17 Weight 1 for Levels $N = 2p$ with Primes $p \geq 5$

17.1 Eta Products for Fricke Groups

For primes $p \geq 5$ there are exactly four new holomorphic eta products of weight 1 for the Fricke group $\Gamma^*(2p)$, namely,

$$\left[\frac{2^2,\, p^2}{1,\, 2p}\right], \qquad \left[\frac{1^2,\, (2p)^2}{2,\, p}\right], \qquad [2,\, p]\,, \qquad [1,\, 2p]\,.$$

By Theorem 8.1, each of them is a product of two simple theta series. All of them have denominator 8 if $p \equiv 1 \bmod 3$, while for $p \equiv -1 \bmod 3$ the denominators are 8 for the first and second, and 24 for the remaining two eta products. Some of the identities in this subsection are mentioned in [65]. We begin with the discussion of the case $p = 5$, where we will meet theta series on the fields with discriminants 40, -40 and -4:

Example 17.1 *Let $\mathcal{J}_{10}$ be the system of integral ideal numbers for $\mathbb{Q}(\sqrt{-10})$ as defined in Example 7.2. The residues of $1+\sqrt{-10}$, $\sqrt{5}$ and -1 modulo 4 can be chosen as generators of $(\mathcal{J}_{10}/(4))^\times \simeq Z_4 \times Z_2^2$. Four characters $\psi_{\delta,\nu}$ on $\mathcal{J}_{10}$ with period 4 are given by*

$$\psi_{\delta,\nu}(1+\sqrt{-10}) = \delta\nu i, \qquad \psi_{\delta,\nu}(\sqrt{5}) = \delta, \qquad \psi_{\delta,\nu}(-1) = 1$$

with $\delta, \nu \in \{1, -1\}$. The residues of $2-\nu i$, $5+2\nu i$, $1-4\nu i$ and νi modulo $4+12\nu i = 4(1+\nu i)(2+\nu i)$ are generators of $(\mathcal{O}_1/(4+12\nu i))^\times \simeq Z_4 \times Z_2^2 \times Z_4$. Characters $\chi_{\delta,\nu}$ on $\mathcal{O}_1$ with periods $4+12\nu i$ are fixed by their values

$$\chi_{\delta,\nu}(2-\nu i) = \delta, \quad \chi_{\delta,\nu}(5+2\nu i) = \delta, \quad \chi_{\delta,\nu}(1-4\nu i) = -1, \quad \chi_{\delta,\nu}(\nu i) = 1.$$

Let ideal numbers $\mathcal{J}_{\mathbb{Q}(\sqrt{10})}$ for $\mathbb{Q}(\sqrt{10})$ be chosen as in Example 7.16. The residues of $1+\sqrt{10}$, $\sqrt{5}$ and -1 modulo 4 generate the group $\left(\mathcal{J}_{\mathbb{Q}(\sqrt{10})}/(4)\right)^\times \simeq Z_4 \times Z_2^2$. Hecke characters ξ_δ on $\mathcal{J}_{\mathbb{Q}(\sqrt{10})}$ with period 4 are given by

$$\xi_\delta(\mu) = \begin{cases} \operatorname{sgn}(\mu) \\ \delta \operatorname{sgn}(\mu) \\ -\operatorname{sgn}(\mu) \end{cases} \qquad \text{for} \qquad \mu \equiv \begin{cases} 1+\sqrt{10} \\ \sqrt{5} \\ -1 \end{cases} \mod 4.$$

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The corresponding theta series of weight 1 *are identical and satisfy*

$$\begin{aligned}\Theta_1\left(40,\,\xi_\delta,\,\tfrac{z}{8}\right) &= \Theta_1\left(-40,\psi_{\delta,\nu},\tfrac{z}{8}\right) = \Theta_1\left(-4,\chi_{\delta,\nu},\tfrac{z}{8}\right)\\ &= \frac{\eta^2(2z)\eta^2(5z)}{\eta(z)\eta(10z)} + \delta\,\frac{\eta^2(z)\eta^2(10z)}{\eta(2z)\eta(5z)}. \qquad (17.1)\end{aligned}$$

The sign transforms of the eta products in (17.1) belong to $\Gamma_0(20)$ and will be discussed in Example 24.10.

Example 17.2 *The residues of* $1+\sqrt{-10}$, $3+\sqrt{-10}$, $3\sqrt{5}+2\sqrt{-2}$ *and* -1 *modulo* 12 *can be chosen as generators of* $(\mathcal{J}_{10}/(12))^\times \simeq Z_8\times Z_4\times Z_2^2$. *Eight characters* $\varphi_{\delta,\varepsilon,\nu}$ *on* $\mathcal{J}_{10}$ *with period* 12 *are given by*

$$\varphi_{\delta,\varepsilon,\nu}(1+\sqrt{-10}) = -\delta\varepsilon, \qquad \varphi_{\delta,\varepsilon,\nu}(3+\sqrt{-10}) = \nu i,$$

$$\varphi_{\delta,\varepsilon,\nu}(3\sqrt{5}+2\sqrt{-2}) = \delta, \qquad \varphi_{\delta,\varepsilon,\nu}(-1) = 1$$

with $\delta,\varepsilon,\nu\in\{1,-1\}$. *The residues of* $\sqrt{3}-\nu\sqrt{-2}$, $1+\nu\sqrt{-6}$, $7-4\nu\sqrt{-6}$ *and* -1 *modulo* $12+4\nu\sqrt{-6} = 4\sqrt{3}(\sqrt{3}+\nu\sqrt{-2})$ *can be chosen as generators of* $(\mathcal{J}_6/(12+4\nu\sqrt{-6}))^\times \simeq Z_8\times Z_4\times Z_2^2$. *Characters* $\rho_{\delta,\varepsilon,\nu}$ *on* $\mathcal{J}_6$ *with periods* $12+4\nu\sqrt{-6}$ *are given by*

$$\rho_{\delta,\varepsilon,\nu}(\sqrt{3}-\nu\sqrt{-2}) = \delta, \qquad \rho_{\delta,\varepsilon,\nu}(1+\nu\sqrt{-6}) = \varepsilon,$$

$$\rho_{\delta,\varepsilon,\nu}(7-4\nu\sqrt{-6}) = -1, \qquad \rho_{\delta,\varepsilon,\nu}(-1) = 1.$$

The residues of $4+\sqrt{15}$, $\sqrt{5}$, $1+2\sqrt{15}$ *and* -1 *modulo* $M = 4(3+\sqrt{15})$ *are generators of* $\left(\mathcal{J}_{\mathbb{Q}(\sqrt{15})}/(M)\right)^\times \simeq Z_4^2\times Z_2^2$. *Hecke characters* $\xi_{\delta,\varepsilon}$ *on* $\mathcal{J}_{\mathbb{Q}(\sqrt{15})}$ *with period* M *are given by*

$$\xi_{\delta,\varepsilon}(\mu) = \begin{cases} \operatorname{sgn}(\mu) \\ \delta\operatorname{sgn}(\mu) \\ \delta\varepsilon\operatorname{sgn}(\mu) \\ -\operatorname{sgn}(\mu)\end{cases} \quad \textit{for} \quad \mu \equiv \begin{cases} 4+\sqrt{15} \\ \sqrt{5} \\ 1+2\sqrt{15} \\ -1 \end{cases} \mod M.$$

The corresponding theta series of weight 1 *are identical and decompose as*

$$\begin{aligned}\Theta_1\left(60,\,\xi_{\delta,\varepsilon},\,\tfrac{z}{24}\right) &= \Theta_1\left(-40,\varphi_{\delta,\varepsilon,\nu},\tfrac{z}{24}\right) = \Theta_1\left(-24,\rho_{\delta,\varepsilon,\nu},\tfrac{z}{24}\right)\\ &= f_1(z) + \delta\, f_5(z) + 2\varepsilon\, f_7(z) - 2\delta\varepsilon\, f_{11}(z), \qquad (17.2)\end{aligned}$$

where the components f_j *are normalized integral Fourier series with denominator* 24 *and numerator classes* j *modulo* 24, *and where* f_7, f_{11} *are eta products,*

$$f_7(z) = \eta(2z)\eta(5z), \qquad f_{11}(z) = \eta(z)\eta(10z). \qquad (17.3)$$

The components f_1, f_5 will be identified with linear combinations of eta products in Example 27.9. We note that the period $4(3+\sqrt{15})$ of $\xi_{\delta,\varepsilon}$ and its conjugate $4(3-\sqrt{15})$ are associates; their quotient is the unit $-4-\sqrt{15}$.

The eta products of weight 1 for $\Gamma^*(14)$ combine to eigenforms which are Hecke theta series for the field $\mathbb{Q}(\sqrt{-14})$:

Example 17.3 *Let $\mathcal{J}_{14}$ be the system of integral ideal numbers for $\mathbb{Q}(\sqrt{-14})$ as defined in Example 7.7, where $\Lambda = \Lambda_{14} = \sqrt{\sqrt{2}+\sqrt{-7}}$ is a root of $\Lambda^8 + 10\Lambda^4 + 81 = 0$. The residues of Λ, $\sqrt{-7}$ and -1 modulo 4 can be chosen as generators of $(\mathcal{J}_{14}/(4))^\times \simeq Z_8 \times Z_2^2$. Eight characters $\chi_{\delta,\varepsilon,\nu}$ on $\mathcal{J}_{14}$ with period 4 are fixed by their values*

$$\chi_{\delta,\varepsilon,\nu}(\Lambda) = \tfrac{1}{\sqrt{2}}(\delta+\nu i), \qquad \chi_{\delta,\varepsilon,\nu}(\sqrt{-7}) = -\varepsilon, \qquad \chi_{\delta,\varepsilon,\nu}(-1) = 1$$

with $\delta, \varepsilon, \nu \in \{1,-1\}$. The corresponding theta series of weight 1 decompose as

$$\Theta_1\left(-56, \chi_{\delta,\varepsilon,\nu}, \tfrac{z}{8}\right) = f_1(z) + \delta\sqrt{2}\, f_3(z) + \delta\varepsilon\sqrt{2}\, f_5(z) - \varepsilon\, f_7(z) \tag{17.4}$$

where the components f_j are normalized integral Fourier series with denominator 8 and numerator classes j modulo 8. All of them are eta products,

$$f_1 = \left[\frac{2^2, 7^2}{1, 14}\right], \qquad f_3 = [2,7], \qquad f_5 = [1,14], \qquad f_7 = \left[\frac{1^2, 14^2}{2, 7}\right]. \tag{17.5}$$

The results for level 22 are similar to those for level 10. They are even more complete since in Example 17.5, in an analogue to (17.2), we can identify all the components of a theta series with (linear combinations of) eta products, which, however, do not all belong to the Fricke group:

Example 17.4 *Let $\mathcal{J}_{22}$ be the system of integral ideal numbers for $\mathbb{Q}(\sqrt{-22})$ as defined in Example 7.2. The residues of $1+\sqrt{-22}$ and $\sqrt{11}$ modulo 4 can be chosen as generators of $(\mathcal{J}_{22}/(4))^\times \simeq Z_4^2$. Four characters $\psi_{\delta,\nu}$ on $\mathcal{J}_{22}$ with period 4 are given by*

$$\psi_{\delta,\nu}(1+\sqrt{-22}) = \delta\nu i, \qquad \psi_{\delta,\nu}(\sqrt{11}) = \delta$$

with $\delta, \nu \in \{1,-1\}$. The residues of $3-\nu\sqrt{-2}$, $1+8\nu\sqrt{-2}$ and -1 modulo $4(3+\nu\sqrt{-2})$ can be chosen as generators of $(\mathcal{O}_2/(12+4\nu\sqrt{-2}))^\times \simeq Z_{20} \times Z_2^2$. Characters $\chi_{\delta,\nu}$ on $\mathcal{O}_2$ with periods $4(3+\nu\sqrt{-2})$ are fixed by their values

$$\chi_{\delta,\nu}(3-\nu\sqrt{-2}) = \delta, \qquad \chi_{\delta,\nu}(1+8\nu\sqrt{-2}) = -1, \qquad \chi_{\delta,\nu}(-1) = 1.$$

The residues of $2+\sqrt{11}$, $1+2\sqrt{11}$ and -1 modulo $M = 4(3+\sqrt{11})$ are generators of $(\mathbb{Z}[\sqrt{11}]/(M))^\times \simeq Z_4 \times Z_2^2$. Hecke characters ξ_δ on $\mathbb{Z}[\sqrt{11}]$ with period M are given by

$$\xi_\delta(\mu) = \begin{cases} \mathrm{sgn}(\mu) \\ -\mathrm{sgn}(\mu) \end{cases} \quad \text{for} \quad \mu \equiv \begin{cases} 1+2\sqrt{11} \\ 2+\sqrt{11},\ -1 \end{cases} \mod M.$$

The corresponding theta series of weight 1 *are identical and satisfy*

$$\begin{aligned}\Theta_1\left(44,\,\xi_\delta,\,\tfrac{z}{8}\right) &= \Theta_1\left(-88,\psi_{\delta,\nu},\tfrac{z}{8}\right) = \Theta_1\left(-8,\chi_{\delta,\nu},\tfrac{z}{8}\right)\\ &= \frac{\eta^2(2z)\eta^2(11z)}{\eta(z)\eta(22z)} + \delta\,\frac{\eta^2(z)\eta^2(22z)}{\eta(2z)\eta(11z)}.\end{aligned} \tag{17.6}$$

Similarly as before in Example 17.2, the character period $4(3+\sqrt{11})$ and its conjugate $4(3-\sqrt{11})$ are associates.

Example 17.5 *The residues of* $1+\sqrt{-22}$, $3+\sqrt{-22}$ *and* $\sqrt{11}$ *modulo* 12 *can be chosen as generators of* $(\mathcal{J}_{22}/(12))^\times \simeq Z_8 \times Z_4^2$. *Eight characters* $\rho_{\delta,\varepsilon,\nu}$ *on* $\mathcal{J}_{22}$ *with period* 12 *are fixed by their values*

$$\rho_{\delta,\varepsilon,\nu}(1+\sqrt{-22}) = \varepsilon, \qquad \rho_{\delta,\varepsilon,\nu}(3+\sqrt{-22}) = -\delta\nu i,$$
$$\rho_{\delta,\varepsilon,\nu}(\sqrt{11}) = \delta\varepsilon$$

with $\delta, \varepsilon, \nu \in \{1,-1\}$. *Let* $\mathcal{J}_{66}$ *be the system of integral ideal numbers for* $\mathbb{Q}(\sqrt{-66})$ *as defined in Example* 7.10, *with* $\Lambda = \Lambda_{66} = \sqrt{\sqrt{3}+\sqrt{-22}}$. *The residues of* Λ, $1+\sqrt{-66}$, 5 *and* $\sqrt{-11}$ *modulo* $4\sqrt{3}$ *can be chosen as generators of* $(\mathcal{J}_{66}/(4\sqrt{3}))^\times \simeq Z_8 \times Z_4 \times Z_2^2$, *where* $\Lambda^4(1+\sqrt{-66})^2 \equiv -1 \bmod 4\sqrt{3}$. *Eight characters* $\varphi_{\delta,\varepsilon,\nu}$ *on* $\mathcal{J}_{66}$ *with period* $4\sqrt{3}$ *are given by*

$$\varphi_{\delta,\varepsilon,\nu}(\Lambda) = \nu, \qquad \varphi_{\delta,\varepsilon,\nu}(1+\sqrt{-66}) = -\varepsilon\nu,$$
$$\varphi_{\delta,\varepsilon,\nu}(5) = -1, \qquad \varphi_{\delta,\varepsilon,\nu}(\sqrt{-11}) = \delta\varepsilon.$$

The residues of $2-\varepsilon\sqrt{3}$, 23, $17+8\varepsilon\sqrt{3}$, $11-2\varepsilon\sqrt{3}$ *and* -1 *modulo* $M_\varepsilon = 4(3-5\varepsilon\sqrt{3})$ *can be chosen as generators of* $(\mathbb{Z}[\sqrt{3}]/(M_\varepsilon))^\times \simeq Z_{20}\times Z_2^4$. *Hecke characters* $\xi_{\delta,\varepsilon}$ *on* $\mathbb{Z}[\sqrt{3}]$ *with period* M_ε *are given by*

$$\xi_{\delta,\varepsilon} = \begin{cases} \mathrm{sgn}(\mu) \\ -\delta\,\mathrm{sgn}(\mu) \\ -\mathrm{sgn}(\mu)\end{cases} \quad \textit{for} \quad \mu \equiv \begin{cases} 2-\varepsilon\sqrt{3},\ 23,\ 17+8\varepsilon\sqrt{3} \\ 11-2\varepsilon\sqrt{3} \\ -1 \end{cases} \bmod M_\varepsilon\,.$$

The corresponding theta series of weight 1 *are identical and decompose as*

$$\begin{aligned}\Theta_1\left(12,\,\xi_{\delta,\varepsilon},\,\tfrac{z}{24}\right) &= \Theta_1\left(-88,\rho_{\delta,\varepsilon,\nu},\tfrac{z}{24}\right) = \Theta_1\left(-264,\varphi_{\delta,\varepsilon,\nu},\tfrac{z}{24}\right)\\ &= f_1(z) + \delta\varepsilon\, f_{11}(z) + 2\delta\, f_{13}(z) + 2\varepsilon\, f_{23}(z)\,,\end{aligned} \tag{17.7}$$

where the components f_j *are normalized integral Fourier series with denominator* 24 *and numerator classes* j *modulo* 24 *which are eta products or linear combinations thereof,*

$$\begin{aligned} f_1(z) &= \frac{\eta(z)\eta^2(11z)}{\eta(22z)} + 2\frac{\eta^2(2z)\eta(22z)}{\eta(z)},\\ f_{11}(z) &= \frac{\eta^2(z)\eta(11z)}{\eta(2z)} + 2\frac{\eta(2z)\eta^2(22z)}{\eta(11z)},\end{aligned} \tag{17.8}$$

$$f_{13}(z) = \eta(2z)\eta(11z), \qquad f_{23}(z) = \eta(z)\eta(22z). \tag{17.9}$$

The eta products in (17.8) will appear once more in Example 17.21 in the components of another theta series.

For level $N = 26$ we find six eigenforms which are theta series and involve, besides the four eta products, two components which are not identified with eta products. Here for the first time we meet a field with class number 6:

Example 17.6 *Let $\mathcal{J}_{26}$ be the system of integral ideal numbers for $\mathbb{Q}(\sqrt{-26})$ as defined in Example 7.14, where $\Lambda = \Lambda_{26} = \sqrt[3]{1+\sqrt{-26}}$ is a root of the polynomial $X^6 - 2X^3 + 27$. The residues of Λ and $\sqrt{-13}$ modulo 4 can be chosen as generators of the group $(\mathcal{J}_{26}/(4))^\times \simeq Z_{12} \times Z_4$. Eight characters $\psi_{\delta,\varepsilon,\nu}$ on $\mathcal{J}_{26}$ with period 4 are given by*

$$\psi_{\delta,\varepsilon,\nu}(\Lambda) = \tfrac{1}{2}(\varepsilon\sqrt{3} + \nu i), \qquad \psi_{\delta,\varepsilon,\nu}(\sqrt{-13}) = \delta$$

with $\delta, \varepsilon, \nu \in \{1, -1\}$. The characters $\varphi_{\delta,\nu} = \psi^3_{\delta,\varepsilon,\nu}$ on $\mathcal{J}_{26}$ with period 4 are defined by

$$\varphi_{\delta,\nu}(\Lambda) = \nu i, \qquad \varphi_{\delta,\nu}(\sqrt{-13}) = \delta.$$

The residues of $3 - 2\nu i$, $5 - 6\nu i$, $1 + 10\nu i$ and νi modulo $4(1+\nu i)(3+2\nu i) = 4 + 20\nu i$ can be chosen as generators of $(\mathcal{O}_1/(4+20\nu i))^\times \simeq Z_{12} \times Z_2^2 \times Z_4$. Characters $\chi_{\delta,\nu}$ on $\mathcal{O}_1$ with periods $4(1+5\nu i)$ are given by

$$\chi_{\delta,\nu}(3-2\nu i) = \delta, \qquad \chi_{\delta,\nu}(5-6\nu i) = -\delta, \qquad \chi_{\delta,\nu}(1+10\nu i) = \delta, \qquad \chi_{\delta,\nu}(\nu i) = 1.$$

Let the ideal numbers $\mathcal{J}_{\mathbb{Q}(\sqrt{26})}$ be given as in Example 7.16. The residues of $1+\sqrt{26}$, $\sqrt{13}$ and -1 modulo 4 are generators of $\left(\mathcal{J}_{\mathbb{Q}(\sqrt{26})}/(4)\right)^\times \simeq Z_4 \times Z_2^2$. Define characters ξ_δ modulo 4 on $\mathcal{J}_{\mathbb{Q}(\sqrt{26})}$ by

$$\xi_\delta(\mu) = \begin{cases} \operatorname{sgn}(\mu) \\ \delta\operatorname{sgn}(\mu) \\ -\operatorname{sgn}(\mu) \end{cases} \quad \text{for} \quad \mu \equiv \begin{cases} 1+\sqrt{26} \\ \sqrt{13} \\ -1 \end{cases} \mod 4.$$

The corresponding theta series of weight 1 satisfy the identities

$$\Theta_1\left(-104, \psi_{\delta,\varepsilon,\nu}, \tfrac{z}{8}\right) = f_1(z) + \varepsilon\sqrt{3}\,f_3(z) + \delta\,f_5(z) - \delta\varepsilon\sqrt{3}\,f_7(z), \quad (17.10)$$

$$\Theta_1\left(104, \xi_\delta, \tfrac{z}{8}\right) = \Theta_1\left(-104, \varphi_{\delta,\nu}, \tfrac{z}{8}\right) = \Theta_1\left(-4, \chi_{\delta,\nu}, \tfrac{z}{8}\right) = g_1(z) - \delta\,g_5(z), \quad (17.11)$$

where the components f_j, g_j are integral Fourier series with denominator 8 and numerator classes j modulo 8 which are normalized with the exception of g_5. Those for $j = 1, 5$ are linear combinations of eta products,

$$f_1 = \left[\frac{2^2, 13^2}{1, 26}\right] + [1, 26], \qquad f_5 = [2, 13] + \left[\frac{1^2, 26^2}{2, 13}\right], \quad (17.12)$$

$$g_1 = \left[\frac{2^2, 13^2}{1, 26}\right] - 2\,[1, 26], \qquad g_5 = 2\,[2, 13] - \left[\frac{1^2, 26^2}{2, 13}\right]. \quad (17.13)$$

Similar results for the sign transforms of the eta products in (17.12), (17.13) will be given in Example 22.18.

In the following two examples we describe theta series which contain the eta products of weight 1 for $\Gamma^*(34)$ in their components: The sign transforms of the eta products in (17.16) belong to $\Gamma_0(68)$ and will be discussed in Example 22.10.

Example 17.7 *Let $\mathcal{J}_{34}$ be the system of integral ideal numbers for $\mathbb{Q}(\sqrt{-34})$ as defined in Example 7.7, where $\Lambda_{34} = \sqrt{2\sqrt{2}+\sqrt{-17}}$ is a root of the polynomial $X^8 - 18X^4 + 625$. The residues of Λ_{34} and $1+\sqrt{-34}$ modulo $4\sqrt{2}$ can be chosen as generators of the group $(\mathcal{J}_{34}/(4))^\times \simeq Z_8 \times Z_4$, where $\Lambda_{34}^4 \equiv -1 \bmod 4$. Eight characters $\varphi_{\delta,\nu}$ and $\rho_{\delta,\nu}$ on $\mathcal{J}_{34}$ with period 4 are fixed by their values*

$$\varphi_{\delta,\nu}(\Lambda_{34}) = \delta i, \quad \varphi_{\delta,\nu}(1+\sqrt{-34}) = -\delta\nu i,$$

$$\rho_{\delta,\nu}(\Lambda_{34}) = \delta, \quad \rho_{\delta,\nu}(1+\sqrt{-34}) = \nu i$$

with $\delta, \nu \in \{1,-1\}$. Let $\mathcal{J}_{17}$ be the system of integral ideal numbers for $\mathbb{Q}(\sqrt{-17})$ as defined in Example 7.9, where $\Lambda_{17} = \sqrt{(1+\sqrt{-17})/\sqrt{2}}$ is a root of the polynomial $X^8 + 16X^4 + 81$. The residues of Λ_{17}, $1+2\sqrt{-17}$ and 3 modulo 4 can be chosen as generators of the group $(\mathcal{J}_{17}/(4\sqrt{2}))^\times \simeq Z_{16} \times Z_2^2$, where $\Lambda_{17}^8 \equiv -1 \bmod 4\sqrt{2}$. Four characters $\psi_{\delta,\nu}$ on $\mathcal{J}_{17}$ with period $4\sqrt{2}$ are given by

$$\psi_{\delta,\nu}(\Lambda_{17}) = \nu, \qquad \psi_{\delta,\nu}(1+2\sqrt{-17}) = \delta\nu, \qquad \psi_{\delta,\nu}(3) = -1.$$

The residues of $2+\nu i$, $3+8\nu i$, $7-2\nu i$ and νi modulo $4(1+\nu i)(4-\nu i) = 20+12\nu i$ are generators of $(\mathcal{O}_1/(20+12\nu i))^\times \simeq Z_{16} \times Z_2^2 \times Z_4$. Characters $\chi_{\delta,\nu}$ on $\mathcal{O}_1$ with periods $4(5+3\nu i)$ are given by

$$\chi_{\delta,\nu}(2+\nu i) = \delta, \qquad \chi_{\delta,\nu}(3+8\nu i) = -1, \qquad \chi_{\delta,\nu}(7-2\nu i) = \delta, \qquad \chi_{\delta,\nu}(\nu i) = 1.$$

The residues of $1-\delta\sqrt{2}$, $5+\delta\sqrt{2}$ and -1 modulo $M_\delta = 4(5-2\delta\sqrt{2})$ are generators of $(\mathbb{Z}[\sqrt{2}]/(M_\delta))^\times \simeq Z_{16} \times Z_4 \times Z_2$. Define characters ξ_δ^* modulo M_δ on $\mathbb{Z}[\sqrt{2}]$ by

$$\xi_\delta^*(\mu) = \begin{cases} -\delta\,\mathrm{sgn}(\mu) \\ -\mathrm{sgn}(\mu) \end{cases} \quad \textit{for} \quad \mu \equiv \begin{cases} 1-\delta\sqrt{2},\ 5+\delta\sqrt{2} \\ -1 \end{cases} \bmod M_\delta.$$

Let $\mathcal{J}_{\mathbb{Q}(\sqrt{34})}$ be given as in Example 7.18. The residues of $\Lambda = \sqrt{3+\sqrt{34}}$ and -1 modulo 4 are generators of $\left(\mathcal{J}_{\mathbb{Q}(\sqrt{34})}/(4)\right)^\times \simeq Z_8 \times Z_2$. Define characters ξ_δ modulo 4 on $\mathcal{J}_{\mathbb{Q}(\sqrt{34})}$ by

$$\xi_\delta(\mu) = \begin{cases} \delta\,\mathrm{sgn}(\mu) \\ -\mathrm{sgn}(\mu) \end{cases} \quad \textit{for} \quad \mu \equiv \begin{cases} \Lambda \\ -1 \end{cases} \bmod 4.$$

The theta series of weight 1 *for* ξ_δ^*, $\varphi_{\delta,\nu}$, $\psi_{\delta,\nu}$ *are identical, and those for* ξ_δ, $\rho_{\delta,\nu}$, χ_δ *are identical, and we have the decompositions*

$$\begin{aligned}\Theta_1\left(8,\,\xi_\delta^*,\,\tfrac{z}{8}\right) &= \Theta_1\left(-136,\varphi_{\delta,\nu},\tfrac{z}{8}\right)\\ &= \Theta_1\left(-68,\psi_{\delta,\nu},\tfrac{z}{8}\right) \;=\; f_1(z) \,+\, 2\delta\, f_7(z),\end{aligned} \tag{17.14}$$

$$\begin{aligned}\Theta_1\left(136,\,\xi_\delta,\,\tfrac{z}{8}\right) &= \Theta_1\left(-136,\rho_{\delta,\nu},\tfrac{z}{8}\right)\\ &= \Theta_1\left(-4,\chi_{\delta,\nu},\tfrac{z}{8}\right) \;=\; g_1(z) \,+ 2\delta\, g_5(z)\end{aligned} \tag{17.15}$$

where the components f_j, g_j *are normalized integral Fourier series with denominator* 8 *and numerator classes* j *modulo* 8, *and where* f_1, g_1 *are linear combinations of eta products,*

$$f_1 = \left[\frac{2^2,17^2}{1,34}\right] + \left[\frac{1^2,34^2}{2,17}\right], \qquad g_1 = \left[\frac{2^2,17^2}{1,34}\right] - \left[\frac{1^2,34^2}{2,17}\right]. \tag{17.16}$$

Example 17.8 *Let* $\mathcal{J}_{34}$ *be given as in the preceding example. The residues of* Λ_{34}, $3+\sqrt{-34}$ *and* $3\sqrt{2}+\sqrt{-17}$ *modulo* 12 *can be chosen as generators of the group* $(\mathcal{J}_{34}/(12))^\times \simeq Z_{16}\times Z_4^2$, *where* $(3+\sqrt{-34})^2(3\sqrt{2}+\sqrt{-17})^2 \equiv -1 \bmod 12$. *Sixteen characters* $\chi_{\delta,\varepsilon,\nu,\sigma}$ *on* $\mathcal{J}_{34}$ *with period* 12 *are fixed by their values*

$$\chi_{\delta,\varepsilon,\nu,\sigma}(\Lambda_{34}) = \tfrac{1}{\sqrt{2}}(\delta - \nu\sigma i), \qquad \chi_{\delta,\varepsilon,\nu,\sigma}(3+\sqrt{-34}) = -\varepsilon\sigma i,$$

$$\chi_{\delta,\varepsilon,\nu,\sigma}(3\sqrt{2}+\sqrt{-17}) = \sigma i$$

with $\delta,\varepsilon,\nu,\sigma\in\{1,-1\}$. *The corresponding theta series of weight* 1 *decompose as*

$$\begin{aligned}\Theta_1\left(-136,\chi_{\delta,\varepsilon,\nu,\sigma},\tfrac{z}{24}\right) &= h_1(z) \,+\, \delta\sqrt{2}\,h_5(z) \,+\, \nu\sqrt{2}\,h_7(z) \,+\, 2\delta\nu\, h_{11}(z)\\ &\quad - \delta\varepsilon\sqrt{2}\,h_{13}(z) \,+ \varepsilon\, h_{17}(z)\\ &\quad - 2\delta\varepsilon\nu\, h_{19}(z) \,+\, \varepsilon\nu\sqrt{2}\,h_{23}(z),\end{aligned} \tag{17.17}$$

where the components h_j *are normalized integral Fourier series with denominator* 24 *and numerator classes* j *modulo* 24, *and where* h_{11}, h_{19} *are eta products,*

$$h_{11}(z) = \eta(z)\eta(34z), \qquad h_{19}(z) = \eta(2z)\eta(17z). \tag{17.18}$$

For $\Gamma^*(38)$ there are four eta products of weight 1 whose denominators are 8 and whose numerators occupy all the residue classes modulo 8. But there are no linear combinations of these functions which are eigenforms. We do not pursue the levels $N = 2p$ for larger primes p.

17.2 Cuspidal Eta Products for $\Gamma_0(10)$

For primes $p \geq 7$ there are exactly ten new holomorphic eta products of weight 1 for $\Gamma_0(2p)$. All of them are products of two simple theta series, and in fact only $\eta(z)$, $\eta^2(z)/\eta(2z)$ and $\eta^2(2z)/\eta(z)$ are needed to concoct these ten eta products. Specifically, we have two non-cuspidal eta products

$$\left[1^2, 2^{-1}, p^2, (2p)^{-1}\right], \qquad \left[1^{-1}, 2^2, p^{-1}, (2p)^2\right]$$

and eight cuspidal ones,

$$\left[2, p^2, (2p)^{-1}\right], \quad \left[1^{-1}, 2^2, p\right], \quad \left[1^2, 2^{-1}, 2p\right], \quad \left[1, p^{-1}, (2p)^2\right],$$
$$\left[1, p^2, (2p)^{-1}\right], \quad \left[1^2, 2^{-1}, p\right], \quad \left[1^{-1}, 2^2, 2p\right], \quad \left[2, p^{-1}, (2p)^2\right].$$

For those in the last line the denominator $t = 24$ does not depend upon p. For $\Gamma_0(10)$ there are, in addition, four non-cuspidal and four cuspidal eta products of weight 1. In this subsection we discuss the 12 cuspidal eta products of level 10. Two of them have denominator 12; they appear in theta series for the fields with discriminants 60, -4 and -15:

Example 17.9 *Let $\mathcal{J}_{15}$ be the system of integral ideal numbers for $\mathbb{Q}(\sqrt{-15})$ as defined in Example* 7.3. *The residues of $\sqrt{-5}$, 7, $2+\sqrt{-15}$ and -1 modulo $2(3+\sqrt{-15})$ can be chosen as generators of $(\mathcal{J}_{15}/(6+2\sqrt{-15}))^\times \simeq Z_4 \times Z_2^3$. Four characters $\psi_{\delta,\nu}$ on $\mathcal{J}_{15}$ with period $2(3+\sqrt{-15})$ are fixed by their values*

$$\psi_{\delta,\nu}(\sqrt{-5}) = \delta i, \qquad \psi_{\delta,\nu}(7) = -1, \qquad \psi_{\delta,\nu}(2+\sqrt{-15}) = \nu, \qquad \psi_{\delta,\nu}(-1) = 1$$

with $\delta, \nu \in \{1, -1\}$. The residues of $2+\nu i$, 7 and νi modulo $6(1+\nu i)(2-\nu i) = 6(3+\nu i)$ can be chosen as generators of $(\mathcal{O}_1/(18+6\nu i))^\times \simeq Z_8 \times Z_4^2$. Characters $\chi_{\delta,\nu}$ on $\mathcal{O}_1$ with periods $6(3+\nu i)$ are given by

$$\chi_{\delta,\nu}(2+\nu i) = \delta i, \qquad \chi_{\delta,\nu}(7) = -1, \qquad \chi_{\delta,\nu}(\nu i) = 1.$$

The residues of $\sqrt{3}+2\sqrt{5}$ and $\sqrt{5}$ modulo $M = 2(3+\sqrt{15})$ are generators of the group $\left(\mathcal{J}_{\mathbb{Q}(\sqrt{15})}/(M)\right)^\times \simeq Z_4^2$, where $(\sqrt{3}+2\sqrt{5})^2 \equiv -1 \bmod M$. Define characters $\widetilde{\xi}_\delta$ modulo M on $\mathcal{J}_{\mathbb{Q}(\sqrt{15})}$ by

$$\widetilde{\xi}_\delta(\mu) = \begin{cases} -\delta i \operatorname{sgn}(\mu) \\ \delta i \operatorname{sgn}(\mu) \end{cases} \quad \text{for} \quad \mu \equiv \begin{cases} \sqrt{3}+2\sqrt{5} \\ \sqrt{5} \end{cases} \bmod M.$$

The corresponding theta series of weight 1 *are identical and decompose as*

$$\begin{aligned} \Theta_1\left(60, \widetilde{\xi}_\delta, \tfrac{z}{12}\right) &= \Theta_1\left(-15, \psi_{\delta,\nu}, \tfrac{z}{12}\right) \\ &= \Theta_1\left(-4, \chi_{\delta,\nu}, \tfrac{z}{12}\right) = f_1(z) + \delta i\, f_5(z), \end{aligned} \tag{17.19}$$

where the components f_j are eta products,

$$f_1(z) = \frac{\eta(2z)\eta^2(5z)}{\eta(10z)}, \qquad f_5(z) = \frac{\eta^2(z)\eta(10z)}{\eta(2z)}. \tag{17.20}$$

The sign transforms of the eta products in (17.20) belong to $\Gamma_0(20)$ and will be considered in Example 24.11.

Let $F_\delta = f_1 + \delta i f_5$ denote the functions given by (17.19), (17.20). The Fricke involution W_{10} acts on F_δ according to $F_\delta(W_{10}z) = -2\sqrt{5}iz\, G_\delta(z)$, where $G_\delta = \left[1^{-1}, 2^2, 5\right] + \delta i \left[1, 5^{-1}, 10^2\right]$ is a linear combination of eta products with denominator $t = 3$. One would expect that the functions G_δ are Hecke eigenforms and representable by theta series. However, although the coefficients of G_δ are multiplicative, they violate the proper recursion formula for powers of the prime 2, and therefore G_δ is not a Hecke theta series. We get an eta–theta identity when we rectify the bad behavior at the prime 2, using the eta products with denominator $t = 12$:

Example 17.10 *Let $\mathcal{J}_{15}$ be given as before in Example 17.9. The residues of $\frac{1}{2}(\sqrt{3} + \nu\sqrt{-5})$ and -1 modulo $\frac{1}{2}(\sqrt{3} + 3\nu\sqrt{-5})$ are generators of $(\mathcal{J}_{15}/(\frac{1}{2}(\sqrt{3}+3\nu\sqrt{-5})))^\times \simeq Z_4 \times Z_2$. Characters $\varphi_{\delta,\nu}$ on $\mathcal{J}_{15}$ with periods $\frac{1}{2}(\sqrt{3}+3\nu\sqrt{-5})$ are given by*

$$\varphi_{\delta,\nu}\left(\tfrac{1}{2}(\sqrt{3}+\nu\sqrt{-5})\right) = \delta i, \qquad \varphi_{\delta,\nu}(-1) = 1$$

with $\delta, \nu \in \{1, -1\}$. The residues of $2 - \nu i$ and νi modulo $3(2+\nu i)$ generate the group $(\mathcal{O}_1/(6+3\nu i))^\times \simeq Z_8 \times Z_4$. Characters $\rho_{\delta,\nu}$ on $\mathcal{O}_1$ with periods $3(2+\nu i)$ are given by

$$\rho_{\delta,\nu}(2-\nu i) = \delta i, \qquad \rho_{\delta,\nu}(\nu i) = 1.$$

The residue of $\sqrt{5}$ modulo $\sqrt{3}$ is a generator of $\left(\mathcal{J}_{\mathbb{Q}(\sqrt{15})}/(\sqrt{3})\right)^\times \simeq Z_4$. Hecke characters ξ_δ on $\mathcal{J}_{\mathbb{Q}(\sqrt{15})}$ modulo $\sqrt{3}$ are given by $\xi_\delta(\mu) = \delta i\,\mathrm{sgn}(\mu)$ for $\mu \equiv \sqrt{5} \bmod \sqrt{3}$. The corresponding theta series of weight 1 are identical and decompose as

$$\begin{aligned}\Theta_1\left(60, \xi_\delta, \tfrac{z}{3}\right) &= \Theta_1\left(-15, \varphi_{\delta,\nu}, \tfrac{z}{3}\right) \\ &= \Theta_1\left(-4, \rho_{\delta,\nu}, \tfrac{z}{3}\right) = g_1(z) + \delta i\, g_2(z), \end{aligned} \tag{17.21}$$

where the components g_j are normalized integral Fourier series with denominator 3 and numerator classes j modulo 3 which are linear combinations of eta products,

$$\begin{aligned} g_1(z) &= \frac{\eta(z/2)\eta^2(5z)}{\eta(5z/2)} + \frac{\eta^2(2z)\eta(20z)}{\eta(4z)}, \\ g_2(z) &= \frac{\eta^2(z)\eta(5z/2)}{\eta(z/2)} - \frac{\eta(4z)\eta^2(10z)}{\eta(20z)}. \end{aligned} \tag{17.22}$$

The eta products in (17.22) have expansions to powers $e\left(\frac{z}{6}\right)^n$. But in the linear combinations all coefficients at odd n vanish thanks to coincidences of coefficients of the eta products with denominators 3 and 12, and hence the expansions of g_1, g_2 proceed to powers $e\left(\frac{z}{3}\right)^n$. Much simpler formulae for g_1, g_2 will be given in Example 24.5 in terms of the sign transforms of $\left[1^{-1}, 2^2, 5\right]$, $\left[1, 5^{-1}, 10\right]$ which belong to $\Gamma_0(20)$.

There are eight eta products with denominator $t = 24$, and we find eight theta series which involve these eta products in their components. The results will be described in the following three examples where we will exhibit the two remarkable identities (17.29), (17.30) connecting eta products of levels 10 and 2.

Example 17.11 *Let $\mathcal{J}_{30}$ be the system of integral ideal numbers for $\mathbb{Q}(\sqrt{-30})$ as defined in Example 7.5. The residues of $1+\sqrt{-30}$, $\sqrt{5}+\sqrt{-6}$, $2\sqrt{10}+\sqrt{-3}$ and -1 modulo $4\sqrt{-3}$ can be chosen as generators of the group $(\mathcal{J}_{30}/(4\sqrt{-3}))^\times \simeq Z_4^2 \times Z_2^2$. Eight characters $\psi = \psi_{\delta,\varepsilon,\nu}$ on $\mathcal{J}_{30}$ with period $4\sqrt{-3}$ are fixed by their values*

$$\psi(1+\sqrt{-30}) = \nu, \quad \psi(\sqrt{5}+\sqrt{-6}) = \delta\nu i, \quad \psi(2\sqrt{10}+\sqrt{-3}) = \varepsilon\nu, \quad \psi(-1) = 1$$

with $\delta, \varepsilon, \nu \in \{1, -1\}$. The residues of $2-\nu i$, $3-2\nu i$, 11, $11+6\nu i$ and νi modulo $12(1-\nu i)(2+\nu i) = 12(3-\nu i)$ can be chosen as generators of $(\mathcal{O}_1/(36-12\nu i))^\times \simeq Z_8 \times Z_4 \times Z_2^2 \times Z_4$. Characters $\chi = \chi_{\delta,\varepsilon,\nu}$ on $\mathcal{O}_1$ with periods $12(3-\nu i)$ are given by

$$\chi(2-\nu i) = \delta i, \quad \chi(3-2\nu i) = \varepsilon, \quad \chi(11) = -1, \quad \chi(11+6\nu i) = \varepsilon, \quad \chi(\nu i) = 1.$$

Let ideal numbers $\mathcal{J}_{\mathbb{Q}(\sqrt{30})}$ be given as in Example 7.19. The residues of $1+\sqrt{30}$, $\sqrt{3}+\sqrt{10}$ and -1 modulo $4\sqrt{3}$ can be chosen as generators of $\left(\mathcal{J}_{\mathbb{Q}(\sqrt{30})}/(4\sqrt{3})\right)^\times \simeq Z_4^2 \times Z_2$. Hecke characters $\xi_{\delta,\varepsilon}$ on $\mathcal{J}_{\mathbb{Q}(\sqrt{30})}$ with period $4\sqrt{3}$ are defined by

$$\xi_{\delta,\varepsilon}(\mu) = \begin{cases} -\delta i \,\mathrm{sgn}(\mu) \\ \delta\varepsilon i \,\mathrm{sgn}(\mu) \\ -\mathrm{sgn}(\mu) \end{cases} \quad \text{for} \quad \mu \equiv \begin{cases} 1+\sqrt{30} \\ \sqrt{3}+\sqrt{10} \\ -1 \end{cases} \mod 4\sqrt{3}.$$

The corresponding theta series of weight 1 are identical and decompose as

$$\begin{aligned} \Theta_1\left(120, \xi_{\delta,\varepsilon}, \tfrac{z}{24}\right) &= \Theta_1\left(-120, \psi_{\delta,\varepsilon,\nu}, \tfrac{z}{24}\right) = \Theta_1\left(-4, \chi_{\delta,\varepsilon,\nu}, \tfrac{z}{24}\right) \\ &= f_1(z) + \delta i\, f_5(z) + 2\varepsilon\, f_{13}(z) - 2\delta\varepsilon i\, f_{17}(z), \end{aligned} \tag{17.23}$$

where the components f_j are normalized integral Fourier series with denominator 24 and numerator classes j modulo 24. All of them are eta products,

$$\begin{aligned} f_1 &= \left[\frac{1, 5^2}{10}\right], \qquad f_5 = \left[\frac{1^2, 5}{2}\right], \\ f_{13} &= \left[\frac{2^2, 10}{1}\right], \qquad f_{17} = \left[\frac{2, 10^2}{5}\right]. \end{aligned} \tag{17.24}$$

We will meet the theta series (17.23) once more in Examples 24.13, 24.16, 27.10 when we will find identities relating the components f_j with eta products on $\Gamma_0(20)$ and on $\Gamma^*(40)$. The sign transforms of the eta products in (17.24) will be identified with components of theta series in Example 24.19.

Example 17.12 *Let the generators of $(\mathcal{O}_1/(36-12\nu i))^\times$ be chosen as before in Example 17.11. Characters $\rho_{\delta,\nu}$ on $\mathcal{O}_1$ with periods $12(3-\nu i)$ are fixed by their values*

$$\rho_{\delta,\nu}(2-\nu i) = \delta i, \quad \rho_{\delta,\nu}(3-2\nu i) = \nu, \quad \rho_{\delta,\nu}(11) = -1, \quad \rho_{\delta,\nu}(11+6\sigma i) = -\nu$$

and $\rho_{\delta,\nu}(\nu i) = 1$ with $\delta, \nu \in \{1, -1\}$. Let the generators of $(\mathcal{J}_6/(12+4\nu\sqrt{-6}))^\times$ be chosen as in Example 17.2. Characters $\varphi_{\delta,\nu}$ on $\mathcal{J}_6$ with periods $4(3+\nu\sqrt{-6})$ are given by

$$\varphi_{\delta,\nu}(\sqrt{3}-\nu\sqrt{-2}) = \delta i, \qquad \varphi_{\delta,\nu}(1+\nu\sqrt{-6}) = \nu,$$

$$\varphi_{\delta,\nu}(7-4\nu\sqrt{-6}) = -1, \qquad \varphi_{\delta,\nu}(-1) = 1.$$

The residues of $1+\sqrt{6}$, 7, $5-4\sqrt{6}$ and -1 modulo $M = 4(3-\sqrt{6})(1-\sqrt{6})$ can be chosen as generators of $(\mathbb{Z}[\sqrt{6}]/(M))^\times \simeq Z_4^2 \times Z_2^2$. Hecke characters ξ_δ on $\mathbb{Z}[\sqrt{6}]$ with period M are given by

$$\xi_\delta(\mu) = \begin{cases} \delta i \,\mathrm{sgn}(\mu) \\ \mathrm{sgn}(\mu) \\ -\mathrm{sgn}(\mu) \end{cases} \quad \text{for} \quad \mu \equiv \begin{cases} 1+\sqrt{6} \\ 7 \\ 5-4\sqrt{6},\ -1 \end{cases} \mod M.$$

The corresponding theta series of weight 1 satisfy

$$\Theta_1\left(24, \xi_\delta, \tfrac{z}{24}\right) = \Theta_1\left(-4, \rho_{\delta,\nu}, \tfrac{z}{24}\right) = \Theta_1\left(-24, \varphi_{\delta,\nu}, \tfrac{z}{24}\right) = g_1(z) + \delta i\, g_5(z)\,, \tag{17.25}$$

where the components g_j are normalized integral Fourier series with denominator 24 and numerator classes j modulo 24 which are linear combinations of eta products,

$$g_1 = 3\left[\frac{2, 5^4}{1, 10^2}\right] - 2\left[\frac{2^4, 5}{1^2, 10}\right], \qquad g_5 = \left[\frac{1^4, 10}{2^2, 5}\right] + 6\left[\frac{1, 10^4}{2, 5^2}\right]. \tag{17.26}$$

Example 17.13 *Let ξ_δ, $\chi_{\delta,\nu}$ and $\psi_{\delta,\nu}$ be the characters on $\mathbb{Z}[\sqrt{6}]$, $\mathcal{O}_1$ and $\mathcal{J}_6$, respectively, as defined in Example 10.5. The corresponding theta series of weight 1 satisfy*

$$\begin{aligned} \Theta_1\left(24, \xi_\delta, \tfrac{z}{24}\right) &= \Theta_1\left(-4, \chi_{\delta,\nu}, \tfrac{z}{24}\right) \\ &= \Theta_1\left(-24, \psi_{\delta,\nu}, \tfrac{z}{24}\right) \\ &= h_1(z) + 2\delta i\, h_5(z)\,, \end{aligned} \tag{17.27}$$

where the components h_j are normalized integral Fourier series with denominator 24 and numerator classes j modulo 24 which are linear combinations of eta products,

$$h_1 = 5\left[\frac{2,5^4}{1,10^2}\right] - 4\left[\frac{2^4,5}{1^2,10}\right], \qquad h_5 = \left[\frac{1^4,10}{2^2,5}\right] + 5\left[\frac{1,10^4}{2,5^2}\right]. \tag{17.28}$$

Observe that the same four eta products show up in (17.26) and in (17.28). When we compare the results in Examples 10.5 and 17.13, we obtain the remarkable eta identities

$$5\,\frac{\eta(2z)\eta^4(5z)}{\eta(z)\eta^2(10z)} - 4\,\frac{\eta^4(2z)\eta(5z)}{\eta^2(z)\eta(10z)} = \frac{\eta^3(z)}{\eta(2z)}, \tag{17.29}$$

$$\frac{\eta^4(z)\eta(10z)}{\eta^2(2z)\eta(5z)} + 5\,\frac{\eta(z)\eta^4(10z)}{\eta(2z)\eta^2(5z)} = \frac{\eta^3(2z)}{\eta(z)}. \tag{17.30}$$

Playing around with these formulae yields

$$\begin{aligned}\left[\frac{2,5^4}{1,10^2}\right] &= \left[\frac{1^3}{2}\right] + 4\left[\frac{10^3}{5}\right],\\ \left[\frac{2^4,5}{1^2,10}\right] &= \left[\frac{1^3}{2}\right] + 5\left[\frac{10^3}{5}\right],\end{aligned} \tag{17.31}$$

$$\begin{aligned}\left[\frac{1^4,10}{2^2,5}\right] &= 5\left[\frac{5^3}{10}\right] - 4\left[\frac{2^3}{1}\right],\\ \left[\frac{1,10^4}{2,5^2}\right] &= \left[\frac{2^3}{1}\right] - \left[\frac{5^3}{10}\right].\end{aligned} \tag{17.32}$$

These identities tell that the eta products of level 10 on the left hand sides are combinations of products of two simple theta series. For example, let $\alpha(n)$ and $\beta(n)$ for $n \equiv 1 \bmod 24$ denote the coefficients of $\left[1^{-1}, 2, 5^4, 10^{-2}\right]$ and of $\left[1^{-2}, 2^4, 5, 10^{-1}\right]$. Then

$$\alpha(n) = \beta(n) = \sum_{x>0,\, y\in\mathbb{Z},\, x^2+24y^2=n} (-1)^y \left(\frac{12}{x}\right)$$

whenever $5 \nmid n$.

17.3 Non-cuspidal Eta Products for $\Gamma_0(10)$

The non-cuspidal eta products of weight 1 for $\Gamma_0(10)$ have denominators 1 and 4, three at a time in each case. Two of those with denominator 4 combine to an Eisenstein series which is well known from Examples 10.6 and 15.11:

Example 17.14 *Let* χ_0 *denote the principal character on* $\mathcal{O}_1$ *with period* $1+i$. *Then*

$$\begin{aligned}\Theta_1\left(-4,\chi_0,\tfrac{z}{4}\right) &= \sum_{n \text{ odd}}\left(\sum_{d|n}\left(\tfrac{-1}{d}\right)\right)e\left(\tfrac{nz}{4}\right)\\ &= \frac{1}{4}\left(5\,\frac{\eta(2z)\eta^3(5z)}{\eta(z)\eta(10z)} - \frac{\eta^3(z)\eta(10z)}{\eta(2z)\eta(5z)}\right) \qquad (17.33)\end{aligned}$$

and

$$\sum_{n \text{ odd}}\left(\sum_{d|n}\left(\tfrac{-1}{d}\right)\right)e\left(\tfrac{5nz}{4}\right) = \frac{1}{4}\left(\frac{\eta(2z)\eta^3(5z)}{\eta(z)\eta(10z)} - \frac{\eta^3(z)\eta(10z)}{\eta(2z)\eta(5z)}\right) = \frac{\eta^4(10z)}{\eta^2(5z)}.$$

Comparing the results from Examples 10.6, 15.11, 17.14 yields the eta identities

$$\left[\frac{2^4}{1^2}\right] = \frac{1}{4}\left(5\left[\frac{2,5^3}{1,10}\right] - \left[\frac{1^3,10}{2,5}\right]\right) = \left[\frac{8^5}{2,16^2}\right] + 2\left[\frac{4^2,16^2}{2,8}\right]. \qquad (17.34)$$

Multiplying (17.30) with $\eta(2z)/\eta(z)$ gives another identity for $[1^{-2},2^4]$ which together with (17.34) implies

$$\left[\frac{2,5^3}{1,10}\right] = \left[\frac{2^4}{1^2}\right] - \left[\frac{10^4}{5^2}\right], \qquad \left[\frac{1^3,10}{2,5}\right] = \left[\frac{2^4}{1^2}\right] - 5\left[\frac{10^4}{5^2}\right]. \qquad (17.35)$$

The eta product with denominator 4 and numerator 3 is the sign transform of an eta product for $\Gamma^*(20)$. It is a component in two Eisenstein series which are theta series on the field with discriminant -20. The other component is a linear combination of two eta products for $\Gamma_0(20)$ whose sign transforms belong to $\Gamma^*(20)$ and which will appear in Examples 24.1 and 24.3.

Example 17.15 *Let* ψ_1 *and* ψ_{-1} *denote the trivial and the non-trivial character on* $\mathcal{J}_5$ *with period* $\sqrt{2}$, *respectively. Then for* $\delta \in \{1,-1\}$ *we have*

$$\Theta_1\left(-20,\psi_\delta,\tfrac{z}{4}\right) = \sum_{n>0 \text{ odd}}\left(\left(\tfrac{\delta}{n}\right)\sum_{d|n}\left(\tfrac{-20}{d}\right)\right)e\left(\tfrac{nz}{4}\right) = f_1(z) + 2\delta\, f_3(z), \qquad (17.36)$$

where the components f_j *are normalized integral Fourier series with denominator* 4 *and numerator classes* j *modulo* 4 *which are eta products or linear combinations thereof,*

$$f_1 = \left[\frac{4^2,10^5}{2,5^2,20^2}\right] + \left[\frac{2^5,20^2}{1^2,4^2,10}\right], \qquad f_3 = \left[\frac{2^2,10^2}{1,5}\right]. \qquad (17.37)$$

We will meet the series in (17.36) again in Example 24.27, and then we get other identifications of f_1, f_3 with eta products. We note that the components f_1 and f_3 of the theta series come from the summation on $\mathcal{O}_5$ and on $\mathcal{J}_5 \setminus \mathcal{O}_5$, respectively. The characters ψ_δ in Example 17.15 take the values $\psi_\delta(x + y\sqrt{-5}) = 1$ for $x \not\equiv y \bmod 2$, $\psi_\delta((x + y\sqrt{-5})/\sqrt{2}) = \delta$ for $x \equiv y \equiv 1 \bmod 2$, and $\psi_\delta(\mu) = 0$ if $\mu\overline{\mu}$ is even.

The eta product $[1, 2^{-1}, 5^{-1}, 10^3]$ with denominator 1 is identified with an Eisenstein series. Its coefficients are multiplicative, but they violate the proper Hecke recursions at powers of the prime 2, and therefore this function is not a theta series. Its sign transform is both a theta series and an Eisenstein series, as will be shown in Example 24.31. The coefficients of $[1, 2^{-1}, 5^{-1}, 10^3]$ and of $[1^{-1}, 2^3, 5, 10^{-1}]$ coincide at all indices n for which $5 \nmid n$, and the difference of these two functions is an Eisenstein series which is well known from Example 10.6:

Example 17.16 *We have the identities*

$$\frac{\eta(z)\eta^3(10z)}{\eta(2z)\eta(5z)} = \sum_{n=1}^{\infty}(-1)^{n-1}\Bigg(\sum_{d|n,\ 5\nmid d}\left(\frac{-1}{d}\right)\Bigg)e(nz), \tag{17.38}$$

$$\begin{aligned}\frac{\eta^3(\frac{2z}{5})\eta(z)}{\eta(\frac{z}{5})\eta(2z)} - \frac{\eta(\frac{z}{5})\eta^3(2z)}{\eta(\frac{2z}{5})\eta(z)} &= 1 - \sum_{n=1}^{\infty}\Bigg((-1)^{n-1}\left(\tfrac{-1}{n}\right)\sum_{d|n}\left(\tfrac{-1}{d}\right)\Bigg)e(nz) \\ &= \frac{\eta^4(z)}{\eta^2(2z)}. \end{aligned} \tag{17.39}$$

Finally, the eta product $[1^2, 2^{-1}, 5^2, 10^{-1}]$ with denominator 1 is not identified with a constituent of an Eisenstein or theta series. But it is a difference of eta products of level 20. This will be deduced in Example 24.4 from an identity for its sign transform which belongs to $\Gamma^*(20)$.

17.4 Eta Products for $\Gamma_0(14)$

The 8 cuspidal eta products of weight 1 for $\Gamma_0(14)$ combine nicely and make up 8 eigenforms which are Hecke theta series. The precise results are stated in the following two examples:

Example 17.17 *Let $\mathcal{J}_{21}$ be the system of integral ideal numbers for $\mathbb{Q}(\sqrt{-21})$ as defined in Example* 7.6*. The residues of $\frac{1}{\sqrt{2}}(\sqrt{3} + \sqrt{-7})$ and $\sqrt{3} + 2\sqrt{-7}$ modulo $2\sqrt{6}$ can be chosen as generators of $(\mathcal{J}_{21}/(2\sqrt{6}))^\times \simeq$*

$Z_8 \times Z_4$, where $(\sqrt{3}+2\sqrt{-7})^2 \equiv -1 \bmod 2\sqrt{6}$. Four characters $\chi_{\delta,\varepsilon}$ on $\mathcal{J}_{21}$ with period $2\sqrt{6}$ are fixed by their values

$$\chi_{\delta,\varepsilon}\big(\tfrac{1}{\sqrt{2}}(\sqrt{3}+\sqrt{-7})\big) = \tfrac{1}{\sqrt{2}}(\varepsilon+\delta i), \qquad \chi_{\delta,\varepsilon}(\sqrt{3}+2\sqrt{-7}) = \delta$$

with $\delta, \varepsilon \in \{1,-1\}$. The corresponding theta series of weight 1 decompose as

$$\Theta_1\left(-84, \chi_{\delta,\varepsilon}, \tfrac{z}{12}\right) = f_1(z) + \delta i\sqrt{2}\, f_5(z) + \varepsilon i\, f_7(z) + \delta\varepsilon\sqrt{2}\, f_{11}(z) \quad (17.40)$$

where the components f_j are normalized integral Fourier series with denominator 12 and numerator classes j modulo 12. All of them are eta products,

$$\begin{aligned} f_1 &= \left[\frac{2,7^2}{14}\right], \qquad f_5 = \left[\frac{2^2,7}{1}\right], \\ f_7 &= \left[\frac{1^2,14}{2}\right], \qquad f_{11} = \left[\frac{1,14^2}{7}\right]. \end{aligned} \quad (17.41)$$

Example 17.18 *Let $\mathcal{J}_{21}$ be given as before in Example 17.17, and let $\mathcal{J}_{42}$ be the system of ideal numbers for $\mathbb{Q}(\sqrt{-42})$ as defined in Example 7.5. The residues of $\frac{1}{\sqrt{2}}(\sqrt{3}+\sqrt{-7})$, $\sqrt{3}+2\sqrt{-7}$, $1+2\sqrt{-21}$ and -1 modulo $4\sqrt{6}$ can be chosen as generators of the group $(\mathcal{J}_{21}/(4\sqrt{6}))^\times \simeq Z_8 \times Z_4 \times Z_2^2$. Eight characters $\rho = \rho_{\delta,\varepsilon,\nu}$ on $\mathcal{J}_{21}$ with period $4\sqrt{6}$ are given by*

$$\rho\left(\frac{\sqrt{3}+\sqrt{-7}}{\sqrt{2}}\right) = \nu i, \quad \rho(\sqrt{3}+2\sqrt{-7}) = -\varepsilon i,$$

$$\rho(1+2\sqrt{-21}) = -\delta\varepsilon\nu, \quad \rho(-1) = 1$$

with $\delta, \varepsilon, \nu \in \{1,-1\}$. The residues of $1+\sqrt{-42}$, $\sqrt{6}-\sqrt{-7}$ and $2\sqrt{2}+\sqrt{-21}$ modulo $4\sqrt{3}$ can be chosen as generators of $(\mathcal{J}_{42}/(4\sqrt{3}))^\times \simeq Z_4^3$, where $(2\sqrt{2}+\sqrt{-21})^2 \equiv -1 \bmod 4\sqrt{3}$. Eight characters $\psi_{\delta,\varepsilon,\nu}$ on $\mathcal{J}_{42}$ with period $4\sqrt{3}$ are given by

$$\psi_{\delta,\varepsilon,\nu}(1+\sqrt{-42}) = \nu, \quad \psi_{\delta,\varepsilon,\nu}(\sqrt{6}-\sqrt{-7}) = -\varepsilon\nu i,$$

$$\psi_{\delta,\varepsilon,\nu}(2\sqrt{2}+\sqrt{-21}) = -\delta\nu.$$

The residues of $1+\delta\sqrt{2}$, $3-\delta\sqrt{2}$, 13 and -1 modulo $M_\delta = 12(3+\delta\sqrt{2})$ are generators of $(\mathbb{Z}[\sqrt{2}]/(M_\delta))^\times \simeq Z_{24} \times Z_4 \times Z_2^2$. Define Hecke characters $\xi_{\delta,\varepsilon}$ on $\mathbb{Z}[\sqrt{2}]$ with period M_δ by

$$\xi_{\delta,\varepsilon}(\mu) = \begin{cases} \delta\,\mathrm{sgn}(\mu) \\ \varepsilon i\,\mathrm{sgn}(\mu) \\ -\mathrm{sgn}(\mu) \end{cases} \quad \textit{for} \quad \mu \equiv \begin{cases} 1+\delta\sqrt{2} \\ 3-\delta\sqrt{2} \\ 13,\ -1 \end{cases} \mod M_\delta.$$

The corresponding theta series of weight 1 are identical and decompose as

$$\begin{aligned} \Theta_1\left(8, \xi_{\delta,\varepsilon}, \tfrac{z}{24}\right) &= \Theta_1\left(-84, \rho_{\delta,\varepsilon,\nu}, \tfrac{z}{24}\right) = \Theta_1\left(-168, \psi_{\delta,\varepsilon,\nu}, \tfrac{z}{24}\right) \\ &= g_1(z) + \varepsilon i\, g_7(z) \\ &\quad + 2\delta\varepsilon i\, g_{17}(z) + 2\delta\, g_{23}(z), \end{aligned} \quad (17.42)$$

where the components g_j are normalized integral Fourier series with denominator 24 and numerator classes j modulo 24. All of them are eta products,

$$g_1 = \left[\frac{1, 7^2}{14}\right], \qquad g_7 = \left[\frac{1^2, 7}{2}\right], \qquad g_{17} = \left[\frac{2^2, 14}{1}\right], \qquad g_{23} = \left[\frac{2, 14^2}{7}\right]. \tag{17.43}$$

We will return to the eta products (17.43) and their sign transforms in Example 23.24. We note that (g_1, g_{17}), (g_7, g_{23}) are pairs of transforms with respect to W_{14}.

One of the non-cuspidal eta products of weight 1 for $\Gamma_0(14)$ is an Eisenstein series and a theta series for the field $\mathbb{Q}(\sqrt{-7})$. The other one has multiplicative coefficients which behave like those of an Eisenstein series at odd indices but do not satisfy the proper Hecke recursions at powers of the prime 2. The corresponding identities in the following example can be deduced directly from (8.5), (8.7) and from the arithmetic in the factorial ring $\mathcal{O}_7$:

Example 17.19 *Let χ_0 and $\widehat{\chi}_0$ denote the principal characters on $\mathcal{O}_7$ with periods $\frac{1}{2}(1+\sqrt{-7})$ and $\frac{1}{2}(1-\sqrt{-7})$, respectively. Then we have*

$$\Theta_1(-7, \chi_0, z) = \Theta_1(-7, \widehat{\chi}_0, z) = \sum_{n=1}^{\infty}\Big(\sum_{2\nmid d|n}\left(\tfrac{-7}{d}\right)\Big)e(nz) \;=\; \frac{\eta^2(2z)\eta^2(14z)}{\eta(z)\eta(7z)}\,. \tag{17.44}$$

Moreover, we have

$$\frac{\eta^2(z)\eta^2(7z)}{\eta(2z)\eta(14z)} = 1 - 2\sum_{n=1}^{\infty}\lambda(n)e(nz)\,, \qquad \textit{where} \qquad \lambda(2^r m) = -(r-1)\sum_{d|m}\left(\tfrac{-7}{d}\right) \tag{17.45}$$

if m is odd and $r \geq 0$.

17.5 Eta Products for $\Gamma_0(22)$

The four eta products with denominator 12 combine to four eigenforms which are theta series for $\mathbb{Q}(\sqrt{-33})$:

Example 17.20 *Let $\mathcal{J}_{33}$ be the system of integral ideal numbers for $\mathbb{Q}(\sqrt{-33})$ as defined in Example 7.6. The residues of $\frac{1}{\sqrt{2}}(\sqrt{3}+\sqrt{-11})$, $\sqrt{-11}$ and -1 modulo $2\sqrt{6}$ can be chosen as generators of the group $(\mathcal{J}_{33}/(2\sqrt{6}))^\times \simeq Z_8 \times Z_2^2$. Eight characters $\chi_{\delta,\varepsilon,\nu}$ on $\mathcal{J}_{33}$ with period $2\sqrt{6}$ are fixed by their values*

$$\chi_{\delta,\varepsilon,\nu}\big(\tfrac{1}{\sqrt{2}}(\sqrt{3}+\sqrt{-11})\big) = \tfrac{1}{\sqrt{2}}(\nu+\varepsilon i), \qquad \chi_{\delta,\varepsilon,\nu}(\sqrt{-11}) = \delta\varepsilon,$$

$$\chi_{\delta,\varepsilon,\nu}(-1) = 1$$

with $\delta, \varepsilon, \nu \in \{1, -1\}$. *The corresponding theta series of weight* 1 *decompose as*

$$\Theta_1\left(-132, \chi_{\delta,\varepsilon,\nu}, \tfrac{z}{12}\right) = f_1(z) + \delta i\sqrt{2}\, f_5(z) + \varepsilon i\sqrt{2}\, f_7(z) + \delta\varepsilon\, f_{11}(z) \quad (17.46)$$

where the components f_j *are normalized integral Fourier series with denominator* 12 *and numerator classes* j *modulo* 12. *All of them are eta products,*

$$\begin{aligned} f_1 &= \left[\frac{2, 11^2}{22}\right], \qquad f_5 = \left[\frac{1, 22^2}{11}\right], \\ f_7 &= \left[\frac{2^2, 11}{1}\right], \qquad f_{11} = \left[\frac{1^2, 22}{2}\right]. \end{aligned} \quad (17.47)$$

The four eta products with denominator 24 make up two of the components of the theta series in Example 17.5 which belong to the Fricke group $\Gamma^*(21)$. Another two linear combinations form two of the components of four eigenforms which are theta series for the fields $\mathbb{Q}(\sqrt{11})$, $\mathbb{Q}(\sqrt{-66})$ and $\mathbb{Q}(\sqrt{-6})$:

Example 17.21 *Let the generators of* $(\mathcal{J}_{66}/(4\sqrt{3}))^\times \simeq Z_8 \times Z_4 \times Z_2^2$ *be chosen as in Example* 17.5, *and define eight characters* $\psi_{\delta,\varepsilon,\nu}$ *on* $\mathcal{J}_{66}$ *with period* $4\sqrt{3}$ *by*

$$\psi_{\delta,\varepsilon,\nu}(\Lambda) = \varepsilon i, \qquad \psi_{\delta,\varepsilon,\nu}(1 + \sqrt{-66}) = -\delta\nu,$$

$$\psi_{\delta,\varepsilon,\nu}(5) = -1, \qquad \psi_{\delta,\varepsilon,\nu}(\sqrt{-11}) = \delta$$

with $\delta, \varepsilon, \nu \in \{1, -1\}$. *The residues of* $1 + \nu\sqrt{-6}$, $\sqrt{3} - 2\nu\sqrt{-2}$, 23 *and* -1 *modulo* $4\sqrt{3}(\sqrt{3} + 2\nu\sqrt{-2}) = 4(3 + 2\nu\sqrt{-6})$ *can be chosen as generators of* $(\mathcal{J}_6/(12 + 8\nu\sqrt{-6}))^\times \simeq Z_{20} \times Z_4 \times Z_2^2$. *Characters* $\rho_{\delta,\varepsilon,\nu}$ *on* $\mathcal{J}_6$ *with periods* $4(3 + 2\nu\sqrt{-6})$ *are given by*

$$\rho_{\delta,\varepsilon,\nu}(1 + \nu\sqrt{-6}) = \delta\varepsilon i, \quad \rho_{\delta,\varepsilon,\nu}(\sqrt{3} - 2\nu\sqrt{-2}) = \delta,$$

$$\rho_{\delta,\varepsilon,\nu}(23) = -1, \quad \rho_{\delta,\varepsilon,\nu}(-1) = 1.$$

The residues of $2+\sqrt{11}$, $10+3\sqrt{11}$, $1+6\sqrt{11}$ *and* -1 *modulo* $M = 12(3+\sqrt{11})$ *are generators of* $(\mathbb{Z}[\sqrt{11}]/(M))^\times \simeq Z_8 \times Z_4 \times Z_2^2$. *Define Hecke characters* $\xi_{\delta,\varepsilon}$ *on* $\mathbb{Z}[\sqrt{11}]$ *with period* M *by*

$$\xi_{\delta,\varepsilon}(\mu) = \begin{cases} \delta\varepsilon i \operatorname{sgn}(\mu) \\ \operatorname{sgn}(\mu) \\ \delta \operatorname{sgn}(\mu) \\ -\operatorname{sgn}(\mu) \end{cases} \quad \textit{for} \quad \mu \equiv \begin{cases} 2 + \sqrt{11} \\ 10 + 3\sqrt{11} \\ 1 + 6\sqrt{11} \\ -1 \end{cases} \mod M.$$

The corresponding theta series of weight 1 *satisfy*

$$\begin{aligned} \Theta_1\left(44, \xi_{\delta,\varepsilon}, \tfrac{z}{24}\right) &= \Theta_1\left(-264, \psi_{\delta,\varepsilon,\nu}, \tfrac{z}{24}\right) = \Theta_1\left(-24, \rho_{\delta,\varepsilon,\nu}, \tfrac{z}{24}\right) \\ &= g_1(z) + 2\varepsilon i\, g_5(z) + 2\delta\varepsilon i\, g_7(z) + \delta\, g_{11}(z), \end{aligned} \quad (17.48)$$

where the components g_j are normalized integral Fourier series with denominator 24 *and numerator classes j modulo* 24*, and where g_1, g_{11} are linear combinations of eta products,*

$$g_1 = \left[\frac{1, 11^2}{22}\right] - 2\left[\frac{2^2, 22}{1}\right], \qquad g_{11} = \left[\frac{1^2, 11}{2}\right] - 2\left[\frac{2, 22^2}{11}\right]. \tag{17.49}$$

The coefficients of the non-cuspidal eta products $[1^2, 2^{-1}, 11^2, 22^{-1}]$ and $[1^{-1}, 2^2, 11^{-1}, 22^2]$ for $\Gamma_0(22)$ are closely related to the arithmetic in $\mathbb{Q}(\sqrt{-11})$. There is a linear combination of one of these eta products and a rescaling of the other one which is identified with an Eisenstein series: We have

$$\begin{aligned} &-\frac{1}{2}\,\frac{\eta^2(z)\eta^2(11z)}{\eta(2z)\eta(22z)} + 2\,\frac{\eta^2(4z)\eta^2(44z)}{\eta(2z)\eta(22z)} \\ &\quad = -\frac{1}{2} + \sum_{n=1}^{\infty}\left((-1)^{n-1}\sum_{d|n}\left(\frac{d}{11}\right)\right)e(nz). \end{aligned} \tag{17.50}$$

17.6 Weight 1 for Levels 26, 34 and 38

The eta products of weight 1 and denominator 24 for $\Gamma_0(26)$ combine neatly to a quadruplet of theta series, similarly as those of levels 10 and 14 in Examples 17.11 and 17.18:

Example 17.22 *Let $\mathcal{J}_{78}$ be the system of ideal numbers for $\mathbb{Q}(\sqrt{-78})$ as defined in Example* 7.5*. The residues of $1+\sqrt{-78}$, $2\sqrt{2}+\sqrt{-39}$ and $2\sqrt{6}+\sqrt{-13}$ modulo $4\sqrt{3}$ can be chosen as generators of $(\mathcal{J}_{78}/(4\sqrt{3}))^\times \simeq Z_4^3$, where $(2\sqrt{6}+\sqrt{-13})^2 \equiv -1 \bmod 4\sqrt{3}$. Eight characters $\psi_{\delta,\varepsilon,\nu}$ on $\mathcal{J}_{78}$ with period $4\sqrt{3}$ are fixed by their values*

$$\psi_{\delta,\varepsilon,\nu}(1+\sqrt{-78}) = \delta\varepsilon\nu, \qquad \psi_{\delta,\varepsilon,\nu}(2\sqrt{2}+\sqrt{-39}) = \nu i,$$

$$\psi_{\delta,\varepsilon,\nu}(2\sqrt{6}+\sqrt{-13}) = -\varepsilon$$

with $\delta, \varepsilon, \nu \in \{1, -1\}$. The residues of $2+\nu i$, 5, $7+12\nu i$, $5+6\nu i$ and νi modulo $12(1+\nu i)(3-2\nu i) = 12(5+\nu i)$ are generators of $(\mathcal{O}_1/(60+12\nu i))^\times \simeq Z_{24}\times Z_4\times Z_2^2\times Z_4$. Characters $\varphi = \varphi_{\delta,\varepsilon,\nu}$ on $\mathcal{O}_1$ with periods $12(5+\nu i)$ are given by

$$\varphi(2+\nu i) = \delta i, \quad \varphi(5) = 1, \quad \varphi(7+12\nu i) = -1,$$

$$\varphi(5+6\nu i) = \varepsilon, \quad \varphi(\nu i) = 1.$$

Let $\mathcal{J}_{\mathbb{Q}(\sqrt{78})}$ be given as in Example 7.19*. The residues of $\sqrt{6}+\sqrt{13}$, $1+2\sqrt{78}$, $\sqrt{13}$ and -1 modulo $M = 4(9+\sqrt{78})$ are generators of $\left(\mathcal{J}_{\mathbb{Q}(\sqrt{78})}/(M)\right)^\times \simeq$*

$Z_4 \times Z_2^3$. Hecke characters $\xi_{\delta,\varepsilon}$ on $\mathcal{J}_{\mathbb{Q}(\sqrt{78})}$ with period M are given by

$$\xi_{\delta,\varepsilon}(\mu) = \begin{cases} -\delta\varepsilon i \,\mathrm{sgn}(\mu) \\ \varepsilon \,\mathrm{sgn}(\mu) \\ -\mathrm{sgn}(\mu) \end{cases} \quad \text{for} \quad \mu \equiv \begin{cases} \sqrt{6}+\sqrt{13} \\ \sqrt{13} \\ 1+2\sqrt{78}, \ \ -1 \end{cases} \mod M.$$

The corresponding theta series of weight 1 satisfy the identities

$$\begin{aligned} \Theta_1\left(312, \xi_{\delta,\varepsilon}, \tfrac{z}{24}\right) &= \Theta_1\left(-312, \psi_{\delta,\varepsilon,\nu}, \tfrac{z}{24}\right) = \Theta_1\left(-4, \varphi_{\delta,\varepsilon,\nu}, \tfrac{z}{24}\right) \\ &= f_1(z) + 2\delta i\, f_5(z) + \varepsilon\, f_{13}(z) \\ &\quad + 2\delta\varepsilon i\, f_{17}(z)\,, \end{aligned} \tag{17.51}$$

where the components f_j are normalized integral Fourier series with denominator 24 and numerator classes j modulo 24. All of them are eta products,

$$\begin{aligned} f_1 &= \left[\frac{1, 13^2}{26}\right], \qquad f_5 = \left[\frac{2^2, 26}{1}\right], \\ f_{13} &= \left[\frac{1^2, 13}{2}\right], \qquad f_{17} = \left[\frac{2, 26^2}{13}\right]. \end{aligned} \tag{17.52}$$

The sign transforms of the eta products (17.52) belong to $\Gamma_0(52)$ and will be handled in Example 22.21.

The other four cuspidal eta products of weight 1 for $\Gamma_0(26)$ are $[2, 13^2, 26^{-1}]$, $[1^2, 2^{-1}, 26]$ with denominator 12 and their Fricke transforms $[1^{-1}, 2^2, 13]$, $[1, 13^{-1}, 26^2]$ with denominator 3. For each pair the numerators are congruent to each other modulo the denominator. Therefore complementing components are needed to obtain eigenforms. In the following example we describe theta series which represent the eigenforms with denominator 12:

Example 17.23 *Let $\mathcal{J}_{39}$ be the system of ideal numbers for $\mathbb{Q}(\sqrt{-39})$ as defined in Example 7.8, where $\Lambda = \Lambda_{39} = \sqrt{\frac{1}{2}(\sqrt{13}+\sqrt{-3})}$ is a root of the polynomial $X^8 - 5X^4 + 16$. The residues of $\frac{1}{2\Lambda}(1+\sqrt{-39})$, $2+\sqrt{-39}$, 5 and -1 modulo $4\sqrt{-3}\Lambda$ can be chosen as generators of the group $(\mathcal{J}_{39}/(4\sqrt{-3}\Lambda))^\times \simeq Z_8 \times Z_2^3$. Eight characters $\varphi_{\delta,\nu}$ and $\psi_{\delta,\nu}$ on $\mathcal{J}_{39}$ with period $4\sqrt{-3}\Lambda$ are defined by*

$$\varphi_{\delta,\nu}\left(\tfrac{1}{2\Lambda}(1+\sqrt{-39})\right) = \delta i, \quad \varphi_{\delta,\nu}(2+\sqrt{-39}) = \nu,$$

$$\varphi_{\delta,\nu}(5) = -1, \quad \varphi_{\delta,\nu}(-1) = 1,$$

$$\psi_{\delta,\nu}\left(\tfrac{1}{2\Lambda}(1+\sqrt{-39})\right) = \nu, \quad \psi_{\delta,\nu}(2+\sqrt{-39}) = \delta\nu,$$

$$\psi_{\delta,\nu}(5) = -1, \quad \psi_{\delta,\nu}(-1) = 1$$

with $\delta, \nu \in \{1, -1\}$. Let $\mathcal{J}_{13}$ be the ideal numbers for $\mathbb{Q}(\sqrt{-13})$ as defined in Example 7.1. The residues of $\frac{1}{\sqrt{2}}(5+\sqrt{-13})$ and $2+\sqrt{-13}$ modulo $6\sqrt{2}$ can *be chosen as generators of* $(\mathcal{J}_{13}/(6\sqrt{2}))^\times \simeq Z_8^2$, *where* $\big((5+\sqrt{-13})/\sqrt{2}\big)^4 \equiv -1 \bmod 6\sqrt{2}$. *Four characters* $\rho_{\delta,\nu}$ *on* $\mathcal{J}_{13}$ *with period* $6\sqrt{2}$ *are given by*

$$\rho_{\delta,\nu}\big((5+\sqrt{-13})/\sqrt{2}\big) = \delta\nu i, \qquad \rho_{\delta,\nu}(2+\sqrt{-13}) = -\nu i.$$

The residues of $2+\nu i$, $6+\nu i$ *and* νi *modulo* $6(1+\nu i)(3-2\nu i) = 6(5+\nu i)$ *are generators of* $(\mathcal{O}_1/(30+6\nu i))^\times \simeq Z_{24}\times Z_4^2$. *Characters* $\chi_{\delta,\nu}$ *on* $\mathcal{O}_1$ *with periods* $6(5+\nu i)$ *are given by*

$$\chi_{\delta,\nu}(2+\nu i) = \delta i, \qquad \chi_{\delta,\nu}(6+\nu i) = -1, \qquad \chi_{\delta,\nu}(\nu i) = 1.$$

The residues of $\frac{1}{\sqrt{2}}(7+\sqrt{39})$, $1+2\sqrt{39}$, 5 *and* -1 *modulo* $M = 4\sqrt{2}(6+\sqrt{39})$ *are generators of* $\big(\mathcal{J}_{\mathbb{Q}(\sqrt{39})}/(M)\big)^\times \simeq Z_8\times Z_2^3$. *Hecke characters* ξ_δ *on* $\mathcal{J}_{\mathbb{Q}(\sqrt{39})}$ *with period* M *are given by*

$$\xi_\delta(\mu) = \begin{cases} \delta i\,\mathrm{sgn}(\mu) \\ \mathrm{sgn}(\mu) \\ -\mathrm{sgn}(\mu) \end{cases} \qquad \textit{for} \qquad \mu \equiv \begin{cases} \frac{1}{\sqrt{2}}(7+\sqrt{39}) \\ 1+2\sqrt{39} \\ 5,\ -1 \end{cases} \bmod M.$$

The residues of $1-2\delta\sqrt{3}$, $4+\delta\sqrt{3}$ *and* -1 *modulo* $M_\delta = 2(9+\delta\sqrt{3})$ *are generators of* $(\mathbb{Z}[\sqrt{3}]/(M_\delta))^\times \simeq Z_{12}\times Z_4\times Z_2$. *Hecke characters* Ξ_δ *on* $\mathbb{Z}[\sqrt{3}]$ *with period* M_δ *are given by*

$$\Xi_\delta(\mu) = -\,\mathrm{sgn}(\mu) \qquad \textit{for} \qquad \mu \equiv 1-2\delta\sqrt{3},\ 4+\delta\sqrt{3},\ -1 \bmod M_\delta.$$

The corresponding theta series of weight 1 *satisfy*

$$\begin{aligned} \Theta_1\left(156,\,\xi_\delta,\,\tfrac{z}{12}\right) &= \Theta_1\left(-39,\varphi_{\delta,\nu},\tfrac{z}{12}\right) \\ &= \Theta_1\left(-4,\chi_{\delta,\nu},\tfrac{z}{12}\right) \\ &= g_1(z) + 2\delta i\, g_5(z), \end{aligned} \tag{17.53}$$

$$\begin{aligned} \Theta_1\left(12,\,\Xi_\delta,\,\tfrac{z}{12}\right) &= \Theta_1\left(-39,\psi_{\delta,\nu},\tfrac{z}{12}\right) \\ &= \Theta_1\left(-52,\rho_{\delta,\nu},\tfrac{z}{12}\right) \\ &= h_1(z) + 2\delta\, h_{11}(z), \end{aligned} \tag{17.54}$$

where g_j *and* h_j *are normalized integral Fourier series with denominator* 12 *and numerator classes* j *modulo* 12. *The components* g_1, h_1 *are linear combinations of eta products,*

$$g_1 = \left[\frac{2,13^2}{26}\right] + \left[\frac{1^2,26}{2}\right], \qquad h_1 = \left[\frac{2,13^2}{26}\right] - \left[\frac{1^2,26}{2}\right]. \tag{17.55}$$

The sign transforms of the eta products in (17.55) will be discussed in Example 22.20, with similar results.

The Fricke involution W_{26} transforms the functions (17.55) into the linear combinations $[1^{-1}, 2^2, 13] \pm [1, 13^{-1}, 26^2]$ of eta products with denominator $t = 3$ and numerators $s \equiv 2 \bmod 3$. One would expect that these functions are components of Hecke eigenforms. Indeed it is easy to construct complementing components with numerator 1 such that the resulting combinations have multiplicative coefficients. However, they violate the proper relations at powers of the prime 2, and hence are not eigenforms of the Hecke operator T_2 and cannot be represented by a theta series. But we get a more complicated representation by sums of two theta series:

Example 17.24 *Let $\varphi_{\delta,\nu}$ and $\psi_{\delta,\nu}$ be the characters on $\mathcal{J}_{39}$ with period $4\sqrt{-3}\Lambda$ as defined in Example 17.23. The residues of Λ, 5 and -1 modulo $\sqrt{-3}\,\overline{\Lambda}^2 = \frac{1}{2}(3 + \sqrt{-39})$ can be chosen as generators of $(\mathcal{J}_{39}/(\frac{1}{2}(3 + \sqrt{-39})))^\times \simeq Z_4 \times Z_2^2$. Characters χ_δ and ρ_δ on $\mathcal{J}_{39}$ with period $\frac{1}{2}(3+\sqrt{-39})$ are fixed by their values*

$$\chi_\delta(\Lambda) = -\delta i, \qquad \chi_\delta(5) = -1, \qquad \chi_\delta(-1) = 1,$$

$$\rho_\delta(\Lambda) = -\delta, \qquad \rho_\delta(5) = -1, \qquad \rho_\delta(-1) = 1$$

with $\delta \in \{1, -1\}$. For the corresponding theta series of weight 1 we have the identities

$$\Theta_1\left(-39, \varphi_{\delta,\nu}, \tfrac{z}{3}\right) + \delta i\,\Theta_1\left(-39, \chi_\delta, \tfrac{2z}{3}\right) = \widetilde{g}_1(z) + \delta i\,\widetilde{g}_2(z), \qquad (17.56)$$

$$\Theta_1\left(-39, \psi_{\delta,\nu}, \tfrac{z}{3}\right) + \delta\,\Theta_1\left(-39, \rho_\delta, \tfrac{2z}{3}\right) = \widetilde{h}_1(z) + \delta\,\widetilde{h}_2(z), \qquad (17.57)$$

where the components $\widetilde{g}_j$, $\widetilde{h}_j$ are normalized integral Fourier series with denominator 3 and numerator classes j modulo 3, and where $\widetilde{g}_2$, $\widetilde{h}_2$ are linear combinations of eta products,

$$\widetilde{g}_2 = \left[\frac{2^2, 13}{1}\right] + \left[\frac{1, 26^2}{13}\right], \qquad \widetilde{h}_2 = \left[\frac{2^2, 13}{1}\right] - \left[\frac{1, 26^2}{13}\right]. \qquad (17.58)$$

The identities continue to hold true when χ_δ, ρ_δ are replaced by the characters $\widehat{\chi}_\delta$, $\widehat{\rho}_\delta$ on $\mathcal{J}_{39}$ with period $\frac{1}{2}(3 - \sqrt{-39})$, which are given by $\widehat{\chi}_\delta(\mu) = \chi_\delta(\overline{\mu})$, $\widehat{\rho}_\delta(\mu) = \rho_\delta(\overline{\mu})$ for $\mu \in \mathcal{J}_{39}$.

One of the non-cuspidal eta products of weight 1 for $\Gamma_0(26)$ is identified with a component of two theta series on $\mathbb{Q}(\sqrt{-13})$. It turns out that the other component is a linear combination of non-cuspidal eta products for $\Gamma_0(52)$:

Example 17.25 *Define characters ψ_{-1} and ψ_1 on $\mathcal{J}_{13}$ with period $\sqrt{2}$ by*

$$\psi_{-1}(\mu) = \left(\frac{-1}{\mu\overline{\mu}}\right), \qquad \psi_1(\mu) = \psi_{-1}^2(\mu) = \chi_0(\mu\overline{\mu})$$

for $\mu \in \mathcal{J}_{13}$*, where* χ_0 *denotes the principal (Dirichlet) character modulo* 2 *on* $\mathbb{Z}$*. The corresponding theta series of weight* 1 *decompose as*

$$\Theta_1\left(-52, \psi_\delta, \tfrac{z}{4}\right) = f_1(z) + 2\delta\, f_3(z), \tag{17.59}$$

where the components f_j *are normalized integral Fourier series with denominator* 4 *and numerator classes* j *modulo* 4 *which are eta products or linear combinations thereof,*

$$f_1 = \left[\frac{4^2, 26^5}{2, 13^2, 52^2}\right] + \left[\frac{2^5, 52^2}{1^2, 4^2, 26}\right], \qquad f_3 = \left[\frac{2^2, 26^2}{1, 13}\right]. \tag{17.60}$$

The identities (17.59), (17.60) can be deduced by elementary arguments from (8.5), (8.8) and from the theory of binary quadratic forms with discriminant -52.

Concerning the cuspidal eta products of weight 1 for $\Gamma_0(34)$, each two of them have denominators 6 and 12, and four of them have denominator 24. We find theta identities only for the latter four:

Example 17.26 *Let* $\mathcal{J}_{102}$ *be the system of ideal numbers for* $\mathbb{Q}(\sqrt{-102})$ *as defined in Example* 7.5*. The residues of* $\sqrt{-17}$*,* $\sqrt{6}+\sqrt{-17}$ *and* $\sqrt{3}+2\sqrt{-34}$ *modulo* $4\sqrt{3}$ *can be chosen as generators of* $(\mathcal{J}_{102}/(4\sqrt{3}))^\times \simeq Z_4^3$*, where* $(\sqrt{3}+2\sqrt{-34})^2 \equiv -1 \bmod 4\sqrt{3}$*. Eight characters* $\psi_{\delta,\varepsilon,\nu}$ *on* $\mathcal{J}_{102}$ *with period* $4\sqrt{3}$ *are defined by*

$$\psi_{\delta,\varepsilon,\nu}(\sqrt{-17}) = -\delta\varepsilon i, \qquad \psi_{\delta,\varepsilon,\nu}(\sqrt{6}+\sqrt{-17}) = \nu i,$$

$$\psi_{\delta,\varepsilon,\nu}(\sqrt{3}+2\sqrt{-34}) = -\delta\nu$$

with $\delta, \varepsilon, \nu \in \{1, -1\}$*. The residue classes of* $2+\nu i$*,* $4+\nu i$*,* $11+6\nu i$*,* 35 *and* νi *modulo* $12(1+\nu i)(4-\nu i) = 12(5+3\nu i)$ *can be chosen as generators of* $(\mathcal{O}_1/(60+36\nu i))^\times \simeq Z_{16}\times Z_8 \times Z_2^2 \times Z_4$*. Characters* $\chi = \chi_{\delta,\varepsilon,\nu}$ *on* $\mathcal{O}_1$ *with periods* $12(5+3\nu i)$ *are given by*

$$\chi(2+\nu i) = \delta i, \qquad \chi(4+\nu i) = -\delta\varepsilon i,$$

$$\chi(11+6\nu i) = \varepsilon, \qquad \chi(35) = -1, \qquad \chi(\nu i) = 1.$$

Let $\mathcal{J}_{\mathbb{Q}(\sqrt{102})}$ *be given as in Example* 7.19*. The residues of* $\sqrt{3}+\sqrt{34}$*,* $1+\sqrt{102}$ *and* -1 *modulo* $4\sqrt{3}$ *are generators of* $\left(\mathcal{J}_{\mathbb{Q}(\sqrt{102})}/(4\sqrt{3})\right)^\times \simeq Z_4^2 \times Z_2$*. Define Hecke characters* $\xi_{\delta,\varepsilon}$ *on* $\mathcal{J}_{\mathbb{Q}(\sqrt{102})}$ *with period* $4\sqrt{3}$ *by*

$$\xi_{\delta,\varepsilon}(\mu) = \begin{cases} \delta\varepsilon i\,\mathrm{sgn}(\mu) \\ -\delta i\,\mathrm{sgn}(\mu) \\ -\mathrm{sgn}(\mu) \end{cases} \quad \textit{for} \quad \mu \equiv \begin{cases} \sqrt{3}+\sqrt{34} \\ 1+\sqrt{102} \\ -1 \end{cases} \bmod 4\sqrt{3}.$$

The corresponding theta series of weight 1 *are identical and decompose as*

$$\begin{aligned}\Theta_1\left(408, \xi_{\delta,\varepsilon}, \tfrac{z}{24}\right) &= \Theta_1\left(-408, \psi_{\delta,\varepsilon,\nu}, \tfrac{z}{24}\right) = \Theta_1\left(-4, \chi_{\delta,\varepsilon,\nu}, \tfrac{z}{24}\right) \\ &= f_1(z) + 2\delta i\, f_5(z) \\ &\quad + 2\varepsilon\, f_{13}(z) - \delta\varepsilon i\, f_{17}(z)\,, \end{aligned} \tag{17.61}$$

where the components f_j *are normalized integral Fourier series with denominator* 24 *and numerator classes* j *modulo* 24*. All of them are eta products,*

$$\begin{aligned} f_1 &= \left[\frac{1, 17^2}{34}\right], \quad f_5 = \left[\frac{2, 34^2}{17}\right], \\ f_{13} &= \left[\frac{2^2, 34}{1}\right], \quad f_{17} = \left[\frac{1^2, 17}{2}\right]. \end{aligned} \tag{17.62}$$

The sign transforms of the eta products in (17.62) will be discussed in Example 22.11.

Each four of the cuspidal eta products of weight 1 for $\Gamma_0(38)$ have denominators $t = 12$ and $t = 24$. For $t = 12$ there is a theta series all of whose components are identified with eta products, while for $t = 24$ there is a theta series with eight components, and only four of them are identified with eta products:

Example 17.27 *Let* $\mathcal{J}_{57}$ *be the system of ideal numbers for* $\mathbb{Q}(\sqrt{-57})$ *as defined in Example* 7.6*. The residues of* $\frac{1}{\sqrt{2}}(\sqrt{3} + \sqrt{-19})$ *and* $\sqrt{3} + 2\sqrt{-19}$ *modulo* $2\sqrt{6}$ *can be chosen as generators of* $(\mathcal{J}_{57}/(2\sqrt{6}))^\times \simeq Z_8 \times Z_4$*, where* $(\sqrt{3} + 2\sqrt{-19})^2 \equiv -1 \bmod 2\sqrt{6}$*. Eight characters* $\varphi_{\delta,\varepsilon,\nu}$ *on* $\mathcal{J}_{57}$ *with period* $2\sqrt{6}$ *are given by*

$$\varphi_{\delta,\varepsilon,\nu}\left(\tfrac{1}{\sqrt{2}}(\sqrt{3} + \sqrt{-19})\right) = \tfrac{1}{\sqrt{2}}(-\delta\varepsilon + \nu i), \qquad \varphi_{\delta,\varepsilon,\nu}(\sqrt{3} + 2\sqrt{-19}) = \delta\nu$$

with $\delta, \varepsilon, \nu \in \{1, -1\}$*. The corresponding theta series of weight* 1 *decompose as*

$$\begin{aligned}&\Theta_1\left(-228, \varphi_{\delta,\varepsilon,\nu}, \tfrac{z}{12}\right) \\ &\quad = g_1(z) + \delta i\sqrt{2}\, g_5(z) + \varepsilon i\, g_7(z) - \delta\varepsilon\sqrt{2}\, g_{11}(z)\,, \end{aligned} \tag{17.63}$$

where the components g_j *are normalized integral Fourier series with denominator* 12 *and numerator classes* j *modulo* 12*. All of them are eta products,*

$$\begin{aligned} g_1 &= \left[\frac{2, 19^2}{38}\right], \quad g_5 = \left[\frac{1, 38^2}{19}\right], \\ g_7 &= \left[\frac{1^2, 38}{2}\right], \quad g_{11} = \left[\frac{2^2, 19}{1}\right]. \end{aligned} \tag{17.64}$$

Example 17.28 *Let* $\mathcal{J}_{114}$ *be the system of ideal numbers for* $\mathbb{Q}(\sqrt{-114})$ *as defined in Example* 7.10 *where* $\Lambda = \Lambda_{114} = \sqrt{\sqrt{6}+\sqrt{-19}}$ *is a root of the polynomial* X^8+26X^4+625. *The residues of* Λ, $\sqrt{2}+\sqrt{-57}$ *and* $2\sqrt{2}+\sqrt{-57}$ *modulo* $4\sqrt{3}$ *can be chosen as generators of* $(\mathcal{J}_{114}/(4\sqrt{3}))^\times \simeq Z_8 \times Z_4^2$, *where* $(2\sqrt{2}+\sqrt{-57})^2 \equiv -1 \bmod 4\sqrt{3}$. *Sixteen characters* $\rho_{\delta,\varepsilon,\nu,\sigma}$ *on* $\mathcal{J}_{114}$ *with period* $4\sqrt{3}$ *are defined by*

$$\rho_{\delta,\varepsilon,\nu,\sigma}(\Lambda) = \tfrac{1}{\sqrt{2}}(\sigma+\nu i), \qquad \rho_{\delta,\varepsilon,\nu,\sigma}(\sqrt{2}+\sqrt{-57}) = \delta,$$

$$\rho_{\delta,\varepsilon,\nu,\sigma}(2\sqrt{2}+\sqrt{-57}) = -\varepsilon\nu\sigma$$

with $\delta,\varepsilon,\nu,\sigma \in \{1,-1\}$. *The corresponding theta series of weight* 1 *decompose as*

$$\begin{aligned}\Theta_1\left(-456, \rho_{\delta,\varepsilon,\nu,\sigma}, \tfrac{z}{24}\right) \;=\;& h_1(z) + \nu i\sqrt{2}\,h_5(z) + \delta\nu i\sqrt{2}\,h_7(z) + 2\delta\,h_{11}(z)\\ &+ \varepsilon\nu\sqrt{2}\,h_{13}(z) + 2\varepsilon i\,h_{17}(z)\\ &+ \delta\varepsilon i\,h_{19}(z) + \delta\varepsilon\nu\sqrt{2}\,h_{23}(z)\,, \end{aligned} \tag{17.65}$$

where the components h_j *are normalized integral Fourier series with denominator* 24 *and numerator classes* j *modulo* 24. *Those for* $j = 1, 11, 17, 19$ *are eta products,*

$$\begin{aligned} h_1 &= \left[\frac{1,19^2}{38}\right], \quad h_{11} = \left[\frac{2,38^2}{19}\right],\\ h_{17} &= \left[\frac{2^2,38}{1}\right], \quad h_{19} = \left[\frac{1^2,19}{2}\right]. \end{aligned} \tag{17.66}$$

The sign transforms of the eta products in (17.66) belong to $\Gamma_0(76)$ and will be identified with components of theta series in Example 21.12.