big)\big).
\end{aligned}$$

But $\Psi_0'(x) = 2Cx$ and $\Psi_0''(x) = 2C$ for $x \in V$, where C is defined in (7.11). By inequality (7.10),

$$\|C\|_{\mathscr{L}(H)} \le \left(\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta}\right)T_0 + \frac{|\lambda|M + \gamma}{2\alpha}T_0^2.$$

Hence,

$$\mathscr{L}\Psi_0(x) - \mathscr{L}_0\Psi_0(x) \le 2\langle Cx, A(x) - A_0x \rangle_H + \tau\big(C\big(B(x)QB^*(x) - B_0xQ(B_0x)^*\big)\big),$$

and we have

$$\begin{aligned}
\mathscr{L}\Psi_0(x) \le \mathscr{L}_0\Psi_0(x) + \|C\|_{\mathscr{L}(H)}\big[&2\|x\|_H\|A(x) - A_0x\|_H \\
&+ \tau\big(B(x)QB^*(x) - B_0xQ(B_0x)^*\big)\big].
\end{aligned}$$

When $T = T_0$, from (7.9) we have

$$\mathscr{L}_0\Psi_0(x) \le -\frac{c}{\alpha_0\beta}\|x\|_H^2 + \frac{|\lambda|M + \gamma}{\alpha}T_0,$$

giving

$$\begin{aligned}
\mathscr{L}\Psi_0(x) &\le -\frac{c}{\alpha_0\beta}\|x\|_H^2 + \frac{|\lambda|M + \gamma}{\alpha}T_0 + \|C\|_{\mathscr{L}(H)}\big(\tilde{\omega}\|x\|_H^2 + k\big) \\
&\le \left(-\frac{c}{\alpha_0\beta} + \tilde{\omega}\|C\|_{\mathscr{L}(H)}\right)\|x\|_H^2 + \frac{|\lambda|M + \gamma}{\alpha}T_0 + k\|C\|_{\mathscr{L}(H)}.
\end{aligned}$$

Now, $-c/(\alpha_0\beta) + \tilde{\omega}\|C\|_{\mathscr{L}(H)} < 0$ if $\tilde{\omega}$ satisfies our original assumption, and we arrive at $\mathscr{L}\Psi_0(x) \le -c_3\Psi_0(x) + k_3$ with $c_3 > 0$. □

Remark 7.1 Note that the function $\Psi_0(x)$ in Theorem 7.8 is the Lyapunov function for the nonlinear equation.

Corollary 7.2 *Suppose that the coefficients of the linear equation* (6.42) *satisfy the coercivity condition* (6.39), *and its solution* $\{X_0^x(t), t \ge 0\}$ *is exponentially ultimately bounded in the m.s.s. Let* $\{X^x(t), t \ge 0\}$ *be a solution of the nonlinear equation* (6.37). *Furthermore, suppose that*

$$A(v) - A_0v \in H \quad \textit{for all } v \in V$$

and that, for $v \in V$,

$$2\|v\|_H \big\|A(v) - A_0 v\big\|_H + \tau\big(B(v)QB^*(v) - B_0 v Q(B_0 v)^*\big) \le k\big(1 + \|v\|_H^2\big) \quad (7.14)$$

for some $k > 0$. *If for* $v \in V$, *as* $\|v\|_H \to \infty$,

$$\begin{aligned} &\big\|A(v) - A_0 v\big\|_H = o\big(\|v\|_H\big) \\ \text{and} \quad & \\ &\tau\big(B(v)QB^*(v) - B_0 v Q(B_0 v)^*\big) = o\big(\|v\|_H^2\big), \end{aligned} \quad (7.15)$$

then $X^x(t)$ *is exponentially ultimately bounded in the m.s.s.*

Proof Under assumption (7.15), for a constant $\tilde{\omega}$ satisfying the condition of Theorem 7.8, there exists an $R > 0$ such that, for all $v \in V$ with $\|v\|_H > R$,

$$2\|v\|_H \big\|A(v) - A_0 v\big\|_H + \tau\big(B(v)QB^*(v) - B_0 v Q(B_0 v)^*\big) \le \tilde{\omega}\|v\|_H^2.$$

For $v \in V$ and $\|v\|_H < R$, by (7.14),

$$\begin{aligned} &2\|v\|_H \big\|A(v) - A_0 v\big\|_H + \tau\big(B(v)QB^*(v) - B_0 v Q(B_0 v)^*\big) \\ &\quad \le k\big(1 + \|v\|_H^2\big) \le k\big(1 + R^2\big). \end{aligned}$$

Hence, we have

$$\begin{aligned} &2\|v\|_H \big\|A(v) - A_0 v\big\|_H + \tau\big(B(v)QB^*(v) - B_0 v Q(B_0 v)^*\big) \\ &\quad \le \tilde{\omega}\|v\|_H^2 + (k+1)R^2. \end{aligned}$$

An appeal to Theorem 7.8 completes the proof. □

Theorem 7.9 *Suppose that the coefficients of the linear equation* (6.42) *satisfy the coercivity condition* (6.39) *and its solution* $\{X_0^x(t), t \ge 0\}$ *is exponentially ultimately stable in the m.s.s. with the function* $t \to E\|X_0^x(t)\|_V^2$ *being continuous for all* $x \in V$. *Let* $\{X^x(t), t \ge 0\}$ *be a solution of the nonlinear equation* (6.37). *If for* $v \in V$,

$$2\|v\|_V \big\|A(v) - A_0 v\big\|_{V^*} + \tau\big(B(v)QB^*(v) - B_0 v Q(B_0 v)^*\big) \le \tilde{\omega}_0\|v\|_V^2 + k_0$$

for some constants $\tilde{\omega}_0$, k_0 *such that*

$$\tilde{\omega} < \frac{c}{(\alpha_0 + 1)\beta\big[\big(\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta}\big) + \big(\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta} + \frac{c}{\beta}\big) + \frac{|\lambda|M}{2\alpha}\big(\frac{1}{\alpha} + \frac{|\lambda|}{\alpha\beta} + \frac{c}{\beta}\big)^2\big]},$$

then $X^x(t)$ *is exponentially ultimately bounded in the m.s.s.*

Proof Let, as before,

$$\Psi_0(x) = \int_0^{T_0}\int_0^t E\big\|X_0^x(t)\big\|_V^2\, ds\, dt$$

with $T_0 = \alpha_0(c|\lambda|/(\alpha\beta) + 1/\alpha) + \frac{c}{\beta}$. For $x \in V$,

$$\begin{aligned}
&\mathscr{L}\Psi_0(x) - \mathscr{L}_0\Psi_0(x) \\
&\quad = \big\langle \Psi_0'(x), A(x) - A_0 x \big\rangle + \frac{1}{2}\operatorname{tr}\big(\Psi_0''(x)\big(B(x)QB^*(x) - B_0 x Q(B_0 x)^*\big)\big)
\end{aligned}$$

with $\Psi_0'(x) = 2\tilde{C}x$ and $\Psi_0''(x) = 2C$, where the operators C and $\tilde{C}$ are defined in (7.11) and (7.12). By inequality (7.10) and the continuity of the embedding $V \hookrightarrow H$,

$$\begin{aligned}
&\|C\|_{\mathscr{L}(H)} \le \left(\frac{1}{\alpha} + \frac{c|\lambda|}{\alpha\beta}\right) T_0 + \frac{|\lambda|M + \gamma}{2\alpha} T_0^2, \\
&\|\tilde{C}\|_{\mathscr{L}(V)} \le \alpha_0 \|C\|_{\mathscr{L}(V)}.
\end{aligned}$$

Hence,

$$\mathscr{L}\Psi_0(x) - \mathscr{L}_0\Psi_0(x) \le 2\langle \tilde{C}x, Ax - A_0 x\rangle_H + \operatorname{tr}\big(C\big(B(x)QB^*(x) - B_0 x Q(B_0 x)^*\big)\big),$$

and we have

$$\begin{aligned}
\mathscr{L}\Psi_0(x) &\le \mathscr{L}_0\Psi_0(x) + 2\|\tilde{C}\|_{\mathscr{L}(V)}\|x\|_V\|Ax - A_0 x\|_{V^*} \\
&\qquad + \operatorname{tr}\big(CB(x)QB^*(x) - B_0 x Q(B_0 x)^*\big) \\
&\le \mathscr{L}_0\Psi_0(x) + \big(\|C\|_{\mathscr{L}(H)} + \|\tilde{C}\|_{\mathscr{L}(V)}\big)\big(2\|x\|_V\|Ax - A_0 x\|_{V^*} \\
&\qquad + \tau\big(B(x)QB^*(x) - B_0 x Q(B_0 x)^*\big)\big).
\end{aligned}$$

Since $s \to E\|X_0^x(s)\|_V^2$ is a continuous function, we obtain from earlier relations for $\mathscr{L}_0\Psi_0(x)$ that

$$\mathscr{L}_0\Psi_0(x) \le -\frac{c}{\beta}\|x\|_V^2 + \frac{|\lambda|M + \gamma}{\alpha} T_0.$$

Hence,

$$\begin{aligned}
\mathscr{L}\Psi_0(x) &\le -\frac{c}{\beta}\|x\|_V^2 + \frac{|\lambda|M}{\alpha} T_0 + \big(\|C\|_{\mathscr{L}(H)} + \|\tilde{C}\|_{\mathscr{L}(V)}\big)\big(\tilde{\omega}_0\|x\|_V^2 + k_0\big) \\
&\le \left(-\frac{c}{\beta} + \tilde{\omega}_0\big(\|C\|_{\mathscr{L}(H)} + \|\tilde{C}\|_{\mathscr{L}(V)}\big)\right)\|x\|_V^2 \\
&\qquad + k_0\left(\|C\|_{\mathscr{L}(H)} + \|\tilde{C}\|_{\mathscr{L}(V)} + \frac{|\lambda|M + \gamma}{\alpha} T_0\right).
\end{aligned}$$

Since, with the condition on $\tilde{\omega}_0$, $-c/\beta + \tilde{\omega}_0(\|C\|_{\mathscr{L}(H)} + \|\tilde{C}\|_{\mathscr{L}(V)}) < 0$, we see that conditions analogous to those of Theorem 7.1 are satisfied by Ψ_0, giving the result. □

Corollary 7.3 *Suppose that the coefficients of the linear equation* (6.42) *satisfy the coercivity condition* (6.39) *and its solution* $\{X_0^x(t),\ t \ge 0\}$ *is exponentially ultimately bounded in the m.s.s. with the function* $t \to E\|X_0^x(t)\|_V^2$ *being continuous for all* $x \in V$. *Let* $\{X^x(t),\ t \ge 0\}$ *be a solution of the nonlinear equation* (6.37). *If for* $v \in V$, *as* $\|v\|_V \to \infty$,

$$\big\|A(v) - A_0 v\big\|_{V^*} = o\big(\|v\|_V\big)$$

and (7.16)

$$\tau\big(B(v)QB^*(v) - B_0 v Q(B_0 v)^*\big) = o\big(\|v\|_V^2\big),$$

then $X^x(t)$ *is exponentially ultimately bounded in the m.s.s.*

Proof We shall use Theorem 7.9. Under assumption (7.16), for a constant $\tilde{\omega}_0$ satisfying the condition of Theorem 7.9, there exists an $R > 0$ such that, for all $v \in V$ with $\|v\|_V > R$,

$$2\|v\|_V \big\|A(v) - A_0 v\big\|_{V^*} + \tau\big(B(v)QB^*(v) - B_0 v Q(B_0 v)^*\big) \le \tilde{\omega}_0 \|v\|_V^2.$$

Using that $\|A(v)\|_{V^*}$, $\|A_0(v)\|_{V^*} \le a_1 \|v\|_V$ and $\|B(v)\|_{\mathscr{L}(K,H)}$, $\|B_0 v\|_{\mathscr{L}(K,H)} \le b_1 \|v\|_V$, we have, for $v \in V$ such that $\|v\|_H < R$,

$$\begin{aligned}
&2\|v\|_V \big\|A(v) - A_0 v\big\|_{V^*} + \tau\big(B(v)QB^*(v) - B_0 v Q(B_0 v)^*\big) \\
&\le 4a_1 \|v\|_V^2 + \big(\big\|B(v)\big\|_{\mathscr{L}(K,H)}^2 + \|B_0 v\|_{\mathscr{L}(K,H)}^2\big)\,\mathrm{tr}(Q) \\
&\le \big(4a_1 + 2b_1^2\,\mathrm{tr}(Q)\big)\|v\|_V^2 \\
&\le \big(4a_1 + 2b_1^2\,\mathrm{tr}(Q)\big)R^2.
\end{aligned}$$

Hence, for $v \in V$,

$$\begin{aligned}
&2\|v\|_V \big\|A(v) - A_0 v\big\|_{V^*} + \tau\big(B(v)QB^*(v) - B_0 v Q(B_0 v)^*\big) \\
&\quad \le \tilde{\omega}_0 \|v\|_V^2 + \big(4a_1 + 2b_1^2\,\mathrm{tr}(Q)\big)R^2.
\end{aligned}$$

An appeal to Theorem 7.9 completes the proof. □

7.3 Abstract Cauchy Problem, Stability and Exponential Ultimate Boundedness

We present an analogue of a result of Zakai and Miyahara for the infinite-dimensional case.

Definition 7.3 A linear operator $A : V \to V^*$ is called *coercive* if it satisfies the following *coercivity condition*: for some $\alpha > 0$, $\gamma, \lambda \in \mathbb{R}$, and all $v \in V$,

$$2\langle v, Av\rangle \le \lambda \|v\|_H^2 - \alpha \|v\|_V^2 + \gamma. \tag{7.17}$$

Proposition 7.1 *Consider a stochastic evolution equation*,

$$\begin{cases} dX(t) = A_0 X(t)\,dt + F(X(t))\,dt + B(X(t))\,dW_t, \\ X(0) = x \in H, \end{cases} \tag{7.18}$$

with the coefficients A_0 and F satisfying the following conditions:

(1) $A_0 : V \to V^*$ *is coercive.*
(2) $F : V \to H$, $B : V \to \mathscr{L}(K,H)$, *and there exists a constant $K > 0$ such that for all $v \in V$*,

$$\|F(v)\|_H^2 + \|B(v)\|_{\mathscr{L}(K,H)}^2 \le K\big(1 + \|v\|_H^2\big).$$

(3) *There exists a constant $L > 0$ such that for all $v, v' \in V$*,

$$\|F(v) - F(v')\|_H^2 + \operatorname{tr}\big(\big(B(v) - B(v')\big)Q\big(B^*(v) - B^*(v')\big)\big) \le L\|v - v'\|_H^2.$$

(4) *For $v \in V$, as $\|v\|_H \to \infty$*,

$$\|F(v)\|_H = o\big(\|v\|_H\big), \quad \|B(v)\|_{\mathscr{L}(K,H)} = o\big(\|v\|_H\big).$$

If the classical solution $\{u^x(t), t \ge 0\}$ of the abstract Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} = A_0 u(t), \\ u(0) = x \in H, \end{cases} \tag{7.19}$$

is exponentially stable (*or even exponentially ultimately bounded*), *then the solution of* (7.18) *is exponentially ultimately bounded in the m.s.s.*

Proof Let $A(v) = A_0 v + F(v)$ for $v \in V$. Since $F(v) \in H$,

$$\begin{aligned} &2\big\langle v, A(v)\big\rangle + \operatorname{tr}\big(B(v)QB^*(v)\big) \\ &\quad = 2\langle v, A_0 v\rangle + 2\big\langle v, F(v)\big\rangle + \operatorname{tr}\big(B(v)QB^*(v)\big) \\ &\quad \le \lambda\|v\|_H^2 - \alpha\|v\|_V^2 + 2\|v\|_H\|F(v)\|_H + \|B(v)\|_{\mathscr{L}(K,H)}^2 \operatorname{tr}(Q) \\ &\quad \le \lambda'\|v\|_H^2 - \alpha\|v\|_H^2 + \gamma \end{aligned}$$

for some constants λ' and γ. Hence, the evolution equation (7.18) satisfies the coercivity condition (6.39). Under assumption (2)

$$\begin{aligned} \|F(v)\|_H^2 + \operatorname{tr}\big(B(v)QB^*(v)\big) &\le \|F(v)\|_H^2 + \operatorname{tr}(Q)\|B(v)\|_{\mathscr{L}(K,H)}^2 \\ &\le \big(1 + \operatorname{tr}(Q)\big)K\big(1 + \|v\|_H^2\big), \end{aligned}$$

so that condition (7.14) holds, and since

$$\big\|F(v)\big\|_H = o\big(\|v\|_H\big) \quad \text{and} \quad \tau\big(B(v)QB^*(v)\big) = o\big(\|v\|_H^2\big) \quad \text{as } \|v\|_H \to \infty,$$

Corollary 7.2 gives the result. □

Example 7.2 (Stochastic Heat Equation) Let S^1 be the unit circle realized as the interval $[-\pi, \pi]$ with identified points $-\pi$ and π. Denote by $W^{1,2}(S^1)$ the Sobolev space on S^1 and by $W(t,\xi)$ the Brownian sheet on $[0,\infty) \times S^1$, see Exercise 7.2. Let $\kappa > 0$ be a constant, and f and b be real-valued functions. Consider the following SPDE:

$$\begin{cases} \dfrac{\partial X(t)}{\partial t}(\xi) = \dfrac{\partial^2 X(t)}{\partial \xi^2}(\xi) - \kappa f(X(t)(\xi)) + b(X(t)(\xi))\dfrac{\partial^2 W}{\partial t \partial \xi}, \\ X(0)(\cdot) = x(\cdot) \in L^2(S^1). \end{cases} \tag{7.20}$$

Let $H = L^2(S^1)$ and $V = W^{1,2}(S^1)$. Consider

$$A_0(x) = \left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - \kappa\right)x$$

and mappings F, B defined for $\xi \in S^1$ and $x, y \in V$ by

$$F(x)(\xi) = f\big(x(\xi)\big), \quad \big(B(x)y\big)(\xi) = \big\langle b\big(x(\cdot)\big), y(\cdot)\big\rangle_{L^2(S^1)}.$$

Let

$$\|x\|_H = \left(\int_{S^1} x^2(\xi)\,d\xi\right)^{1/2} \quad \text{for } x \in H,$$

$$\|x\|_V = \left(\int_{S^1}\left(x^2(\xi) + \left(\frac{\mathrm{d}x(\xi)}{\mathrm{d}\xi}\right)^2\right)d\xi\right)^{1/2} \quad \text{for } x \in V.$$

Then we obtain the equation

$$dX(t) = A_0 X(t)\,dt + F\big(X(t)\big)\,dt + B\big(X(t)\big)\,d\tilde{W}_t,$$

where $\tilde{W}_t$ is a cylindrical Wiener process defined in Exercise 7.2. We have

$$\begin{aligned} 2\big\langle x, A_0(x)\big\rangle &= -2\|x\|_V^2 + (-2\kappa + 2)\|x\|_H^2 \\ &\le -2\|x\|_H^2 + (-2\kappa + 2)\|x\|_H^2 = -2\kappa\|x\|_H^2. \end{aligned}$$

By Theorem 6.3(a), with $\Lambda(x) = \|x\|_H^2$, the solution of (7.19) is exponentially stable. If we assume that f and b are Lipschitz continuous and bounded, then conditions (1)–(3) of Proposition 7.1 are satisfied. Using representation (2.35) of the stochastic integral with respect to a cylindrical Wiener process, we can conclude that the solution of the stochastic heat equation (7.20) is exponentially ultimately bounded in the m.s.s.

Exercise 7.2 Let S^1 be the unit circle realized as the interval $[-\pi,\pi]$ with identified points $-\pi$ and π. Denote by $\{f_j(\xi)\}$ an ONB in $L^2(S^1)$ and consider

$$W(t,\zeta)=\sum_{j=1}^{\infty} w_j(t)\int_{-\pi}^{\zeta} f_j(\xi)\,d\xi, \quad t\ge 0, -\pi\le\zeta\le\pi, \tag{7.21}$$

where w_j are independent Brownian motions defined on $\{\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\ge 0},P\}$. Show that the series (7.21) converges P-a.s. and that

$$\operatorname{Cov}\big(W(t_1,\zeta_1)W(t_2,\zeta_2)\big)=(t_1\wedge t_2)(\zeta_1\wedge\zeta_2).$$

Conclude that the Gaussian random field $W(\cdot,\cdot)$ has a continuous version. This continuous version is called the *Brownian sheet* on S^1.

Now, let $\Phi(t)$ be an adapted process with values in $L^2(S^1)$ (identified with $\mathscr{L}(L^2(S^1),\mathbb{R})$) and satisfying

$$E\int_0^{\infty}\|\Phi(t)\|^2_{L^2(S^1)}\,dt<\infty.$$

Consider a standard cylindrical Brownian motion $\tilde{W}_t$ in $L^2(S^1)$ defined by

$$\tilde{W}_t(k)=\sum_{j=1}^{\infty} w_j(t)\langle k,f_j\rangle_{L^2(S^1)}.$$

Show that the cylindrical stochastic integral process

$$\int_0^t \Phi(s)\,d\tilde{W}_s \tag{7.22}$$

is well defined in $L^2(\Omega,\mathbb{R})$.

On the other hand, for an elementary processes of the form

$$\Phi(t,\xi)=1_{[0,t]}(s)1_{[-\pi,\zeta]}(\xi), \tag{7.23}$$

define

$$\Phi\cdot W=\int_0^{\infty}\int_{S^1}\Phi(s,\xi)W(ds,d\xi). \tag{7.24}$$

Clearly $\Phi\cdot W=W(t,\zeta)$. Extend the integral $\Phi\cdot W$ to general processes. Since

$$\Phi\cdot W=\int_0^{\infty}\Phi(s)\,d\tilde{W}_s$$

for elementary processes (7.23), conclude that the integrals are equal for general processes as well.

Example 7.3 Consider the following SPDE driven by a real-valued Brownian motion:

$$\begin{cases} d_t u(t,x) = \left(\alpha^2 \dfrac{\partial^2 u(t,x)}{\partial x^2} + \beta \dfrac{\partial u(t,x)}{\partial x} + \gamma u(t,x) + g(x)\right) dt \\ \qquad\qquad + \left(\sigma_1 \dfrac{\partial u(t,x)}{\partial x} + \sigma_2 u(t,x)\right) dW_t, \\ u(0,x) = \varphi(x) \in L^2\big((-\infty,\infty)\big) \cap L^1\big((-\infty,+\infty)\big), \end{cases} \tag{7.25}$$

where we use the symbol d_t to signify that the differential is with respect to t. Let $H = L^2((-\infty,\infty))$ and $V = W_0^{1,2}((-\infty,\infty))$ with the usual norms

$$\|v\|_H = \left(\int_{-\infty}^{+\infty} v^2\,dx\right)^{1/2}, \quad v \in H,$$

$$\|v\|_V = \left(\int_{-\infty}^{+\infty} \left(v^2 + \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2\right) dx\right)^{1/2}, \quad v \in V.$$

Define the operators $A : V \to V^*$ and $B : V \to \mathscr{L}(H)$ by

$$A(v) = \alpha^2 \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + \beta \frac{\mathrm{d}v}{\mathrm{d}x} + \gamma v + g, \quad v \in V,$$

$$B(v) = \sigma_1 \frac{\mathrm{d}v}{\mathrm{d}x} + \sigma_2 v, \quad v \in V.$$

Suppose that $g \in L^2((-\infty,\infty)) \cap L^1((-\infty,\infty))$. Then, using integration by parts, we obtain for $v \in V$,

$$\begin{aligned} &2\big\langle v, A(v)\big\rangle + \operatorname{tr}\big(Bv(Bv)^*\big) \\ &\quad = 2\left\langle v, \alpha^2 \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + \beta^2 \frac{\mathrm{d}v}{\partial x} + \gamma v + g\right\rangle + \left\|\sigma_1 \frac{\mathrm{d}v}{\mathrm{d}x} + \sigma_2 v\right\|_H^2 \\ &\quad = \big(-2\alpha^2 + \sigma_1^2\big)\|v\|_V^2 + \big(2\gamma + \sigma_2^2 + 2\alpha^2 - \sigma_1^2\big)\|v\|_H^2 + 2\langle v, g\rangle_H \\ &\quad \le \big(-2\alpha^2 + \sigma_1^2\big)\|v\|_V^2 + \big(2\gamma + \sigma_2^2 + 2\alpha^2 - \sigma_1^2 + \varepsilon\big)\|v\|_H^2 + \frac{1}{\varepsilon}\|g\|_H^2 \end{aligned}$$

for any $\varepsilon > 0$. Similarly, for $u, v \in V$,

$$\begin{aligned} &2\big\langle u - v, A(u) - A(v)\big\rangle + \operatorname{tr}\big(B(u-v)\big(B(u-v)\big)^*\big) \\ &\qquad \le \big(-2\alpha^2 + \sigma_1^2\big)\|u - v\|_V^2 + \big(2\gamma + \sigma_2^2 + 2\alpha^2 - \sigma_1^2\big)\|u - v\|_H^2. \end{aligned}$$

If $-2\alpha^2 + \sigma_1^2 < 0$, then the coercivity and weak monotonicity conditions, (6.39) and (6.40), hold, and we know from Theorem 4.7 that there exists a unique strong solution $u^\varphi(t)$ to (7.25) in $L^2(\Omega, C([0,T],H)) \cap \boldsymbol{M}^2([0,T],V)$. Taking the Fourier transform yields

$$\begin{aligned} d_t\hat{u}^\varphi(t,\lambda) &= \big(-\alpha^2\lambda^2\hat{u}^\varphi(t,\lambda) + i\lambda\beta\hat{u}^\varphi(t,\lambda) + \gamma\hat{u}^\varphi(t,\lambda) + \hat{g}(\lambda)\big)\,dt \\ &\quad + \big(i\sigma_1\lambda\hat{u}^\varphi(t,\lambda) + \sigma_2\hat{u}^\varphi(t,\lambda)\big)\,dW_t \\ &= \big(\big(-\alpha^2\lambda^2 + i\lambda\beta + \gamma\big)\hat{u}^\varphi(t,\lambda) + \hat{g}(\lambda)\big)\,dt \\ &\quad + (i\sigma_1\lambda + \sigma_2)\hat{u}^\varphi(t,\lambda)\,dW_t. \end{aligned}$$

For fixed λ,

$$\begin{aligned} a &= -\alpha^2\lambda^2 + i\lambda\beta + \gamma, \\ b &= \hat{g}(\lambda), \\ c &= i\sigma_1\lambda + \sigma_2. \end{aligned}$$

By simple calculation (see Exercise 7.3),

$$\begin{aligned} E\big|\hat{u}^\varphi(t,\lambda)\big|^2 &= E\big|\hat{\varphi}(\lambda)\big|^2 + 2\mathrm{Re}\bigg(\frac{b\overline{b} + \overline{b}\hat{\varphi}(\lambda)(a+\overline{a}+c\overline{c})}{(a+\overline{a}+c\overline{c})(\overline{a}+c\overline{c})}\mathrm{e}^{(a+\overline{a}+c\overline{c})}\bigg) \\ &\quad - 2\mathrm{Re}\bigg(\frac{\overline{b}(a\hat{\varphi}(\lambda)+b)}{a(\overline{a}+c\overline{c})}\mathrm{e}^{at}\bigg) + 2\mathrm{Re}\bigg(\frac{\beta\overline{b}}{a(a+\overline{a}+c\overline{c})}\bigg). \end{aligned} \tag{7.26}$$

By Plancherel's theorem,

$$E\big\|u^\varphi(t)\big\|_H^2 = \int_{-\infty}^{+\infty} E\big|\hat{u}^\varphi(t,\lambda)\big|^2\,d\lambda$$

and

$$\begin{aligned} E\big\|u^\varphi(t)\big\|_V^2 &= E\big\|u^\varphi(t)\big\|_H^2 + E\bigg\|\frac{\mathrm{d}}{\mathrm{d}x}u^\varphi(t,x)\bigg\|_H^2 \\ &= \int_{-\infty}^{+\infty}\big(1+\lambda^2\big)E\big|\hat{u}^\varphi(t,\lambda)\big|^2 d\lambda. \end{aligned}$$

For a suitable $T > 0$,

$$\begin{aligned} \Psi(\varphi) &= \int_0^T\int_0^t E\big\|u^\varphi(s)\big\|_V^2\,ds\,dt \\ &= \int_{-\infty}^{+\infty}\big(1+\lambda^2\big)\int_0^T\int_0^t E\big\|\hat{u}(s,\lambda)\big\|^2\,ds\,dt\,d\lambda. \end{aligned}$$

Thus it is difficult to compute a Lyapunov function explicitly. In view of Remark 7.1, it is enough to compute a Lyapunov function of the linear SPDE

$$d_t u(t,x) = \left(\alpha^2 \frac{\partial^2 u(t,x)}{\partial x^2} + \beta \frac{\partial u(t,x)}{\partial x} + \gamma u(t,x)\right) du + \left(\sigma_1 \frac{\partial u(t,x)}{\partial x} + \sigma_2 u(t,x)\right) dW_t.$$

Define the operators $A_0 : V \to V^*$ and $B_0 : V \to \mathscr{L}(H)$ by

$$A_0(v) = \alpha^2 \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + \beta \frac{\mathrm{d}v}{\mathrm{d}x} + \gamma v, \quad v \in V,$$
$$B_0(v) = B(v), \quad v \in V$$

(since B is already linear). Taking the Fourier transform and solving explicitly, we obtain that the solution is the geometric Brownian motion

$$\hat{u}_0^{\varphi}(t,\lambda) = \hat{\varphi}(\lambda) \mathrm{e}^{at - \frac{1}{2}c^2 t + cW_t},$$
$$E\left|\hat{u}_0^{\varphi}(t,\lambda)\right|^2 = \left|\hat{\varphi}(\lambda)\right|^2 \mathrm{e}^{(a+\overline{a}+c\overline{c})t}.$$

The function $t \to E\|u_0^{\varphi}(t)\|_V^2$ is continuous for all $\varphi \in V$,

$$\left\|A(v) - A_0(v)\right\|_{V^*} = \|g\|_{V^*} = o\left(\|v\|_V\right) \quad \text{as } \|v\|_V \to \infty,$$

and

$$\tau\left(B(v) Q B^*(v) - (B_0 v) Q (B_0 v)^*\right)) = 0.$$

Thus, if $\{u_0(t), t \geq 0\}$ is exponentially ultimately bounded in the m.s.s., then the Lyapunov function $\Psi_0(\varphi)$ of the linear system is the Lyapunov function of the nonlinear system, and

$$\begin{aligned}
\Psi_0(\varphi) &= \int_{-\infty}^{+\infty} \left(1+\lambda^2\right) \left(\int_0^T \int_0^t E\left|\hat{u}_0(s,\lambda)\right|^2 ds\, dt\right) d\lambda \\
&= \int_{-\infty}^{+\infty} \left\{ \left(1+\lambda^2\right) \left|\hat{\varphi}(\lambda)\right|^2 \left(\frac{\exp\{(-2\alpha^2 + \sigma_1^2)\lambda^2 + 2\gamma + \sigma_2^2)T\}}{((-2\alpha^2 + \sigma_1^2)\lambda^2 + 2\gamma + \sigma_2^2)^2} \right) \right. \\
&\quad \left. - \frac{T}{(-2\alpha^2 + \sigma_1^2)\lambda^2 + 2\gamma + \sigma_2^2} - \frac{1}{(-2\alpha^2 + \sigma_1^2)\lambda^2 + 2\gamma + \sigma_2^2} \right\} d\lambda.
\end{aligned}$$

Using Theorem 7.8, we can conclude that the solution of the nonlinear system is exponentially ultimately bounded in the m.s.s.

Exercise 7.3 Complete the computations in (7.26).

Example 7.4 Consider an equation of the form

$$\begin{cases} dX_t = AX(t)\,dt + F(X(t))\,dt + B(X(t))\,dW_t, \\ X(0) = x \in H, \end{cases}$$

where F and B satisfy the conditions of Proposition 7.1. This example is motivated by the work of Funaki. If $-A$ is coercive, a typical case being $A = \Delta$, we conclude that the solution of the deterministic linear equation is exponentially stable since the Laplacian has negative eigenvalues. Thus, the solution of the deterministic equation is exponentially bounded, and hence, by Proposition 7.1, the solution of the nonlinear equation above is exponentially ultimately bounded in the m.s.s.

Example 7.5 Let $\mathscr{O} \subseteq \mathbb{R}^n$ be a bounded open domain with smooth boundary. Assume that $H = L^2(\mathscr{O})$ and $V = W_0^{1,2}(\mathscr{O})$, the Sobolev space. Suppose that $\{W_q(t,x); t \geq 0, x \in 0\}$ is an H-valued Wiener process with associated covariance operator Q, given by a continuous symmetric nonnegative definite kernel $q(x,y) \in L^2(\mathscr{O} \times \mathscr{O})$, $q(x,x) \in L^2(\mathscr{O})$,

$$(Qf)(x) = \int_{\mathscr{O}} q(x,y) f(y)\,dy.$$

By Mercer's theorem [41], there exists an orthonormal basis $\{e_j\}_{j=1}^{\infty} \subset L^2(\mathscr{O})$ consisting of eigenfunctions of Q such that

$$q(x,y) = \sum_{j=1}^{\infty} \lambda_j e_j(x) e_j(y)$$

with $\mathrm{tr}(Q) = \int_{\mathscr{O}} q(x,x)\,dx = \sum_{j=1}^{\infty} \lambda_j < \infty$.

Let $-A$ be a linear strongly elliptic differential operator of second order on $\mathscr{O}$, and $B(u) : L^2(\mathscr{O}) \to L^2(\mathscr{O})$ with $B(u)f(\cdot) = u(\cdot)f(\cdot)$. By Garding's inequality, $-A$ is coercive (see [63], Theorem 7.2.2). Then the infinite-dimensional problem is as follows:

$$d_t u(t,x) = Au(t,x)\,dt + u(t,x)\,d_t W_q(t,x),$$

and we choose $\Lambda(v) = \|v\|_H^2$ for $v \in W_0^{1,2}(\mathscr{O})$. We shall check conditions under which Λ is a Lyapunov function. With $\mathscr{L}$ defined in (6.15), using the spectral representation of $q(x,y)$, we have

$$\begin{aligned} \mathscr{L}\big(\|v\|_H^2\big) &= 2\langle v, Av\rangle + \mathrm{tr}\big(B(v)QB^*(v)\big) \\ &= 2\langle v, Av\rangle + \int_{\mathscr{O}} q(x,x) v^2(x)\,dx. \end{aligned}$$

Let

$$\lambda_0 = \sup\left\{\frac{\mathscr{L}(\|v\|_H^2)}{\|v\|_H^2},\ v \in W_0^{1,2}(\mathscr{O}),\ \|v\|_H^2 \neq 0\right\}$$
$$= \sup\left\{\frac{2\langle v, Av\rangle + \langle Qv, v\rangle_H}{\|v\|_H^2},\ v \in W_0^{1,2}(\mathscr{O}),\ \|v\|_H^2 \neq 0\right\}.$$

If $\lambda_0 < 0$, then, by Theorem 6.4, the solution is exponentially stable in the m.s.s.

Consider the nonlinear equation in $\mathscr{O}$,

$$\begin{cases} d_t u(t,x) = \tilde{A}(x, u(t,x))\,dt + \tilde{B}(u(t,x))d_t W_q(t,x), \\ u(0,x) = \varphi(x), \quad u(t,x)|_{\partial\mathscr{O}} = 0. \end{cases} \tag{7.27}$$

Assume that

$$\tilde{A}(x,v) = Av + \alpha_1(x,v), \quad \tilde{B}(x,v) = B(v) + \alpha_2(x,v),$$

where $\alpha_i(x,v)$ satisfy the Lipschitz-type condition

$$\sup_{x\in\mathscr{O}} \left|\alpha_i(x,v_1) - \alpha_i(x,v_2)\right| < c\|v_1 - v_2\|_H,$$

so that the nonlinear equation (7.27) has a unique strong solution. Under the assumption

$$\alpha_i(x,0) = 0,$$

zero is a solution of (7.27), and if

$$\sup_{x\in\mathscr{O}} \left|\alpha_i(x,v)\right| = o\big(\|v\|_H\big), \quad \|v\|_H \to 0,$$

then, by Theorem 6.14, the strong solution of the nonlinear equation (7.27) is exponentially stable in the m.s.s.

On the other hand, let us consider the operator A as above and F and B satisfying the conditions of Proposition 7.1. Then, under the condition

$$\sup\left\{\frac{2\langle v, Av\rangle}{\|v\|_H^2},\ u \in W_0^{1,2}(\mathscr{O}),\ \|v\|_H^2 \neq 0\right\} < 0,$$

the solution of the abstract Cauchy problem (7.19), with A_0 replaced by A, is exponentially stable, and we conclude that the solution of the equation

$$\begin{cases} dX(t) = AX(t)\,dt + F(X(t))\,dt + B(X(t))\,dW_t, \\ X(0) = x \in H, \end{cases}$$

is ultimately exponentially bounded in the m.s.s.

Consider now the SSDE (3.1) and assume that A is the infinitesimal generator of a pseudo-contraction C_0-semigroup $\{S(t),\ t \ge 0\}$ on H (see Chap. 3) with the coefficients $F : H \to H$ and $B : H \to \mathscr{L}(K, H)$, independent of t and ω. We assume that F and B are in general nonlinear mappings satisfying the linear growth condition (A3) and the Lipschitz condition (A4) (see Sect. 3.3). In addition, the initial condition is assumed deterministic, so that (3.1) takes the form

$$\begin{cases} dX(t) = (AX(t) + F(X(t)))\,dt + B(X(t))\,dW_t, \\ X(0) = x \in H. \end{cases} \tag{7.28}$$

By Theorem 3.5, there exists a unique continuous mild solution.

Using Corollary 7.1, we now have the following analogue of Proposition 7.1.

Proposition 7.2 *Suppose that the classical solution $\{u^x(t),\ t \ge 0\}$ of the abstract Cauchy problem (7.19) is exponentially stable (or even exponentially ultimately bounded) and, as $\|h\|_H \to \infty$,*

$$\big\|F(h)\big\|_H = o\big(\|h\|_H\big),$$
$$\big\|B(h)\big\|_{\mathscr{L}(K,H)} = o\big(\|h\|_H\big),$$

then the mild solution of (7.28) is exponentially ultimately bounded in the m.s.s.

7.4 Ultimate Boundedness and Invariant Measure

We are interested in the behavior of the law of a solution to an SDE as $t \to \infty$. Let us begin with a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\ge 0}, P)$ and an H-valued time-homogeneous Markov process $X^{\xi_0}(t)$, $X^{\xi_0}(0) = \xi_0$, where ξ_0 is $\mathscr{F}_0$-measurable random variable with distribution μ^{ξ_0}. Assume that its associated semigroup P_t is Feller. We can define for $A \in \mathscr{B}(H)$, the *Markov transition probabilities*

$$P(t, x, A) = P_t 1_A(x), \quad x \in H.$$

Since a regular conditional distribution of $X^{\xi_0}(t)$ exists (Theorem 3, Vol. I, Sect. 1.3 in [25]), we have that

$$P(t, x, A) = P\big(X^{\xi_0}(t) \in A | \xi_0 = x\big) = \int_H P\big(X^{\xi_0} \in A | \xi_0 = x\big)\,\mu^{\xi_0}(dx), \quad x \in H.$$

Then, for a bounded measurable function f on H $(f \in B_b(H))$,

$$(P_t f)(x) = \int_H f(y) P(t, x, dy). \tag{7.29}$$

The Markov property (3.52) takes the form

$$E\big(f\big(X^{\xi_0}(t+s)\big)\big|\mathscr{F}^{X_t^{\xi_0}}\big) = (P_s f)\big(X_t^{\xi_0}\big) = \int_H f(y)P\big(s, X^{\xi_0}(t), dy\big),$$

so that the transition probability $P(t,x,A)$ is a *transition function for a time-homogeneous Markov process* $X^{\xi_0}(t)$.

We observe that the following *Chapman–Kolmogorov* equation holds for Markov transition probabilities

$$P(t+s,x,A) = \int_H P(t,y,A)P(s,x,dy), \tag{7.30}$$

which follows from the semigroup property of P_t, (3.58) applied to $\varphi(x) = 1_A(x)$ and from the fact that $P(t,x,dy)$ is the conditional law of $X^{\xi_0}(t)$.

Exercise 7.4 Show (7.30).

Let us now define an invariant probability measure and state a general theorem on its existence.

Definition 7.4 We say that a probability measure μ on H is invariant for a time-homogeneous Markov process $X^x(t)$ with the related Feller semigroup $\{P_t,\ t \geq 0\}$ defined by (7.29) if for all $A \in \mathscr{B}(H)$,

$$\mu(A) = \int_H P(t,x,A)\mu(dx),$$

or equivalently, since H is a Polish space, if for all $f \in C_b(H)$,

$$\int_H (P_t f)\,d\mu = \int_H f(y)\,d\mu.$$

Let μ be a probability measure on H and define

$$\mu_n(A) = \frac{1}{t_n}\int_0^{t_n}\int_H P(t,x,A)\,dt\,\mu(dx) \tag{7.31}$$

for a sequence $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}_+$, $t_n \to \infty$. In particular, for a real-valued bounded Borel-measurable function $f(x)$ on H, we have

$$\int_H f(x)\,\mu_n(dx) = \frac{1}{t_n}\int_0^{t_n}\int_H\int_H f(y)P(t,x,dy)\,\mu(dx)\,dt. \tag{7.32}$$

Theorem 7.10 *If ν is weak limit of a subsequence of $\{\mu_n\}$, then ν is an invariant measure.*

Proof We can assume without loss of generality that $\mu_n \Rightarrow \nu$. Observe that, by the Fubini theorem and the Chapman–Kolmogorov equation,

$$\begin{aligned}
\int_H (P_t f)(x)\nu(dx) &= \lim_{n\to\infty}\int_H (P_t f)(x)\,\mu_n(dx)\\
&= \lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\int_H\int_H (P_t f)(y)P(s,x,dy)\,\mu(dx)\,ds\\
&= \lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\int_H (P_{t+s} f)(x)\,\mu(dx)\,ds\\
&= \lim_{n\to\infty}\left[\frac{1}{t_n}\left\{\int_0^{t_n}\int_H (P_s f)(x)\,\mu(dx)\,ds\right.\right.\\
&\quad \left.\left. + \int_{t_n}^{t_n+t}\int_H (P_s f)(x)\,\mu(dx)\,ds - \int_0^t\int_H (P_s f)(x)\,\mu(dx)\,ds\right\}\right].
\end{aligned}$$

Since $\|P_s f(x_0)\|_H \le \|f(x_0\|_H$, the last two integrals are bounded by a constant, and hence, using (7.32),

$$\begin{aligned}
\int_H (P_t f)(x)\,\nu(dx) &= \lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\int_H (P_s f)(x_0)\,\mu(dx)\,ds\\
&= \lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\int_H\int_H f(y)P(s,x,dy)\,\mu(dx)\,ds\\
&= \lim_{n\to\infty}\int_H f(x)\,\mu_n(dx) = \int_H f(x)\,\nu(dx).
\end{aligned}$$

□

Corollary 7.4 *If the sequence* $\{\mu_n\}$ *is relatively compact, then an invariant measure exists.*

Exercise 7.5 Show that if, as $t \to \infty$, the laws of $X^x(t)$ converge weakly to a probability measure μ, then μ is an invariant measure for the corresponding semigroup P_t.

We shall now consider applications of the general results on invariant measures to SPDEs. In case where $\{X^{\xi_0}(t),\ t \ge 0\}$ is a solution of an SDE with a random initial condition ξ_0, taking in (7.31) $\mu = \mu^{\xi_0}$, the distribution of ξ_0, gives

$$P\big(X^{\xi_0}(t) \in A\big) = \int_H P(t,x,A)\,\mu^{\xi_0}(dx). \tag{7.33}$$

Thus, properties of the solution can be used to obtain tightness of the measures μ_n.

Exercise 7.6 Prove (7.33).

Before we apply the result on ultimate boundedness to obtain the existence of an invariant measure, let us consider some examples.

Example 7.6 (Navier–Stokes Equation [76]) Let $\mathscr{D} \subseteq \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\mathscr{D}$. Consider the equation

$$\begin{cases} \dfrac{\partial v_i(t,x)}{\partial t} + \displaystyle\sum_{j=1}^{2} v_j \dfrac{\partial v_i(t,x)}{\partial x_j} = -\dfrac{1}{\rho}\dfrac{\partial P(t,x)}{\partial x_i} + \nu \displaystyle\sum_{j=1}^{2} \dfrac{\partial^2 v_i(t,x)}{\partial x_j^2} + \sigma_i \dot{W}_t^i(x), \\ \displaystyle\sum_{j=1}^{2} \dfrac{\partial v_j}{\partial x_j} = 0, \quad x \in \mathscr{D},\ \nu > 0, \\ \quad i = 1,2. \end{cases} \tag{7.34}$$

Let $C_0^\infty = \{v \in C_0^\infty(\mathscr{D}) \times C_0^\infty(\mathscr{D});\ \nabla v = 0\}$, with ∇ denoting the gradient. Let $H = \overline{C_0^\infty}$ in $L^2(\mathscr{D}) \times L^2(\mathscr{D})$, and $V = \{v : W_0^{1,2}(\mathscr{D}) \times W_0^{1,2}(\mathscr{D}), \nabla v = 0\}$. Then $V \subseteq H \subseteq V^*$ is a Gelfand triplet, and the embedding $V \hookrightarrow H$ is compact.

It is known [76] that

$$L^2(\mathscr{D}) \times L^2(\mathscr{D}) = H \oplus H^\perp,$$

where $H^\perp$ is characterized by $H^\perp = \{v : v = \nabla(p) \text{ for some } p \in W^{1,2}(\mathscr{D})\}$.

Denote by Π the orthogonal projection of $L^2(\mathscr{D}) \times L^2(\mathscr{D})$ onto $H^\perp$, and for $v \in C_0^\infty$, define

$$A(v) = \nu \Pi \Delta v - \Pi\big[(v \cdot \nabla) v\big].$$

Then A can be extended as a continuous operator form V to V^*.

Equation (7.34) can be recast as an evolution equation in the form

$$\begin{cases} dX(t) = A(X(t))\,dt + \sigma\,dW_t, \\ X(0) = \xi, \end{cases}$$

where W_t is an H-valued Q-Wiener process, and $\xi \in V$ a.e. is an $\mathscr{F}_0$-measurable H-valued random variable. It is known (see [76]) that the above equation has a unique strong solution $\{u^\xi(t),\ t \ge 0\}$ in $C([0,T],H) \cap L^2([0,T],V)$, which is a homogeneous Markov and Feller process, satisfying for $T < \infty$,

$$E\big\|u^\xi(T)\big\|_H^2 + \nu \int_0^T \sum_{i=1}^{2} \left\| \frac{\partial u^\xi(t)}{\partial x_i} \right\|_H^2 dt \le E\|\xi\|_H^2 + \frac{T}{2}\operatorname{tr}(Q).$$

Using the fact that $\|u^\xi(t)\|_V$ is equivalent to $(\sum_{i=1}^2 \|\frac{\partial u(\xi)}{\partial x_i}\|_H^2)^{1/2}$, we have

$$\sup_T \frac{1}{T}\int_0^T E\big(\big\|u^\xi(t)\big\|_V^2\big)\,dt \le \frac{c}{2\nu}\operatorname{tr}(Q)$$

with some constant c. By the Chebychev inequality,

$$\lim_{R\to\infty} \sup_T \frac{1}{T}\int_0^T P\big(\big\|u^\xi(t)\big\|_V > R\big)\,dt = 0.$$

Hence, for $\varepsilon > 0$, there exists an R_ε such that

$$\sup_T \frac{1}{T}\int_0^T P\big(\|u^\xi(t)\|_V > R_\varepsilon\big)\,dt < \varepsilon.$$

Thus, as $t_n \to \infty$,

$$\sup_n \frac{1}{t_n}\int_H\int_0^{t_n} P\big(t, x, \widetilde{B}_{R_\varepsilon}\big)\,dt\,\mu^\xi(dx) < \varepsilon,$$

where $\widetilde{B}_{R_\varepsilon}$ is the image of the set $\{v \in V;\ \|v\|_V > R_\varepsilon\}$ under the compact embedding $V \hookrightarrow H$, and μ^ξ is the distribution of ξ on H. Since $\widetilde{B}_{R_\varepsilon}$ is a complement of a compact set, we can use Prokhorov's theorem and Corollary 7.4 to conclude that an invariant measure exists. Note that its support is in V, by the weak convergence.

Example 7.7 (Linear equations with additive noise [79]) Consider the mild solution of the equation

$$\begin{cases} dX(t) = AX(t)\,dt + dW_t, \\ X(0) = x \in H, \end{cases}$$

where A is an infinitesimal generator of a strongly continuous semigroup $\{S(t), t \geq 0\}$ on H. Denote

$$Q_t = \int_0^t S(r)QS^*(r)\,dr,$$

and assume that $\mathrm{tr}(Q_t) < \infty$. We know from Theorems 3.1 and 3.2 that

$$X(t) = S(t)x + \int_0^t S(t-s)\,dW_s \tag{7.35}$$

is the mild solution of the above equation. The stochastic convolution $\int_0^t S(t-s)\,dW_s$ is an H-valued Gaussian process with covariance

$$Q_t = \int_0^t S(u)QS^*(u)\,du$$

for any t. The Gaussian process $X(t)$ is also Markov and Feller, and it is called an *Ornstein–Uhlenbeck process*. The probability measure μ on H is invariant if for $f \in C_b(H)$ and any $t \geq 0$,

$$\begin{aligned}\int_H f(x)\,\mu(dx) &= \int_H E\big(f\big(X^x(t)\big)\big)\,\mu(dx)\\ &= \int_H Ef\left(S(t)x + \int_0^t S(t-s)\,dW_s\right)\mu(dx).\end{aligned}$$

For $f(x) = \mathrm{e}^{i\langle\lambda,x\rangle_H}$, $\lambda \in H$, we obtain

$$\hat{\mu}(\lambda) = \hat{\mu}\big(S^*(t)\lambda\big)\mathrm{e}^{-\frac{1}{2}\langle Q_t\lambda,\lambda\rangle_H},$$

where $\hat{\mu}$ denotes the characteristic function of μ. It follows that

$$\big|\hat{\mu}(\lambda)\big| \le \mathrm{e}^{-\frac{1}{2}\langle Q_t,\lambda,\lambda\rangle_H},$$

or

$$\langle Q_t\lambda,\lambda\rangle_H \le -2\ln\big|\hat{\mu}(\lambda)\big| = 2\ln\bigg(\frac{1}{|\hat{\mu}(\lambda)|}\bigg).$$

Since $\hat{\mu}(\lambda)$ is the characteristic function of a measure μ on H, then by Sazonov's theorem [74], for $\varepsilon > 0$, there exists a trace-class operator S_0 on H such that $|\hat{\mu}(\lambda)| \ge 1/2$ whenever $\langle S_0\lambda,\lambda\rangle_H \le 1$. Thus, we conclude that

$$\langle Q_t\lambda,\lambda\rangle_H \le 2\ln 2$$

if $\langle S_0\lambda,\lambda\rangle_H \le 1$. This yields

$$0 \le Q_t \le (2\ln 2)S_0.$$

Hence, $\sup_t \mathrm{tr}(Q_t) < \infty$.

On the other hand, if $\sup_t \mathrm{tr}(Q_t) < \infty$, let us denote by $\overline{P}$ the limit in trace norm of Q_t and observe that

$$S(t)\overline{P}S^*(t) = \int_0^\infty S(t+r)QS^*(t+r)\,dr = \int_t^\infty S(u)QS(u)\,du = \overline{P} - Q_t.$$

Thus,

$$\frac{1}{2}\big\langle S(t)\overline{P}S^*(t)\lambda,\lambda\big\rangle_H = \frac{1}{2}\big\langle \overline{P}\lambda,\lambda\big\rangle_H - \frac{1}{2}\langle Q_t\lambda,\lambda\rangle_H,$$

implying

$$\mathrm{e}^{-\frac{1}{2}\langle\overline{P}\lambda,\lambda\rangle_H} = \mathrm{e}^{-\frac{1}{2}\langle\overline{P}S^*(t)\lambda,S^*(t)\lambda\rangle_H}\mathrm{e}^{-\frac{1}{2}\langle Q_t\lambda,\lambda\rangle_H}.$$

In conclusion, μ with the characteristic functional $\mathrm{e}^{-\frac{1}{2}\langle\overline{P}\lambda,\lambda\rangle_H}$ is an invariant measure. We observe that the invariant measure exists for the Markov process $X(t)$ defined in (7.35) if and only if $\sup_t \mathrm{tr}(Q_t) < \infty$. Also, if $S(t)$ is an exponentially stable semigroup (i.e., $\|S(t)\|_{\mathscr{L}(H)} \le M\mathrm{e}^{-\mu t}$ for some positive constants M and μ) or if $S_t x \to 0$ for all $x \in H$ as $t \to \infty$, then the Gaussian measure with covariance $\overline{P}$ is the invariant (Maxwell) probability measure.

Let $\{X(t), t \ge 0\}$ be exponentially ultimately bounded in the m.s.s., then, clearly,

$$\limsup_{t\to\infty} E\big\|X(t)\big\|_H^2 \le M < \infty \quad \text{for all } x \in H. \tag{7.36}$$

Definition 7.5 A stochastic process $X(t)$ satisfying condition (7.36) is called ultimately bounded in the m.s.s.

7.4.1 Variational Equations

We focus our attention now on the variational equation with a deterministic initial condition,

$$\begin{cases} dX(t) = A(X(t))\,dt + B(X(t))\,dW_t, \\ X(0) = x \in H, \end{cases} \tag{7.37}$$

which is driven by a Q-Wiener process W_t. The coefficients $A : V \to V^*$ and $B : V \to \mathscr{L}(K, H)$ are independent of t and ω, and they satisfy the linear growth, coercivity (C), and weak monotonicity (WM) conditions (6.38), (6.39), (6.40). By Theorem 4.8 and Remark 4.2 the solution is a homogeneous Markov process, and the associated semigroup is Feller.

We note that in Theorem 7.5, we give conditions for exponential ultimate boundedness in the m.s.s. in terms of the Lyapunov function. Assume that $\Psi : H \to \mathbb{R}$ satisfies the conditions of Theorem 6.10 (Itô's formula) and define

$$\mathscr{L}\psi(u) = \big\langle \psi'(u), A(u)\big\rangle + (1/2)\,\mathrm{tr}\big(\psi''(u)B(u)QB^*(u)\big). \tag{7.38}$$

Let $\{X^x(t), t \ge 0\}$ be the solution of (7.37). We apply Itô's formula to $\Psi(X^x(t))$, take the expectation, and use condition (2) of Theorem 7.5 to obtain

$$\begin{aligned} E\Psi\big(X^x(t)\big) - E\Psi\big(X^x(t')\big) &= E\int_{t'}^{t} \mathscr{L}\Psi\big(X^x(s)\big)\,ds \\ &\le \int_{t'}^{t} \big(-c_3 E\Psi\big(X^t(s)\big) + k_3\big)\,ds. \end{aligned}$$

Let $\Phi(t) = E\Psi(X^x(t))$, then $\Phi(t)$ is continuous, so that

$$\Phi'(t) \le -c_3\Phi(t) + k_3.$$

Hence,

$$E\Psi\big(X_t^x\big) \le \frac{k_3}{c_3} + \left(\Psi(x) - \frac{k_3}{c_3}\right)\mathrm{e}^{-c_3 t}.$$

Assuming that $\Psi(x) \ge c_1\|x\|_H^2 - k_1$, we obtain

$$c_1 E\big\|X^x(t)\big\|_H^2 - k_1 \le \frac{k_3}{c_3} + \left(c_2\|x\|_H^2 - \frac{k_3}{c_3}\right)\mathrm{e}^{-c_3 t}.$$

Thus we have proved the following:

Proposition 7.3 *Let $\Psi : H \to \mathbb{R}$ satisfy conditions* (1)–(5) *of Theorem* 6.10 *and assume that condition* (2) *of Theorem* 7.5 *holds and that $c_1\|x\|_H^2 - k_1 \le \Psi(x)$ for $x \in H$ and some constants $c_1 > 0$ and $k_1 \in \mathbb{R}$. Then*

$$\limsup_{t\to\infty} E\big\|X^x(t)\big\|_H^2 \le \frac{1}{c_1}\left(k_1 + \frac{k_3}{c_3}\right).$$

In particular, $\{X^x(t), t \ge 0\}$ is ultimately bounded.

Let us now state the theorem connecting the ultimate boundedness with the existence of invariant measure.

Theorem 7.11 *Let $\{X^x(t), t \geq 0\}$ be a solution of (7.37). Assume that the embedding $V \hookrightarrow H$ is compact. If $X^x(t)$ is ultimately bounded in the m.s.s., then there exists an invariant measure μ for $\{X^x(t), t \geq 0\}$.*

Proof Applying Itô's formula to the function $\|x\|_H^2$ and using the coercivity condition, we have

$$E\big\|X^x(t)\big\|_H^2 - \|x\|_H^2 = \int_0^t E\mathscr{L}\big\|X^x(t)\big\|_H^2\,ds \leq \lambda\int_0^t E\big\|X^x(s)\big\|_H^2 ds - \alpha\int_0^t E\big\|X^x(s)\big\|_V^2 + \gamma t$$

with $\mathscr{L}$ defined in (7.38). Hence,

$$\int_0^t E\big\|X^x(s)\big\|_V^2\,ds \leq \frac{1}{\alpha}\bigg(\lambda\int_0^t E\big\|X^x(s)\big\|_H^2\,ds + \|x\|_H^2 + \gamma t\bigg).$$

Therefore,

$$\frac{1}{T}\int_0^T P\big(\big\|X^x(t)\big\|_V > R\big)\,dt \leq \frac{1}{T}\int_0^T \frac{E\|X^x(t)\|_V^2}{R^2}\,dt \leq \frac{1}{\alpha R^2}\frac{1}{T}\bigg(|\lambda|\int_0^T E\big\|X^x(t)\big\|_H^2\,dt + \|x\|_H^2 + \gamma T\bigg).$$

Now, by (7.36), $E\|X^x(t)\|_H^2 \leq M$ for $t \geq T_0$ and some $T_0 \geq 0$. But

$$\sup_{t\leq T_0} E\big\|X^x(t)\big\|_H^2 \leq M'$$

by Theorem 4.7, so that

$$\begin{aligned}
&\lim_{R\to\infty}\sup_T \frac{1}{T}\int_0^T P\big(\big\|X^x(t)\big\|_V > R\big)\,dt \\
&\leq \lim_{R\to\infty}\sup_T \frac{|\lambda|}{\alpha R^2}\frac{1}{T}\bigg(\int_0^{T_0} E\big\|X^x(t)\big\|^2\,dt + \int_{T_0}^T E\big\|X^x(t)\big\|_H^2\,dt\bigg) \\
&\leq \lim_{R\to\infty}\sup_T \frac{|\lambda|}{\alpha R^2}\bigg(\frac{T_0}{T}M' + \frac{T-T_0}{T}M\bigg) \\
&\leq \lim_{R\to\infty}\frac{|\lambda|}{\alpha R^2}\big(M' + M\big), \quad 0 \leq T_0 \leq T.
\end{aligned}$$

Hence, given $\varepsilon > 0$, there exists an R_ε such that

$$\sup_T \frac{1}{T}\int_0^T P\big(\big\|X^x(t)\big\|_V > R_\varepsilon\big)\,dt < \varepsilon.$$

By the assumption that the embedding $V \hookrightarrow H$ is compact, the set $\{v \in V : \|v\|_V \le R_\varepsilon\}$ is compact in H, and the result is proven. □

Remark 7.2 Note that a weaker condition on the second moment of $X^x(t)$, i.e.,

$$\sup_{T>T_0} \frac{1}{T}\int_0^T E\|X^x(t)\|_H^2\,dt < M \quad \text{for some } T_0 \ge 0,$$

is sufficient to carry out the proof of Theorem 7.11.

In Examples 7.2–7.6, we consider equations whose coefficients satisfy the conditions imposed on the coefficients of (7.37) and the embedding $V \hookrightarrow H$ is compact, so that an invariant measure exists if the solution is ultimately bounded in the m.s.s.

Theorem 7.12 *Suppose that $V \hookrightarrow H$ is compact and the solution of $\{X^x(t), t \ge 0\}$ of (7.37) is ultimately bounded in the m.s.s. Then any invariant measure μ satisfies*

$$\int_V \|x\|_V^2\,\mu(dx) < \infty.$$

Proof Let $f(x) = \|x\|_V^2$ and $f_n(x) = 1_{[0,n]}(f(x))$. Now $f_n(x) \in L^1(V,\mu)$. We use the ergodic theorem for a Markov process with an invariant measure (see [78], p. 388). This gives

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T (P_t f_n)(x)\,dt = f_n^*(x) \quad \mu\text{-a.e.}$$

and $E_\mu f_n^* = E_\mu f_n$, where $E_\mu f_n = \int_V f_n(x)\,\mu(dx)$.

By the assumption of ultimate boundedness, we have, as in the proof of Theorem 7.11,

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T E\|X^x(t)\|_V^2\,dt \le \frac{C|\lambda|}{\alpha}, \quad C < \infty.$$

Hence,

$$\begin{aligned} f_n^*(x) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T (P_t f_n)(x)\,dt \\ &\le \limsup_{T\to\infty} \frac{1}{T}\int_0^T \big(P_t f(x)\big)\,dt \\ &= \limsup_{T\to\infty} \frac{1}{T}\int_0^T E\|X^x(t)\|_V^2 dt \le \frac{C|\lambda|}{\alpha}. \end{aligned}$$

But $f_n(x) \uparrow f(x)$, so that

$$E_\mu f = \lim_{n\to\infty} E_\mu f_n = \lim_{n\to\infty} E_\mu f_n^* \le \frac{C|\lambda|}{\alpha}.$$ □

Remark 7.3 (a) For parabolic Itô equations, one can easily derive the result using $\Psi(x)=\|x\|_H^2$ and Theorem 7.11.

(b) Note that if $\mu_n \Rightarrow \mu$ and the support of μ_n is in V with the embedding $V \hookrightarrow H$ being compact, then by the weak convergence the support of μ is in V by the same argument as in Example 7.6.

Let us now consider the problem of uniqueness of the invariant measure.

Theorem 7.13 *Suppose that for* ε, δ, *and* $R>0$, *there exists a constant* $T_0(\varepsilon,\delta,R) > 0$ *such that for* $T \ge T_0$,

$$\frac{1}{T}\int_0^T P\big(\big\|X^x(t)-X^y(t)\big\|_V \ge \delta\big)\,dt < \varepsilon$$

for all $x, y \in V_R = \{v \in V : \|v\|_V \le R\}$ *with the embedding* $V \hookrightarrow H$ *being compact. If there exists an invariant measure* μ *for a solution of* (7.37), $\{X^{x_0}(t), t \ge 0\}$, $X(0)=x_0$, *with support in* V, *then it is unique.*

Proof Suppose that μ, ν are invariant measures with support in V. We need to show that

$$\int_H f(x)\mu(dx) = \int_H f(x)\nu(dx)$$

for f uniformly continuous bounded on H, since such functions form a determining class.

For $G \in \mathscr{B}(H)$, define

$$\mu_T^x(G) = \frac{1}{T}\int_0^T P\big(X^x(t)\in G\big)\,dt, \quad x \in H,\ T>0.$$

Then, using invariance of μ and ν, we have

$$\begin{aligned}
&\left|\int_H f(x)\,\mu(dx) - \int_H f(x)\,\nu(dx)\right| \\
&= \left|\int_H\int_H f(x)\big[\mu_T^y(dx)\mu(dy) - \mu_T^z(dx)\nu(dz)\big]\right| \\
&\le \int_{H\times H}\left|\int_H f(x)\,\mu_T^y(dx) - \int_H f(x)\mu_T^z(dx)\right|\mu(dy)\nu(dz).
\end{aligned}$$

Let

$$F(y,z) = \left|\int_H f(x)\mu_T^y(dx) - \int_H f(x)\mu_T^z(dx)\right|.$$

Then, using the fact that μ, ν have the supports in V, we have

$$\left|\int_H f(x)\,\mu(dx) - \int_H f(x)\,\nu(dx)\right| \le \int_{V\times V}\big|F(y,z)\big|\,\mu(dy)\nu(dz).$$

Let $V_R^c = V \setminus V_R$ and choose $R > 0$ such that

$$\mu\big(V_R^c\big) + \nu\big(V_R^c\big) < \varepsilon.$$

Then,

$$\left|\int_H f(x)\,\mu(dx) - \int_H f(x)\,\nu(dx)\right| \le \int_{V_R\times V_R} \big|F(y,z)\big|\,\mu(dy)\nu(dz) + \big(4\varepsilon + 2\varepsilon^2\big)M,$$

where $M = \sup_{x\in H}|f(x)|$. But for $\delta > 0$,

$$\begin{aligned}
&\int_{V_R\times V_R} \big|F(y,z)\big|\mu(dy)\nu(dz) \\
&\le \int_{V_R\times V_R} \left\{\frac{1}{T}\int_0^T E\big|f\big(X^y(t)\big) - f\big(X^z(t)\big)\big|\,\mu(dy)\nu(dz)\right\} \\
&\le 2M \sup_{y,z\in V_R} \frac{1}{T}\int_0^T P\big(\big\|X^y(t) - X^z(t)\big\|_V > \delta\big) + \sup_{\substack{y,z\in V_R\\ \|y-z\|<\delta}} \big|f(y) - f(z)\big| \\
&\le 2M\varepsilon + \varepsilon
\end{aligned}$$

for $T \ge T_0$, since f is uniformly continuous.

Using the last inequality and the bound for $|\int_H f(x)\,\mu(dx) - \int_H f(x)\,\nu(dx)|$, we obtain the result. □

Let us now give a condition on the coefficients of the SDE (7.37) which guarantees the uniqueness of the invariant measure. We have proved in Theorem 7.11 (see Remark 7.3), that the condition

$$\sup_{T>T_0}\left\{\frac{1}{T}\int_0^T E\big\|X^x(t)\big\|_H^2\,dt\right\} \le M \quad \text{for some } T_0 \ge 0$$

implies that there exists an invariant measure to the strong solution $\{X^x(t), t \ge 0\}$, whose support is in V.

Theorem 7.14 *Suppose that* $V \hookrightarrow H$ *is compact, the coefficients of* (7.37) *satisfy the coercivity condition* (6.39), *and that for* $u, v \in V$,

$$2\big\langle u - v, A(u) - A(v)\big\rangle + \big\|B(u) - B(v)\big\|^2_{\mathscr{L}_2(K_Q,H)} \le -c\|u - v\|_V^2,$$

where the norm $\|\cdot\|_{\mathscr{L}_2(K_Q,H)}$ *is the Hilbert–Schmidt norm defined in* (2.7). *Assume that the solution* $\{X^x(t), t \ge 0\}$ *of* (7.37) *is ultimately bounded in the m.s.s. Then there exists a unique invariant measure.*

Proof By Itô's formula, we have, for $t > 0$,

$$E\big\|X^x(t)\big\|_H^2 = \|x\|_H^2 + 2E\int_0^t \big\langle X^x(s), A\big(X^x(s)\big)\big\rangle\,ds + E\int_0^t \big\|B\big(X^x(s)\big)\big\|^2_{\mathscr{L}_2(K_Q,H)}\,ds.$$

Using the coercivity condition (C), (6.39), we have

$$E\|X^x(t)\|_H^2 + \alpha E\int_0^t \|X^x(s)\|_V^2\,ds \le \left(\|x\|_H^2 + \gamma t\right) + \lambda E\int_0^t \|X^x(s)\|_H^2\,ds.$$

It follows, similarly as in the proof of Theorem 7.11, that

$$\sup_{T>T_0}\frac{1}{T}\int_0^T E\|X^x(s)\|_V^2\,ds \le \frac{|\gamma| + \|x\|_H^2/T_0}{\alpha} + \frac{|\lambda|}{\alpha}\sup_{T>T_0}\int_0^T E\|X^x(s)\|_H^2\,ds.$$

By the Chebychev inequality, we know that

$$\frac{1}{T}\int_0^T P\left(\|X^x(s)\|_V > R\right) \le \frac{1}{R^2}\left\{\frac{1}{T}\int_0^T E\|X^x(s)\|_V^2\,ds\right\}.$$

Hence, using the arguments in Example 7.6, an invariant measure exists and is supported on V. To prove the uniqueness, let $X^{x_1}(t), X^{x_2}(t)$ be two solutions with initial values x_1, x_2. We apply Itô's formula to $X(t) = X^{x_1}(t) - X^{x_2}(t)$ and obtain

$$\begin{aligned} E\|X(t)\|_H^2 \le \|x_1 - x_2\|_H^2 + 2E\int_0^t \left\langle X(s) - A\left(X^{x_1}(s)\right) - A\left(X^{x_2}(s)\right)\right\rangle ds \\ + E\int_0^t \left\|B\left(X^{x_1}(s)\right) - B\left(X^{x_2}(s)\right)\right\|_{\mathscr{L}_2(K_Q,H)}^2 ds. \end{aligned}$$

Using the assumption, we have

$$E\|X(t)\|_H^2 \le \|x_1 - x_2\|_H^2 - c\int_0^t E\|X(s)\|_V^2\,ds,$$

which implies that

$$\int_0^t E\|X(s)\|_V^2 \le \frac{1}{c}\|x_1 - x_2\|_H^2.$$

It now suffices to refer to the Chebychev inequality and Theorem 7.13 to complete the proof. □

7.4.2 Semilinear Equations Driven by a Q-Wiener Process

Let us consider now the existence of an invariant measure for a mild solution of a semilinear SDE with deterministic initial condition

$$\begin{cases} dX(t) = (AX(t) + F(X(t)))\,dt + B(X(t))\,dW_t, \\ X(0) = x \in H, \end{cases} \tag{7.39}$$

where A is the infinitesimal generator of a pseudo-contraction C_0-semigroup $S(t)$ on H, and the coefficients $F : H \to H$ and $B : H \to \mathscr{L}(K, H)$, independent of t and ω, are in general nonlinear mappings satisfying the linear growth condition (A3) and the Lipschitz condition (A4) in Sect. 3.3. We know from Theorem 3.6 that the solution is a homogeneous Markov process and from Theorem 3.7 that it is continuous with respect to the initial condition, so that the associated semigroup is Feller.

We studied a special case in Example 7.7. Here we look at the existence under the assumption of exponential boundedness in the m.s.s. We will use the Lyapunov function approach developed earlier in Theorem 7.8 and Corollary 7.3. We first give the following proposition.

Proposition 7.4 *Suppose that the mild solution $\{X^x(t)\}$ of (7.39) is ultimately bounded in the m.s.s. Then any invariant measure ν of the Markov process $\{X^x(t), t \ge 0\}$ satisfies*

$$\int_H \|y\|_H^2 \nu(dy) \le M,$$

where M is as in (7.36).

The proof is similar to the proof of Theorem 7.12 and is left to the reader as an exercise.

Exercise 7.7 Prove Proposition 7.4.

Theorem 7.15 *Suppose that the solution $\{X^x(t), t \ge 0\}$ of (7.39) is ultimately bounded in the m.s.s. If for all $R > 0$, $\delta > 0$, and $\varepsilon > 0$, there exists $T_0 = T_0(R, \delta, \varepsilon) > 0$ such that for all $t \ge T_0$,*

$$P\left(\left\|X^x(t) - X^y(t)\right\|_H > \delta\right) < \varepsilon \quad \text{for } x, y \in B_H(R) \tag{7.40}$$

with $B_H(R) = \{x \in H, \|x\| \le R\}$, then there exists at most one invariant measure for the Markov process $X^x(t)$.

Proof Let μ_i, $i = 1, 2$, be two invariant measures. Then, by Proposition 7.4, for each $\varepsilon > 0$, there exists $R > 0$ such that $\mu_i(H \setminus B_H(R)) < \varepsilon$. Let f be a bounded weakly continuous function on H. We claim that there exists a constant $T = T(\varepsilon, R, f) > 0$ such that

$$\left|P_t f(x) - P_t f(y)\right| \le \varepsilon \quad \text{for } x, y \in B_H(R) \quad \text{if } t \ge T.$$

Let C be a weakly compact set in H. The weak topology on C is given by the metric

$$d(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \left|\langle e_k, x - y\rangle_H\right|, \qquad x, y \in C, \tag{7.41}$$

where $\{e_k\}_{k=1}^{\infty}$ in an orthonormal basis in H.

By the ultimate boundedness, there exists $T_1 = T_1(\varepsilon, R) > 0$ such that for $T \geq T_1$,

$$P\big(X^x(t) \in B_H(R)\big) > 1 - \varepsilon/2 \quad \text{for } x \in B_H(R).$$

Now f is uniformly continuous w.r.t. the metric (7.41) on $B_H(R)$. Hence, there exists $\delta' > 0$ such that $x, y \in H_R$ with $d(x, y) < \delta'$ imply that $|f(x) - f(y)| \leq \delta$, and there exists $J > 0$ such that

$$\sum_{k=J+1}^{\infty} \frac{1}{2^k} \big|\langle e_k, x - y\rangle_H\big| \leq \delta'/2 \quad \text{for } x, y \in B_H(R).$$

Since $P(|\langle e_k, X^x(t) - X^y(t)\rangle| > \delta) \leq P(\|X^x(t) - X^y(t)\|_H > \delta)$, by the given assumption we can choose $T_2 \geq T_1$ such that for $t \geq T_2$,

$$P\left\{\sum_{k=1}^{J}\big(\langle e_k, X^x(t)\rangle - \langle e_k, X^y(t)\rangle\big)^2 > \delta'/2\right\} \geq 1 - \varepsilon/3 \tag{7.42}$$

for $x, y \in B_H(R)$. Hence, for $t \geq T_2$,

$$\begin{aligned}
&P\big\{\big|f\big(X^x(t)\big) - f\big(X^y(t)\big)\big| \leq \delta\big\} \\
&\quad \geq P\big\{X^x(t), X^y(t) \in B_H(R), d\big(X^x(t), X^y(t)\big) \leq \delta'\big\} \\
&\quad \geq P\left\{X^x(t), X^y(t) \in B_H(R), \sum_{k=1}^{J} \frac{1}{2^k}\big|\big\langle e_k, X^x(t) - X^y(t)\big\rangle_H\big| \leq \delta'/2\right\} \\
&\quad \geq P\big\{X^x(t), X^y(t) \in B_H(R), \big|\big\langle e_k, X^x(t) - X^y(t)\big\rangle_H\big| \leq \delta'/2, k = 1, \dots, J\big\} \\
&\quad \geq 1 - \varepsilon/3 - \varepsilon/3 - \varepsilon/3 = 1 - \varepsilon,
\end{aligned}$$

since the last probability above is no smaller than that in (7.42).

Now, with $M_0 = \sup|f(x)|$, given $\varepsilon > 0$, choose T so that for $t \geq T$,

$$P\big(\big|f\big(X^x(t)\big) - f\big(X^y(t)\big)\big| \leq \varepsilon/2\big) \geq 1 - \frac{\varepsilon}{4M_0}.$$

Then

$$E\big|f\big(X^x(t)\big) - f\big(X^y(t)\big)\big| \leq \frac{\varepsilon}{2} + 2M_0\frac{\varepsilon}{4M_0} = \varepsilon.$$

Note that for invariant measures μ_1, μ_2,

$$\int_H f(x)\mu_i(dx) = \int_H (P_t f)(x)\mu_i(dx), \quad i = 1, 2.$$

For $t \geq T$, we have

$$\left|\int_H f(x)\mu_1(dx) - \int_H f(y)\mu_2(dy)\right|$$

$$
\begin{aligned}
&= \left| \int_H \int_H \bigl[f(x) - f(y) \bigr] \mu_1(dx) \mu_2(dy) \right| \\
&= \left| \int_H \int_H \bigl[(P_t f)(x) - (P_t f)(y) \bigr] \mu_1(dx) \mu_2(y) \right| \\
&= \left| \left(\int_{B_H(R)} + \int_{H \setminus B_H(R)} \right) \left(\int_{B_H(R)} + \int_{H \setminus B_H(R)} \right) \right. \\
&\qquad \left. \times \bigl[(P_t f)(x) - (P_t f)(y) \bigr] \mu_1(dx) \mu_2(dy) \right| \\
&\le \varepsilon + 2(2M_0)\varepsilon + 2M_0 \varepsilon^2.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$
\int_H f(x)\, \mu_1(dx) = \int_H f(x)\, \mu_2(dx).
$$

□

In case we look at the solution to (7.39), whose coefficients satisfy the linear growth and Lipschitz conditions (A3) and (A4) of Sect. 3.1 in Chap. 3, we conclude that under assumption (7.40) and conditions for exponential ultimate boundedness, there exists at most one invariant measure.

Note that in the problem of existence of the invariant measure, the relative weak compactness of the sequence μ_n in Theorem 7.10 is crucial. In the variational case, we achieved this condition, under ultimate boundedness in the m.s.s., assuming that the embedding $V \hookrightarrow H$ is compact. For mild solutions, Ichikawa [33] and Da Prato and Zabczyk [11], give sufficient conditions. Da Prato and Zabczyk use a factorization technique introduced in [10]. We start with the result in [32].

Theorem 7.16 *Assume that A is a self-adjoint linear operator with eigenvectors $\{e_k\}_{k=1}^{\infty}$ forming an orthonormal basis in H and that the corresponding eigenvalues $-\lambda_k \downarrow -\infty$ as $k \to \infty$. Let the mild solution of* (7.39) *satisfy*

$$
\frac{1}{T} \int_0^T E \bigl\| X^x(s) \bigr\|_H^2 \, ds \le M \bigl(1 + \|x\|_H^2 \bigr). \tag{7.43}
$$

Then there exists an invariant measure for the Markov semigroup generated by the solution of (7.39).

Proof The proof depends on the following lemma.

Lemma 7.2 *Under the conditions of Theorem* 7.16, *the set of measures*

$$
\mu_t(\cdot) = \frac{1}{t} \int_0^t P(s, x, \cdot) \quad \textit{for } t \ge 0
$$

with $P(s, x, A) = P(X^x(s) \in A)$ is relatively weakly compact.

Proof Let $y_k(t) = \langle X^x(t), e_k\rangle_H$. Then, by a well-known result about the weak compactness ([25], Vol. I, Chap. VI, Sect. 2, Theorem 2), we need to show that the expression

$$\frac{1}{T}\int_0^T \left(\sum_{k=1}^{\infty} E y_k^2(t)\right) dt$$

is uniformly convergent in T.

Let $S(t)$ be the C_0-semigroup generated by A. Since $S(t)e_k = \mathrm{e}^{-\lambda_k t} e_k$ for each k, $y_k(t)$ satisfies

$$\begin{aligned}
y_k(t) &= \mathrm{e}^{-\lambda_k t} x_k^0 + \int_0^t \mathrm{e}^{-\lambda_k (t-s)} \big\langle e_k, F\big(X^x(s)\big)\big\rangle_H \, ds \\
&\quad + \int_0^t \mathrm{e}^{-\lambda_k (t-s)} \big\langle e_k, B\big(X^x(s)\big)\, dW(s)\big\rangle_H, \\
E y_k^2(t) &\le 3\mathrm{e}^{-2\lambda_k t}\big(x_k^0\big)^2 + 3E\left|\int_0^t \mathrm{e}^{-\lambda_k (t-s)} \big\langle e_k, F\big(X^x(s)\big)\big\rangle_H \, ds\right|^2 \\
&\quad + 3E\left|\int_0^t \mathrm{e}^{-\lambda_k (t-s)} \big\langle e_k, B\big(X^x(s)\big)\, dW_s\big\rangle\right|^2.
\end{aligned}$$

For N large enough, so that $\lambda_N > 0$, and any $m > 0$, using Exercise 7.8 and assumption (7.43), we have

$$\begin{aligned}
&\sum_{k=N}^{N+m} \frac{1}{T}\int_0^T E\left|\int_0^t \mathrm{e}^{-\lambda_k (t-s)} \big\langle e_k, F\big(X^x(s)\big)\big\rangle_H \, ds\right|^2 dt \\
&\quad \le \frac{1}{2\varepsilon T}\int_0^T \int_0^t \mathrm{e}^{2(-\lambda_k+\varepsilon)(t-s)} \left|\big\langle e_k, F\big(X^x(s)\big)\big\rangle_H\right|^2 ds\, dt \\
&\quad = \frac{1}{T}\int_0^T \int_r^T \mathrm{e}^{2(-\lambda_k+\varepsilon)(t-s)}\, dt \left|\big\langle e_k, F\big(X^x(s)\big)\big\rangle_H\right|^2 ds \\
&\quad \le \frac{\int_0^T E\|F(X^x(s))\|_H^2\, ds}{4\varepsilon(\lambda_N - \varepsilon)T} \le \frac{c_1(1+\|x\|_H^2)}{\varepsilon(\lambda_N-\varepsilon)}
\end{aligned}$$

for some constants $\varepsilon > 0$ and $c_1 > 0$.

Utilizing the Hölder inequality, we also have that

$$\begin{aligned}
&\sum_{N}^{N+m} \frac{1}{T}\int_0^T E\left|\int_0^t \mathrm{e}^{-\lambda_k (t-s)} \big\langle e_k, B\big(X^x(s)\big)\, dW(s)\big\rangle\right|^2 dt \\
&\quad \le \frac{\mathrm{tr}(Q)\int_0^T E\|B(X^x(t))\|_{\mathscr{L}(K,H)}^2\, dt}{2\lambda_N T} \le \frac{c_2\,\mathrm{tr}(Q)(1+\|x\|_H^2)}{\lambda_N}
\end{aligned}$$

for some constant $c_2 > 0$. Thus,

$$\sum_{N}^{N+m} \frac{1}{T} \int_0^T E y_k^2(t)\, dt \le \frac{3\|x\|_H^2}{2\lambda_N}$$
$$+ 3(c_1 + c_2)\big(1 + \|x\|_H^2\big)\left[\frac{1}{\delta(\lambda_N - \delta)} + \frac{\operatorname{tr}(Q)}{\lambda_N}\right].$$

Thus the condition in [25] holds. □

The proof of Theorem 7.16 is an immediate consequence of the lemma. □

Exercise 7.8 Let $p > 1$, and let g be a nonnegative locally p-integrable function on $[0, \infty)$. Then for all $\varepsilon > 0$ and real d,

$$\left(\int_0^t e^{d(t-r)} g(r)\, dr\right)^p \le \left(\frac{1}{q\varepsilon}\right)^{p/q} \int_0^t e^{p(d+\varepsilon)(t-r)} g^p(r)\, dr,$$

where $1/p + 1/q = 1$.

7.4.3 Semilinear Equations Driven by a Cylindrical Wiener Process

We finally present a result in [12], which uses an innovative technique to prove the tightness of the laws $\mathscr{L}(X^x(t))$. We start with the problem

$$\begin{cases} dX(t) = (AX(t) + F(X(t)))\, dt + B(X(t))\, d\tilde{W}_t, \\ X(0) = x \in H, \end{cases} \tag{7.44}$$

where $\tilde{W}_t$ is a cylindrical Wiener process in a separable Hilbert space K. Assume that the coefficients and the solution satisfy the following hypotheses.

Hypothesis (DZ) Let conditions (DZ1)–(DZ4) of Sect. 3.10 hold, and, in addition, assume that:

(DZ5) $\{S(t),\ t > 0\}$ is a compact semigroup.
(DZ6) For all $x \in H$ and $\varepsilon > 0$, there exists $R > 0$ such that for every $T \ge 1$,

$$\frac{1}{T} \int_0^T P\big(\|X^x(t)\|_H > R\big)\, dt < \varepsilon,$$

where $\{X^x(t),\ t \ge 0\}$ is a mild solution of (7.44).

Remark 7.4 (a) Condition (DZ6) holds if $\{X^x(t),\ t \ge 0\}$ is ultimately bounded in the m.s.s.

(b) In the special case where W_t is a Q-Wiener process, we can replace B with $\tilde{B} = BQ^{1/2}$.

Theorem 7.17 *Under Hypothesis* (DZ), *there exists an invariant measure for the mild solution of* (7.44).

Proof We recall the factorization formula used in Lemma 3.3. Let $x \in H$, and

$$Y^x(t) = \int_0^t (t-s)^{-\alpha} S(t-s) B\big(X^x(s)\big)\, dW_s.$$

Then

$$X^x(1) = S(1)x + G_1 F\big(X^x(\cdot)\big)(1) + \frac{\sin \pi\alpha}{\pi} G_\alpha Y^x(\cdot)(1) \quad P\text{-a.s.}$$

By Lemma 3.12, the compactness of the semigroup $\{S(t),\ t \ge 0\}$ implies that the operators G_α defined by

$$G_\alpha f(t) = \int_0^t (t-s)^{\alpha-1} S(t-s) f(s)\, ds, \quad f \in L^p\big([0,T], H\big),$$

are compact from $L^p([0,T],H)$ into $C([0,T],H)$ for $p \ge 2$ and $1/p < \alpha \le 1$. Consider $\gamma : H \times L^p([0,1],H) \times L^p([0,1],H) \to H$,

$$\gamma(y, f, g) = S(1)y + G_1 f(1) + G_\alpha g(1).$$

Then γ is a compact operator, and hence, for $r > 0$, the set

$$K(r) = \bigg\{ x \in H : x = S(1)y + G_1 f(1) + G_\alpha g(1),$$
$$\|y\|_H \le r, \|f\|_{L^p} \le r, \|g\|_{L^p} \le \frac{r\pi}{\sin \pi\alpha} \bigg\}$$

is relatively compact in H.

We now need the following lemma.

Lemma 7.3 *Assume that* $p > 2$, $\alpha \in (1/p, 1/2)$, *and that Hypothesis* (DZ) *holds. Then there exists a constant* $c > 0$ *such that for* $r > 0$ *and all* $x \in H$ *with* $\|x\|_H \le r$,

$$P\big(X^x(1) \in K(r)\big) \ge 1 - cr^{-p}\big(1 + \|x\|_H^p\big).$$

Proof By Lemma 3.13, using Hypothesis (DZ3), we calculate

$$E \int_0^1 \big\|Y^x(s)\big\|_H^p\, ds$$

$$
\begin{aligned}
&= E\int_0^1 \left\| \int_0^s (s-u)^{-\alpha} S(s-u) B\big(X^x(u)\big)\, dW_u \right\|_H^p ds \\
&\le kE\int_0^1 \left(\int_0^s (s-u)^{-2\alpha} \big\| S(s-u) B\big(X^x(u)\big)\big\|_{\mathscr{L}_2(K,H)}^2 du \right)^{p/2} ds \\
&\le k2^{p/2} E\int_0^1 \left(\int_0^s (s-u)^{-2\alpha} \mathscr{K}^2(s-u)\big(1+\big\|X^x(u)\big\|_H^2\big)\, du \right)^{p/2} ds.
\end{aligned}
$$

By (3.103) and Exercise 3.7,

$$
\begin{aligned}
E\int_0^1 \big\|Y^x(s)\big\|_H^p ds &\le k2^{p/2}\left(\int_0^1 t^{-2\alpha}\mathscr{K}^2(t)\,dt\right)^{p/2} E\int_0^1 \big(1+\big\|X^x(u)\big\|_H^2\big)^{p/2} du \\
&\le k_1\big(1+\|x\|_H^p\big) \quad \text{for some } k_1>0.
\end{aligned}
$$

Also, using Hypothesis (DZ2), we get

$$
E\int_0^1 \big\|F\big(X^x(u)\big)\big\|_H^p du \le k_2\big(1+\|x\|_H^p\big), \quad x\in H.
$$

By the Chebychev inequality,

$$
\begin{aligned}
&P\left(\big\|Y^x(\cdot)\big\|_{L^p} \le \frac{\pi r}{\sin\alpha\pi}\right) \\
&\quad \ge 1 - r^{-p}\frac{\sin^p\alpha\pi}{\pi^p} E\big\|Y^x(\cdot)\big\|_{L^p}^p \ge 1 - r^{-p}\pi^{-p}k_1\big(1+\|x\|_H^p\big) \\
&P\big(\big\|F\big(X^x(\cdot)\big)\big\|_{L^p}\le r\big) \\
&\quad \ge 1 - r^{-p}E\big(\big\|F\big(X^x(\cdot)\big)\big\|_{L^p}^p\big) \ge 1 - r^{-p}k_2\big(1+\|x\|_H^p\big),
\end{aligned}
$$

giving

$$
\begin{aligned}
P\big(X^x(1)\in K(r)\big) &\ge P\left(\left\{\big\|Y^x(\cdot)\big\|_{L^p} \le \frac{\pi r}{\sin\alpha\pi}\right\} \cap \big\{\big\|F\big(X^x(\cdot)\big)\big\|_{L^p}\le r\big\}\right) \\
&\ge 1 - r^{-p}\big(\pi^{-p}k_1 + k_2\big)\big(1+\|x\|_H^p\big).
\end{aligned}
$$

□

We continue the proof of Theorem 7.17.

For any $t>1$ and $r>r_1>0$, by the Markov property (recall Proposition 3.4) and Lemma 7.3, we have

$$
\begin{aligned}
&P\big(X^x(t)\in K(r)\big) = P\big(t,x,K(r)\big) \\
&\quad = \int_H P\big(1,y,K(r)\big)P(t-1,x,dy) \\
&\quad \ge \int_{\|y\|_H\le r_1} P\big(1,y,K(r)\big)P(t-1,x,dy)
\end{aligned}
$$

$$\geq \big(1 - c\big(r^{-p}\big(1 + r_1^p\big)\big)\big) \int_{\|y\|_H \leq r_1} P(t-1, x, dy)$$
$$= \big(1 - c\big(r^{-p}\big(1 + r_1^p\big)\big)\big) P\big(\big\|X^x(t-1)\big\|_H \leq r_1\big),$$

giving

$$\frac{1}{T}\int_0^T P\big(X^x(t) \in K(r)\big)\, dt \geq 1 - cr^{-p}\big(1 + r_1^p\big)\frac{1}{T}\int_0^T P\big(\big\|X^x(t)\big\|_H \leq r_1\big)\, dt.$$

If we choose r_1 according to condition (DZ6) and take $r > r_1$ sufficiently large, we obtain that $\frac{1}{T}\int_0^T P(t, x, \cdot)\, dt$ is relatively compact, ensuring the existence of an invariant measure. □

7.5 Ultimate Boundedness and Weak Recurrence of the Solutions

In Sect. 4.3 we proved the existence and uniqueness for strong variational solutions, and in Sect. 4.4 we showed that they are strong Markov and Feller processes. We will now study weak (positive) recurrence of the strong solution of (7.45), which is (exponentially) ultimately bounded in the m.s.s.

The weak recurrence property to a bounded set was considered in [59] for the solutions of SDEs in the finite dimensions and in [33] for solutions of stochastic evolution equations in a Hilbert space. This section is based on the work of R. Liu [51].

Let us consider a strong solution of the variational equation

$$\begin{cases} dX(t) = A(X(t))\, dt + B(X(t))\, dW_t, \\ X(0) = x \in H. \end{cases} \tag{7.45}$$

We start with the definition of weak recurrence.

Definition 7.6 A stochastic process $X(t)$ defined on H is *weakly recurrent* to a compact set if there exists a compact set $C \subset H$ such that

$$P^x\big(X(t) \in C \text{ for some } t \geq 0\big) = 1 \quad \text{for all } x \in H,$$

where P^x is the conditional probability under the condition $X(0) = x$. The set C is called a *recurrent region*. From now on *recurrent* means *recurrent to a compact set*.

Theorem 7.18 *Suppose that $V \hookrightarrow H$ is compact and the coefficients of* (7.45) *satisfy the coercivity and the weak monotonicity conditions* (6.39) *and* (6.40). *If its solution $\{X^x(t), t \geq 0\}$ is ultimately bounded in the m.s.s., then it is weakly recurrent.*

Proof We prove the theorem using a series of lemmas.

Lemma 7.4 *Let $\{X(t),\, t \geq 0\}$ be a strong Markov process in H. If there exists a positive Borel-measurable function $\rho : H \to \mathbb{R}_+$, a compact set $C \subset H$, and a constant $\delta > 0$ such that*

$$P^x\big(X\big(\rho(x)\big) \in C\big) \geq \delta \quad \textit{for all } x \in H,$$

then $X^x(t)$ is weakly recurrent with the recurrence region C.

Proof For a fixed $x \in H$, let $\tau_1 = \rho(x)$, $\Omega_1 = \{\omega :\ X(\tau_1) \notin C\}$, $\tau_2 = \tau_1 + \rho(X(\tau_1))$, $\Omega_2 = \{\omega : X(\tau_2) \notin C\}$, $\tau_3 = \tau_2 + \rho(X(\tau_2))$, etc. Define $\Omega_\infty = \bigcap_{i=1}^\infty \Omega_i$. Since

$$\big\{\omega : X(t,\omega) \notin C \ \text{ for any } \ t \geq 0\big\} \subset \Omega_\infty,$$

it suffices to show that $P^x(\Omega_\infty) = 0$. Note that

$$P^x(\Omega_1) < 1 - \delta < 1.$$

Since $\rho : H \to \mathbb{R}_+$ is Borel measurable and τ_i is a stopping time for each i, we can use the strong Markov property to get

$$\begin{aligned}
P^x(\Omega_1 \cap \Omega_2) &= E^x\big(E^x\big(1_{\Omega_1}(\omega)1_{\Omega_2}(\omega)|\mathscr{F}_{\tau_1}\big)\big)\\
&= E^x\big(1_{\Omega_1}(\omega)E^x\big(1_{\Omega_2}(\omega)|\mathscr{F}_{\tau_1}\big)\big)\\
&= E^x\big(1_{\Omega_1}(\omega)E^x\big(1_{\Omega_2}(\omega)|X(\tau_1)\big)\big)\\
&= E^x\big(1_{\Omega_1}(\omega)P^{X(\tau_1)}\big(\big\{\omega : X\big(\rho(\tau_1)\big) \notin C\big\}\big)\big).
\end{aligned}$$

But, by the assumption,

$$P^{X(\tau_1)}\big(\big\{\omega :\ X\big(\rho\big(X\big(\tau_1(\omega)\big)\big)\big) \notin C\big\}\big) < 1 - \delta,$$

so that

$$P^x(\Omega_1 \cap \Omega_2) < (1 - \delta)^2.$$

By repeating the above argument, we obtain

$$P^x\left(\bigcap_{i=1}^n \Omega_i\right) < (1 - \delta)^n,$$

which converges to zero, and this completes the proof. □

Lemma 7.5 *Let $\{X(t), t \geq 0\}$ be a continuous strong Markov process. If there exists a positive Borel-measurable function γ defined on H, a closed set C, and a constant $\delta > 0$ such that*

$$\int_{\gamma(x)}^{\gamma(x)+1} P^x\big(X(t) \in C\big)\, dt \geq \delta \quad \textit{for all } x \in H, \tag{7.46}$$

then, there exists a Borel-measurable function $\rho : H \to \mathbb{R}_+$ *such that* $\gamma(x) \le \rho(x) \le \gamma(x) + 1$ *and*

$$P^x\big(X\big(\rho(x)\big) \in C\big) \ge \delta \quad \textit{for all } x \in H. \tag{7.47}$$

Proof By the assumption (7.46), there exists $t_x \in [\gamma(x), \gamma(x) + 1)$ such that

$$P^x\big(X(t_x) \in C\big) \ge \delta.$$

Define

$$\rho(x) = \inf\big\{t \in \big[\gamma(x), \gamma(x) + 1\big) : P^x\big(\big\{\omega : X(t, \omega) \in C\big\}\big) \ge \delta\big\}.$$

Since the mapping $t \to X(t)$ is continuous and the characteristic function of a closed set is upper semicontinuous, we have that the function

$$t \to P^x\big(X(t) \in C\big)$$

is upper semicontinuous for each x. Hence,

$$P^x\big(X\big(\rho(x)\big) \in C\big) \ge \delta.$$

We need to show that the function $x \to \rho(x)$ is Borel measurable. Let us define $\mathscr{B}_t(H) = \mathscr{B}(H)$, for $t > 0$. Since $\{X(t), 0 \le t \le T\}$ is a Feller process, the map $\Theta : (t, x) \to P^x(\omega : X(t) \in C)$ from $([0, T] \times H, \mathscr{B}([0, T] \times H))$ to $(\mathbb{R}^1, \mathscr{B}(\mathbb{R}^1))$ is measurable (see [54], [27]). Hence, Θ is a progressively measurable process with respect to $\{\mathscr{B}_t(H)\}$. By Corollary 1.6.12 in [16], $x \to \rho(x)$ is Borel measurable. □

Let us now introduce some notation. Let $B_r = \{v \in V : \|v\|_V \le r\}$ be a sphere in V with the radius r, centered at 0, and let $\overline{B}_r$ be its closure in $(H, \|\cdot\|_H)$. For $A \subset H$, denote its interior in $(H, \|\cdot\|_H)$ by A^0. If $B_r^c = H \setminus B_r$, then $(\overline{B}_r)^c = (B_r^c)^0$.

Lemma 7.6 *Suppose that the coefficients of* (7.45) *satisfy the coercivity condition* (6.39) *and, in addition, that its solution* $\{X^x(t),\ t \ge 0\}$ *exists and is ultimately bounded in the m.s.s. Then there exists a positive Borel-measurable function* ρ *on* H *such that*

$$P^x\big(\big\{\omega : X\big(\rho(x), \omega\big) \in \overline{B}_r\big\}\big) \ge 1 - \frac{1}{\alpha r^2}\big(|\lambda| M_1 + M_1 + |\gamma|\big), \quad x \in H, \tag{7.48}$$

and

$$P^x\big(\big\{\omega : X\big(\rho(x), \omega\big) \in \big(B_r^c\big)^0\big\}\big) \le \frac{1}{\alpha r^2}\big(|\lambda| M_1 + M_1 + |\gamma|\big), \quad x \in H, \tag{7.49}$$

where α, λ, γ *are as in the coercivity condition, and* $M_1 = M + 1$ *with* M *as in the ultimate boundedness condition* (7.36).

Proof Since $\limsup_{t\to\infty} E^x\|X(t)\|_H^2 \le M < M_1$ for all $x \in H$, there exist positive numbers $\{T_x,\ x \in H\}$ such that

$$E^x \|X(t)\|_H^2 \le M \quad \text{for } t \ge T_x.$$

Hence, we can define

$$\gamma(x) = \inf\bigl\{t : E^x \|X(s)\|_H^2 \le M_1 \text{ for all } s \ge t\bigr\}.$$

Since $t \to E^x\|X(t)\|_H^2$ is continuous, $E^x\|X(\gamma(x))\|_H^2 \le M_1$. The set

$$\begin{aligned}\bigl\{x : \gamma(x) \le t\bigr\} &= \bigl\{x : E^x \|X(s)\|_H^2 \le M_1 \text{ for all } s \ge t\bigr\}\\ &= \bigcap_{\substack{s\ge t\\ s\in Q}} \bigl\{x : E^x \|X(s)\|_H^2 \le M_1\bigr\}\end{aligned}$$

is in $\mathscr{B}(H)$, since the function $x \to E^x\|X(s)\|^2$ is Borel measurable. Using Itô's formula (4.37) for $\|x\|_H^2$, then taking the expectations on both sides, and applying the coercivity condition (6.39), we arrive at

$$\begin{aligned}&E^x \bigl\|X\bigl(\gamma(x)+1\bigr)\bigr\|_H^2 - E^x \bigl\|X\bigl(\gamma(x)\bigr)\bigr\|_H^2\\ &= E^x \int_{\gamma(x)}^{\gamma(x)+1} \bigl(2\bigl\langle X(s), A\bigl(X(s)\bigr)\bigr\rangle + \operatorname{tr}\bigl(B\bigl(X(s)\bigr)Q\bigl(B\bigl(X(s)\bigr)^*\bigr)\bigr)\,ds\\ &\le \lambda \int_{\gamma(x)}^{\gamma(x+1)} E^x \bigl\|X(s)\bigr\|_H^2\,ds - \alpha \int_{\gamma(x)}^{\gamma(x)+1} E^x \bigl\|X(s)\bigr\|_V^2\,ds + \gamma.\end{aligned}$$

It follows that

$$\int_{\gamma(x)}^{\gamma(x)+1} E\bigl\|X(s)\bigr\|_V^2\,ds \le \frac{1}{\alpha}\bigl(|\lambda|M_1 + M_1 + |\gamma|\bigr).$$

Using Chebychev's inequality, we get

$$\int_{\gamma(x)}^{\gamma(x+1)} P^x\bigl(\bigl\{\omega : \bigl\|X(t,\omega)\bigr\|_V > r\bigr\}\bigr)\,dt \le \frac{1}{\alpha r^2}\bigl(|\lambda|M_1 + M_1 + |\gamma|\bigr).$$

Hence,

$$\int_{\gamma(x)}^{\gamma(x)+1} P^x\bigl(\bigl\{\omega : X(t,\omega) \in \bigl(B_r^c\bigr)^0\bigr\}\bigr) \le \frac{1}{\alpha r^2}\bigl(|\lambda|M_1 + M_1 + |\gamma|\bigr),$$

and consequently,

$$\int_{\gamma(x)}^{\gamma(x)+1} P^x\bigl(\bigl\{\omega : X(t,\omega) \in \overline{B}_r\bigr\}\bigr)\,dt \ge 1 - \frac{1}{\alpha r^2}\bigl(|\lambda|M_1 + M_1 + |\gamma|\bigr).$$

Using Lemma 7.5, we can claim the existence of a positive Borel-measurable function $\rho(x)$ defined on H such that $\gamma(x) \le \rho(x) \le \gamma(x) + 1$, and (7.48) and then (7.49) follow for $r > 0$ and for all $x \in H$. □

We now conclude the proof of Theorem 7.18. Using (7.48), we can choose r large enough such that

$$P^x\big(\big\{\omega : X\big(\rho(x), \omega\big) \in \overline{B}_r\big\}\big) \ge \frac{1}{2} \quad \text{for } x \in H.$$

Since the mapping $V \hookrightarrow H$ is compact, the set $\overline{B}_r$ is compact in H, giving that $X(t)$ is weakly recurrent to $\overline{B}_r$ by Lemma 7.4. □

Definition 7.7 An H-valued stochastic process $\{X(t),\ t \ge 0\}$ is called *weakly positive recurrent to a compact set* if there exists a compact set $C \subset H$ such that $X(t)$ is weakly recurrent to C and the first hitting time to C,

$$\tau = \inf\big\{t \ge 0 : X(t) \in C\big\},$$

has finite expectation for any $x = X(0) \in H$.

Theorem 7.19 *Suppose that $V \hookrightarrow H$ is compact and the coefficients of* (7.45) *satisfy the coercivity condition* (6.39) *and the monotonicity condition* (6.40). *If its solution $\{X^x(t),\ t \ge 0\}$ is exponentially ultimately bounded in the m.s.s., then it is weakly positively recurrent.*

Proof We know that

$$E^x\big\|X(t)\big\|_H^2 \le c e^{-\beta t}\|x\|_H^2 + M \quad \text{for all } x \in H.$$

Let $M_1 = M + 1$, and $w(r) = \frac{1}{\beta}\ln(1 + cr^2)$, $r \in \mathbb{R}$. Then we have

$$E^x\big\|X(t)\big\|_H^2 \le M_1 \quad \text{for } x \in H \text{ and } t \ge w\big(\|x\|_H\big),$$

and

$$\sum_{l=1}^{\infty} \frac{w((l+1)N)}{l^2} < \infty \quad \text{for any } N \ge 0. \tag{7.50}$$

Let $K = (1 + \Delta)\sqrt{|\lambda| M_1 + M_1 + |\gamma|}/\sqrt{\alpha}$, and let us define the sets

$$E_0 = \overline{B}_K,$$
$$E_l = \overline{B}_{(l+1)K} - \overline{B}_{lK} = \overline{B}_{(l+1)K} \cap \big(B_{lK}^c\big)^0 \quad \text{for } l \ge 1,$$

where B_r is a sphere in V with the radius r, centered at 0. We denote $w'(l) = w(lK\alpha_0) + 1$ with α_0 such that $\|x\|_H \le \alpha_0\|x\|_V$ for all $x \in V$. As in the proof of

Lemma 7.6, there exists a Borel-measurable function $\rho(x)$ defined on H satisfying $w(\|x\|_H) \le \rho(x) \le w(\|x\|_H) + 1$, and

$$P^x\big(\big\{\omega : X\big(\rho(x), \omega\big) \in \big(B^c_{lK}\big)^0\big\}\big) \le \frac{1}{\alpha(lK)^2}\big(|\lambda| M_1 + M_1 + |\gamma|\big) \le \frac{1}{l^2(1+\Delta)^2} \quad \text{for all } x \in H. \tag{7.51}$$

Let

$$\tau_1 = \rho(x), \quad x_1(\omega) = X(\tau_1, \omega), \quad \Omega_1 = \big\{\omega : x_1(\omega) \notin E_0\big\},$$
$$\tau_2 = \tau_1 + \rho\big(x_1(\omega)\big), \quad x_2(\omega) = X(\tau_2, \omega), \quad \Omega_2 = \big\{\omega : x_2(\omega) \notin E_0\big\}, \quad \ldots,$$

and so on. Let $\Omega_\infty = \bigcap_{i=1}^\infty \Omega_i$. As in the proof of Lemma 7.4,

$$P^x\left(\bigcap_{i=1}^\infty \Omega_i\right) = 0.$$

Hence, Ω differs from

$$\bigcup_{i=1}^\infty \Omega_i^c = \bigcup_{i=1}^\infty \big\{\omega : x_i(\omega) \in E_0\big\}$$

by at most a set of P^x-measure zero. Let

$$A_i = \Omega_i^c - \bigcup_{j=1}^{i-1} \big\{\omega : x_j(\omega) \in E_0\big\} = \big\{\omega : x_1(\omega) \notin E_0, \ldots, x_{i-1} \notin E_0,\ x_i \in E_0\big\}.$$

Then Ω differs from $\bigcup_{i=1}^\infty A_i$ by at most a set of P^x-measure zero. For $i \ge 2$, let us further partition

$$A_i = \bigcup_{j_1, j_2, \ldots, j_{n-1}} A_i^{j_1, \ldots, j_{i-1}},$$

where

$$A_i^{j_1, \ldots, j_{i-1}} = \big\{\omega : x_1(\omega) \in E_{j_1}, \ldots, x_{i-1}(\omega) \in E_{j_{i-1}}, x_i(\omega) \in E_0\big\}.$$

Let $\tau(\omega)$ be first hitting time to E_0. Then for $\omega \in A_1 = \Omega_1^c$,

$$\tau(\omega) \le \rho(x) \le w\big(\|x\|_H\big) + 1,$$

and for $\omega \in A_i^{j_1, \ldots, j_{i-1}}$,

$$\tau(\omega) \le \tau_i(\omega) \le \tau_{i-1}(\omega) + \rho\big(x_{i-1}(\omega)\big).$$

Moreover, for $\omega \in A_i^{j_1,\dots,j_{i-1}}$,

$$x_{i-1}(\omega) \in E_{j_{i-1}} \subset \overline{B}_{(j_{i-1}+1)K}.$$

Hence,

$$\|x_{i-1}(\omega)\|_H \le \alpha_0 \|x_{i-1}(\omega)\|_V \le \alpha_0 (j_{i-1}+1)K,$$

giving

$$\rho\big(x_{i-1}(\omega)\big) \le w\big(\|x_{i-1}(\omega)\|_H\big) + 1 \le w\big(\alpha_0(j_{i-1}+1)K\big) + 1 = w'(j_{i-1}+1)$$

and

$$\tau(\omega) \le \tau_{i-1} + w'(j_{i-1}+1).$$

Using induction,

$$\tau(\omega) \le w\big(\|x\|_H\big) + 1 + w'(j_1+1) + \cdots + w'(j_{i-1}+1).$$

By the strong Markov property,

$$\begin{aligned}
P^x\big(A_i^{j_1,\dots,j_{i-1}}\big) &= P^x\big(\big\{\omega : x_1(\omega) \in E_{j_1}, \dots, x_{i-1}(\omega) \in E_{j_{i-1}}, x_i(\omega) \in E_0\big\}\big)\\
&\le P^x\big(\big\{\omega : x_1(\omega) \in E_{j_1}, \dots, x_{i-1}(\omega) \in E_{j_{i-1}}\big\}\big)\\
&= P^x\big(\big\{\omega : x_1(\omega) \in E_{j_1}, \dots, x_{i-2}(\omega) \in E_{j_{i-2}}\big\} \cap \big\{x_{i-1}(\omega) \in E_{j_{i-1}}\big\}\big)\\
&\le E^x\Big\{1_{\{\omega : x_1(\omega) \in E_{j_1}, \dots, x_{i-2} \in E_{j_{i-2}}\}} P^{x_{i-2}(\omega)}\big(\big\{\tilde{\omega} : X\big(\rho\big(x_{i-2}(\omega)\big), \tilde{\omega}\big) \in E_{j_{i-1}}\big\}\big)\Big\}.
\end{aligned}$$

Since $E_{j_{i-1}} = \overline{B}_{(j_{i-1}+1)K} \cap (B^c_{j_{i-1}K})^0$, we get by (7.51)

$$\begin{aligned}
&P^{x_{i-2}(\omega)}\big(\tilde{\omega} : X\big(\rho\big(x_{i-2}(\omega)\big), \tilde{\omega}\big) \in E_{j_{i-1}}\big)\\
&\quad \le P^{x_{i-2}}\big(\tilde{\omega} : X\big(\rho\big(x_{i-2}(\omega)\big), \tilde{\omega}\big) \in \big(B^c_{j_{i-1}K}\big)^0\big)\\
&\quad \le \frac{1}{j_{i-1}^2 (1+\Delta)^2}.
\end{aligned}$$

Hence,

$$P^x\big(A_i^{j_1,\dots,j_{i-1}}\big) \le \frac{1}{j_{i-1}^2(1+\Delta)^2} P^x\big(\big\{\omega : x_1(\omega) \in E_{j_1}, \dots, x_{i-2}(\omega) \in E_{j_{i-2}}\big\}\big).$$

By induction,

$$P^x\big(A_i^{j_1,\dots,j_{i-1}}\big) \le \frac{1}{(1+\Delta)^{2(i-1)}} \frac{1}{j_1^2 \cdots j_{i-1}^2},$$

which implies that $P^x(A_i) < 1$, for Δ large enough.

Now

$$
\begin{aligned}
E^x(\tau) \le & \sum_{i,j_1,\dots,j_{i-1}\ge 1} P^x\left(A_i^{j_1,\dots,j_{i-1}}\right) \\
& \times \left[w'\left(\|x\|_H\right) + 1 + w'(j_1+1) + \cdots + w'(j_{i-1}+1)\right] \\
& \le w\left(\|x\|_H\right) + 1 + \left(\sum_{i=2}^{\infty} \frac{1}{(1+\Delta)^{2(i-1)}}\right. \\
& \left.\left(\sum_{j_1,\dots,j_{i-1}\ge 1} \frac{w'(\|x\|_H) + 1 + w'(j_1+1) + \cdots + w'(j_{i-1}+1)}{j_1^2\cdots j_{i-1}^2}\right)\right. \\
& = w\left(\|x\|_H\right) + 1 + \left(\sum_{i=2}^{\infty} \frac{1}{(1+\Delta)^{2(i-1)}} \left(w\left(\|x\|_H\right)+1\right)\right. \\
& \left\{\left(\sum_{j_1,\dots,j_{i-1}\ge 1} \frac{1}{j_1^2\cdots j_{i-1}^2}\right) + (i-1) \sum_{j_1,\dots,j_{i-1}\ge 1} \frac{w'(j_1+1)}{j_1^2\cdots j_{i-1}^2}\right\} \\
& = \left(w\left(\|x\|_H\right)+1\right)\left(1 + \sum_{i=2}^{\infty}\left(\frac{A}{(1+\Delta)^2}\right)^{i-1}\right) \\
& + \frac{B}{(1+\Delta)^2} \sum_{i=2}^{\infty}\left(\frac{A}{(1+\Delta)^2}\right)^{i-2} (i-1),
\end{aligned}
$$

where $A = \sum_{l=1}^{\infty} \frac{1}{l^2}$, and $B = \sum_{l=1}^{\infty} \frac{1}{l^2} w'(l+1)$, with both series converging due to (7.50).

Consequently, $E^x(\tau)$ is finite for Δ large enough. The set E_0 is compact since the embedding $V \hookrightarrow H$ is compact. □

We have given precise conditions using a Lyapunov function for exponential ultimate boundedness in the m.s.s. We can thus obtain sufficient conditions for weakly (positive) recurrence of the solutions in terms of a Lyapunov function.

We close with important examples of stochastic reaction–diffusion equations. Let $\mathscr{O} \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\mathscr{O}$, and p be a positive integer. Let $V = W^{1,2}(\mathscr{O})$ and $H = W^{0,2}(\mathscr{O}) = L^2(\mathscr{O})$. We know that $V \hookrightarrow H$ is a compact embedding. Let

$$
A_0(x) = \sum_{|\alpha|\le 2p} a_\alpha(x) \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}},
$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multiindex. If A_0 is strongly elliptic, then by Garding inequality ([63], Theorem 7.2.2) A_0 is coercive.

Example 7.8 (Reaction–diffusion equation) Consider a parabolic Itô equation

$$\begin{cases} dX(t,x) = A_0 X(t,x)\,dt + f(X(t,x))\,dt + B(t,x))\,dW_t, \\ X(0,x) = \varphi(x) \in H, \quad X|_{\partial\mathscr{O}} = 0, \end{cases} \tag{7.52}$$

where A_0, f, and B satisfy the following conditions:

(1) $A_0 : V \to V^*$ is a strongly elliptic operator.
(2) $f : H \to H$ and $B : H \to \mathscr{L}(K,H)$ satisfy

$$\|f(h)\|_H^2 + \|B(h)\|_{\mathscr{L}(K,H)}^2 \le K\big(1 + \|h\|_H^2\big), \quad h \in H.$$

(3) $\|f(h_1) - f(h_2)\|_H^2 + \operatorname{tr}((B(h_1) - B(h_2))Q(B(h_1) - B(h_2))^*) \le \lambda\|h_1 - h_2\|$, $h_1, h_2 \in H$.

If the solution to the equation

$$du(t,x) = A_0 u(t,x)\,dt$$

is exponentially ultimately bounded and, as $\|h\|_H \to \infty$,

$$\|f(h)\|_H = o\big(\|h\|_H\big),$$
$$\|B(h)\|_{\mathscr{L}(K,H)} = o\big(\|h\|_H\big),$$

then the strong variational solution of (7.52) is exponentially ultimately bounded in the m.s.s. by Proposition 7.1, and consequently it is weakly positive recurrent.

Example 7.9 (Reaction–diffusion equation) Consider the following one-dimensional parabolic Itô equation

$$\begin{cases} dX(t,x) = \left(\alpha^2 \dfrac{\partial^2 X}{\partial x^2} + \beta \dfrac{\partial X}{\partial x} + \gamma X + g(x)\right) dt + \left(\sigma_1 \dfrac{\partial X}{\partial x} + \sigma_2 X\right) dW_t, \\ u(0,x) = \varphi(x) \in L^2(\mathscr{O}) \cap L^1(\mathscr{O}), \quad X|_{\partial\mathscr{O}} = 0, \end{cases} \tag{7.53}$$

where $\mathscr{O} = (0,1)$, and W_t is a standard Brownian motion.

Similarly as in Example 7.3, if $-2\alpha^2 + \sigma_1^2 < 0$, then the coercivity and weak monotonicity conditions (6.39) and (6.40) hold, and Theorem 4.7 implies the existence of a unique strong solution.

With $\Lambda(v) = \|v\|_H^2$ and $\mathscr{L}$ defined by (6.15), we get

$$\mathscr{L}\Lambda(v) \le \big(-2\alpha^2 + \sigma_1^2\big)\left\|\frac{\mathrm{d}v}{\mathrm{d}x}\right\|_H^2 + \big(2\gamma + \sigma_2^2 + \epsilon\big)\|v\|_H^2 + \frac{1}{\epsilon}\|g\|_H^2.$$

Since $\|\frac{\mathrm{d}v}{\mathrm{d}x}\|_H^2 \ge \|v\|_H^2$ (see Exercise 7.9), we have

$$\mathscr{L}\Lambda(v) \le \big(-2\alpha^2 + \sigma_1^2 + 2\gamma + \sigma 2_2 + \epsilon\big)\|v\|_H^2 + \frac{1}{\epsilon}\|g\|_H^2.$$

Hence, if $-2\alpha^2 + \sigma_1^2 + 2\gamma + \sigma_2^2 < 0$, then the strong variational solution of (7.53) is exponentially ultimately bounded by Theorem 7.5, and hence it is weakly positive recurrent.

Exercise 7.9 Let $f \in W^{0,2}((a,b))$. Prove the Poincaré inequality

$$\int_a^b f^2(x)\,dx \le (b-a)^2 \int_a^b \left(\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right)^2 dx.$$