Chapter 6
Stability Theory for Strong and Mild Solutions

6.1 Introduction

Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a Banach space, and let us consider the Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t), \quad 0 < t < T, \\ u(0) = x \in \mathfrak{X}. \end{cases} \tag{6.1}$$

We know that if A generates a C_0-semigroup $\{S(t),\, t \geq 0\}$, then the mild solution of the Cauchy problem (6.1) is given by

$$u^x(t) = S(t)x.$$

If $\mathfrak{X}$ is finite-dimensional, with a scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{X}}$, Lyapunov proved the equivalence of the following three conditions:

(1) $\|u^x(t)\|_{\mathfrak{X}} \leq c_0 \|x\|_{\mathfrak{X}} \mathrm{e}^{-rt}$, $r, c_0 > 0$.
(2) $\max\{\mathrm{Re}(\lambda) : \det(\lambda I - A) = 0\} < 0$.
(3) There exists a positive definite matrix R satisfying
  (i) $c_1 \|x\|^2_{\mathfrak{X}} \leq \langle Rx, x\rangle_{\mathfrak{X}} \leq c_2 \|x\|^2_{\mathfrak{X}}$, $x \in \mathfrak{X}$, $c_1, c_2 > 0$,
  (ii) $A^* R + RA = -I$.

If condition (1) is satisfied, then the mild solution $\{u^x(t),\, t \geq 0\}$ of the Cauchy problem (6.1) is said to be *exponentially stable*.

To prove that (1) implies (3), the matrix R is constructed using the equation

$$\langle Rx, x\rangle_{\mathfrak{X}} = \int_0^\infty \left\|u^x(t)\right\|^2_{\mathfrak{X}} \, dt.$$

When $\mathfrak{X}$ is infinite-dimensional, then the interesting examples of PDEs result in an unbounded operator A. In this case, if we replace condition (2) by

(2') $\max\{\mathrm{Re}(\lambda) : \lambda \in \sigma(A)\} < 0$,

with $\sigma(A)$ denoting the spectrum of A, the equivalence of (1) and (2') fails ((2') $\not\Rightarrow$ (1)) due to the failure of the spectral mapping theorem (refer to [63], p. 117),

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unless we make more restrictive assumptions on A (e.g., A is analytic). A sufficient condition for exponential stability is given in [63], p. 116:

$$\int_0^\infty \|S(t)x\|_{\mathfrak{X}}^p\,dt < \infty, \quad \text{for } p > 1.$$

In our PDE examples, we need $p = 2$ and $\mathfrak{X} = H$, a real separable Hilbert space. In this case, condition (1) alone implies that R, given by

$$\langle Rx, y\rangle_H = \int_0^\infty \langle u^x(t), u^y(t)\rangle_H\,dt,$$

exists as a bilinear form, and in fact, the equivalence of conditions (1) and (3) above can be proved (see [13]). We now consider the Cauchy problem in a real Hilbert space,

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t), & 0 < t < T, \\ u(0) = x \in H. \end{cases} \tag{6.2}$$

Theorem 6.1 *Let $(H, \langle\cdot,\cdot\rangle_H)$ be a real Hilbert space. The following conditions are equivalent*:

(1) *The solution of the Cauchy problem* (6.2) $\{u^x(t),\ t \geq 0\}$ *is exponentially stable.*
(2) *There exists a nonnegative symmetric operator R such that for $x \in \mathscr{D}(A)$*,

$$A^*Rx + RAx = -x.$$

Proof Define $\langle Rx, y\rangle_H$ as above. Using condition (1), we have

$$\langle Rx, x\rangle_H = \int_0^\infty \|S(t)x\|_H^2\,dt < \infty. \tag{6.3}$$

Clearly, R is nonnegative definite and symmetric. Now, for $x, y \in H$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle RS(t)x, S(t)y\rangle = \langle RAS(t)x, S(t)y\rangle_H + \langle RS(t)x, AS(t)y\rangle_H.$$

But

$$\langle RS(t)x, S(t)y\rangle_H = \int_t^\infty \langle S(u)x, S(u)y\rangle_H\,du$$

by the semigroup property. Hence, we obtain

$$\begin{aligned} &\langle RAS(t)x, S(t)y\rangle_H + \langle RS(t)x, AS(t)y\rangle_H \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty \langle S(u)x, S(u)y\rangle_H\,du \\ &= -\langle S(t)x, S(t)y\rangle_H, \end{aligned}$$

since $S(t)$ is strongly continuous. Thus, if $x \in \mathscr{D}(A)$, then

$$\big\langle\big(RA + A^*R\big)x, y\big\rangle_H = -\langle x, y\rangle_H,$$

giving (2).

From the above calculations condition (2) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\langle RS(t)x, S(t)x\big\rangle_H = -\big\|S(t)x\big\|_H^2.$$

Hence,

$$\begin{aligned}\int_0^t \big\|S(u)x\big\|_H^2\,du &= \langle Rx, x\rangle_H - \big\langle RS(t)x, S(t)x\big\rangle_H\\ &\le \langle Rx, x\rangle_H.\end{aligned}$$

Thus, $\int_0^\infty \|S(t)x\|_H^2\,dt < \infty$.

We know that $S(t)x \to 0$ as $t \to \infty$ for each x (see Exercise 6.1). Hence, by the uniform boundedness principle, for some constant M, we have $\|S(t)\|_{\mathscr{L}(H)} \le M$ for all $t \ge 0$.

Consider the map $T : H \to L^2(\mathbb{R}_+, H)$, $Tx = S(t)x$. Then T is a closed linear operator on H. Using the closed graph theorem, we have

$$\int_0^\infty \big\|S(t)x\big\|_H^2\,dt \le c^2\|x\|_H^2.$$

Let $0 < \rho < M^{-1}$ and define

$$t_x(\rho) = \sup\big\{t : \big\|S(s)x\big\|_H > \rho\|x\|_H, \text{ for all } 0 \le s \le t\big\}.$$

Since $\|S(t)x\|_H \to 0$ as $t \to \infty$, we have that $t_x(\rho) < \infty$ for each $x \in H$, $t_x(\rho)$ is clearly positive, and

$$t_x(\rho)\rho^2\|x\|_H^2 \le \int_0^{t_x(\rho)} \big\|S(t)x\big\|_H^2\,dt \le c^2\|x\|_H^2,$$

giving $t_x(\rho) \le (c/\rho)^2 = t_0$.

For $t > t_0$, using the definition of $t_x(\rho)$, we have

$$\begin{aligned}\big\|S(t)x\big\|_H &\le \big\|S\big(t - t_x(\rho)\big)\big\|_{\mathscr{L}(H)}\big\|S\big(t_x(\rho)\big)x\big\|_H\\ &\le M\rho\|x\|_H.\end{aligned}$$

Let $\beta = M\rho < 1$ and $t_1 > t_0$ be fixed. For $0 < s < t_1$, let $t = nt_1 + s$. Then

$$\begin{aligned}\big\|S(t)\big\|_{\mathscr{L}(H)} &\le \big\|S(nt_1)\big\|_{\mathscr{L}(H)}\big\|S(s)\big\|_{\mathscr{L}(H)}\\ &\le M\big\|S(t_1)\big\|_{\mathscr{L}(H)}^n \le M\beta^n \le M'\mathrm{e}^{-\mu t},\end{aligned}$$

where $M' = M/\beta$ and $\mu = -(1/t_1)\log\beta > 0$. □

In particular, we have proved the following corollary.

Corollary 6.1 *If $S(\cdot)x \in L^2(\mathbb{R}_+, H)$ for all x in a real separable Hilbert space H, then*

$$\|S(t)\|_{\mathscr{L}(H)} \le c\,\mathrm{e}^{-rt}, \quad \textit{for some } r > 0.$$

Exercise 6.1 (a) Find a continuous function $f(t)$ such that $\int_0^\infty (f(t))^2\,dt < \infty$ but $\lim_{t\to\infty} f(t) \ne 0$.

(b) Show that if $\int_0^\infty \|S(t)x\|_H^2\,dt < \infty$ for every $x \in H$, then $\lim_{t\to\infty} \|S(t)x\|_H = 0$ for every $x \in H$.

Hint: recall that $\|S(t)\|_{\mathscr{L}(H)} \le M\mathrm{e}^{\alpha t}$. Assume that $\|S(t_j)x\|_H > \delta$ for some sequence $t_j \to \infty$. Then, $\|S(t)x\|_H \ge \delta(Me)^{-1}$ on $[t_j - \alpha^{-1}, t_j]$.

We note that $\langle Rx, x\rangle_H$ does not play the role of the Lyapunov function, since in the infinite-dimensional case, $\langle Rx, x\rangle_H \ge c_1\|x\|^2$ with $c_1 > 0$ does not hold (see Example 6.1). We shall show that if A generates a pseudo-contraction semigroup, then we can produce a Lyapunov function related to R. The function Λ in Theorems 6.2 and 6.3 is called the *Lyapunov function*. Let us recall that $\{S(t),\ t \ge 0\}$ is a pseudo-contraction semigroup if there exists $\omega \in \mathbb{R}$ such that

$$\|S(t)\|_{\mathscr{L}(H)} \le \mathrm{e}^{\omega t}.$$

Theorem 6.2 (a) *Let $\{u^x(t)\, t \ge 0\}$ be a mild solution to the Cauchy problem* (6.2). *Suppose that there exists a real-valued function Λ on H satisfying the following conditions*:

(1) $c_1\|x\|_H^2 \le \Lambda(x) \le c_2\|x\|_H^2$ *for* $x \in H$,
(2) $\langle \Lambda'(x), Ax\rangle_H \le -c_3\Lambda(x)$ *for* $x \in \mathscr{D}(A)$,

where c_1, c_2, c_3 are positive constants. Then the solution $u^x(t)$ is exponentially stable.

(b) *If the solution $\{u^x(t)\, t \ge 0\}$ to the Cauchy problem* (6.2) *is exponentially stable and A generates a pseudo-contraction semigroup, then there exists a real-valued function Λ on H satisfying conditions* (1) *and* (2) *in part* (a).

Proof (a) Consider $\mathrm{e}^{c_3 t}\Lambda(u^x(t))$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{e}^{c_3 t}\Lambda\big(u^x(t)\big)\big) = c_3\mathrm{e}^{c_3 t}\Lambda\big(u^x(t)\big) + \mathrm{e}^{c_3 t}\big\langle \Lambda'\big(u^x(t)\big), Au^x(t)\big\rangle_H.$$

Hence,

$$\mathrm{e}^{c_3 t}\Lambda\big(u^x(t)\big) - \Lambda(x) = \int_0^t \mathrm{e}^{c_3 s}\big\{c_3\Lambda\big(u^x(s)\big) + \big\langle \Lambda'\big(u^x(s)\big), Au^x(s)\big\rangle_H\big\}\,ds.$$

It follows, by condition (2), that

$$\mathrm{e}^{c_3 t}\Lambda\big(u^x(t)\big) \le \Lambda(x).$$

Using (1), we have

$$c_1\left\|u^x(t)\right\|_H^2 \le \mathrm{e}^{-c_3 t}\Lambda(x) \le c_2 \mathrm{e}^{-c_3 t}\|x\|_H^2,$$

proving (a).

(b) Conversely, we first observe that for $\Psi(x) = \langle Rx, x\rangle_H$ with R defined in (6.3), we have $\Psi'(x) = 2Rx$ by the symmetry of R. Since $R = R^*$, we can write

$$\begin{aligned}\left\langle \Psi'(x), Ax\right\rangle_H &= \langle Rx, Ax\rangle_H + \langle Rx, Ax\rangle_H = \left\langle A^* Rx, x\right\rangle_H + \langle x, RAx\rangle_H \\ &= \left\langle A^* Rx + RAx, x\right\rangle_H = -\|x\|_H^2.\end{aligned}$$

Consider now, for some $\alpha > 0$ (to be determined later),

$$\Lambda(x) = \langle Rx, x\rangle_H + \alpha\|x\|_H^2.$$

Clearly $\Lambda(x)$ satisfies condition (1) in (a). Since $S(t)$ is a pseudo-contraction semigroup, there exists a constant λ (assumed positive WLOG) such that (see Exercise 3.5)

$$\langle x, Ax\rangle_H \le \lambda\|x\|_H^2, \quad x \in \mathscr{D}(A). \tag{6.4}$$

We calculate

$$\left\langle \Lambda'(x), Ax\right\rangle_H = \left\langle \Psi'(x), Ax\right\rangle_H + 2\alpha\langle x, Ax\rangle_H = \|x\|_H^2(2\alpha\lambda - 1).$$

Choosing α small enough, so that $2\alpha\lambda < 1$, and using condition (1), we obtain (2) in (a). □

Let us now consider the case of a coercive operator A (see condition (6.5)), with a view towards applications to PDEs. For this, we recall some concepts from Part I.

We have a Gelfand triplet of real separable Hilbert spaces

$$V \hookrightarrow H \hookrightarrow V^*,$$

where the embeddings are continuous. The space V^* is the continuous dual of V, with the duality on $V \times V^*$ denoted by $\langle\cdot,\cdot\rangle$ and satisfying

$$\langle v, h\rangle = \langle v, h\rangle_H$$

if $h \in H$.

Assume that V is dense in H. We shall now construct a Lyapunov function for determining the exponential stability of the solution of the Cauchy problem (6.2), where $A : V \to V^*$ is a linear bounded operator satisfying the coercivity condition

$$2\langle v, Av\rangle \le \lambda\|v\|_H^2 - \alpha\|v\|_V^2, \quad v \in V,\ \lambda \in \mathbb{R},\ \alpha > 0. \tag{6.5}$$

We note that the following *energy equality* in [72] holds for solutions $u^x(t) \in L^2([0,T],V) \cap C([0,T],H)$:

$$\big\|u^x(t)\big\|_H^2 - \|x\|_H^2 = 2\int_0^t \big\langle u^x(s), Au^x(s)\big\rangle\, ds. \tag{6.6}$$

We now state our theorem.

Theorem 6.3 (a) *The solution of the Cauchy problem* (6.2) *with a coercive coefficient* A *is exponentially stable if there exists a real-valued function* Λ *that is Fréchet differentiable on* H, *with* Λ *and* Λ' *continuous, locally bounded on* H, *and satisfying the following conditions*:

(1) $c_1\|x\|_H^2 \le \Lambda(x) \le c_2\|x\|_H^2$.
(2) *For* $x \in V$, $\Lambda'(x) \in V$, *and the function*

$$V \ni x \to \big\langle \Lambda'(x), v^*\big\rangle \in \mathbb{R}$$

is continuous for any $v^* \in V^*$.
(3) *For* $x \in V$, $\langle \Lambda'(x), Ax\rangle \le -c_3\Lambda(x)$, *where* c_1, c_2, c_3 *are positive constants.*

In particular, if

$$2\big\langle \Lambda'(x), Ax\big\rangle_H = -\|x\|_V^2,$$

then condition (3) *is satisfied.*

(b) *Conversely, if the solution to the Cauchy problem* (6.2) *is exponentially stable, then the real-valued function*

$$\Lambda(x) = \int_0^\infty \big\|u^x(t)\big\|_V^2\, dt \tag{6.7}$$

satisfies conditions (1)–(3) *in part* (a).

Proof Note that for $t, t' \ge 0$,

$$\Lambda\big(u^x(t)\big) - \Lambda\big(u^x(t')\big) = \int_{t'}^t \frac{\mathrm{d}}{\mathrm{d}s}\Lambda\big(u^x(s)\big)\, ds.$$

But, using (2) and (3), we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\Lambda\big(u^x(s)\big) = \big\langle \Lambda'\big(u^x(s)\big), Au^x(s)\big\rangle \le -c_3\Lambda\big(u^x(s)\big).$$

Denoting $\Phi(t) = \Lambda(u^x(t))$, we can then write

$$\Phi'(t) \le -c_3\Phi(t)$$

or, equivalently, $d(\Phi(t)\mathrm{e}^{c_3 t})/dt \le 0$, giving $\Phi(t)\mathrm{e}^{c_3 t} \le \Phi(0)$.

Using condition (1), we have

$$c_1 \|u^x(t)\|_H^2 \le \Lambda\big(u^x(t)\big) \le \Lambda(x)\mathrm{e}^{-c_3 t} \le c_2\|x\|_H^2 \mathrm{e}^{-c_3 t}.$$

To prove (b), we observe that, by the energy equality,

$$\begin{aligned}\|u^x(t)\|_H^2 &= \|x\|_H^2 + 2\int_0^t \big\langle Au^x(s), u^x(s)\big\rangle\, ds \\ &\le \|x\|_H^2 + |\lambda| \int_0^t \|u^x(s)\|_H^2\, ds - \alpha \int_0^t \|u^x(s)\|_V^2\, ds.\end{aligned}$$

Hence,

$$\|u^x(t)\|_H^2 + \alpha \int_0^t \|u^x(s)\|_V^2\, ds \le \|x\|_H^2 + |\lambda| \int_0^t \|u^x(s)\|_H^2\, ds.$$

Letting $t \to \infty$ and using the fact that

$$\|u^x(t)\|_H^2 \le c\|x\|_H^2 \mathrm{e}^{-\gamma t} \quad (\gamma > 0),$$

we obtain

$$\int_0^\infty \|u^x(s)\|_V^2\, ds \le \frac{1}{\alpha}\left(1 + \frac{|\lambda| c}{2\gamma}\right)\|x\|_H^2.$$

Define

$$\Lambda(x) = \int_0^\infty \|u^x(s)\|_V^2\, ds,$$

then $\Lambda(x) \le c_2\|x\|_H^2$. Let $x,\ y \in H$ and consider

$$T(x, y) = \int_0^\infty \big(u^x(t), u^y(t)\big)_V\, dt.$$

Using the fact that $u^x(s) \in L^2([0,\infty), V)$ and the Schwarz inequality, we can see that $T(x, y)$ is a continuous bilinear form on V, which is continuous on H. Hence, $T(x, y) = \langle \tilde{C}x, y\rangle_H$. Since $\Lambda'(x) = 2\tilde{C}x$ (by identifying H with H^*), we can see that Λ and Λ' are locally bounded and continuous on H. By the continuity of the embedding $V \hookrightarrow H$, we have that for $v,\ v' \in V$, $T(v, v') = \langle Cv, v'\rangle_V$ for some bounded linear operator C on V, and property (2) in (a) follows. Now,

$$\|u^x(t)\|_H^2 - \|x\|_H^2 = 2\int_0^t \big\langle Au^x(s), u^x(s)\big\rangle\, ds.$$

But $|\langle u^x(s), Au^x(s)\rangle| \le c_2'\|u^x(s)\|_V^2$, giving

$$\|u^x(t)\|_H^2 - \|x\|_H^2 \ge -2c_2' \int_0^t \|u^x(s)\|_V^2\, ds.$$

Let $t \to \infty$; then $\|u^x(t)\|_H^2 \to 0$, so that

$$-\|x\|_H^2 \geq -2c_2' \Lambda(x),$$

implying $\Lambda(x) \geq c_1 \|x\|_H^2$ for $c_1 = 1/(2c_2')$.

It remains to prove that $\Lambda(x)$ satisfies condition (3) in (a).

Note that

$$\Lambda\big(u^x(t)\big) = \int_0^\infty \big\|u^{u^x(t)}(s)\big\|_V^2 \, ds.$$

By the uniqueness of solution,

$$u^{u^x(t)}(s) = u^x(t+s).$$

Hence,

$$\Lambda\big(u^x(t)\big) = \int_0^\infty \big\|u^x(t+s)\big\|_V^2 \, ds = \int_t^\infty \big\|u^x(s)\big\|_V^2 \, ds.$$

Observe that

$$\frac{\mathrm{d}}{\mathrm{d}s} \Lambda\big(u^x(s)\big) = \big\langle \Lambda' u^x(s), A u^x(s) \big\rangle.$$

Since the map $\Lambda' : V \to H$ is continuous, we can write

$$\Lambda\big(u^x(t)\big) - \Lambda(x) = \int_0^t \big\langle \Lambda'\big(u^x(s)\big), A u^x(s) \big\rangle \, ds = -\int_0^t \big\|u^x(s)\big\|_V^2 \, ds.$$

By the continuity of the embedding $V \hookrightarrow H$, we have $\|x\|_H \leq c_0 \|x\|_V$, $x \in V$, $c_0 > 0$, and hence,

$$\int_0^t \big\langle \Lambda'\big(u^x(s)\big), A u^x(s) \big\rangle \, ds \leq -\frac{1}{c_0^2} \int_0^t \big\|u^x(s)\big\|_H^2 \, ds.$$

Now divide both sides by t and let $t \to 0$. Since Λ' is continuous and $u^x(\cdot) \in C([0, T], H)$, we get

$$\big\langle \Lambda'(x), A x \big\rangle \leq -\frac{1}{c_0^2} \|x\|_H^2.$$

□

The following example shows that in the infinite-dimensional case, if we define

$$\Lambda(x) = \int_0^\infty \big\|u^x(t)\big\|_H^2 \, dt,$$

then $\Lambda(x)$ does not satisfy the lower bound in condition (2) of (a) of Theorem 6.3.

Example 6.1 Consider a solution of the following equation

$$\begin{cases} d_t u(t,x) = a^2 \dfrac{\partial^2 u}{\partial x^2} \, dt + \left(b \dfrac{\partial u}{\partial t} + cu \right) dt, \\ u(0,x) = \varphi(x) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}). \end{cases} \tag{6.8}$$

Here, $H = L^2(\mathbb{R})$, and V is the Sobolev space $W^{1,2}(\mathbb{R})$. We denote by $\hat{\varphi}(\lambda)$ the Fourier transform of $\varphi(x)$ and use the similar notation $\hat{u}(t,\lambda)$ for the Fourier transform of $u(t,x)$. Then (6.8) can be written as follows:

$$\begin{cases} \dfrac{\mathrm{d}\hat{u}(t,\lambda)}{\mathrm{d}t} = -a^2\lambda^2\hat{u}(t,\lambda) + (ib\lambda + c)\hat{u}(t,\lambda), \\ \hat{u}(0,\lambda) = \hat{\varphi}(\lambda). \end{cases} \tag{6.9}$$

The solution is

$$\hat{u}^{\varphi}(t,\lambda) = \hat{\varphi}(\lambda)\exp\bigl\{\bigl(-a^2\lambda^2 + ib\lambda + c\bigr)t\bigr\}.$$

By Plancherel's theorem, $\|u^{\varphi}(t,\cdot)\|_H = \|\hat{u}^{\varphi}(t,\cdot)\|_H$, so that

$$\begin{aligned} \bigl\|u^{\varphi}(t,\cdot)\bigr\|_H^2 &= \int_{-\infty}^{\infty} \bigl|\hat{\varphi}(\lambda)\bigr|^2 \exp\bigl\{\bigl(-2a^2\lambda^2 + 2c\bigr)t\bigr\}\,d\lambda \\ &\le \|\varphi\|_H^2 \exp\{\gamma t\} \quad (\gamma = 2c). \end{aligned}$$

For $c < 0$, we obtain an exponentially stable solution.

Take $A = -2a^2$, $B = 2c$. Then

$$\Lambda(\varphi) = \int_0^{\infty}\int_{-\infty}^{\infty} \bigl|\hat{\varphi}(\lambda)\bigr|^2 \exp\bigl\{-\bigl(A\lambda^2 + B\bigr)t\bigr\}\,d\lambda\,dt = \int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\lambda)|^2}{A\lambda^2 + B}\,d\lambda$$

does not satisfy $\Lambda(\varphi) \ge c_1\|\varphi\|_H^2$ (see condition (1) in part (a) of Theorem 6.3).

In the next section, we consider the stability problem for infinite-dimensional stochastic differential equations using the Lyapunov function approach. We shall show that the fact that a Lyapunov function for the linear case is bounded below can be used to study the stability for nonlinear stochastic PDEs.

6.2 Exponential Stability for Stochastic Differential Equations

We recall some facts from Part I. Consider the following stochastic differential equation in H:

$$\begin{cases} dX(t) = (AX(t) + F(X(t)))\,dt + B(X(t))\,dW_t, \\ X(0) = x \in H, \end{cases} \tag{6.10}$$

where

(1) A is the generator of a C_0-semigroup $\{S(t),\ t \ge 0\}$ on H.
(2) W_t is a K-valued $\mathscr{F}_t$-Wiener process with covariance Q.
(3) $F: H \to H$ and $B: H \to \mathscr{L}(K,H)$ are Bochner-measurable functions satisfying

$$\bigl\|F(x)\bigr\|_H^2 + \operatorname{tr}\bigl(B(x)QB^*(x)\bigr) \le \ell\bigl(1 + \|x\|_H^2\bigr),$$

$$\bigl\|F(x) - F(y)\bigr\|_H^2 + \operatorname{tr}\bigl(\bigl(B(x) - B(y)\bigr)Q\bigl(B(x) - B(y)\bigr)^*\bigr) \le \mathscr{K}\|x - y\|_H^2.$$

Then (6.10) has a unique $\mathscr{F}_t$-adapted mild solution (Chap. 3, Theorem 3.5), which is a Markov process (Chap. 3, Theorem 3.6) and depends continuously on the initial condition (Chap. 3, Theorem 3.7). That is, the integral equation

$$X(t) = S(t)x + \int_0^t S(t-s)F\big(X(s)\big)\,ds + \int_0^t S(t-s)B\big(X(s)\big)\,dW_s \tag{6.11}$$

has a solution in $C([0,T], L^{2p}((\Omega,\mathscr{F},P),H))$, $p \ge 1$. Here $\mathscr{F} = \sigma(\bigcup_{t\ge 0}\mathscr{F}_t)$.

In addition, the solution of (6.11) can be approximated by solutions X_n obtained by using Yosida approximations of the operator A in the following manner.

Recall from (1.22), Chap. 1, that for $n \in \rho(A)$, the resolvent set of A, $R(n,A)$ denotes the resolvent of A at n, and if $R_n = nR(n,A)$, then $A_n = AR_n$ are the Yosida approximations of A. The approximating semigroup is $S_n(t) = e^{tA_n}$. Consider the strong solution X_n^x of

$$\begin{cases} dX(t) = (A_n X(t) + F(X(t)))\,dt + B(X(t))\,dW_t, \\ X(0) = x \in H. \end{cases} \tag{6.12}$$

Then $X_n^x \in C([0,T], L^{2p}((\Omega,\mathscr{F},P),H))$, $p \ge 1$, by Theorem 3.5 in Chap. 3. By Proposition 3.2 in Chap. 3, for $p \ge 1$,

$$\lim_{n\to\infty} \sup_{0\le t\le T} E\big(\big\|X_n^x(t) - X^x(t)\big\|_H^{2p}\big) = 0, \tag{6.13}$$

where $X^x(t)$ is the solution of (6.11).

We also recall the Itô formula, Theorem 2.9 in Chap. 2, for strong solutions of (6.10). Let $C^2_{b,\mathrm{loc}}([0,T]\times H)$ denote the space of twice differentiable functions $\Psi : [0,T]\times H \to \mathbb{R}$ with locally bounded and continuous partial derivatives Ψ_t, Ψ_x, and Ψ_{xx}. Let $X^x(t)$ be a strong solution of (6.10), and $\Psi \in C^2_{b,\mathrm{loc}}([0,T]\times H)$. Then, with $x \in \mathscr{D}(A)$,

$$\begin{aligned} \Psi\big(t, X^x(t)\big) - \Psi(0,x) &= \int_0^t \big(\Psi_t\big(s, X^x(s)\big) + \mathscr{L}\Psi\big(s, X^x(s)\big)\big)\,ds \\ &\quad + \int_0^t \big\langle \Psi_x\big(s, X^x(s)\big), B\big(X^x(s)\big)\,dW_s\big\rangle_H, \end{aligned} \tag{6.14}$$

where

$$\mathscr{L}\Psi(t,x) = \big\langle \Psi_x(t,x), Ax + F(x)\big\rangle_H + \frac{1}{2}\mathrm{tr}\big(\Psi_{xx}(t,x)B(x)QB^*(x)\big). \tag{6.15}$$

Clearly (6.14) is valid for strong solutions of (6.12), with $x \in H$ and A replaced by A_n in (6.15).

We are ready to discuss the stability of mild solutions of (6.10).

Definition 6.1 Let $\{X^x(t),\ t \ge 0\}$ be a mild solution of (6.10). We say that $X^x(t)$ is *exponentially stable in the mean square sense* (*m.s.s.*) if for all $t \ge 0$ and $x \in H$,

$$E\left\|X^x(t)\right\|_H^2 \le c\mathrm{e}^{-\beta t}\|x\|_H^2, \quad c, \beta > 0. \tag{6.16}$$

It is convenient to denote by $C_{2p}^2(H)$, with $p \ge 1$, the subspace of $C^2(H)$ consisting of functions $f : H \to \mathbb{R}$ whose first two derivatives satisfy the following growth condition:

$$\left\|f'(x)\right\|_H \le C\|x\|_H^{2p} \quad \text{and} \quad \left\|f''(x)\right\|_{\mathscr{L}(H)} \le C\|x\|_H^{2p}$$

for some constant $C \ge 0$.

Theorem 6.4 *The mild solution of* (6.10) *is exponentially stable in the m.s.s. if there exists a function* $\Lambda : H \to \mathbb{R}$ *satisfying the following conditions*:

(1) $\Lambda \in C_{2p}^2(H)$.
(2) *There exist constants* $c_1, c_2 > 0$ *such that*

$$c_1\|x\|_H^2 \le \Lambda(x) \le c_2\|x\|_H^2 \quad \textit{for all } x \in H.$$

(3) *There exists a constant* $c_3 > 0$ *such that*

$$\mathscr{L}\Lambda(x) \le -c_3\Lambda(x) \quad \textit{for all } x \in \mathscr{D}(A)$$

with $\mathscr{L}\Lambda(x)$ *defined in* (6.15).

Proof Assume first that the initial condition $x \in \mathscr{D}(A)$. Let $X_n^x(t)$ be the mild solution of Theorem 3.5 in Chap. 3 to the approximating equation

$$\begin{cases} dX(t) = AX(t) + R_nF(X(t))\,dt + R_nB(X(t))\,dW_t, \\ X(0) = x \in \mathscr{D}(A), \end{cases} \tag{6.17}$$

that is,

$$X_n^x(t) = S(t)x + \int_0^t S(t-s)R_nF\big(X_n^x(s)\big)\,ds + \int_0^t S(t-s)R_nB\big(X_n^x(s)\big)\,dW_s$$

with R_n defined in (1.21). We note that (6.17) is an alternative to (6.12) in approximating the mild solution of (6.10) with strong solutions. This technique preserves the operator A and we have used it in the proof of Theorem 3.11.

Since $x \in \mathscr{D}(A)$ and $R_n : H \to \mathscr{D}(A)$, the solution $X_n^x(t) \in \mathscr{D}(A)$. Moreover, since the initial condition is deterministic, Theorem 3.5 in Chap. 3 guarantees that $X_n^x \in \tilde{\mathscr{H}}_2$. Then, the linear growth of B and the boundedness of R_n and $S(t)$ implies that the conditions of Theorem 3.2 are met, so that $X_n^x(t)$ is a strong solution of (6.17). We apply Itô's formula (6.14) to $\mathrm{e}^{c_3t}\Lambda(X_n^x(t))$ and take the expectations, to obtain

$$\mathrm{e}^{c_3t}E\Lambda\big(X_n^x(t)\big) - \Lambda\big(X_n^x(0)\big) = E\int_0^t \mathrm{e}^{c_3s}\big(c_3\Lambda\big(X_n^x(s)\big) + \mathscr{L}_n\Lambda\big(X_n^x(s)\big)\big)\,ds,$$

where

$$\mathscr{L}_n\Lambda(x) = \big\langle \Lambda'(x), Ax + R_n F(x)\big\rangle_H + \frac{1}{2}\operatorname{tr}\big(\Lambda''(x)\big(R_n B(x)\big) Q \big(R_n B(x)\big)^*\big). \tag{6.18}$$

By condition (3),

$$c_3\Lambda(x) + \mathscr{L}_n\Lambda(x) \le -\mathscr{L}\Lambda(x) + \mathscr{L}_n\Lambda(x).$$

The RHS of the above equals to

$$\begin{aligned}
&\big\langle \Lambda'(x), (R_n - I)F(x)\big\rangle_H \\
&\quad + \frac{1}{2}\operatorname{tr}\big\{\Lambda''(x)\big[\big(R_n B(x)\big) Q \big(R_n B(x)\big)^* - B(x) Q \big(B(x)\big)^*\big]\big\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&e^{c_3 t} E\Lambda\big(X_n^x(t)\big) - \Lambda(x) \\
&\quad \le E\int_0^t e^{c_3 s}\Big\{\big\langle \Lambda'\big(X_n^x(s)\big), (R_n - I)F\big(X_n^x(s)\big)\big\rangle_H \\
&\qquad + \frac{1}{2}\operatorname{tr}\big\{\Lambda''\big(X_n^x(s)\big)\big[\big(R_n B\big(X_n^x(s)\big)\big) Q \big(R_n B\big(X_n^x(s)\big)\big)^* \\
&\qquad\quad - B\big(X_n^x(s)\big) Q \big(B\big(X_n^x(s)\big)\big)^*\big]\big\}\Big\}\, ds.
\end{aligned} \tag{6.19}$$

In order to pass to the limit, we need to show that

$$\sup_{0\le t\le T} E\big\|X_n^x(t) - X^x(t)\big\|_H^2 \to 0. \tag{6.20}$$

Consider

$$\begin{aligned}
&E\big\|X_n^x(t) - X^x(t)\big\|_H^2 \\
&\quad \le E\bigg\|\int_0^t S(t-s)\big(R_n F\big(X_n^x(s)\big) - F\big(X^x(s)\big)\big)\, ds \\
&\qquad + \int_0^t S(t-s)\big(R_n B\big(X_n^x(s)\big) - B\big(X^x(s)\big)\big)\, dW_s\bigg\|_H^2 \\
&\quad \le C\bigg\{E\bigg\|\int_0^t S(t-s) R_n\big(F\big(X_n^x(s)\big) - F\big(X^x(s)\big)\big)\, ds\bigg\|_H^2 \\
&\qquad + E\int_0^t \big\|S(t-s) R_n\big(B\big(X_n^x(s)\big) - B\big(X^x(s)\big)\big)\big\|_{\mathscr{L}_2(K_Q,H)}^2\, ds \\
&\qquad + E\bigg\|\int_0^t S(t-s)(R_n - I)F\big(X^x(s)\big)\, ds\bigg\|_H^2 \\
&\qquad + E\int_0^t \big\|S(t-s)(R_n - I)B\big(X^x(s)\big)\big\|_{\mathscr{L}_2(K_Q,H)}^2\, ds\bigg\}.
\end{aligned}$$

The first two summands are bounded by $C\mathscr{K} E\int_0^t \|X_n^x(s) - X^x(s)\|_H^2$ for $n > n_0$ (n_0 sufficiently large), where C depends on $\sup_{0\le t\le T}\|S(t)\|_{\mathscr{L}(H)}$ and $\sup_{n>n_0}\|R_n\|_{\mathscr{L}(H)}$, and $\mathscr{K}$ is the Lipschitz constant.

By the properties of R_n, the integrand in the third summand converges to zero, and, by (2.17) in Lemma 2.2, Chap. 2, the integrand in the fourth summand converges to zero. Both integrands are bounded by $C\ell\|X^x(s)\|_H^2$ for some constant C depending on the norms of $S(t)$ and R_n, similar as above, and the constant ℓ in the linear growth condition. By the Lebesgue DCT, the third and fourth summands can be bounded uniformly in t by $\varepsilon_n(T) \to 0$.

An appeal to Gronwall's lemma completes the argument.

The convergence in (6.20) allows us to choose a subsequence $X_{n_k}^x$ such that

$$X_{n_k}^x(t) \to X^x(t), \quad 0 \le t \le T, \quad P\text{-a.s.}$$

We will denote such a subsequence again by X_n^x.

Now we use assumption (1), the continuity and local boundedness of Λ', the continuity of F, the uniform boundedness of $\|R_n\|_{\mathscr{L}(H)}$, and the convergence $(R_n - I)x \to 0$ to conclude that

$$E\int_0^t \mathrm{e}^{c_3 s}\bigl\langle \Lambda'\bigl(X_n^x(s)\bigr), (R_n - I)F\bigl(X_n^x(s)\bigr)\bigr\rangle_H\, ds \to 0$$

by the Lebesgue DCT. Now, using Exercise 2.19, we have

$$\begin{aligned}
&\operatorname{tr}\bigl\{\Lambda''\bigl(X_n^x(s)\bigr)\bigl(R_nB\bigl(X_n^x(s)\bigr)\bigr)Q\bigl(R_nB\bigl(X_n^x(s)\bigr)\bigr)^*\bigr\}\\
&\quad= \operatorname{tr}\bigl\{\bigl(R_nB\bigl(X_n^x(s)\bigr)\bigr)^*\Lambda''\bigl(X_n^x(s)\bigr)\bigl(R_nB\bigl(X_n^x(s)\bigr)\bigr)Q\bigr\}\\
&\quad= \sum_{j=1}^{\infty}\lambda_j\bigl\langle \Lambda''\bigl(X_n^x(s)\bigr)\bigl(R_nB\bigl(X_n^x(s)\bigr)\bigr)f_j, \bigl(R_nB\bigl(X_n^x(s)\bigr)\bigr)f_j\bigr\rangle_H
\end{aligned}$$

with

$$\begin{aligned}
&\bigl\langle \Lambda''\bigl(X_n^x(s)\bigr)\bigl(R_nB\bigl(X_n^x(s)\bigr)\bigr)f_j, \bigl(R_nB\bigl(X_n^x(s)\bigr)\bigr)f_j\bigr\rangle_H\\
&\quad\to \bigl\langle \Lambda''\bigl(X^x(s)\bigr)B\bigl(X^x(s)\bigr)f_j, B\bigl(X^x(s)\bigr)f_j\bigr\rangle_H.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\operatorname{tr}\bigl\{\Lambda''\bigl(X_n^x(s)\bigr)\bigl(R_nB\bigl(X_n^x(s)\bigr)\bigr)Q\bigl(R_nB\bigl(X_n^x(s)\bigr)\bigr)^*\bigr\}\\
&\quad\to \operatorname{tr}\bigl\{\Lambda''\bigl(X^x(s)\bigr)B\bigl(X^x(s)\bigr)Q\bigl(B\bigl(X^x(s)\bigr)\bigr)^*\bigr\}.
\end{aligned}$$

Obviously,

$$\begin{aligned}
&\operatorname{tr}\bigl\{\Lambda''\bigl(X_n^x(s)\bigr)B\bigl(X_n^x(s)\bigr)Q\bigl(B\bigl(X_n^x(s)\bigr)\bigr)^*\bigr\}\\
&\quad\to \operatorname{tr}\bigl\{\Lambda''\bigl(X^x(s)\bigr)B\bigl(X^x(s)\bigr)Q\bigl(B\bigl(X^x(s)\bigr)\bigr)^*\bigr\}.
\end{aligned}$$

Now we use assumption (1), the continuity and local boundedness of Λ' and Λ'', the growth condition on F and B, and the fact that

$$\sup_{0\le t\le T} E\left\|X_n^x(s)\right\|_H^2<\infty,$$

and apply Lebesgue's DCT to conclude that the right-hand side in (6.19) converges to zero.

By the continuity of Λ and (6.20), we obtain

$$e^{c_3 t} E\Lambda\big(X^x(t)\big)\le \Lambda(x),$$

and finally, by condition (2),

$$E\left\|X^x(t)\right\|_H^2\le \frac{c_2}{c_1}e^{-c_3 t}\|x\|_H^2,\quad x\in\mathscr{D}(A). \tag{6.21}$$

We recall that the mild solution $X^x(t)$ depends continuously on the initial condition $x\in H$ in the following way (Lemma 3.7):

$$\sup_{t\le T} E\left\|X^x(t)-X^y(t)\right\|_H^2\le c_T\|x-y\|_H^2,\quad T>0.$$

Then for $t\le T$,

$$\begin{aligned}E\left\|X^x(t)\right\|_H^2&\le E\left\|X^y(t)\right\|_H^2+E\left\|X^x(t)-X^y(t)\right\|_H^2\\&\le \frac{c_2}{c_1}e^{-c_3 t}\|y\|_H^2+c_T\|x-y\|_H^2\\&\le \frac{c_2}{c_1}e^{-c_3 t}2\|x-y\|_H^2+\frac{c_2}{c_1}e^{-c_3 t}2\|x\|_H^2+c_T\|x-y\|_H^2\end{aligned}$$

for all $y\in\mathscr{D}(A)$, forcing inequality (6.21) to hold for all $x\in H$, since $\mathscr{D}(A)$ is dense in H. □

The function Λ defined in Theorem 6.4, satisfying conditions (1)–(3), is called a *Lyapunov function*.

We now consider the linear case of (6.10) with $F\equiv 0$ and $B(x)=B_0x$, where $B_0\in\mathscr{L}(H,\mathscr{L}(K,H))$, and $\|B_0x\|\le d\|x\|_H$,

$$\begin{cases}dX(t)=AX(t)\,dt+B_0X(t)\,dW_t,\\ X(0)=x\in H.\end{cases} \tag{6.22}$$

Mild solutions are solutions of the corresponding integral equation

$$X(t)=S(t)x+\int_0^t S(t-s)B_0X(s)\,dW_s. \tag{6.23}$$

The concept of exponential stability in the m.s.s. for mild solutions of (6.22) obviously transfers to this case. We show that the existence of a Lyapunov function is a necessary condition for stability of mild solutions of (6.22). The following notation will be used:

$$\mathscr{L}_0\Psi(x) = \big\langle \Psi'(x), Ax\big\rangle_H + \frac{1}{2}\operatorname{tr}\big(\Psi''(x)(B_0x)Q(B_0x)^*\big). \tag{6.24}$$

Theorem 6.5 *Assume that A generates a pseudo-contraction semigroup of operators $\{S(t),\ t \geq 0\}$ on H and that the mild solution of* (6.22) *is exponentially stable in the m.s.s. Then there exists a function $\Lambda_0(x)$ satisfying conditions* (1) *and* (2) *of Theorem* 6.4 *and the condition*

(3') $\mathscr{L}_0\Lambda_0(x) \leq -c_3\Lambda_0(x),\ x \in \mathscr{D}(A)$, *for some* $c_3 > 0$.

Proof Let

$$\Lambda_0(x) = \int_0^\infty E\big\|X^x(t)\big\|_H^2\,dt + \alpha\|x\|_H^2,$$

where the value of the constant $\alpha > 0$ will be determined later. Note that $X^x(t)$ depends on x linearly. The exponential stability in the m.s.s. implies that

$$\int_0^\infty E\big\|X^x(t)\big\|_H^2\,dt < \infty.$$

Hence, by the Schwarz inequality,

$$T(x,y) = \int_0^\infty E\big\langle X^x(t), X^y(t)\big\rangle_H\,dt$$

defines a continuous bilinear form on $H \times H$, and there exists a symmetric bounded linear operator $\tilde{T} : H \to H$ such that

$$\langle \tilde{T}x, x\rangle_H = \int_0^\infty E\big\|X^x(t)\big\|_H^2\,dt.$$

Let

$$\Psi(x) = \langle \tilde{T}x, x\rangle_H.$$

Using the same arguments, we define bounded linear operators on H by

$$\big\langle \tilde{T}(t)x, x\big\rangle_H = \int_0^t E\big\|X^x(s)\big\|_H^2\,ds.$$

Consider solutions $\{X_n^x(t),\ t \geq 0\}$ to the following equation:

$$\begin{cases} dX(t) = A_nX(t)\,dt + B_0X(t)\,dW_t, \\ X(0) = x \in H, \end{cases}$$

obtained using the Yosida approximations of A. Just as above, we have continuous bilinear forms T_n, symmetric linear operators $\tilde{T}_n(t)$, and real-valued continuous functions $\Psi_n(t)$, defined for X_n,

$$T_n(t)(x,y) = \int_0^t E\langle X_n^x(u), X_n^y(u)\rangle_H \, du,$$

$$\langle \tilde{T}_n(t)x, x\rangle_H = \int_0^t E\|X_n^x(u)\|_H^2 \, du,$$

$$\Psi_n(t)(x) = \langle \tilde{T}_n(t)x, x\rangle_H.$$

We have

$$\Psi_n(t)\big(X_n^x(s)\big) = \left(\int_0^t E\|X_n^y(u)\|_H^2 \, du\right)\bigg|_{y=X_n^x(s)}.$$

Let $\varphi : H \to \mathbb{R}$, $\varphi(h) = \|h\|_H^2$, and

$$\big(\tilde{P}_t\varphi\big)(x) = E\varphi\big(X^x(t)\big)$$

be the transition semigroup. Using the uniqueness of the solution, the Markov property (3.59) yields

$$\begin{aligned} E\Psi_n(t)\big(X_n^x(s)\big) &= E\int_0^t (\tilde{P}_u\varphi)\big(X_n^x(s)\big)\, du \\ &= E\int_0^t E\big(\varphi\big(X_n^x(u+s)\big)\big|\mathscr{F}_s^{X_n^x}\big)\, du \\ &= \int_0^t E\|X_n^x(u+s)\|_H^2 \, du \\ &= \Psi_n(t+s)(x) - \Psi_n(s)(x). \end{aligned} \tag{6.25}$$

With t and n fixed, we use the Itô formula for the function $\Psi_n(t)(x)$, then take the expectation of both sides to arrive at

$$E\big(\Psi_n(t)\big(X_n^x(s)\big)\big) = \Psi_n(t)(x) + \int_0^s E\big(\mathscr{L}_n\Psi_n(t)\big(X_n^x(u)\big)\big)\, du, \tag{6.26}$$

where

$$\mathscr{L}_n\Psi_n(t)(x) = 2\langle \tilde{T}_n(t)x, A_n x\rangle_H + \operatorname{tr}\big(\tilde{T}_n(t)(B_0x)Q(B_0x)^*\big).$$

Putting (6.25) and (6.26) together, we have

$$\Psi_n(t+s)(x) - \Psi_n(s)(x) = \int_0^s E\big(\mathscr{L}_n\Psi_n(t)\big(X_n^x(u)\big)\big)\, du + \Psi_n(t)(x).$$

Rearranging the above, dividing by s, and taking the limit as $s \to 0$ give

$$\frac{\Psi_n(t+s)(x) - \Psi_n(t)(x)}{s} = \frac{1}{s}\int_0^s E\big(\mathscr{L}_n\Psi_n(t)\big(X_n^x(u)\big)\big)\, du + \frac{\Psi_n(s)(x)}{s}. \tag{6.27}$$

We fix n and t, and intend to take the limit in (6.27) as $s \to 0$.

The processes $X_n^x(u)$ are continuous in the mean-square, since

$$\begin{aligned} &E\left\|X_n^x(u) - X_n^x(v)\right\|_H^2 \\ &\quad \le 2\big(\|A_n\|^2_{\mathscr{L}(H)} + \|B_0\|^2_{\mathscr{L}(H,\mathscr{L}(K,H))}\big)\,\mathrm{tr}(Q)\int_u^v E\left\|X_n^x(r)\right\|_H^2\,dr. \end{aligned}$$

Hence,

$$\lim_{s\to 0}\frac{\Psi_n(s)(x)}{s} = \lim_{s\to 0}\frac{1}{s}\int_0^s E\left\|X_n^x(u)\right\|_H^2\,du = \|x\|_H^2. \tag{6.28}$$

Now consider

$$\begin{aligned} &E\mathscr{L}_n\Psi_n(t)\big(X_n^x(u)\big) \\ &\quad = E\big(2\langle \tilde{T}_n(t)X_n^x(u), A_nX_n^x(u)\rangle_H\big) + E\big(\mathrm{tr}\big(\tilde{T}_n(t)\big(B_0X_n^x(u)\big)Q\big(B_0X_n^x(u)\big)^*\big)\big). \end{aligned}$$

Since

$$\lim_{u\to 0} A_nX_n^x(u) = A_nx, \qquad \lim_{u\to 0}\tilde{T}_n(t)X_n^x(u) = \tilde{T}_n(t)x,$$

and

$$\left|\langle \tilde{T}_n(t)X_n^x(u), A_nX_n^x(u)\rangle\right| \le \left\|\tilde{T}_n(t)\right\|_{\mathscr{L}(H)}\|A_n\|_{\mathscr{L}(H)}\left\|X_n^x(u)\right\|_H^2 \in L^1(\Omega),$$

the Lebesgue DCT gives

$$\lim_{u\to 0} E\big(2\langle \tilde{T}_n(t)X_n^x(u), A_nX_n^x(u)\rangle_H\big) = 2\langle \tilde{T}_n(t)x, A_nx\rangle_H.$$

For the term involving the *trace*, we simplify the notation and denote

$$\Phi_n(u) = B_0X_n^x(u) \quad \text{and} \quad x_n^j(u) = \Phi_n(u)f_j,$$

where $\{f_j\}_{j=1}^\infty$ is an ONB in K that diagonalizes the covariance operator Q. Using Exercise 2.19, we have

$$\begin{aligned} \mathrm{tr}\big(\tilde{T}_n(t)\Phi_n(u)Q\big(\Phi_n(u)\big)^*\big) &= \mathrm{tr}\big(\big(\Phi_n(u)\big)^*\tilde{T}_n(t)\Phi_n(u)Q\big) \\ &= \sum_{j=1}^\infty \lambda_j\langle \tilde{T}_n(t)\Phi_n(u)f_j, \Phi_n(u)f_j\rangle_H \\ &= \sum_{j=1}^\infty \lambda_j\langle \tilde{T}_n(t)x_n^j(u), x_n^j(u)\rangle_H \\ &= \sum_{j=1}^\infty \lambda_j\int_0^t E\left\|X_n^{x_n^j(u)}(s)\right\|_H^2\,ds. \end{aligned} \tag{6.29}$$

Denote $x^j = (B_0 x) f_j$. Since B_0 is continuous, as $u \to 0$, $B_0 X_n^x(u) \to B_0 x$ in $\mathscr{L}(K, H)$, so that $x_n^j(u) \to x^j$ in H. By the continuity of the solution X_n with respect to the initial condition (Chap. 3, Lemma 3.7),

$$\sup_{0 \le s \le T} E \left\| X_n^{x_n^j(u)}(s) - X_n^{x^j}(s) \right\|_H^2 \to 0 \quad \text{as } u \to 0,$$

so that, by Lebesgue's DCT and by reversing the calculations in (6.29),

$$\begin{aligned}
&\sum_{j=1}^{\infty} \lambda_j \int_0^t E \left\| X_n^{x_n^j(u)}(s) \right\|_H^2 ds \\
&\quad \to \sum_{j=1}^{\infty} \lambda_j \int_0^t E \left\| X_n^{x^j}(s) \right\|_H^2 ds = \operatorname{tr}\big(\tilde{T}_n(t)(B_0 x) Q (B_0 x)^*\big).
\end{aligned}$$

Summarizing, we proved that

$$\frac{\mathrm{d}\Psi_n(t)(x)}{\mathrm{d}t} = \mathscr{L}_n \Psi_n(t)(x) + \|x\|_H^2.$$

In the next step, we fix t and allow $n \to \infty$. By the mean-square continuity of $X_n^x(t)$ and the definition of $\langle \tilde{T}_n(t)x, x\rangle_H$ and $\langle \tilde{T}(t)x, x\rangle_H$, we can calculate the derivatives below, and the convergence follows from condition (6.13):

$$\frac{\mathrm{d}\Psi_n(t)(x)}{\mathrm{d}t} = E \left\| X_n^x(t) \right\|_H^2 \to E \left\| X^x(t) \right\|_H^2 = \frac{\mathrm{d}\Psi(t)(x)}{\mathrm{d}t}.$$

Now, we need to show that as $n \to \infty$, for $x \in \mathscr{D}(A)$,

$$\mathscr{L}_n \big\langle \tilde{T}_n(t)x, x \big\rangle_H \to \mathscr{L}_0 \big\langle \tilde{T}(t)x, x \big\rangle_H. \tag{6.30}$$

Consider

$$\begin{aligned}
&\left| \big\langle \tilde{T}_n(t)x, A_n x \big\rangle_H - \big\langle \tilde{T}(t)x, Ax \big\rangle_H \right| \\
&\quad \le \left\| \tilde{T}_n(t)x \right\|_H \left\| (A_n - A)x \right\|_H + \left| \big\langle \big(\tilde{T}_n(t) - \tilde{T}(t)\big)x, Ax \big\rangle_H \right| \to 0.
\end{aligned}$$

Since (6.13) implies that

$$\lim_{n \to \infty} E \int_0^T \left\| X_n^x(u) - X^x(u) \right\|_H^2 du = 0, \tag{6.31}$$

we thus have the weak convergence of $T_n(t)x$ to $T(t)x$, and, further, by the Banach–Steinhaus theorem, we deduce that $\sup_n \|T_n(t)\|_{\mathscr{L}(H)} < \infty$. Using calculations similar as in (6.29), we have

$$\begin{aligned}\operatorname{tr}\big(\tilde{T}_n(t)(B_0x)Q(B_0x)^*\big) &= \sum_{j=1}^{\infty}\lambda_j\big\langle\tilde{T}_n(t)x^j, x^j\big\rangle_H \\ &\to \sum_{j=1}^{\infty}\lambda_j\big\langle\tilde{T}(t)x^j, x^j\big\rangle_H \\ &= \operatorname{tr}\big(\tilde{T}(t)B_0xQ(B_0x)^*\big),\end{aligned}$$

by Lebesgue's DCT, proving the convergence in (6.30). Summarizing, we have

$$\frac{\mathrm{d}\langle\tilde{T}(t)x, x\rangle_H}{\mathrm{d}t} = \mathscr{L}_0\big\langle\tilde{T}(t)x, x\big\rangle_H + \|x\|_H^2.$$

We will now let $t \to \infty$. Then, by the exponential stability condition,

$$\frac{\mathrm{d}\langle\tilde{T}(t)x, x\rangle_H}{\mathrm{d}t} = E\big\|X^x(t)\big\|_H^2 \to 0.$$

Since $\langle\tilde{T}(t)x, x\rangle_H \to \langle\tilde{T}x, x\rangle_H$, using the weak convergence of $\tilde{T}(t)x$ to $\tilde{T}x$ and the Lebesgue DCT, exactly as above, we obtain that

$$\begin{aligned}&\mathscr{L}_0\big\langle\tilde{T}(t)x, x\big\rangle_H = 2\big\langle\tilde{T}(t)x, Ax\big\rangle_H + \operatorname{tr}\big(\tilde{T}(t)B_0xQ(B_0x)^*\big) \\ &\quad\to 2\langle\tilde{T}x, Ax\rangle_H + \operatorname{tr}\big(\tilde{T}B_0xQ(B_0x)^*\big) = \mathscr{L}_0\langle\tilde{T}x, x\rangle_H.\end{aligned}$$

In conclusion,

$$\mathscr{L}_0\Psi(x) = -\|x\|_H^2, \quad x \in \mathscr{D}(A).$$

Now, Λ_0 satisfies conditions (1) and (2). To prove condition (3'), let us note that, as in Sect. 6.1, since $\|S(t)\| \le \mathrm{e}^{\omega t}$, inequality (6.4) is valid for some constant $\lambda > 0$. Hence,

$$\mathscr{L}_0\|x\|_H^2 = 2\langle x, Ax\rangle_H + \operatorname{tr}\big((B_0x)Q(B_0x)^*\big) \le \big(2\lambda + d^2\operatorname{tr}Q\big)\|x\|_H^2 \tag{6.32}$$

gives

$$\mathscr{L}_0\Lambda_0(x) \le -\|x\|_H^2 + \alpha\big(2\lambda + d^2\operatorname{tr}(Q)\big)\|x\|_H^2 \le -c_3\Lambda_0(x),$$

$c_3 > 0$, by choosing α small enough. □

Remark 6.1 For the nonlinear equation (6.10), we need to assume $F(0) = 0$ and $B(0) = 0$ to assure that zero is a solution. In this case, if the solution $\{X^x(t),\ t \ge 0\}$ is exponentially stable in the m.s.s., we can still construct

$$\Lambda(x) = \int_0^{\infty} E\big\|X^x(t)\big\|_H^2\,dt + \alpha\|x\|_H^2.$$

We however do not know if it satisfies condition (1) of Theorem 6.4. If we assume that it does, then one can show, as in Theorem 6.5, that it satisfies condition (2). Then we can prove that $\Lambda(x)$ also satisfies condition (3).

First, observe that for $\Psi(x) = \langle Rx, x\rangle_H$, as before,

$$\mathscr{L}\Psi(x) = -\|x\|_H^2$$

and

$$\begin{aligned}\mathscr{L}\Lambda(x) &= \mathscr{L}\Psi(x) + \alpha\mathscr{L}\|x\|_H^2 \\ &= -\|x\|_H^2 + \alpha\big(2\langle x, Ax + F(x)\rangle_H + \operatorname{tr}\big(B(x)QB^*(x)\big)\big),\end{aligned}$$

noting the form of the infinitesimal generator $\mathscr{L}$ of the Markov process $X^x(t)$. We obtain

$$\mathscr{L}\Lambda(x) \le -\|x\|_H^2 + 2\alpha\lambda\|x\|_H^2 + \alpha\big(2\langle x, F(x)\rangle_H + \operatorname{tr}\big(B(x)QB^*(x)\big)\big).$$

Now using the fact that $F(0) = 0$, $B(0) = 0$, and the Lipschitz property of F and B, we obtain

$$\mathscr{L}\Lambda(x) \le -\|x\|_H^2 + \alpha\big(2\lambda + 2\mathscr{K} + \mathscr{K}^2\operatorname{tr}(Q)\big)\|x\|_H^2.$$

Hence, for α small enough, condition (3) follows from condition (2).

As shown in Part I, the differentiability with respect to the initial value requires stringent assumptions on the coefficients F and B. In order to make the result more applicable, we provide another technique that uses first-order approximation. We use trace norm of a difference of nonnegative definite operators in the approximation condition. Recall that for any trace-class operator T, we defined the trace norm in (2.1) by

$$\tau(T) = \operatorname{tr}\big((TT^*)^{1/2}\big).$$

Note (see [68]) that for a trace-class operator T and a bounded operator S,

(a) $|\operatorname{tr}(T)| \le \tau(T)$,
(b) $\tau(ST) \le \|S\|\tau(T)$ and $\tau(TS) \le \|S\|\tau(T)$.

Theorem 6.6 *Assume that A generates a pseudo-contraction semigroup of operators $\{S(t), t \ge 0\}$ on H. Suppose that the solution $\{X_0^x(t), t \ge 0\}$ of the linear equation (6.22) is exponentially stable in the m.s.s. Then the solution $\{X^x(t), t \ge 0\}$ of (6.10) is exponentially stable in the m.s.s. if*

$$2\|x\|_H\big\|F(x)\big\|_H + \tau\big(B(x)QB^*(x) - B_0xQ(B_0x)^*\big) \le \frac{\beta}{2c}\|x\|_H^2. \tag{6.33}$$

Proof Let $\Lambda_0(x) = \langle \tilde{T}x, x\rangle_H + \alpha\|x\|_H^2$, as in the proof of Theorem 6.5. Note that $\langle \tilde{T}x, x\rangle_H = E\int_0^\infty \|X_0^x(t)\|_H^2\,dt$, so that

$$\langle \tilde{T}x, x\rangle_H \le \int_0^\infty c\mathrm{e}^{-\beta t}\|x\|_H^2\,dt = \frac{c}{\beta}\|x\|_H^2.$$

Hence, $\|\tilde{T}\|_{\mathscr{L}(H)} \le c/\beta$. Clearly Λ_0 satisfies conditions (1) and (2) of Theorem 6.4. It remains to prove that

$$\mathscr{L}\Lambda_0(x) \le -c_3\Lambda_0(x).$$

Consider

$$\begin{aligned}
&\mathscr{L}\Lambda_0(x) - \mathscr{L}_0\Lambda_0(x)\\
&\quad = \big\langle \Lambda_0'(x), F(x)\big\rangle_H + \frac{1}{2}\operatorname{tr}\big(\Lambda_0''(x)\big(B(x)QB^*(x) - (B_0x)Q(B_0x)^*\big)\big)\\
&\quad \le 2\big\langle (\tilde{T}+\alpha)x, F(x)\big\rangle_H + \tau\big((\tilde{T}+\alpha)\big(B(x)QB^*(x) - (B_0x)Q(B_0x)^*\big)\big)\\
&\quad \le \big(\|\tilde{T}\|_{\mathscr{L}(H)} + \alpha\big)\big(2\|x\|_H\big\|F(x)\big\|_H + \tau\big(B(x)QB^*(x) - (B_0x)Q(B_0x)^*\big)\big)\\
&\quad \le \left(\frac{1}{2} + \alpha\frac{\beta}{2c}\right)\|x\|_H^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathscr{L}\Lambda_0(x) &\le \mathscr{L}_0\Lambda_0(x) + \left(\frac{1}{2} + \alpha\frac{\beta}{2c}\right)\|x\|_H^2\\
&\le -\|x\|_H^2 + \alpha\big(2\lambda + d^2\operatorname{tr}(Q)\big)\|x\|_H^2 + \left(\frac{1}{2} + \frac{\alpha\beta}{2c}\right)\|x\|_H^2.
\end{aligned}$$

For α small enough, we obtain condition (3) in Theorem 6.4 using condition (2). □

We now consider stability in probability of the zero solution of (6.10).

Definition 6.2 Let $\{X^x(t)\}_{t\ge 0}$ be the mild solution of (6.10) with $F(0) = 0$ and $B(0) = 0$ (assuring that zero is a solution). The *zero solution* of (6.10) is called *stable in probability* if for any $\varepsilon > 0$,

$$\lim_{\|x\|_H\to 0} P\Big(\sup_{t\ge 0}\big\|X^x(t)\big\|_H > \varepsilon\Big) = 0. \tag{6.34}$$

Once a Lyapunov function satisfying conditions (1) and (2) of Theorem 6.4 is constructed, the following theorem provides a technique for proving condition (6.34).

Theorem 6.7 *Let $X^x(t)$ be the solution of* (6.10). *Assume that there exists a function $\Psi \in C_{2p}^2(H)$ having the following properties*:

(1) $\Psi(x) \to 0$ *as* $\|x\|_H \to 0$.
(2) $\inf_{\|x\|_H > \varepsilon} \Psi(x) = \lambda_\varepsilon > 0$.
(3) $\mathscr{L}\Psi(x) \le 0$, *when* $x \in \mathscr{D}(A)$ *and* $\|x\|_H < \delta$ *for some* $\delta > 0$.

Then, $\{X^x(t),\ t \ge 0\}$ *satisfies condition* (6.34).

Proof The proof is similar to the proof of Theorem 6.4. We assume first that the initial condition $x \in \mathscr{D}(A)$ and consider strong solutions $X_n^x(t)$ of the approximating equations (6.17), $n = 1, 2, \ldots,$

$$X_n^x(t) = S(t)x + \int_0^t S(t-s) R_n F\big(X_n^x(s)\big)\,ds + \int_0^t S(t-s) R_n B\big(X_n^x(s)\big)\,dW_s.$$

Denote $B_\varepsilon = \{x \in H : \|x\| < \varepsilon\}$ and let

$$\tau_\varepsilon = \inf\big\{t : \big\|X^x(t)\big\|_H > \varepsilon\big\} \quad \text{and} \quad \tau_\varepsilon^n = \inf\big\{t : \big\|X_n^x(t)\big\|_H > \varepsilon\big\}.$$

Applying Itô's formula to $\Psi(X_n^x(t))$ and taking the expectations yield

$$E\psi\big(X_n^x\big(t \wedge \tau_\varepsilon^n\big)\big) - \psi(x) = E\int_0^{t\wedge\tau_\varepsilon} \mathscr{L}_n \Psi\big(X_n^x(s)\big)\,ds,$$

where

$$\mathscr{L}_n \Psi(x) = \big\langle \Psi'(x), Ax + R_n F(x)\big\rangle_H + \frac{1}{2}\operatorname{tr}\big(\Psi''(x)\big(R_n B(x)\big) Q \big(R_n B(x)\big)^*\big).$$

Let $\varepsilon < \delta$. Then for $x \in B_\varepsilon$, using condition (3), we get

$$\begin{aligned}
\mathscr{L}_n \Psi(x) &\le -\mathscr{L}\Psi(x) + \mathscr{L}_n \Psi(x) \\
&= \big\langle \Psi'(x), (R_n - I) F(x)\big\rangle_H \\
&\quad + \frac{1}{2}\operatorname{tr}\big\{\Psi''(x)\big[\big(R_n B(x)\big) Q \big(R_n B(x)\big)^* - B(x) Q \big(B(x)\big)^*\big]\big\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&E\Psi\big(X_n^x(t \wedge \tau_\varepsilon^n)\big) - \Psi(x) \\
&\quad \le E\int_0^{t\wedge\tau_\varepsilon^n} \big\{\big\langle \Psi'\big(X_n^x(s)\big), (R_n - I) F\big(X_n^x(s)\big)\big\rangle_H \\
&\qquad + \frac{1}{2}\operatorname{tr}\big\{\Psi''\big(X_n^x(s)\big)\big[\big(R_n B\big(X_n^x(s)\big)\big) Q \big(R_n B\big(X_n^x(s)\big)\big)^* \\
&\qquad - B\big(X_n^x(s)\big) Q \big(B\big(X_n^x(s)\big)\big)^*\big]\big\}\big\}\,ds.
\end{aligned} \tag{6.35}$$

Using (6.20) and passing to the limit, as in the proof of Theorem 6.4, show that the RHS in (6.35) converges to zero. Using condition (2), we conclude that for $x \in \mathscr{D}(A) \cap B_\varepsilon$ and any n,

$$\Psi(x) \ge E\big(\Psi\big(X_n^x\big(t \wedge \tau_\varepsilon^n\big)\big)\big) \ge \lambda_\varepsilon P\big(\tau_\varepsilon^n < t\big). \tag{6.36}$$

By the a.s. (and hence weak) convergence of τ_ε^n to τ_ε, we have that

$$\Psi(x) \geq \lambda_\varepsilon \liminf_{n\to\infty} P\big(\tau_\varepsilon^n < t\big) \geq \lambda_\varepsilon P(\tau_\varepsilon < t).$$

To remove the restriction that $x \in \mathscr{D}(A)$, recall that

$$\sup_{0\leq t\leq T} E\big\|X^x(t) - X^y(t)\big\|_H^2 \to 0 \quad \text{as } \|y - x\|_H \to 0.$$

We can select a sequence $y_n \to x$, $y_n \in \mathscr{D}(A)$, such that $X^{y_n}(t) \to X^x(t)$ a.s. for all t. Now using the assumptions on Ψ and the Lebesgue DCT, we obtain (6.36) for all $x \in H$. Inequality (6.36), together with conditions (2) and (1), implies that for $x \in B_\varepsilon$,

$$P\Big(\sup_{t\geq 0}\big\|X_t^x\big\|_H > \varepsilon\Big) \leq \frac{\Psi(x)}{\lambda_\varepsilon} \to 0, \quad \|x\|_H \to 0,$$

giving (6.34). □

The following results are now obvious from Theorems 6.5 and 6.6.

Theorem 6.8 *Assume that A generates a pseudo-contraction semigroup of operators $\{S(t),\ t \geq 0\}$ on H. If the solution $X_0^x(t)$ of the linear equation* (6.22) *is exponentially stable in the m.s.s., then the zero solution of* (6.22) *is stable in probability.*

Theorem 6.9 *Assume that A generates a pseudo-contraction semigroup of operators $\{S(t),\ t \geq 0\}$ on H. If the solution $X_0^x(t)$ of the linear equation* (6.22) *is exponentially stable in the m.s.s. and condition* (6.33) *holds for a sufficiently small neighborhood of $x = 0$, then the zero solution of* (6.10) *is stable in probability.*

We note that the exponential stability gives degenerate invariant measures. To obtain nondegenerate invariant measures, we use a more general concept introduced in Chap. 7.

6.3 Stability in the Variational Method

We consider a Gelfand triplet of real separable Hilbert spaces

$$V \hookrightarrow H \hookrightarrow V^*.$$

The space V^* is the continuous dual of V, V is dense in H, and all embeddings are continuous. With $\langle\cdot,\cdot\rangle$ denoting the duality between V and V^*, we assume that for $h \in H$,

$$\langle v, h\rangle = \langle v, h\rangle_H.$$

Let K be a real separable Hilbert space, and Q a nonnegative definite trace-class operator on K. We consider $\{W_t,\ t \ge 0\}$, a K-valued Q-Wiener process defined on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}\}_{t\ge 0}, P)$.

Let $\boldsymbol{M}^2([0,T],V)$ denote the space of all V-valued measurable processes satisfying

(1) $u(t,\cdot)$ is $\mathscr{F}_t$-measurable,
(2) $E\int_0^T \|u(t,\omega)\|_V^2\,dt < \infty$.

Throughout this section we consider the following equation:

$$\begin{cases} dX(t) = A(X(t))\,dt + B(X(t))\,dW_t, \\ X(0) = x \in H, \end{cases} \tag{6.37}$$

where A and B are in general nonlinear mappings, $A : V \to V^*$, $B : V \to \mathscr{L}(K,H)$, and

$$\big\|A(v)\big\|_{V^*} \le a_1\|v\|_V \quad \text{and} \quad \big\|B(v)\big\|_{\mathscr{L}(K,H)} \le b_1\|v\|_V, \quad v \in V, \tag{6.38}$$

for some positive constants a_1, b_1.

We recall the coercivity and weak monotonicity conditions from Chap. 4, which we impose on the coefficients of (6.37)

(C) (Coercivity) There exist $\alpha > 0$, γ, $\lambda \in \mathbb{R}$ such that for all $v \in V$,

$$2\big\langle v, A(v)\big\rangle + \operatorname{tr}\big(B(v)QB^*(v)\big) \le \lambda\|v\|_H^2 - \alpha\|v\|_V^2 + \gamma. \tag{6.39}$$

(WM) (Weak Monotonicity) There exists $\lambda \in \mathbb{R}$ such that for all $u, v \in V$,

$$\begin{aligned} 2\big\langle u - v, A(u) - A(v)\big\rangle + \operatorname{tr}\Big(\big(B(u) - B(v)\big)Q\big(B(u) - B(v)\big)^*\Big) \\ \le \lambda\|u - v\|_H^2. \end{aligned} \tag{6.40}$$

Since conditions (6.38), (6.39), and (6.40) are stronger than the assumptions in Theorem 4.7 (also in Theorem 4.4) of Chap. 4, we conclude that there exists a unique strong solution $\{X^x(t),\ t \ge 0\}$ of (6.37) such that

$$X^x(\cdot) \in L^2\Big(\Omega, C\big([0,T],H\big)\Big) \cap \boldsymbol{M}^2\big([0,T],V\big).$$

Furthermore, the solution $X^x(t)$ is Markovian, and the corresponding semigroup has the Feller property.

The major tool we will use will be the Itô formula due to Pardoux [62]. It was introduced in Part I, Sect. 4.2, Theorem 4.3, for the function $\Psi(u) = \|u\|_H^2$.

Theorem 6.10 (Itô Formula) *Suppose that* $\Psi : H \to \mathbb{R}$ *satisfies the following conditions*:

(1) Ψ *is twice Fréchet differentiable, and* Ψ, Ψ', Ψ'' *are locally bounded.*
(2) Ψ *and* Ψ' *are continuous on* H.
(3) *For all trace-class operators* T *on* H, $\operatorname{tr}(T\Psi''(\cdot)) : H \to \mathbb{R}$ *is continuous.*

(4) *If* $v \in V$, *then* $\Psi'(v) \in V$, *and for any* $v' \in V^*$, *the function* $\langle \Psi'(\cdot), v' \rangle : V \to \mathbb{R}$ *is continuous.*

(5) $\|\Psi'(v)\|_V \le c_0(1 + \|v\|_V)$ *for some constant* $c_0 > 0$ *and any* $v \in V$.

Let $X^x(t)$ *be a solution of* (6.37) *in* $L^2(\Omega, C([0,T], H)) \cap \boldsymbol{M}^2([0,T], V)$. *Then*

$$\Psi\big(X^x(t)\big) = \Psi(x) + \int_0^t \mathscr{L}\Psi\big(X^x(s)\big)\,ds + \int_0^t \big\langle \Psi'\big(X^x(s)\big), B\big(X^x(s)\big)dW_s\big\rangle_H,$$

where

$$\mathscr{L}\Psi(u) = \big\langle \Psi'(u), A(u)\big\rangle + \frac{1}{2}\operatorname{tr}\big(\Psi''(u)B(u)QB^*(u)\big).$$

We extend the notion of exponential stability in the m.s.s. to the variational case.

Definition 6.3 We say that the strong solution of the variational equation (6.37) in the space $L^2(\Omega, C([0,T], H)) \cap \boldsymbol{M}^2([0,T], V)$ is *exponentially stable in the m.s.s.* if it satisfies condition (6.16) in Definition 6.1.

The following is the analogue of Theorem 6.4 in the variational context. The proof for a strong solution is a simplified version of the proof of Theorem 6.4 and is left to the reader as an exercise.

Theorem 6.11 *The strong solution of the variational equation* (6.37) *in the space* $L^2(\Omega, C([0,T], H)) \cap \boldsymbol{M}^2([0,T], V)$ *is exponentially stable in the m.s.s. if there exists a function* Ψ *satisfying conditions* (1)–(5) *of Theorem* 6.10, *and the following two conditions hold*:

(1) $c_1\|x\|_H^2 \le \Psi(x) \le c_2\|x\|_H^2$, $c_1, c_2 > 0$, $x \in H$.

(2) $\mathscr{L}\Psi(v) \le -c_3\Psi(v)$, $c_3 > 0$, $v \in V$, *with* $\mathscr{L}$ *defined in Theorem* 6.10.

Exercise 6.2 Prove Theorem 6.11.

We now consider the linear problem analogous to (6.37). Let $A_0 \in \mathscr{L}(V, V^*)$ and $B_0 \in \mathscr{L}(V, \mathscr{L}(K, H))$. In order to construct a Lyapunov function directly from the solution, we assume a more restrictive coercivity condition

(C') (Coercivity) There exist $\alpha > 0$, $\lambda \in \mathbb{R}$ such that for all $v \in V$,

$$2\langle v, A_0 v\rangle + \operatorname{tr}\big((B_0 v)Q(B_0 v)^*\big) \le \lambda\|v\|_H^2 - \alpha\|v\|_V^2. \tag{6.41}$$

We denote by $\mathscr{L}_0$ the operator $\mathscr{L}$ with A and B replaced by A_0 and B_0. Consider the following linear problem:

$$\begin{cases} dX(t) = A_0 X(s)\,ds + B_0(X(s))\,dW_s, \\ X(0) = x \in H. \end{cases} \tag{6.42}$$

Theorem 6.12 *Under the coercivity condition* (6.41), *the solution of the linear equation* (6.42) *is exponentially stable in the m.s.s. if and only if there exists a* *function Ψ satisfying conditions* (1)–(5) *of Theorem* 6.10 *and conditions* (1) *and* (2) *of Theorem* 6.11.

Remark 6.2 A function Ψ satisfying conditions in Theorem 6.12 is called a *Lyapunov function*.

Proof It remains to prove the necessity. By the Itô formula applied to $\|x\|_H^2$, taking expectations, and using condition (6.41), we have

$$\begin{aligned}
E\big\|X^x(t)\big\|_H^2 &= \|x\|_H^2 + 2E\int_0^t \big\langle A_0X^x(s), X^x(s)\big\rangle\,ds \\
&\quad + E\int_0^t \operatorname{tr}\big(B_0X^x(s)Q\big(B_0X^x(s)\big)^*\big)\,ds \\
&\le \|x\|_H^2 + |\lambda|\int_0^t E\big\|X^x(s)\big\|_H^2\,ds - \alpha\int_0^t E\big\|X^x(s)\big\|_V^2\,ds \\
&\le \|x\|_H^2\big(1+|\lambda|c\big)\int_0^t \mathrm{e}^{-\beta s}\,ds - \alpha\int_0^t E\big\|X^x(s)\big\|_V^2\,ds
\end{aligned}$$

by exponential stability in the m.s.s. Let $t\to\infty$. Then

$$\int_0^\infty E\big\|X^x(s)\big\|_V^2\,ds \le \frac{1}{\alpha}\big(1+|\lambda|c/\beta\big)\|x\|_H^2.$$

Define

$$T(x,y) = \int_0^\infty E\big\langle X^x(s), X^y(s)\big\rangle_V\,ds.$$

Then, by the preceding inequality and the Schwarz inequality, it is easy to see that T is a continuous bilinear form on $H\times H$. Since the embedding $V\hookrightarrow H$ is continuous, T is also a continuous bilinear form on $V\times V$. This fact can be used to show that conditions (1)–(5) of Theorem 6.10 are satisfied by the function $\Psi(x)=T(x,x)$. Clearly, $\Psi(x)\le c_2\|x\|_H^2$. To prove the lower bound on $\Psi(x)$, we observe that

$$\mathscr{L}_0\|v\|_H^2 = 2\langle v, A_0v\rangle + \operatorname{tr}\big((B_0v)Q(B_0v)^*\big),$$

so that, for some constants m, c_0',

$$\big|\mathscr{L}_0\|v\|_H^2\big| \le c_0\|v\|_V^2 + m\operatorname{tr}(Q)\|v\|_V^2 \le c_0'\|v\|_V^2.$$

Again, by Itô's formula, after taking the expectations, we obtain that

$$\begin{aligned}
E\big\|X^x(t)\big\|_H^2 - \|x\|_H^2 &= \int_0^t E\mathscr{L}_0\big\|X^x(s)\big\|_H^2\,ds \\
&\ge -c_0'\int_0^t E\big\|X^x(s)\big\|_V^2\,ds.
\end{aligned}$$

As $t \to \infty$, using exponential stability in the m.s.s., we can see that

$$\Psi(x) \ge c_1 \|x\|_H^2,$$

where $c_1 = 1/c_0'$.

To prove the last condition, observe that, similarly as in (6.25), the uniqueness of the solution and the Markov property (3.59) yield

$$\begin{aligned} E\Psi\big(X^x(t)\big) &= \int_0^\infty E\big\|X^x(s+t)\big\|_V^2\,ds \\ &= \int_t^\infty E\big\|X^x(s)\big\|_V^2\,ds \\ &\le \int_0^\infty E\big\|X^x(s)\big\|_V^2\,ds - k\int_0^t E\big\|X^x(s)\big\|_H^2\,ds, \end{aligned}$$

since $k\|x\|_H^2 \le \|x\|_V^2$ for some constant k. Hence, by taking the derivatives of both sides at $t = 0$, we get

$$\mathscr{L}_0\Psi(x) \le -k\|x\|_H^2 \le -\frac{k}{c_2}\Psi(x).$$

□

Remark 6.3 Note that in case where $t \to E\|X^x(t)\|_V^2$ is continuous at zero, in the last step of the proof of Theorem 6.12, we obtain that $\mathscr{L}_0\Psi(v) = -\|v\|_V^2$ for $v \in V$.

Let us now state analogues of Theorem 6.6 for the solutions in variational case.

Theorem 6.13 *Let $\{X_0(t)\}_{t\ge 0}$ be the solution of the linear equation (6.42) with the coefficients satisfying condition (6.41). Assume that the function $t \to E\|X_0(t)\|_V^2$ is continuous and that the solution $X_0(t)$ is exponentially stable in the m.s.s. If for a sufficiently small constant c,*

$$2\|v\|_V \big\|A(v) - A_0 v\big\|_{V^*} + \tau\big(B(v)QB^*(v) - B_0 v Q(B_0 v)^*\big) \le c\|v\|_V^2, \tag{6.43}$$

then the strong solution of (6.37) is exponentially stable in the m.s.s.

For the zero solution of (6.37) to be stable in probability, it is enough to assume (6.43) for $v \in (V, \|\cdot\|_V)$ in a sufficiently small neighborhood of zero.

Theorem 6.14 *Let $\{X_0(t)\}_{t\ge 0}$ be the solution of the linear equation (6.42) with the coefficients satisfying condition (6.41). Assume that the solution $X_0(t)$ is exponentially stable in the m.s.s. Let for $v \in V$, $A(v) - A_0 v \in H$. If for a sufficiently small constant c,*

$$2\|v\|_H \big\|A(v) - A_0 v\big\|_H + \tau\big(B(v)QB^*(v) - B_0 v Q(B_0 v)^*\big) \le c\|v\|_H^2, \tag{6.44}$$

then the strong solution of (6.37) is exponentially stable in the m.s.s.

For the zero solution of (6.37) to be stable in probability, it is enough to assume (6.44) for $v \in (H, \|\cdot\|_H)$ in a sufficiently small neighborhood of zero.

Exercise 6.3 Verify that Theorem 6.7 holds for 6.37 (replacing (6.10)), under additional assumptions (1)–(5) of Theorem 6.10.

Exercise 6.4 Prove Theorems 6.13 and 6.14.

Remark 6.4 Using an analogue of Theorem 6.7 with the function Ψ satisfying conditions (1)–(5) of Theorem 6.10, we can also prove conclusions in Theorems 6.8 and 6.9 for (6.37) and its linear counterpart (6.42) under conditions (6.43) and (6.44).

Appendix: Stochastic Analogue of the Datko Theorem

Theorem 6.15 *Let A generate a pseudo-contraction C_0 semigroup $\{S(t),\, t \geq 0\}$ on a real separable Hilbert space H, and $B : H \to \mathscr{L}(K, H)$. A mild solution $\{X^x(t),\, t \geq 0\}$ of the stochastic differential equation (6.22) is exponentially stable in the m.s.s. if and only if there exists a nonnegative definite operator $R \in \mathscr{L}(H)$ such that*

$$\mathscr{L}_0 \langle Rx, y\rangle_H = -\langle x, y\rangle_H \quad \textit{for all } x, y \in H,$$

where $\mathscr{L}_0$ is defined in (6.24).

Proof The necessity part was already proved in Sect. 6.2, Theorem 6.5, with

$$\langle Rx, y\rangle_H = \int_0^\infty E\big\langle X^x(t), X^y(t)\big\rangle_H \, dt,$$

which, under stability assumption, is well defined by the Schwarz inequality. To prove the sufficiency, assume that R as postulated exists; then

$$2\langle Rx, Ay\rangle_H = -\big\langle \big(I + \Delta(R)\big)x, y\big\rangle_H, \tag{6.45}$$

where $\Delta(R) = \operatorname{tr}(R(B_0 x) Q (B_0 x)^*)\, I$. The operator $I + \Delta(R)$ is invertible, so that we get

$$2\big\langle R\big(I + \Delta(R)\big)^{-1} x, y\big\rangle_H = \langle x, y\rangle_H .$$

By Corollary 6.1,

$$\big\| S(t) \big\|_{\mathscr{L}(H)} \leq M e^{-\lambda t}, \quad \lambda > 0.$$

We consider the solutions $\{X_n^x(t),\, t \geq 0\}$ obtained by using the Yosida approximations $A_n = A R_n$ of A. Let us apply Itô's formula to $\langle R X_n^x(t), X_n^x(t)\rangle_H$ and take the expectations of both sides to arrive at

$$E\big\langle RX_n^x(t), X_n^x(t)\big\rangle_H = \langle Rx, x\rangle_H + 2E\int_0^t \big\langle RX_n^x(s), A_n X_n^x(s)\big\rangle_H\, ds$$

$$+ E\int_0^t \big\langle \Delta(R)X_n^x(s), X_n^x(s)\big\rangle_H\, ds.$$

From (6.45) with $y = R_n X_n^x$ it follows that

$$2\big\langle RX_n^x(s), AR_n X_n^x(s)\big\rangle_H = -\big\langle \Delta(R)X_n^x(s), R_n X_n^x(s)\big\rangle_H - \big\langle X_n^x(s), R_n X_n^x(s)\big\rangle_H.$$

Hence,

$$E\big\langle RX_n^x(t), X_n^x(t)\big\rangle_H = \langle Rx, x\rangle_H - E\int_0^t \big\langle X_n^x(s), R_n X_n^x(s)\big\rangle_H\, ds$$

$$+ E\int_0^t \big\langle \Delta(R)X_n^x(s), X_n^x(s) - R_n X_n^x(s)\big\rangle_H\, ds.$$

We let $n \to \infty$ and use the fact that $\sup_n \sup_{t\le T} E\|X_n^x(t)\|_H^2 < \infty$ to obtain

$$E\big\langle RX^x(t), X^x(t)\big\rangle_H = \langle Rx, x\rangle_H - \int_0^t E\big\|X^x(s)\big\|_H^2\, ds.$$

Let $\Xi(t) = E\langle RX^x(t), X^x(t)\rangle_H$. Then

$$\Xi(t) \le \|R\|_{\mathscr{L}(H)} E\big\|X^x(t)\big\|_H^2$$

and

$$\Xi'(t) = -E\big\|X^x(t)\big\|_H^2 \le \frac{-1}{\|R\|_{\mathscr{L}(H)}}\Xi(t),$$

so that

$$\Xi(t) \le \langle Rx, x\rangle_H \mathrm{e}^{\frac{-1}{\|R\|_{\mathscr{L}(H)}}t},$$

since $\Xi(0) = \langle Rx, x\rangle_H$. Hence,

$$E\big\|X^x(t)\big\|_H^2 \le 2\big\|S(t)x\big\|_H^2 + 2E\bigg\|\int_0^t S(t-s)BX^x(s)\,dW_s\bigg\|_H^2$$

$$\le 2M^2\mathrm{e}^{-2\lambda t}\|x\|_H^2 + 2\,\mathrm{tr}(Q)M^2\|B\|_{\mathscr{L}(H)}^2 \int_0^t \mathrm{e}^{-2\lambda(t-s)} E\big\|X^x(s)\big\|_H^2\, ds.$$

We complete the proof by using

$$\langle Rx, x\rangle_H \le \|R\|_{\mathscr{L}(H)}\|x\|_H^2 \quad \text{and} \quad E\big\|X^x(s)\big\|_H^2 = -\Xi'(s). \qquad \square$$

As shown in Sect. 6.1, Example 6.1, $x \to \langle Rx, x\rangle$, however, is not a Lyapunov function, so that we cannot study stability of nonlinear equations using Theorem 6.15.