9 Level 1: The Full Modular Group

Clearly, the only holomorphic eta product of weight k for the full modular group is $\eta^{2k}(z)$. Lacunary powers of the eta function have been studied by Serre [129] exhaustively. Here we will present theta series representations for some modular forms on the full modular group, including Serre's results on powers of $\eta(z)$.

9.1 Weights $k = 1$, $k \equiv 1 \bmod 4$ and $k \equiv 1 \bmod 6$

From Euler's series (1.2) for $\eta(z)$ we obtain

$$\eta^2(z) = \sum_{n\equiv 1 \bmod 12} a_2(n) e\left(\frac{nz}{12}\right) \quad \text{with} \quad a_2(n) = \sum_{x,y>0,\, x^2+y^2=2n} \left(\frac{12}{xy}\right). \tag{9.1}$$

The representation of $\eta^2(z)$ as a theta series for the Gaussian number field has been known to Weber, Ramanujan [115] and Hecke [50]. Hecke [50] also discovered a representation as a theta series on the real quadratic field $\mathbb{Q}(\sqrt{3})$. Schoeneberg [121] observed that there is also a representation on $\mathbb{Q}(\sqrt{-3})$. We state the result in Example 9.1. For the notations $\mathcal{O}_d$, Z_m, etc., we refer to the Index of Notations. In particular, we recall that $\omega = e\left(\frac{1}{6}\right) = \frac{1}{2}(1+\sqrt{-3})$.

We recall from Theorem 5.3 that Hecke characters on real quadratic fields always occur in pairs corresponding to the field automorphism of algebraic conjugation. In the following description of $\eta^2(z)$ we write down only one of these characters. We will do so throughout this monograph whenever a real quadratic field comes into play. This will cause some asymmetry between the real and imaginary cases in the appearance of our identities. (Look for Example 23.16 for a particularly apparent case of asymmetry.) We emphasize that symmetry can be restored by algebraic conjugation according to Theorem 5.3.

Example 9.1 *The residues of* $2+i$ *and* $2+3i$ *modulo* 6 *can be chosen as generators for the group* $(\mathcal{O}_1/(6))^\times \simeq Z_8 \times Z_2$. *A pair of characters* χ_ν *on*

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$\mathcal{O}_1$ with period 6 is fixed by the values

$$\chi_\nu(2+i) = \nu i, \qquad \chi_\nu(2+3i) = -1$$

with $\nu \in \{1,-1\}$. We have $(2+i)^2(2+3i) \equiv -i \bmod 6$. The residues of $1+2\omega$, $1-4\omega$ and ω modulo $4(1+\omega)$ can be chosen as generators for the group $\mathcal{O}_3/(4+4\omega)^\times \simeq Z_2 \times Z_2 \times Z_6$. A pair of characters ψ_ν on $\mathcal{O}_3$ with period $4(1+\omega)$ is fixed by the values

$$\psi_\nu(1+2\omega) = -\nu, \qquad \psi_\nu(1-4\omega) = -1, \qquad \psi_\nu(\omega) = 1$$

with $\nu \in \{1,-1\}$. A Hecke character ξ on $\mathbb{Z}[\sqrt{3}]$ with period $2\sqrt{3}$ is given by

$$\xi(\mu) = \begin{cases} \operatorname{sgn}(\mu) \\ -\operatorname{sgn}(\mu) \end{cases} \quad \textit{for} \quad \mu \equiv \begin{cases} 1, \ \ 2+\sqrt{3} \\ -1, -2+\sqrt{3} \end{cases} \quad \bmod 2\sqrt{3}.$$

The corresponding theta series of weight 1 satisfy

$$\Theta_1\left(12, \xi, \frac{z}{12}\right) = \Theta_1\left(-4, \chi_\nu, \frac{z}{12}\right) = \Theta_1\left(-3, \psi_\nu, \frac{z}{12}\right) = \eta^2(z). \tag{9.2}$$

By Theorems 5.3, 5.1 and Sect. 1.3, the functions $\Theta_1(\xi, z)$, $\Theta_1(\chi_\nu, z)$, $\Theta_1(\psi_\nu, z)$ and $\eta^2(z)$ are modular forms of weight 1 and level 12^2. Therefore the matching of small initial segments of their Fourier expansions suffices to prove (9.2). We have

$$a_2(p) = \chi_\nu(\mu) + \chi_\nu(\overline{\mu}) = 0 \quad \text{for primes} \quad p = \mu\overline{\mu} \equiv 5 \bmod 12, \ \mu \in \mathcal{O}_1,$$

$$a_2(p) = \psi_\nu(\mu) + \psi_\nu(\overline{\mu}) = 0 \quad \text{for primes} \quad p = \mu\overline{\mu} \equiv 7 \bmod 12, \ \mu \in \mathcal{O}_3.$$

For primes $p \equiv 1 \bmod 12$ the representation of $\eta^2(z)$ by χ_ν on $\mathcal{O}_1$ shows that

$$a_2(p) = \begin{cases} 2 \\ -2 \end{cases} \quad \text{if} \quad p = \begin{cases} 36x^2 + y^2 \\ 9x^2 + 4y^2 \end{cases} \tag{9.3}$$

for some $x, y \in \mathbb{Z}$. This tells us that the representation of primes by quadratic forms of discriminant -144 is governed by the coefficients of the modular form $\eta^2(z)$. The representation of primes by quadratic forms x^2+Ny^2 is studied in the monograph [27] and in several papers, for example [56], [58], [66]. More results on this topic will be given in Corollaries 10.3, 11.2, 11.10, 12.2, 12.5, 12.7, (12.16), and in a remark after Example 12.10. At the end of Sect. 9.2 we will prove that

$$a_2(p) = \begin{cases} 2 \\ -2 \end{cases} \quad \text{if} \quad p = x^2 + 4xy + 16y^2 \quad \text{with} \quad y > 0, x \equiv \begin{cases} 1 \\ -1 \end{cases} \bmod 4. \tag{9.4}$$

The criterion (9.3) can be read off from Fig. 9.1 which displays the values of χ_ν and ψ_ν within period meshes of these characters; here dots stand for positions with character value 0.

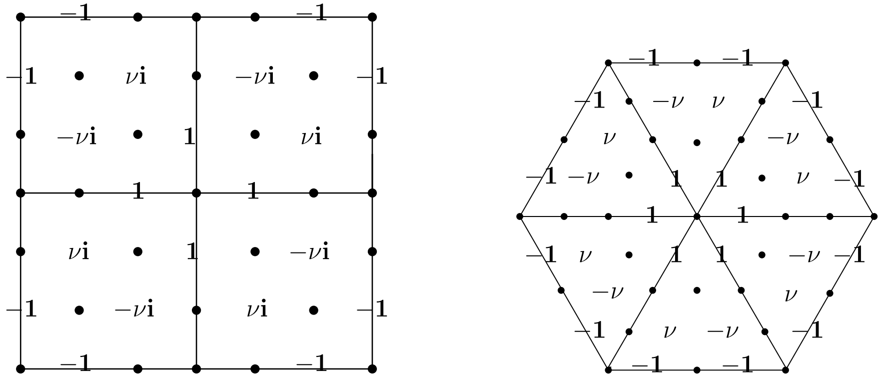

Figure 9.1: Values of the characters χ_ν and ψ_ν in period meshes

The theta series of weight k for χ_ν and ψ_ν are not identically 0 for $k \equiv 1 \bmod 4$ and $k \equiv 1 \bmod 6$, respectively. We have seen that for weight $k = 1$ both values of the sign ν yield the same theta series. The reason is that $\chi_\nu(\mu)+\chi_\nu(\overline{\mu}) = 0$ for $\mu \in \mathcal{O}_1$, $\mu\overline{\mu} \equiv 5 \bmod 8$ and $\psi_\nu(\mu) + \psi_\nu(\overline{\mu}) = 0$ for $\mu \in \mathcal{O}_3$, $\mu\overline{\mu} \equiv 7 \bmod 12$. We will meet many more examples for this phenomenon—the next one in Example 10.5. For $k > 1$ different signs ν yield different modular forms. They are identified with linear combinations of Eisenstein series and powers of $\eta(z)$:

Example 9.2 *The theta series for the characters χ_ν and ψ_ν in Example 9.1 satisfy*

$$\Theta_5\left(\chi_\nu, \frac{z}{12}\right) = E_4(z)\eta^2(z) - 48\nu\eta^{10}(z), \tag{9.5}$$

$$\Theta_9\left(\chi_\nu, \frac{z}{12}\right) = E_4^2(z)\eta^2(z) + 672\nu E_4(z)\eta^{10}(z), \tag{9.6}$$

$$\Theta_{13}\left(\chi_\nu, \frac{z}{12}\right) = E_4^3(z)\eta^2(z) - 20592\nu E_4^2(z)\eta^{10}(z) - 6912000\eta^{26}(z), \tag{9.7}$$

$$\Theta_7\left(\psi_\nu, \frac{z}{12}\right) = E_6(z)\eta^2(z) + 360\sqrt{-3}\nu\eta^{14}(z), \tag{9.8}$$

$$\Theta_{13}\left(\psi_\nu, \frac{z}{12}\right) = E_6^2(z)\eta^2(z) - 102960\sqrt{-3}\nu E_6(z)\eta^{14}(z) + 9398592\eta^{26}(z). \tag{9.9}$$

All these identities (or equivalent versions) are known from Serre [129]. The identities (9.5) and (9.8) were found by van Lint [89]. As a consequence, these authors obtain the lacunarity of certain powers of $\eta(z)$: From (9.5) and (9.8) it follows that

$$\eta^{10}(z) = -\frac{1}{96}\left(\Theta_5\left(\chi_1, \frac{z}{12}\right) - \Theta_5\left(\chi_{-1}, \frac{z}{12}\right)\right)$$

and

$$\eta^{14}(z) = \frac{1}{720\sqrt{-3}} \left(\Theta_7 \left(\psi_1, \frac{z}{12} \right) - \Theta_7 \left(\psi_{-1}, \frac{z}{12} \right) \right)$$

are linear combinations of Hecke theta series, and hence are lacunary. In the same way, (9.7) and (9.9) show that $E_4^3\eta^2 - 6912000\eta^{26}$ and $E_6^2\eta^2 + 9398592\eta^{26}$ are linear combinations of two Hecke theta series. Since $E_4^3 - E_6^2 = 1728\eta^{24}$, it follows that $\eta^{26}(z)$ is a linear combination of four Hecke theta series, and hence is lacunary.

9.2 Weights $k = 2$ and $k \equiv 2 \bmod 6$

The expansion of $\eta^4(z)$ can be written as

$$\eta^4(z) = \sum_{n \equiv 1 \bmod 6} a_4(n) e\left(\frac{nz}{6} \right)$$

with

$$a_4(n) = \sum_{j,l>0,\ j+l=2n} a_2(j)a_2(l) = \sum_{x,y>0,\, x^2+3y^2=4n} \left(\frac{12}{x} \right) \left(\frac{-1}{y} \right) y.$$

The second expression for $a_4(n)$ comes from Euler's and Jacobi's formulae for $\eta(z)$ and $\eta^3(z)$. Theorem 5.1 together with certain non-vanishing values of $a_4(n)$ implies that $D = -3$ is the only conceivable discriminant for a theta series representation of $\eta^4(z)$. Such a representation exists indeed:

Example 9.3 *The group $(\mathcal{O}_3/(2+2\omega))^\times \simeq Z_6$ is generated by the residue of ω modulo $2+2\omega$. A character ψ on $\mathcal{O}_3$ with period $2(1+\omega)$ is fixed by the value*

$$\psi(\omega) = \overline{\omega}.$$

The corresponding theta series are not identically 0 for weights $k \equiv 2 \bmod 6$ and satisfy

$$\Theta_2 \left(\psi, \frac{z}{6} \right) = \eta^4(z), \tag{9.10}$$

$$\Theta_8 \left(\psi, \frac{z}{6} \right) = E_6(z)\eta^4(z), \tag{9.11}$$

$$\Theta_{14} \left(\psi, \frac{z}{6} \right) = E_6^2(z)\eta^4(z) + 616896\eta^{28}(z), \tag{9.12}$$

$$\Theta_{20} \left(\psi, \frac{z}{6} \right) = E_6^3(z)\eta^4(z) - 116375616 E_6(z)\eta^{28}(z). \tag{9.13}$$

The identity (9.10) is equivalent to identities given by Mordell [97] and Petersson [111], and Mordell dates it back to Klein and Fricke. We draw some consequences for the coefficients $a_4(p)$ of $\eta^4(z)$ at primes $p \equiv 1 \bmod 6$. From

Sect. 5.3 it is clear that the coefficients are multiplicative and satisfy the recursion

$$a_4(p^{r+1}) = a_4(p)a_4(p^r) - pa_4(p^{r-1})$$

for all primes $p > 3$. Another identity involving $\eta^4(z)$ and the character ψ will appear in Example 15.14.

Corollary 9.4 *For primes $p \equiv 1 \bmod 6$ the coefficients $a_4(p)$ of $\eta^4(z)$ have the following properties*:

(1) *We have*

$$a_4(p) \equiv 2 \bmod 6,$$

whence $a_4(p) \geq 2$ or $a_4(p) \leq -4$.

(2) *We have*

$$|a_4(p)| \leq 2\sqrt{p-3}$$

with equality if and only if $p = 4x^2 + 3$ for some $x \in \mathbb{Z}$.

(3) *We have*

$$a_4(p) = \begin{cases} 2 \\ -4 \end{cases} \quad \text{if and only if} \quad p = \begin{cases} 3v^2 + 1 \\ 3v^2 + 4 \end{cases}$$

for some $v \in \mathbb{Z}$.

(4) *Every odd prime divisor q of $a_4(p)$ satisfies $\left(\frac{3p}{q}\right) = 1$.*

Proof. Let $p \equiv 1 \bmod 6$ be given. Since p is split in the factorial ring $\mathcal{O}_3$, we have

$$p = \mu\overline{\mu} = x^2 + xy + y^2$$

where $\mu = x + y\omega \in \mathcal{O}_3$ is unique up to associates and conjugates, which are $\pm\mu = \pm(x + y\omega)$, $\pm\omega\mu = \pm(-y + (x + y)\omega)$, $\pm\omega^2\mu = \pm(-(x + y) + x\omega)$, $\pm\overline{\omega} = \pm((x+y)-y\omega)$, $\pm\omega\overline{\mu} = \pm(y+x\omega)$, $\pm\omega^2\overline{\mu} = \pm(-x+(x+y)\omega)$. We have $x \not\equiv y \bmod 3$ since otherwise p would be a multiple of 3. If $\varepsilon\mu \equiv \mu \bmod 2+2\omega$ for some unit $\varepsilon \in \mathcal{O}_3^\times$ then since μ and $2+2\omega$ are relatively prime, it follows that $\varepsilon = 1$. Therefore exactly one of the six associates of μ is congruent to 1 modulo $2+2\omega$, and therefore we may assume that

$$\mu = x + y\omega \equiv 1 \bmod 2 + 2\omega.$$

This implies that y is even, x is odd and $x - 1 \equiv y \bmod 3$. (We use that 2 is prime in O_3 and that an element $a + b\omega \in \mathcal{O}_3$ is a multiple of the prime element $1 + \omega$ if and only if $a \equiv b \bmod 3$.) We can interchange μ and $\overline{\mu}$, if necessary, and assume that $y > 0$. Thus we get a unique μ satisfying

$$y > 0, \quad y \text{ even}, \quad x \text{ odd}, \quad x \equiv y + 1 \bmod 3.$$

Then we have $\psi(\mu) = \psi(\overline{\mu}) = 1$, and from

$$\eta^4(6z) = \Theta_2(\psi, z) = \frac{1}{6} \sum_{\mu \in \mathcal{O}_3} \psi(\mu)\mu e(\mu\overline{\mu}z)$$

we obtain

$$a_4(p) = \mu + \overline{\mu} = 2x + y.$$

Thus $a_4(p)$ is even and $a_4(p) \equiv 3y + 2 \equiv 2 \bmod 3$, hence $a_4(p) \equiv 2 \bmod 6$. In particular, $a_4(p)$ cannot take the values $-3, -2, \ldots, 1$. This proves part (1).

Since $|\mu| = \sqrt{p}$ we can write

$$\mu = \sqrt{p}e^{i\alpha}, \qquad a_4(p) = \mu + \overline{\mu} = 2\sqrt{p}\cos\alpha$$

with $0 \leq \alpha < 2\pi$. The absolute value of the cosine is maximal if α is as close to 0, π or 2π as possible. This means that $|y|$ is as small as possible. Since $y = 2$ is the smallest possible value, we get the largest values of $a_4(p)$ for $\mu = x + 2\omega$, and then we have

$$p = x^2 + 2x + 4 = (x+1)^2 + 3, \qquad a_4(p) = 2(x+1) = 2\sqrt{p-3}.$$

This proves (2).

From $4p = (2x + y)^2 + 3y^2 = a_4(p)^2 + 3y^2$ we obtain $p = \left(\frac{1}{2}a_4(p)\right)^2 + 3v^2$ with $v = \frac{1}{2}y \in \mathbb{N}$. Inserting the values 2 and -4 for $a_4(p)$ yields the assertion (3).—We note that there are similar criteria for any value of $a_4(p)$.

Let q be an odd prime divisor of $a_4(p) = 2x + y$. Then $y \equiv -2x \bmod q$, hence

$$p = x^2 + xy + y^2 \equiv (1 - 2 + 4)x^2 \equiv 3x^2 \bmod q$$

and $3p \equiv (3x)^2 \bmod q$. Thus $3p$ is a square modulo q, which proves (4). □

We illustrate the results in Table 9.1, presenting values of $a_4(p)$ and $\mu = x + y\omega$ for small primes p. An asterisk $*$ or a cross # at p indicate that $|a_4(p)| = 2\sqrt{p-3}$ or $a_4(p) \in \{2, -4\}$, respectively.

With the proof of Corollary 9.4 at hand, it is easy now to prove the criterion (9.4):

Proof of (9.4). Let a prime $p \equiv 1 \bmod 12$ be given. As in the proof of Corollary 9.4, we write p uniquely in the form $p = \mu\overline{\mu}$ with

$$\mu = x + y\omega, \quad y > 0, \quad y \text{ even}, \quad x \text{ odd}, \quad x \equiv y + 1 \bmod 3.$$

We have $y \equiv 0 \bmod 4$ because of $p \equiv 1 \bmod 4$. We put $y = 4v$ and obtain $p = x^2 + xy + y^2 = \frac{1}{4}((2x + y)^2 + 3y^2) = (x + 2v)^2 + 12v^2$, hence

$$p = x^2 + 4xv + 16v^2.$$

Table 9.1: Coefficients of $\eta^4(z)$ at primes p

p	$a_4(p)$	μ	p	$a_4(p)$	μ	p	$a_4(p)$	μ
$7^{*,\#}$	-4	$-3+2\omega$	127	20	$7+6\omega$	277	26	$7+12\omega$
$13^{\#}$	2	$-1+4\omega$	139	-16	$-13+10\omega$	283	32	$13+6\omega$
19^{*}	8	$3+2\omega$	$151^{\#}$	-4	$-9+14\omega$	307	-16	$-17+18\omega$
$31^{\#}$	-4	$-5+6\omega$	157	14	$1+12\omega$	313	-22	$-19+16\omega$
37	-10	$-7+4\omega$	163	8	$-3+14\omega$	331	32	$11+10\omega$
43	8	$1+6\omega$	181	26	$11+4\omega$	337	-34	$-21+8\omega$
61	14	$5+4\omega$	$193^{\#}$	2	$-7+16\omega$	349	14	$-3+20\omega$
67^{*}	-16	$-9+2\omega$	199^{*}	-28	$-15+2\omega$	$367^{\#}$	-4	$-13+22\omega$
73	-10	$-9+8\omega$	211	-16	$-15+14\omega$	373	38	$17+4\omega$
$79^{\#}$	-4	$-7+10\omega$	223	-28	$-17+6\omega$	379	8	$-7+22\omega$
97	14	$3+8\omega$	229	-22	$-17+12\omega$	397	-34	$-23+12\omega$
103^{*}	20	$9+2\omega$	241	14	$-1+16\omega$	409	38	$15+8\omega$
$109^{\#}$	2	$-5+12\omega$	271	-28	$-19+10\omega$	421	-22	$-21+20\omega$

From (9.2) and the definition of the characters ψ_ν we conclude that $a_2(p) = 2$ if and only if $\psi_\nu(\mu) = 1$, which holds if and only if $\mu \equiv 1 \bmod 4(1+\omega)$. This in turn is equivalent with $x \equiv 1 \bmod 4$. Writing y instead of v we obtain (9.4). □

9.3 Weights $k = 3$ and $k \equiv 3 \bmod 4$

From Jacobi's identity (8.15) we obtain

$$\eta^6(z) = \sum_{n\equiv 1 \bmod 4} a_6(n) e\left(\frac{nz}{4}\right) \qquad \text{with} \qquad a_6(n) = \sum_{x,y>0,x^2+y^2=2n} \left(\frac{-1}{xy}\right) xy. \tag{9.14}$$

A theta series representation exists for the discriminant -4 only:

Example 9.5 *A character χ on the Gaussian number ring $\mathcal{O}_1$ with period 2 is defined by the Legendre symbol*

$$\chi(x+iy) = \left(\frac{-1}{x^2-y^2}\right)$$

for $x \not\equiv y \bmod 2$. The corresponding theta series are not identically 0 for *weights* $k \equiv 3 \bmod 4$. *They satisfy the identities*

$$\Theta_3\left(\chi, \frac{z}{4}\right) = \eta^6(z), \tag{9.15}$$

$$\Theta_7\left(\chi, \frac{z}{4}\right) = E_4(z)\eta^6(z), \tag{9.16}$$

$$\Theta_{11}\left(\chi, \frac{z}{4}\right) = E_4^2(z)\eta^6(z), \tag{9.17}$$

$$\Theta_{15}\left(\chi, \frac{z}{4}\right) = E_4^3(z)\eta^6(z) - 153600\eta^{30}(z), \tag{9.18}$$

$$\Theta_{19}\left(\chi, \frac{z}{4}\right) = E_4^4(z)\eta^6(z) + 1843200E_4(z)\eta^{30}(z), \tag{9.19}$$

$$\Theta_{23}\left(\chi, \frac{z}{4}\right) = E_4^5(z)\eta^6(z) + 69734400E_4^2(z)\eta^{30}(z). \tag{9.20}$$

As with (9.10), an equivalent version of (9.15) has been known since Mordell [97].—We get consequences similar to those in Corollary 9.4:

Corollary 9.6 *The coefficients* $a_6(p)$ *of* $\eta^6(z)$ *at primes* $p \equiv 1 \bmod 4$ *have the following properties*:

(1) *We have* $a_6(p) \equiv 2p \bmod 16$*,* $a_6(p) \equiv 2 \bmod 8$ *and*

$$-2p + 4 \leq a_6(p) \leq 2p - 16.$$

(2) *We have* $a_6(p) \geq 10$ *or* $a_6(p) \leq -6$*, and we have*

$$|a_6(p)| \geq 2\sqrt{2p-1}$$

with equality if and only if $p = 2x^2 + 2x + 1$ *for some* $x \in \mathbb{N}$.

(3) *If* q *is an odd prime divisor of* $a_6(p)$ *then*

$$\left(\frac{p}{q}\right) = \left(\frac{2}{q}\right) = (-1)^{(q^2-1)/8}.$$

If $\left(\frac{p}{3}\right) = -1$*, i.e., if* $p \equiv 5 \bmod 12$*, then* 3 *divides* $a_6(p)$*. If* $\left(\frac{p}{5}\right) = -1$ *then* 5 *divides* $a_6(p)$*.*

(4) *Every prime divisor* q *of* $a_6(p)$ *satisfies* $q \leq \sqrt{2p-1}$*. If* $q = \sqrt{2p-1} > 0$ *is an integer then* $|a_6(p)| = 2q$*.*

Proof. Let $p \equiv 1 \bmod 4$ be a prime. Then p is split in $\mathcal{O}_1$, and we have $p = \mu\overline{\mu} = x^2 + y^2$ for a unique element

$$\mu = x + iy \in \mathcal{O}_1 \quad \text{with} \quad y > 0, \quad y \text{ even}, \quad x > 0, \quad x \text{ odd}.$$

Then $\chi(\mu) = \chi(\overline{\mu}) = 1$, and (9.15) implies

$$a_6(p) = \mu^2 + \overline{\mu}^2 = 2(x^2 - y^2) = 2p - 4y^2 = 4x^2 - 2p.$$

Since y is even and $p \equiv 1 \bmod 4$, this implies the congruences in (1), and from $4y^2 \geq 16$, $4x^2 \geq 4$ we get the inequalities in (1).

The smallest values of $|a_6(p)| = 2|x^2 - y^2|$ are attained when $x = y + \delta$, $\delta \in \{1, -1\}$. In this case, $p = 2y^2 + 2\delta y + 1 = 2\big((y + \frac{1}{2}\delta)^2 + \frac{1}{4}\big)$ and $a_6(p) = 2(y+\delta)^2 - 2y^2 = 2\delta(2y + \delta)$. Since we assume that y is positive and even, we get $a_6(p) \geq 10$ if $\delta = 1$ and $a_6(p) \leq -6$ if $\delta = -1$. Moreover, for the minimal absolute value of $a_6(p)$ we get

$$|a_6(p)| = 4\left(y + \tfrac{1}{2}\delta\right) = 4\sqrt{\tfrac{p}{2} - \tfrac{1}{4}} = 2\sqrt{2p - 1}.$$

Thus we obtain $|a_6(p)| \geq 2\sqrt{2p-1}$ with equality if and only if $p = 2y^2 \pm 2y + 1$ for some positive even y. The case of the minus sign is reduced to the plus sign since $2y^2 - 2y + 1 = 2(y-1)^2 + 2(y-1) + 1$, and we can replace y by $x = y - 1$. Thus we have proved (2).

Let q be an odd prime divisor of $a_6(p)$. Then we get $2p = 2x^2 + 2y^2 = a_6(p) + 4y^2 \equiv (2y)^2 \bmod q$. This implies $\left(\frac{2p}{q}\right) = 1$, hence $\left(\frac{p}{q}\right) = \left(\frac{2}{q}\right)$. For the primes $q \in \{3, 5\}$ we can prove the converse of this criterion:

We suppose that $\left(\frac{p}{3}\right) = -1$. Then $3 \nmid x$ and $3 \nmid y$, since otherwise $p = x^2 + y^2$ would be a square modulo 3. It follows that $3|(x - y)$ or $3|(x + y)$, and hence $a_6(p) = 2(x^2 - y^2)$ is a multiple of 3. Now we suppose that $\left(\frac{p}{5}\right) = -1$. Then $5 \nmid x$, $5 \nmid y$, $x \not\equiv 2y \bmod 5$ and $x \not\equiv 3y \bmod 5$, since otherwise p would be a square modulo 5. It follows that $x \equiv y \bmod 5$ or $x \equiv -y \bmod 5$, and hence $a_6(p)$ is a multiple of 5. We have proved (3).

Clearly $q = 2$ satisfies the inequality in (4). So we assume that q is an odd prime divisor of $a_6(p) = 2(x - y)(x + y)$. Then q divides one of the factors. From $x > 0$, $y > 0$ we obtain

$$q \leq x + y = \sqrt{x^2 + 2xy + y^2} \leq \sqrt{2x^2 + 2y^2 - 1} = \sqrt{2p - 1}.$$

Now we suppose that $2p = q^2 + 1$ for some integer $q > 0$. Then

$$p = \tfrac{1}{2}(q^2 + 1) = \left(\tfrac{1}{2}(q + 1)\right)^2 + \left(\tfrac{1}{2}(q - 1)\right)^2,$$

hence $x = \frac{1}{2}(q - 1)$, $y = \frac{1}{2}(q + 1)$ or vice versa, according to the residue of q modulo 4. It follows that

$$|a_6(p)| = 2(x + y)|x - y| = 2q.$$

This proves (4). More precisely, from (1) we get $a_6(p) = 2q$ for $q \equiv 1 \bmod 4$ and $a_6(p) = -2q$ for $q \equiv -1 \bmod 4$. □

The list of primes p of the form $p = \frac{1}{2}(q^2 + 1)$ with $q \leq 101$ is

$$5, 13, 41, 61, 113, 181, 313, 421, 613, 761, 1013, 1201, 1301, 1741, 1861,$$

$$2113, 2381, 2521, 3121, 3613, 4513, 5101.$$

9.4 Weights $k = 4$ and $k \equiv 1 \bmod 3$

For the coefficients $a_8(n)$ in

$$\eta^8(z) = \sum_{n \equiv 1 \bmod 3} a_8(n) e\left(\frac{nz}{3}\right)$$

there is no such formula as in (9.1), etc., coming from the multiplication of two simple theta series of half-integral weights. But there is a representation as a Hecke theta series. The only conceivable discriminant is $D = -3$:

Example 9.7 *A character ψ on $\mathcal{O}_3$ with period $1+\omega$ is defined by the Legendre symbol*

$$\psi(x + y\omega) = \left(\frac{x-y}{3}\right).$$

The corresponding theta series are not identically 0 for weights $k \equiv 4 \bmod 6$. They satisfy the identities

$$\Theta_4\left(\psi, \frac{z}{3}\right) = \eta^8(z), \tag{9.21}$$

$$\Theta_{10}\left(\psi, \frac{z}{3}\right) = E_6(z)\eta^8(z), \tag{9.22}$$

$$\Theta_{16}\left(\psi, \frac{z}{3}\right) = E_6^2(z)\eta^8(z) - 31752\eta^{32}(z), \tag{9.23}$$

$$\Theta_{22}\left(\psi, \frac{z}{3}\right) = E_6^3(z)\eta^8(z) - 2095632 E_6(z)\eta^{32}(z). \tag{9.24}$$

As with (9.10) and (9.15), an equivalent version of (9.21) was known to Mordell [97]. Again, we list some arithmetical consequences for the coefficients of $\eta^8(z)$:

Corollary 9.8 *The coefficients $a_8(p)$ of $\eta^8(z)$ at primes $p \equiv 1 \bmod 6$ have the following properties*:

(1) *We have $a_8(p) \equiv 2 \bmod 18$.*

(2) *We have $a_8(p) \geq 3p-1$ or $a_8(p) \leq -(6p-8)$, with equality if and only if $p = 3y^2+3y+1$ or $p = 3y^2+6y+4$ with some $y \in \mathbb{N}$, respectively.*

(3) *We have*

$$|a_8(p)| \leq (p-3)\sqrt{4p-3}$$

with equality if and only if $p = x^2+x+1$ for some $x \in \mathbb{N}$.

(4) *If q is an odd prime divisor of $a_8(p)$ then $\left(\frac{3p}{q}\right) = 1$. If $\left(\frac{p}{5}\right) = -1$ then 5 divides $a_8(p)$. If $\left(\frac{p}{7}\right) = -1$ then 7 divides $a_8(p)$.*

(5) *Every prime divisor q of $a_8(p)$ satisfies $q \leq \sqrt{4p-3}$. If $p = x^2+x+1$ for some integer x then $a_8(p)$ is a multiple of the positive integer $q = \sqrt{4p-3}$.*

Proof. Let $p \equiv 1 \bmod 6$ be a prime. Then $p = \mu\overline{\mu} = x^2 + xy + y^2$ for some $\mu = x + y\omega \in \mathcal{O}_3$ which is unique up to associates and conjugates. We have $x \not\equiv y \bmod 3$, and from (9.21) we obtain

$$\begin{aligned} a_8(p) &= \psi(\mu)\left(\mu^3 + \overline{\mu}^3\right) = \psi(\mu)(\mu + \overline{\mu})(\omega\mu + \overline{\omega\mu})(\overline{\omega}\mu + \omega\overline{\mu}) \\ &= \left(\frac{x-y}{3}\right)(2x+y)(x-y)(x+2y). \end{aligned}$$

At least one of the factors on the right is even, and hence $a_8(p)$ is even. By an appropriate choice of μ we achieve that $y = 3v$ is a multiple of 3. Then

$$a_8(p) = \left(\frac{x}{3}\right)(2x+3v)(x-3v)(x+6v) \equiv 2x^3\left(\frac{x}{3}\right) \equiv 2 \bmod 9.$$

Thus we have proved (1).

For estimates of $|a_8(p)|$ we choose μ such that $0 < y < x$. This means that

$$\mu = \sqrt{p}e^{i\alpha} \qquad \text{with} \qquad 0 < \alpha < \frac{\pi}{6},$$

and clearly this choice is possible. Then all factors $2x + y$, $x - y$, $x + 2y$ in $a_8(p)$ are positive, with $x - y$ the smallest among them. Moreover, we get

$$|a_8(p)| = 2p\sqrt{p}\cos(3\alpha).$$

Extremal values of $|a_8(p)p^{-3/2}|$ are attained when α is close to $\frac{\pi}{6}$ or 0, or, equivalently, when y is close to x or to 0. For $x = y + 1$ we get

$$p = 3y^2 + 3y + 1, \qquad a_8(p) = (3y+1)(3y+2) = 3p - 1;$$

for $x = y + 2$ we get

$$p = 3y^2 + 6y + 4, \qquad a_8(p) = -2(3y+4)(3y+2) = -6p + 8.$$

This proves the lower estimates for $|a_8(p)|$ in (2) and the criteria for values closest to 0.

For $y = 1$ we obtain $p = x^2 + x + 1$ and

$$\begin{aligned} a_8(p) &= \left(\tfrac{x-1}{3}\right)(2x+1)(x-1)(x+2) = \left(\tfrac{x-1}{3}\right)(x^2+x-2)(2x+1) \\ &= \left(\tfrac{x-1}{3}\right)(p-3)\sqrt{4p-3}. \end{aligned}$$

This implies the upper estimate for $|a_8(p)|$ and the criterion for maximal values in (3).

Let q be an odd prime divisor of $a_8(p)$. Then one of the factors $2x+y$, $x-y$, $x+2y$ is a multiple of q, which implies that $p = x^2+xy+y^2 \equiv 3x^2 \bmod q$ or $p \equiv 3y^2 \bmod q$. Hence $3p$ is a square modulo q, i.e., $\left(\frac{3p}{q}\right) = 1$. For $q \in \{5, 7\}$ the converse of this criterion holds:

Table 9.2: Coefficients of $\eta^8(z)$ at primes p

p	μ	$a_8(p)$	p	μ	$a_8(p)$	p	μ	$a_8(p)$
$7^{*,\#}$	$2+\omega$	20	$127^{\#}$	$7+6\omega$	380	277	$12+7\omega$	$-4030=-2{\cdot}5{\cdot}13{\cdot}31$
$13^{*,\#}$	$3+\omega$	-70	139	$10+3\omega$	2576	283	$13+6\omega$	$5600=2^5\cdot5^2\cdot7$
$19^{\#}$	$3+2\omega$	56	151	$9+5\omega$	1748	307^*	$17+\omega$	$10640=2^4\cdot5\cdot7\cdot19$
31^*	$5+\omega$	308	157^*	$12+\omega$	-3850	313	$16+3\omega$	$10010=2{\cdot}5{\cdot}7{\cdot}11{\cdot}13$
$37^{\#}$	$4+3\omega$	110	163	$11+3\omega$	-3400	$331^{\#}$	$11+10\omega$	$992=2^5\cdot31$
43^*	$6+\omega$	-520	181	$11+4\omega$	3458	337	$13+8\omega$	$-4930=-2{\cdot}5{\cdot}17{\cdot}29$
$61^{\#}$	$5+4\omega$	182	$193^{\#}$	$9+7\omega$	-1150	349	$17+3\omega$	$-11914=-2{\cdot}7{\cdot}23{\cdot}37$
67	$7+2\omega$	-880	199	$13+2\omega$	-5236	367	$13+9\omega$	$4340=2^2\cdot5\cdot7\cdot31$
73^*	$8+\omega$	1190	211^*	$14+\omega$	6032	373	$17+4\omega$	$12350=2\cdot5^2\cdot13\cdot19$
79	$7+3\omega$	884	223	$11+6\omega$	-3220	379	$15+7\omega$	$-8584=-2^3\cdot29\cdot37$
97	$8+3\omega$	-1330	229	$12+5\omega$	4466	$397^{\#}$	$12+11\omega$	$1190=2\cdot5\cdot7\cdot17$
103	$9+2\omega$	1820	241^*	$15+\omega$	-7378	409	$15+8\omega$	$8246=2\cdot7\cdot19\cdot31$
$109^{\#}$	$7+5\omega$	-646	$271^{\#}$	$10+9\omega$	812	421^*	$20+\omega$	$17138=2{\cdot}11{\cdot}19{\cdot}41$

We suppose that $\left(\frac{3p}{5}\right)=1$ or, equivalently, that $\left(\frac{p}{5}\right)=-1$. Then $5\nmid x$, $5\nmid y$ and $5\nmid(x-4y)$, since otherwise $p=x^2+xy+y^2$ would be a square modulo 5. (Observe that 0 is a square modulo 5, too.) Therefore $x\equiv y$ or $x\equiv 2y$ or $x\equiv 3y$ modulo 5. Consequently, one of the factors $2x+y$, $x-y$, $x+2y$ in $a_8(p)$ is a multiple of 5, and we get $5|a_8(p)$. Now we suppose that $\left(\frac{p}{7}\right)=-1$. As before we conclude that $7\nmid x$, $7\nmid y$, $7\nmid(x-2y)$, $7\nmid(x-4y)$, $7\nmid(x+y)$. Therefore $x\equiv y$ or $x\equiv 3y$ or $x\equiv 5y$ modulo 7. Hence one of the factors in $a_8(p)$ is a multiple of 7, and we get $7|a_8(p)$. Thus we have proved (4).

Clearly the prime 2 satisfies the estimate in (5). Let q be an odd prime divisor of $a_8(p)$. As before, we choose μ such that $0<y<x$. Then $2x+y$ is the biggest of the three positive factors in $a_8(p)$. Therefore,

$$q\le 2x+y=\sqrt{4x^2+4x+y^2}=\sqrt{4p-3y^2}\le\sqrt{4p-3}.$$

Now we suppose that $p=x^2+x+1$ for some integer x. Then $4p=(2x+1)^2+3$, hence $q=\sqrt{4p-3}=|2x+y|$ is a positive integer (not necessarily a prime) and a divisor of $a_8(p)$. Thus we have proved (5). □

We illustrate the results in Table 9.2 similar to that in Sect. 9.2. Here, an asterisk $*$ or a cross # at p indicate that $|a_8(p)|=(p-3)\sqrt{4p-3}$ and $\sqrt{4p-3}\in\mathbb{N}$ or that $a_8(p)\in\{3p-1,-6p+8\}$, respectively.

9.5 Weights $k \equiv 0 \bmod 6$

The 12-th power $\eta^{12}(z)$ is a Hecke eigenform. But it has no expansion as a Hecke theta series, since otherwise the coefficients in

$$\eta^{12}(z) = \sum_{n \equiv 1 \bmod 2} a_{12}(n) e\left(\frac{nz}{2}\right)$$

would vanish at all primes in certain arithmetical progressions, which is not the case. Schoeneberg [122] proved that if $a_{12}(n) = 0$ for some odd n then the smallest such n is a prime p and satisfies $p \equiv -1 \bmod 2^8$. Moreover, $a_{12}(n) \equiv \sigma_5(n) \bmod 2^8$ for all n.—The modular form $E_4(z)\eta^4(z)$ of weight 6 is a Hecke theta series:

Example 9.9 *Let $\overline{\psi}$ be the conjugate of the character ψ on $\mathcal{O}_3$ in Example 9.3, having period $2(1+\omega)$ and satisfying $\overline{\psi}(\omega) = \omega$. The corresponding theta series are not identically 0 for weights $k \equiv 0 \bmod 6$ and satisfy the identities*

$$\Theta_6\left(\overline{\psi}, \frac{z}{6}\right) = E_4(z)\eta^4(z), \tag{9.25}$$

$$\Theta_{12}\left(\overline{\psi}, \frac{z}{6}\right) = E_4(z)E_6(z)\eta^4(z), \tag{9.26}$$

$$\Theta_{18}\left(\overline{\psi}, \frac{z}{6}\right) = E_4(z)E_6^2(z)\eta^4(z) - 27687744E_4(z)\eta^{28}(z), \tag{9.27}$$

$$\Theta_{24}\left(\overline{\psi}, \frac{z}{6}\right) = E_4(z)E_6^3(z)\eta^4(z) + 7950446784E_4(z)E_6(z)\eta^{28}(z). \tag{9.28}$$

Remark. Let us write

$$E_4(z)\eta^4(z) = \sum_{n \equiv 1 \bmod 6} c(n) e\left(\frac{nz}{6}\right).$$

The expansions of E_4 and η^4 yield

$$c(n) = a_4(n) + 240 \sum_{j,l>0, 6j+l=n} \sigma_3(j) a_4(l),$$

and hence we have $c(n) \equiv a_4(n) \bmod 240$ for all $n > 0$. Let $p \equiv 1 \bmod 6$ be prime. Then $p = \mu\overline{\mu}$ where we can choose $\mu \in \mathcal{O}_3$ uniquely as in the proof of Corollary 9.4, which implies that $\overline{\psi}(\mu) = \overline{\psi}(\overline{\mu}) = 1$. Therefore, from (9.25) we obtain

$$\begin{aligned} c(p) &= \mu^5 + \overline{\mu}^5 = (\mu + \overline{\mu})(\mu^4 - \mu^3\overline{\mu} + \mu^2\overline{\mu}^2 - \mu\overline{\mu}^3 + \overline{\mu}^4) \\ &= a_4(p)(\mu^4 - \mu^3\overline{\mu} + \mu^2\overline{\mu}^2 - \mu\overline{\mu}^3 + \overline{\mu}^4). \end{aligned}$$

Thus $c(p)$ is a multiple of $a_4(p)$.