16 Levels $N = pq$ with Primes $3 \le p < q$

16.1 Weight 1 for Fricke Groups $\Gamma^*(3q)$

In this and the following two sections we discuss eta products whose levels $N = pq$ are products of two distinct primes p, q, whence the number of divisors of N is 4. In the present section we begin with the case of odd primes $3 \le p < q$. Then the denominator of an eta product of integral weight is different from 8 and 24 (because the sum of an even number of odd integers is even). Remarkably, in this case every new holomorphic eta product of weight 1 belongs to the Fricke group $\Gamma^*(pq)$.

For level $N = 15$ and weight 1, the only new holomorphic eta products are two non-cuspidal eta products $\left[1^{-1}, 3^2, 5^2, 15^{-1}\right]$, $\left[1^2, 3^{-1}, 5^{-1}, 15^2\right]$ and two cuspidal eta products $\left[3, 5\right]$, $\left[1, 15\right]$. Therefore it follows from Theorem 3.9 that $\eta(pz)\eta(qz)$ and $\eta(z)\eta(pqz)$ are the only new holomorphic eta products of weight 1 for levels $N = pq \ne 15$ with distinct odd primes p, q. The results for the levels 15 and 21 are indicated in the Table in [65].

The non-cuspidal eta products of weight 1 and level 15 are identified with Eisenstein series and theta series as follows:

Example 16.1 *Let* 1 *denote the trivial character on* $\mathcal{J}_{15}$*, and let* χ_0 *be the non-trivial character modulo* 1 *on the system* $\mathcal{J}_{15}$ *of ideal numbers for* $\mathbb{Q}(\sqrt{-15})$*, as defined in Example* 7.3*. Then we have the identities*

$$\frac{\eta^2(3z)\eta^2(5z)}{\eta(z)\eta(15z)} = \Theta_1(-15, 1, z) = 1 + \sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{d}{15}\right)\right)e(nz), \tag{16.1}$$

$$\frac{\eta^2(z)\eta^2(15z)}{\eta(3z)\eta(5z)} = \Theta_1(-15, \chi_0, z) = \sum_{n=1}^{\infty} a(n)e(nz), \tag{16.2}$$

where

$$a(3^r m) = (-1)^r\left(\tfrac{m}{3}\right)\sum_{d|m}\left(\tfrac{d}{15}\right) \quad \text{for} \quad r \ge 0,\ 3 \nmid m.$$

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DOI 10.1007/978-3-642-16152-0_16,

The cuspidal eta products of weight 1 and level 15 combine to eigenforms which are theta series for $\mathbb{Q}(\sqrt{-15})$:

Example 16.2 *The residues of* $\frac{1}{2}(\sqrt{3}+\sqrt{-5})$ *and* -1 *modulo* 3 *generate the group* $(\mathcal{J}_{15}/(3)) \simeq Z_6 \times Z_2$. *Four characters* $\psi_{\delta,\nu}$ *on* $\mathcal{J}_{15}$ *with period* 3 *are fixed by their values*

$$\psi_{\delta,\nu}\left(\tfrac{1}{2}(\sqrt{3}+\sqrt{-5})\right) = \zeta = \tfrac{1}{2}(\delta + \nu\sqrt{-3}), \qquad \psi_{\delta,\nu}(-1) = 1$$

with $\delta, \nu \in \{1, -1\}$, *such that* $\zeta^3 = -\delta$. *The corresponding theta series of weight* 1 *satisfy*

$$\Theta_1\left(-15, \psi_{\delta,\nu}, \tfrac{z}{3}\right) = \eta(3z)\eta(5z) + \delta\,\eta(z)\eta(15z). \tag{16.3}$$

For the levels 21 and 33 we can identify some, but not all components of a theta series with eta products:

Example 16.3 *Let* $\mathcal{J}_{21}$ *be the system of ideal numbers for* $\mathbb{Q}(\sqrt{-21})$ *as defined in Example* 7.6. *The residues of* $\frac{1}{\sqrt{2}}(\sqrt{3}+\sqrt{-7})$ *and* $\sqrt{-7}$ *modulo* 6 *can be chosen as generators of* $(\mathcal{J}_{21}/(6))^\times \simeq Z_{12} \times Z_4$. *Eight characters* $\varphi_{\delta,\varepsilon,\nu}$ *on* $\mathcal{J}_{21}$ *with period* 6 *are given by*

$$\varphi_{\delta,\varepsilon,\nu}\left(\tfrac{1}{\sqrt{2}}(\sqrt{3}+\sqrt{-7})\right) = \xi = \tfrac{1}{2}(\delta\sqrt{3} + \nu i), \qquad \varphi_{\delta,\varepsilon,\nu}(\sqrt{-7}) = \varepsilon$$

with $\delta, \varepsilon, \nu \in \{1, -1\}$, *where* ξ *is a primitive* 12*th root of unity for which* $\xi^3 = \nu i$. *The corresponding theta series of weight* 1 *decompose as*

$$\Theta_1\left(-84, \varphi_{\delta,\varepsilon,\nu}, \tfrac{z}{12}\right) = f_1(z) + \delta\sqrt{3}\,f_5(z) + \varepsilon\,f_7(z) - \delta\varepsilon\sqrt{3}\,f_{11}(z), \tag{16.4}$$

where the components f_j *are normalized integral Fourier series with denominator* 12 *and numerator classes* j *modulo* 12. *Those for* $j = 5, 11$ *are eta products,*

$$f_5(z) = \eta(3z)\eta(7z), \qquad f_{11}(z) = \eta(z)\eta(21z). \tag{16.5}$$

Example 16.4 *Let* $\mathcal{J}_{33}$ *be the system of ideal numbers for* $\mathbb{Q}(\sqrt{-33})$ *as defined in Example* 7.6. *The residues of* $\frac{1}{\sqrt{2}}(1+\sqrt{-33})$, $\sqrt{-11}$ *and* -1 *modulo* 6 *can be chosen as generators of the group* $(\mathcal{J}_{33}/(6))^\times \simeq Z_{12} \times Z_2^2$. *Eight characters* $\chi_{\delta,\varepsilon,\nu}$ *on* $\mathcal{J}_{33}$ *with period* 6 *are fixed by their values*

$$\chi_{\delta,\varepsilon,\nu}\left(\tfrac{1}{\sqrt{2}}(1+\sqrt{-33})\right) = \xi = \tfrac{1}{2}(\delta\sqrt{3} + \nu i), \qquad \chi_{\delta,\varepsilon,\nu}(\sqrt{-11}) = \varepsilon,$$

$$\chi_{\delta,\varepsilon,\nu}(-1) = 1$$

with $\delta, \varepsilon, \nu \in \{1, -1\}$, *where* ξ *is a primitive* 12*th root of unity for which* $\xi^3 = \nu i$. *The corresponding theta series of weight* 1 *decompose as*

$$\Theta_1\left(-132, \chi_{\delta,\varepsilon,\nu}, \tfrac{z}{12}\right) = g_1(z) + \delta\sqrt{3}\,g_5(z) + \delta\varepsilon\sqrt{3}\,g_7(z) + \varepsilon\,g_{11}(z), \tag{16.6}$$

where the components g_j are normalized integral Fourier series with denominator 12 *and numerator classes j modulo* 12*. Those for $j = 5, 7$ are eta products,*

$$g_5(z) = \eta(z)\eta(33z), \qquad g_7(z) = \eta(3z)\eta(11z). \tag{16.7}$$

Two linear combinations of the eta products of weight 1 and level 39 are components of theta series:

Example 16.5 *Let $\mathcal{J}_{39}$ be the system of ideal numbers for $\mathbb{Q}(\sqrt{-39})$ as defined in Example* 7.8*, with any choice of the root $\Lambda = \Lambda_{39}$ of the equation $\Lambda^8 - 5\Lambda^4 + 16 = 0$. The residues of Λ and -1 modulo* 3 *can be chosen as generators of $(\mathcal{J}_{39}/(3))^\times \simeq Z_{12} \times Z_2$. Eight characters $\rho_{\delta,\nu}$ and $\widetilde{\rho}_{\delta,\nu}$ on $\mathcal{J}_{39}$ with period* 3 *are given by*

$$\rho_{\delta,\nu}(\Lambda) = \xi = \tfrac{1}{2}(\delta\sqrt{3} + \nu i), \qquad \rho_{\delta,\nu}(-1) = 1,$$

$$\widetilde{\rho}_{\delta,\nu}(\Lambda) = \delta\xi^2 = \tfrac{1}{2}(\delta + \nu i\sqrt{3}), \qquad \widetilde{\rho}_{\delta,\nu}(-1) = 1$$

with $\delta, \nu \in \{1, -1\}$, where $\xi^3 = \nu i$. The corresponding theta series of weight 1 *decompose as*

$$\begin{aligned} \Theta_1\left(-39, \rho_{\delta,\nu}, \tfrac{z}{3}\right) &= h_1(z) + \delta\sqrt{3}\, h_2(z), \\ \Theta_1\left(-39, \widetilde{\rho}_{\delta,\nu}, \tfrac{z}{3}\right) &= \widetilde{h}_1(z) + \delta\, \widetilde{h}_2(z) \end{aligned} \tag{16.8}$$

where the components h_j, $\widetilde{h}_j$ are normalized integral Fourier series with denominator 3 *and numerator classes j modulo* 3*, and where h_2, $\widetilde{h}_2$ are linear combinations of eta products,*

$$h_2 = [3, 13] - [1, 39], \qquad \widetilde{h}_2 = [3, 13] + [1, 39]. \tag{16.9}$$

For level $N = 51$ we have the eta product $\eta(z)\eta(51z)$ with order $\frac{13}{6}$ at ∞ and numerator $s \equiv 1 \bmod 6$. For the construction of eigenforms one would need a complementing and overlapping component with numerator $s = 1$, and therefore we cannot find an eta–theta identity in this case. For levels $N = 3q$ with primes $q \geq 23$ all eta products of weight 1 have orders > 1 at ∞, and therefore there seems to be no chance to identify them with constituents in a theta series. In contrast, the situation for $q = 19$, $N = 57$ is quite favorable and similar to that in Example 16.3:

Example 16.6 *Let $\mathcal{J}_{57}$ be the system of ideal numbers for $\mathbb{Q}(\sqrt{-57})$ as defined in Example* 7.6*. The residues of $\frac{1}{\sqrt{2}}(\sqrt{3} + \sqrt{-19})$ and $\sqrt{-19}$ modulo* 6 *can be chosen as generators of $(\mathcal{J}_{57}/(6))^\times \simeq Z_{12} \times Z_4$. Eight characters $\psi_{\delta,\varepsilon,\nu}$ on $\mathcal{J}_{57}$ with period* 6 *are given by*

$$\psi_{\delta,\varepsilon,\nu}\left(\tfrac{1}{\sqrt{2}}(\sqrt{3} + \sqrt{-19})\right) = \xi = \tfrac{1}{2}(-\delta\varepsilon\sqrt{3} + \nu i), \qquad \psi_{\delta,\varepsilon,\nu}(\sqrt{-19}) = \varepsilon$$

with $\delta, \varepsilon, \nu \in \{1, -1\}$, where $\xi^3 = \nu i$. The corresponding theta series of weight 1 decompose as

$$\Theta_1\left(-228, \psi_{\delta,\varepsilon,\nu}, \tfrac{z}{12}\right) = f_1(z) + \delta\sqrt{3}\, f_5(z) + \varepsilon\, f_7(z) - \delta\varepsilon\sqrt{3}\, f_{11}(z) \quad (16.10)$$

where the components f_j are normalized integral Fourier series with denominator 12 and numerator classes j modulo 12, and where f_5, f_{11} are eta products,

$$f_5(z) = \eta(z)\eta(57z), \qquad f_{11}(z) = \eta(3z)\eta(19z). \quad (16.11)$$

16.2 Weight 1 in the Case $5 \le p < q$

It will be clear now that there are not many levels $N = pq$ with primes $5 \le p < q$ for which our method of exhibiting eta–theta identities for weight 1 is successful. There is a nice result for level 35 where the eta products have denominator 2:

Example 16.7 *Let $\mathcal{J}_{35}$ be the system of ideal numbers for $\mathbb{Q}(\sqrt{-35})$ as defined in Example 7.3. The residue of $\frac{1}{2}(\sqrt{5} + \sqrt{-7})$ modulo 2 generates the group $(\mathcal{J}_{35}/(2))^\times \simeq Z_6$. Four characters $\chi_{\delta,\nu}$ on $\mathcal{J}_{35}$ with period 2 are fixed by their value*

$$\chi_{\delta,\nu}\left(\tfrac{1}{2}(\sqrt{5} + \sqrt{-7})\right) = \zeta = \tfrac{1}{2}(\delta + \nu i\sqrt{3})$$

with $\delta, \nu \in \{1, -1\}$, where $\zeta^3 = -\delta$. The corresponding theta series of weight 1 satisfy

$$\Theta_1\left(-35, \chi_{\delta,\nu}, \tfrac{z}{2}\right) = \eta(5z)\eta(7z) + \delta\, \eta(z)\eta(35z). \quad (16.12)$$

The characters $\chi_{\delta,\nu}$ will appear once more in Example 31.22.

There is a partial result for level 55; a difference of two theta series can be identified with a linear combination of eta products:

Example 16.8 *Let $\mathcal{J}_{55}$ be the system of ideal numbers for $\mathbb{Q}(\sqrt{-55})$ as defined in Example 7.8, with any choice of the root $\Lambda = \Lambda_{55}$ of the equation $\Lambda^8 + 3\Lambda^4 + 16 = 0$. The residues of Λ and $\sqrt{-11}$ modulo 3 can be chosen as generators of the group $(\mathcal{J}_{55}/(3))^\times \simeq Z_{16} \times Z_2$, where $\Lambda^4 \equiv -\sqrt{-55} \bmod 3$, $\Lambda^8 \equiv -1 \bmod 3$. Eight characters $\rho_{\delta,\varepsilon,\nu}$ on $\mathcal{J}_{55}$ with period 3 are given by*

$$\rho_{\delta,\varepsilon,\nu}(\Lambda_{55}) = \tfrac{1}{\sqrt{2}}(\delta + \nu i), \qquad \rho_{\delta,\varepsilon,\nu}(\sqrt{-11}) = \varepsilon$$

with $\delta, \varepsilon, \nu \in \{1, -1\}$. The corresponding theta series of weight 1 satisfy

$$\Theta_1\left(-55, \rho_{1,\varepsilon,\nu}, \tfrac{z}{3}\right) - \Theta_1\left(-55, \rho_{-1,\varepsilon,\nu}, \tfrac{z}{3}\right) = 2\sqrt{2}\left(\eta(5z)\eta(11z) + \varepsilon\, \eta(z)\eta(55z)\right). \quad (16.13)$$

For $N = 65$ and $N = 85$ there are results comparable to those in Examples 16.3, 16.4, 16.6; for $N = 85$ we get another instance for an identity of theta series on different number fields:

Example 16.9 *Let $\mathcal{J}_{65}$ be the system of ideal numbers for $\mathbb{Q}(\sqrt{-65})$ as defined in Example 7.11, with any choice of the root $\Lambda = \Lambda_{65}$ of the equation $\Lambda^8 + 8\Lambda^4 + 81 = 0$. The residues of Λ and $\sqrt{5}$ modulo 2 can be chosen as generators of $(\mathcal{J}_{65}/(2))^\times \simeq Z_8 \times Z_2$. Eight characters $\varphi_{\delta,\varepsilon,\nu}$ on $\mathcal{J}_{65}$ with period 2 are given by*

$$\varphi_{\delta,\varepsilon,\nu}(\Lambda_{65}) = \tfrac{1}{\sqrt{2}}(\varepsilon + \nu i), \qquad \varphi_{\delta,\varepsilon,\nu}(\sqrt{5}) = -\delta$$

with $\delta, \varepsilon, \nu \in \{1, -1\}$. The corresponding theta series of weight 1 decompose as

$$\Theta_1\left(-260, \varphi_{\delta,\varepsilon,\nu}, \tfrac{z}{4}\right) = f_{1,\delta}(z) + \varepsilon\sqrt{2}\, f_{3,\delta}(z), \tag{16.14}$$

where the components $f_{j,\delta}$ are normalized integral Fourier series with denominator 4 and numerator classes j modulo 4, and where $f_{3,\delta}$ are linear combinations of eta products,

$$f_{3,\delta}(z) = \eta(5z)\eta(13z) + \delta\,\eta(z)\eta(65z). \tag{16.15}$$

Example 16.10 *Let $\mathcal{J}_{51}$ and $\mathcal{J}_{85}$ be the systems of ideal numbers for $\mathbb{Q}(\sqrt{-51})$ and $\mathbb{Q}(\sqrt{-85})$ as defined in Examples 7.3 and 7.6. The residues of $\frac{1}{2}(\sqrt{3} - \nu\sqrt{-17})$, $2 + \nu\sqrt{-51}$, 19 and -1 modulo $\frac{1}{2}(\sqrt{3} + \nu\sqrt{-17}) \cdot \sqrt{3} \cdot 4 = 6 + 2\nu\sqrt{-51}$ can be chosen as generators of the group $\left(\mathcal{J}_{51}/(6+2\nu\sqrt{-51})\right)^\times \simeq Z_{24} \times Z_2^3$. Characters $\chi_{\delta,\varepsilon,\nu}$ on $\mathcal{J}_{51}$ with periods $6+2\nu\sqrt{-51}$ are fixed by their values*

$$\chi_{\delta,\varepsilon,\nu}(\tfrac{1}{2}(\sqrt{3} - \nu\sqrt{-17})) = \varepsilon, \quad \chi_{\delta,\varepsilon,\nu}(2 + \nu\sqrt{-51}) = \delta\varepsilon,$$
$$\chi_{\delta,\varepsilon,\nu}(19) = -1, \quad \chi_{\delta,\varepsilon,\nu}(-1) = 1$$

with $\delta, \varepsilon, \nu \in \{1, -1\}$. The residues of $\frac{1}{\sqrt{2}}(\sqrt{5} + \sqrt{-17})$, $\frac{1}{\sqrt{2}}(3 + \sqrt{-85})$ and $\sqrt{-17}$ modulo 6 can be chosen as generators of $(\mathcal{J}_{85}/(6))^\times \simeq Z_8 \times Z_4 \times Z_2$, where $\left(\frac{1}{\sqrt{2}}(\sqrt{5} + \sqrt{-17})\right)^4 \equiv -1 \bmod 6$. Eight characters $\psi_{\delta,\varepsilon,\nu}$ on $\mathcal{J}_{85}$ with period 6 are given by

$$\psi_{\delta,\varepsilon,\nu}\left(\tfrac{1}{\sqrt{2}}(\sqrt{5} + \sqrt{-17})\right) = \delta, \qquad \psi_{\delta,\varepsilon,\nu}\left(\tfrac{1}{\sqrt{2}}(3 + \sqrt{-85})\right) = \nu i,$$

$$\psi_{\delta,\varepsilon,\nu}(\sqrt{-17}) = \varepsilon.$$

Let ideal numbers $\mathcal{J}_{\mathbb{Q}(\sqrt{15})}$ for $\mathbb{Q}(\sqrt{15})$ be chosen as in Example 7.16. The residues of $\sqrt{3} - 2\delta\sqrt{5}$, $8 - \delta\sqrt{15}$ and -1 modulo $M_\delta = 2(3 + 2\delta\sqrt{15})$ are generators of $\left(\mathcal{J}_{\mathbb{Q}(\sqrt{15})}/(M_\delta)\right)^\times \simeq Z_{32} \times Z_2^2$. Hecke characters $\xi_{\delta,\varepsilon}$ with period M_δ are fixed by their values

$$\xi_{\delta,\varepsilon}(\mu) = \begin{cases} -\delta\varepsilon\,\mathrm{sgn}(\mu) \\ \mathrm{sgn}(\mu) \\ -\mathrm{sgn}(\mu) \end{cases} \quad \text{for} \quad \mu \equiv \begin{cases} \sqrt{3} - 2\delta\sqrt{5} \\ 8 - \delta\sqrt{15} \\ -1 \end{cases} \bmod M_\delta.$$

The theta series of weight 1 *for* $\xi_{\delta,\varepsilon}$, $\chi_{\delta,\varepsilon,\nu}$ *and* $\psi_{\delta,\varepsilon,\nu}$ *are identical, and they decompose as*

$$\begin{aligned}\Theta_1\left(60, \xi_{\delta,\varepsilon}, \tfrac{z}{12}\right) &= \Theta_1\left(-51, \chi_{\delta,\varepsilon,\nu}, \tfrac{z}{12}\right) = \Theta_1\left(-340, \psi_{\delta,\varepsilon,\nu}, \tfrac{z}{12}\right)\\ &= f_1(z) + \varepsilon f_5(z) - 2\delta\varepsilon f_7(z) + 2\delta f_{11}(z), \quad (16.16)\end{aligned}$$

where the components f_j *are normalized integral Fourier series with denominator* 12 *and numerator classes* j *modulo* 12, *and where* f_7, f_{11} *are eta products,*

$$f_7(z) = \eta(z)\eta(85z), \qquad f_{11}(z) = \eta(5z)\eta(17z). \qquad (16.17)$$

For $N = 95$ we have two eta products with denominator $t = 1$. They are identified as constituents in three eigenforms which are theta series on the fields with discriminants -19 and -95. For the latter field we need characters with period 1, that is, characters of the ideal class group, so that we could easily avoid ideal numbers:

Example 16.11 *Let* $\mathcal{J}_{95}$ *be the system of ideal numbers for* $K = \mathbb{Q}(\sqrt{-95})$ *as defined in Example* 7.12, *with any choice of the root* $\Lambda = \Lambda_{95}$ *of the equation* $\Lambda^{16} - 13\Lambda^8 + 256 = 0$. *For* $\delta, \nu \in \{1, -1\}$, *define the characters* $\chi_{\delta,\nu}$ *of the ideal class group of* K *by*

$$\chi_{\delta,\nu}(\mu) = \xi^j, \qquad \xi = \tfrac{1}{\sqrt{2}}(\delta + \nu i) \qquad \textit{for} \qquad \mu \in \mathcal{A}_j,$$

$0 \le j \le 7$, *with* $\mathcal{A}_j$ *as given in Example* 7.12. *Let* ρ_ν *be the characters on* $\mathcal{O}_{19}$ *with periods* $\frac{1}{2}(1 + \nu\sqrt{-19})$, *which are given by*

$$\rho_1(\mu) = \left(\tfrac{x-y}{5}\right),$$

$$\rho_{-1}(\mu) = \rho_1(\overline{\mu}) = \left(\tfrac{x+y}{5}\right) \qquad \textit{for} \qquad \mu = \tfrac{1}{2}(x + y\sqrt{-19}) \in \mathcal{O}_{19}.$$

The corresponding theta series of weight 1 *satisfy the identities*

$$\Theta_1\left(-19, \rho_\nu, z\right) = \eta(5z)\eta(19z) - \eta(z)\eta(95z) \qquad (16.18)$$

and

$$\Theta_1\left(-95, \chi_{\delta,\nu}, z\right) = \eta(5z)\eta(19z) + \eta(z)\eta(95z) + \delta\sqrt{2}\, g(z)$$

with an integral Fourier series $g(z) = \sum_{n=1}^{\infty} b(n)e(nz)$. *We have*

$$\begin{aligned}&\tfrac{1}{2}\left(\Theta_1\left(-95, \chi_{1,\nu}, z\right) + \Theta_1\left(-95, \chi_{-1,\nu}, z\right)\right)\\ &\quad = \eta(5z)\eta(19z) + \eta(z)\eta(95z). \qquad (16.19)\end{aligned}$$

For $N = 7 \cdot 13 = 91$ the eta products have denominator $t = 6$ and numerators $s \equiv 5 \bmod 6$, with a result resembling that in Example 16.9:

Example 16.12 *Let* $\mathcal{J}_{91}$ *be the system of ideal numbers for* $\mathbb{Q}(\sqrt{-91})$ *as defined in Example* 7.3. *The residues of* $\frac{1}{2}(\sqrt{7}+\sqrt{-13})$ *and* $\sqrt{7}$ *modulo* 6 *can be chosen as generators of the group* $(\mathcal{J}_{91}/(6))^\times \simeq Z_{24}\times Z_2$, *where* $\left(\frac{1}{2}(\sqrt{7}+\sqrt{-13})\right)^{12} \equiv -1 \bmod 6$. *Eight characters* $\varphi_{\delta,\varepsilon,\nu}$ *on* $\mathcal{J}_{91}$ *with period* 6 *are fixed by their values*

$$\varphi_{\delta,\varepsilon,\nu}\left(\tfrac{1}{2}(\sqrt{7}+\sqrt{-13})\right) = \tfrac{1}{2}(\delta\sqrt{3}+\nu i), \qquad \varphi_{\delta,\varepsilon,\nu}(\sqrt{7}) = -\varepsilon$$

with $\delta,\varepsilon,\nu \in \{1,-1\}$. *The corresponding theta series of weight* 1 *decompose as*

$$\Theta_1\left(-91,\varphi_{\delta,\varepsilon,\nu},\tfrac{z}{6}\right) = f_1(z) - \varepsilon g_1(z) + \delta\sqrt{3}\left(f_5(z)+\varepsilon g_5(z)\right) \tag{16.20}$$

where the components f_j *and* g_j *are normalized integral Fourier series with denominator* 6 *and numerator classes* j *modulo* 6, *and where* f_5, g_5 *are eta products,*

$$f_5(z) = \eta(7z)\eta(13z), \qquad g_5(z) = \eta(z)\eta(91z). \tag{16.21}$$

The eta products of weight 1 for $N = 7\cdot 17$ and $N = 11\cdot 13$ have denominator $t = 1$. There are no linear combinations which are Hecke eigenforms. Some partially multiplicative properties of the coefficients of $[7,17]\pm[1,119]$ and of $[11,13]\pm[1,143]$ are a temptation to look for suitable complements which would make up theta series. Conceivably the fields $\mathbb{Q}(\sqrt{-119})$ and $\mathbb{Q}(\sqrt{-143})$ with class numbers 10 should be considered.

16.3 Weight 2 for Fricke Groups

For the Fricke groups $\Gamma^*(3q)$ with primes $q > 3$ there are five eta products of weight 2,

$$[1,3,q,(3q)], \quad \left[1^2,(3q)^2\right], \quad \left[3^2,q^2\right],$$
$$\left[1^3,3^{-1},q^{-1},(3q)^3\right], \quad \left[1^{-1},3^3,q^3,(3q)^{-1}\right],$$

and all of them are cuspidal. For $\Gamma^*(15)$, in addition, there are two non-cuspidal eta products $\left[1^{-2},3^4,5^4,15^{-2}\right]$ and $\left[1^4,3^{-2},5^{-2},15^4\right]$. We will list some Hecke eigenforms which are linear combinations of these eta products.

For level $N = 15$ the eta product $\eta(z)\eta(3z)\eta(5z)\eta(15z)$ is an eigenform; according to [93] it is the newform which corresponds to the elliptic curve $Y^2 + XY + Y = X^3 + X^2 - 10X - 10$ without complex multiplication.

The functions

$$\frac{\eta^3(3z)\eta^3(5z)}{\eta(z)\eta(15z)} + \tfrac{1}{2}(1+\delta\sqrt{13})\,\eta^2(z)\eta^2(15z)$$
$$-\ \tfrac{1}{2}\varepsilon\,(1+\delta\sqrt{13})\,\eta^2(3z)\eta^2(5z) + \varepsilon\,\frac{\eta^3(z)\eta^3(15z)}{\eta(3z)\eta(5z)}$$

with $\delta, \varepsilon \in \{1, -1\}$ are Hecke eigenforms, but not lacunary. The non-cuspidal eta products combine to a function

$$\frac{1}{2}\left(\frac{\eta^4(3z)\eta^4(5z)}{\eta^2(z)\eta^2(15z)} + \frac{\eta^4(z)\eta^4(15z)}{\eta^2(3z)\eta^2(5z)}\right) = \frac{1}{2} + \sum_{n=1}^{\infty} a(n)e(nz)$$

with multiplicative coefficients $a(n)$ which satisfy $a(p^r) = \sigma_1(p^r)$ for primes $p \neq 3$, $p \neq 5$, and $a(3^r) = 1$, $a(5^r) = 2\sigma_1(5^r) - 1$.

For level $N = 21$ there are Hecke eigenforms

$$\left[\frac{3^3, 7^3}{1, 21}\right] - \left[\frac{1^3, 21^3}{3, 7}\right] + \delta\sqrt{3}\, f_2,$$

$$\left[\frac{3^3, 7^3}{1, 21}\right] + 4\,[1, 3, 7, 21] + \left[\frac{1^3, 21^3}{3, 7}\right] + \delta\sqrt{7}\, \widetilde{f}_2,$$

$$g_1 + 3\delta\left([3^2, 7^2] + [1^2, 21^2]\right),$$

$$\widetilde{g}_1 + \delta\sqrt{13}\left([3^2, 7^2] - [1^2, 21^2]\right)$$

with $\delta \in \{1, -1\}$, where $f_2(z) = \sum_{n\equiv 2 \bmod 3} a_2(n)e\left(\frac{nz}{3}\right)$, $\widetilde{f}_2(z) = \sum_{n\equiv 2 \bmod 3} \widetilde{a}_2(n)e\left(\frac{nz}{3}\right)$ and $g_1(z) = \sum_{n\equiv 1 \bmod 6} b_1(n)e\left(\frac{nz}{6}\right)$, $\widetilde{g}_1(z) = \sum_{n\equiv 1 \bmod 6} \widetilde{b}_1(n)e\left(\frac{nz}{6}\right)$ are normalized integral Fourier series. None of these eigenforms is lacunary. There is, however, another linear combination for level 21 which is a Hecke theta series on the Eisenstein integers:

Example 16.13 *The residues of* 2 *and* ω *modulo* $3(2+\omega)$ *can be chosen as generators of* $(\mathcal{O}_3/(6+3\omega))^\times \simeq Z_6^2$. *A character* χ *on* $\mathcal{O}_3$ *with period* $3(2+\omega)$ *is fixed by its values*

$$\chi(2) = -1, \qquad \chi(\omega) = \overline{\omega}.$$

Let $\widehat{\chi}$ *be the character on* $\mathcal{O}_3$ *with period* $3(2+\overline{\omega})$ *which is given by* $\widehat{\chi}(\mu) = \chi(\overline{\mu})$ *for* $\mu \in \mathcal{O}_3$. *The corresponding theta series of weight* 2 *satisfy*

$$\begin{aligned}
&\Theta_2\left(-3, \chi, \tfrac{z}{3}\right) + (8 - 3\omega)\,\eta^2(7z)\eta^2(21z) \\
&\quad = \Theta_2\left(-3, \widehat{\chi}, \tfrac{z}{3}\right) + (5 + 3\omega)\,\eta^2(7z)\eta^2(21z) \\
&\quad = \frac{\eta^3(3z)\eta^3(7z)}{\eta(z)\eta(21z)} - 3\,\eta(z)\eta(3z)\eta(7z)\eta(21z) \\
&\qquad + \frac{\eta^3(z)\eta^3(21z)}{\eta(3z)\eta(7z)}.
\end{aligned} \tag{16.22}$$

We remark that $\eta^2(7z)\eta^2(21z)$ is, after rescaling, the theta series on $\mathcal{O}_3$, which is known from Example 11.5.

For level $N = 39$ there are four Hecke eigenforms

$$\left[\frac{3^3, 13^3}{1, 39}\right] + \tfrac{1}{2}(1+\delta\sqrt{37})\,\left([3^2, 13^2] - [1^2, 39^2]\right) \\ - \left[\frac{1^3, 39^3}{3, 13}\right] + \varepsilon\sqrt{\tfrac{1}{2}(7+\delta\sqrt{37})}\, f_{2,\delta}$$

with $\delta, \varepsilon \in \{1, -1\}$ where $f_{2,\delta}(z) = \sum_{n\equiv 2 \bmod 3} \alpha_{2,\delta}(n) e\left(\frac{nz}{3}\right)$ are normalized Fourier series whose coefficients are algebraic integers in $\mathbb{Q}(\sqrt{37})$. These functions are not lacunary.

For the Fricke groups $\Gamma^*(pq)$ with primes $5 \le p < q$ there are only three weight 2 eta products,

$$[1, p, q, (pq)], \qquad [p^2, q^2], \qquad [1^2, (pq)^2],$$

and they are cuspidal. For $\Gamma^*(35)$, in addition, there are two non-cuspidal eta products

$$[1^{-1}, 5^3, 7^3, 35^{-1}], \qquad [1^3, 5^{-1}, 7^{-1}, 35^3].$$

For level 35 the linear combinations

$$[5^2, 7^2] + [1^2, 35^2] \qquad \text{and} \qquad [5^2, 7^2] - \tfrac{1}{2}(1+\delta\sqrt{17})\,[1, 5, 7, 35] - [1^2, 35^2]$$

with $\delta \in \{1, -1\}$ are eigenforms; they are not lacunary.

16.4 Cuspidal Eta Products of Weight 2 for $\Gamma_0(15)$

We are able to discuss only a few of the eta products of weight 2 and levels $N = pq$ for primes $3 \le p < q$. Their numbers for $\Gamma_0(15)$ and $\Gamma_0(21)$ are given in Table 16.1. We recall the remark from the beginning of Sect. 16.1, saying that the denominators 8 and 24 cannot occur.

In this subsection we treat the cuspidal eta products of weight 2 for $\Gamma_0(15)$. There are no eigenforms which are linear combinations of the eta products

Table 16.1: Numbers of new eta products of levels 15 and 21 with weight 2

denominator t	1	2	3	4	6	12
$\Gamma_0(15)$ cuspidal	0	4	4	14	12	26
$\Gamma_0(15)$ non-cuspidal	8	0	6	0	0	0
$\Gamma_0(21)$ cuspidal	2	6	6	6	6	22
$\Gamma_0(21)$ non-cuspidal	5	0	5	0	0	0

with denominator $t = 2$,

$$[1^5, 3^{-1}, 5^{-1}, 15], \qquad [1^{-1}, 3, 5^5, 15^{-1}], \qquad [1, 3^2, 5], \qquad [1, 5, 15^2].$$

The cuspidal eta products with denominator $t = 3$ combine to four eigenforms

$$\left[\frac{3, 5^4}{15}\right] + 3\delta i \left[\frac{1, 15^4}{5}\right] + \frac{\varepsilon}{5\sqrt{2}} \left(3(-\delta + 3i) \left[\frac{3^4, 5}{1}\right] + (3\delta + i) \left[\frac{1^4, 15}{3}\right]\right)$$

with $\delta, \varepsilon \in \{1, -1\}$. They are not lacunary.

There are 8 linear combinations of the eta products with denominator $t = 4$ which are theta series. We state the results in the following two examples.

Example 16.14 *Let the generators of* $(\mathcal{O}_1/(12 + 6i))^\times \simeq Z_8 \times Z_2 \times Z_4$ *be chosen as in Example* 12.17. *Two characters* $\chi_{1,\varepsilon}$ *on* $\mathcal{O}_1$ *with period* $6(2 + i)$ *are fixed by their values*

$$\chi_{1,\varepsilon}(2 - i) = \varepsilon \tfrac{1}{\sqrt{2}} (1 + i), \qquad \chi_{1,\varepsilon}(2 + 3i) = 1, \qquad \chi_{1,\varepsilon}(i) = -i$$

with $\varepsilon \in \{1, -1\}$. *Let* $\chi_{-1,\varepsilon}$ *be the characters on* $\mathcal{O}_1$ *with period* $6(2-i)$ *which are given by* $\chi_{-1,\varepsilon}(\mu) = \overline{\chi_{1,\varepsilon}(\overline{\mu})}$ *for* $\mu \in \mathcal{O}_1$. *The corresponding theta series of weight* 2 *satisfy*

$$\begin{aligned} \Theta_2\left(-4, \chi_{\delta,\varepsilon}, \tfrac{z}{4}\right) = \ & \tfrac{1}{15}(1 + \delta\varepsilon i\sqrt{2})\Bigg((4 - 3\delta i)\, \frac{\eta^4(3z)\eta^2(5z)}{\eta(z)\eta(15z)} \\ & + (1 + 3\delta i)\, \frac{\eta^4(z)\eta(5z)}{\eta(3z)} + 5\, \frac{\eta^2(z)\eta(3z)\eta^2(15z)}{\eta(5z)}\Bigg) \\ & + \tfrac{1}{6}\,(2 - \delta\varepsilon i\sqrt{2})\Bigg((1 + 3\delta i)\, \frac{\eta^2(3z)\eta^2(5z)\eta(15z)}{\eta(z)} \\ & + (1 - 3\delta i)\, \frac{\eta^2(z)\eta^4(15z)}{\eta(3z)\eta(5z)} + 2\, \frac{\eta(z)\eta^4(5z)}{\eta(15z)}\Bigg). \end{aligned} \tag{16.23}$$

Example 16.15 *The residues of* $\frac{1}{\sqrt{2}}(1 - \sqrt{-5})$ *and* -1 *modulo* $\sqrt{2}(1 + \sqrt{-5})$ *generate the group* $(\mathcal{J}_5/(\sqrt{2} + \sqrt{-10}))^\times \simeq Z_4 \times Z_2$. *Two characters* $\rho_{1,\varepsilon}$ *on* $\mathcal{J}_5$ *with period* $\sqrt{2}(1 + \sqrt{-5})$ *are given by*

$$\rho_{1,\varepsilon}\left(\tfrac{1}{\sqrt{2}}(1 - \sqrt{-5})\right) = -\varepsilon, \qquad \rho_{1,\varepsilon}(-1) = -1$$

with $\varepsilon \in \{1, -1\}$. *Let* $\rho_{-1,\varepsilon}$ *be the characters on* $\mathcal{J}_5$ *with period* $\sqrt{2}(1 - \sqrt{-5})$ *which are given by* $\rho_{-1,\varepsilon}(\mu) = \rho_{1,\varepsilon}(\overline{\mu})$ *for* $\mu \in \mathcal{J}_5$. *The corresponding theta* *series of weight* 2 *satisfy*

$$\begin{aligned}&\Theta_2\left(-20,\rho_{\delta,\varepsilon},\tfrac{z}{4}\right) + \tfrac{3\varepsilon}{\sqrt{2}}(1-\delta i\sqrt{5})\,\Theta_2\left(-20,\rho_{\delta,\varepsilon},\tfrac{3z}{4}\right)\\ &\quad= \frac{\eta^4(3z)\eta^2(5z)}{\eta(z)\eta(15z)} - \frac{\eta^2(z)\eta(3z)\eta^2(15z)}{\eta(5z)}\\ &\qquad+\delta i\sqrt{5}\left(\frac{\eta^2(3z)\eta^2(5z)\eta(15z)}{\eta(z)} + \frac{\eta^2(z)\eta^4(15z)}{\eta(3z)\eta(5z)}\right)\\ &\qquad+\varepsilon\sqrt{2}\left(\eta^3(z)\eta(15z) - \delta i\sqrt{5}\,\eta(3z)\eta^3(5z)\right).\end{aligned} \tag{16.24}$$

Only eight out of 14 eta products with denominator 4 are involved in the identities in Examples 16.14, 16.15. Another four of these eta products appear in the eigenforms

$$\left[\frac{3^2,5^3}{15}\right] - \delta i\left[\frac{1^3,15^2}{3}\right] + \varepsilon\sqrt{3}\left(\frac{1+\delta i}{\sqrt{2}}\left[\frac{3^3,5^2}{1}\right] - \frac{1-\delta i}{\sqrt{2}}\left[\frac{1^2,15^3}{5}\right]\right),$$

with $\delta,\varepsilon\in\{1,-1\}$, which are not lacunary. We did not find eigenforms involving the remaining two eta products $[1^{-2},3^6,5,15^{-1}]$ and $[1,3^{-1},5^{-2},15^6]$ with denominator 4 in their components.

Now we consider the 12 eta products of level 15, weight 2 and denominator 6. There are eight Hecke eigenforms which are linear combinations of eight of these eta products,

$$\left[\frac{1,3,5^3}{15}\right] + 3\delta\left[\frac{3^3,5,15}{1}\right] + \varepsilon\left(\left[\frac{1^3,5,15}{3}\right] + 3\delta\left[\frac{1,3,15^3}{5}\right]\right)$$

and

$$\left[\frac{3^3,5^2}{15}\right] + \delta i\left[\frac{1^3,15^2}{5}\right] + \varepsilon\sqrt{5}\left(\frac{1+\delta i}{\sqrt{2}}\left[\frac{3^2,5^3}{1}\right] + \frac{1-\delta i}{\sqrt{2}}\left[\frac{1^2,15^3}{3}\right]\right),$$

with $\delta,\varepsilon\in\{1,-1\}$. None of these functions is a Hecke theta series. The remaining four eta products are $\left[1^{-1},3^5,5,15^{-1}\right]$, $[3,5^2,15]$, $[1^2,3,15]$, $\left[1,3^{-1},5^{-1},15^5\right]$. They are the Fricke transforms of the eta products with denominator $t=2$, and there are no linear combinations of these functions which are eigenforms.

Finally we address the 26 eta products of level 15, weight 2 and denominator $t=12$. Applying the Fricke involution W_{15} upon the eta products which constitute the theta series in Examples 16.14, 16.15 yields eight linear combinations of the eta products with denominator 12 which are theta series.

Example 16.16 *Let $\chi_{\delta,\varepsilon}$ be the characters on $\mathcal{O}_1$ with period $6(2+i)$ for $\delta=1$ and with period $6(2-i)$ for $\delta=-1$, as defined in Example 16.14. The* *corresponding theta series of weight* 2 *satisfy*

$$\Theta_2\left(-4, \chi_{\delta,\varepsilon}, \tfrac{z}{12}\right) = \frac{\eta(z)\eta^2(3z)\eta^2(5z)}{\eta(15z)} + \tfrac{9}{5}(1+3\delta i)\,\frac{\eta^4(3z)\eta(15z)}{\eta(z)}$$
$$+ \tfrac{1}{5}(-4+3\delta i)\,\frac{\eta^4(z)\eta^2(15z)}{\eta(3z)\eta(5z)}$$
$$+ \varepsilon\Bigg(\frac{3+\delta i}{\sqrt{2}}\,\frac{\eta^2(3z)\eta^4(5z)}{\eta(z)\eta(15z)} + \frac{3-\delta i}{\sqrt{2}}\,\frac{\eta^2(z)\eta(5z)\eta^2(15z)}{\eta(3z)}$$
$$- 9\delta i\sqrt{2}\,\frac{\eta(3z)\eta^4(15z)}{\eta(5z)}\Bigg). \qquad (16.25)$$

The residues of $\frac{1}{\sqrt{2}}(3+\sqrt{-5})$, $\sqrt{-5}$ *and* -1 *modulo* 6 *generate the group* $(\mathcal{J}_5/(6))^\times \simeq Z_4 \times Z_2^2$. *Four characters* $\psi_{\delta,\varepsilon}$ *on* $\mathcal{J}_5$ *with period* 6 *are given by*

$$\psi_{\delta,\varepsilon}\left(\tfrac{1}{\sqrt{2}}(3+\sqrt{-5})\right) = -\varepsilon, \qquad \psi_{\delta,\varepsilon}(\sqrt{-5}) = -\delta, \qquad \psi_{\delta,\varepsilon}(-1) = -1$$

with $\delta, \varepsilon \in \{1, -1\}$. *The corresponding theta series of weight* 2 *decompose as*

$$\Theta_2\left(-20, \psi_{\delta,\varepsilon}, \tfrac{z}{12}\right) = g_1(z) - \delta i\sqrt{5}\,g_5(z) - 3\varepsilon\sqrt{2}\,g_7(z) - 3\delta\varepsilon i\sqrt{10}\,g_{11}(z), \qquad (16.26)$$

where the components g_j *are normalized integral Fourier series with denominator* 12 *and numerator classes* j *modulo* 12. *They are eta products or linear combinations thereof,*

$$g_1 = \left[\frac{1, 3^2, 5^2}{15}\right] + \left[\frac{1^4, 15^2}{3, 5}\right], \qquad g_5 = \left[\frac{3^2, 5^4}{1, 15}\right] - \left[\frac{1^2, 5, 15^2}{3}\right], \qquad (16.27)$$

$$g_7 = [3^3, 5], \qquad g_{11} = [1, 15^3]. \qquad (16.28)$$

Comparing (16.23) and (16.25) yields a complicated identity among eta products of weight 2 and level 45. We do not write it down here.

There are 18 eta products of level 15, weight 2 and denominator 12 which do not occur in Example 16.16. Among them we could find only 8 linear combinations which are Hecke eigenforms,

$$\left[\frac{1^2, 5^3}{15}\right] + 3\big(1+\delta\sqrt{3}\big)\,[3^2, 5, 15]$$
$$+ \varepsilon i\left(\left[\frac{1^3, 5^2}{3}\right] + 3\big(1+\delta\sqrt{3}\big)\,[1, 3, 15^2]\right)$$
$$+ \nu i\sqrt{6}\sqrt{2+\delta\sqrt{3}}\left(\left[1, 3, 5^2\right] - \tfrac{3}{2}\big(1-\delta\sqrt{3}\big)\left[\frac{3^3, 15^2}{1}\right]\right.$$
$$\left. - \varepsilon i\left(\left[1^2, 5, 15\right] - \tfrac{3}{2}\big(1-\delta\sqrt{3}\big)\left[\frac{3^2, 15^3}{5}\right]\right)\right).$$

These functions are not lacunary.

16.5 Some Eta Products of Weight 2 for $\Gamma_0(21)$

The numbers of eta products of weight 2 for $\Gamma_0(21)$ are listed at the beginning of Sect. 16.4. We discuss only those among them which are involved in theta identities. To begin with, there are two linear combinations of the eta products $[1^2,3,7]$, $[1^{-1},3^2,7^4,21^{-1}]$, $[1^4,3^{-1},7^{-1},21^2]$, $[1,7^2,21]$ with denominator $t=2$ which have multiplicative coefficients but violate the proper recursions at powers of the prime 3. They are identified with linear combinations of two theta series:

Example 16.17 *The residues of $-1+2\omega$ and ω modulo $2(2+\omega)$ generate the group $(\mathcal{O}_3/(4+2\omega))^\times \simeq Z_3 \times Z_6$. A character ψ_1 on $\mathcal{O}_3$ with period $2(2+\omega)$ is defined by*

$$\psi_1(-1+2\omega)=1, \qquad \psi_1(\omega)=\overline{\omega}.$$

Let ψ_{-1} denote the character on $\mathcal{O}_3$ with period $2(2+\overline{\omega})$ which is given by $\psi_{-1}(\mu)=\overline{\psi_1(\overline{\mu})}$. Then for $\delta\in\{1,-1\}$ we have the identity

$$\begin{aligned}\Theta_2\left(-3,\psi_\delta,\tfrac{z}{2}\right) &- 3\delta i\sqrt{3}\,\Theta_2\left(-3,\psi_\delta,\tfrac{3z}{2}\right)\\ &= \tfrac13(1+\delta i\sqrt{3})\,\eta^2(z)\eta(3z)\eta(7z) + \tfrac13(2-\delta i\sqrt{3})\,\frac{\eta^2(3z)\eta^4(7z)}{\eta(z)\eta(21z)}\\ &\quad + \tfrac13\,\frac{\eta^4(z)\eta^2(21z)}{\eta(3z)\eta(7z)} - \tfrac13(1+3\delta i\sqrt{3})\,\eta(z)\eta^2(7z)\eta(21z). \qquad (16.29)\end{aligned}$$

We get simpler results for the eta products with denominator $t=3$. One of the identities involves four eta products with numerators $s\equiv 1 \bmod 3$, the other one two eta products with numerators $s\equiv 2 \bmod 3$:

Example 16.18 *Let the generators of $(\mathcal{O}_3/(6+2\omega))^\times \simeq Z_6^2$ be chosen as in Example 16.13. A character φ_1 on $\mathcal{O}_3$ with period $3(2+\omega)$ is given by*

$$\varphi_1(2)=1, \qquad \varphi_1(\omega)=\overline{\omega}.$$

Let φ_{-1} denote the character on $\mathcal{O}_3$ with period $3(2+\overline{\omega})$, which is given by $\varphi_{-1}(\mu)=\overline{\varphi_1(\overline{\mu})}$. Then for $\delta\in\{1,-1\}$ we have the identity

$$\begin{aligned}\Theta_2\left(-3,\varphi_\delta,\tfrac{z}{3}\right) &= \tfrac14(-1+\delta i\sqrt{3})\,\frac{\eta^4(z)\eta(7z)}{\eta(3z)} + \tfrac14(5-\delta i\sqrt{3})\,\frac{\eta(z)\eta^4(7z)}{\eta(21z)}\\ &\quad + \tfrac34(3-\delta i\sqrt{3})\,\frac{\eta^4(3z)\eta(21z)}{\eta(z)}\\ &\quad - \tfrac34(3+5\delta i\sqrt{3})\,\frac{\eta(3z)\eta^4(21z)}{\eta(7z)}. \qquad (16.30)\end{aligned}$$

Let the generator of $(\mathcal{O}_7/(3))^\times \simeq Z_8$ *be chosen as in Example* 12.3*, and define four characters* $\rho_{\delta,\varepsilon}$ *on* $\mathcal{O}_7$ *with period* 3 *by their value*

$$\rho_{\delta,\varepsilon}\left(\tfrac{1}{2}(1+\sqrt{-7})\right) = \tfrac{1}{\sqrt{2}}\varepsilon(1-\delta i)$$

with $\delta, \varepsilon \in \{1,-1\}$*. The corresponding theta series of weight* 2 *satisfy*

$$\Theta_2\left(-7, \rho_{\delta,\varepsilon}, \tfrac{z}{3}\right) = h_1(z) + \delta\sqrt{7}\,\widetilde{h}_1(z) + \tfrac{1}{\sqrt{2}}\varepsilon i(\sqrt{7}-\delta)\left(\eta^3(3z)\eta(7z) + \delta\sqrt{7}\,\eta(z)\eta^3(21z)\right) \quad (16.31)$$

with normalized integral Fourier series h_1, $\widetilde{h}_1$ *with denominator* 3 *and numerator class* 1 *modulo* 3*.*

The Fricke involution W_{21} transforms the eta products in Example 16.17 into eta products with denominator $t = 6$. For these functions there is a rather simple theta identity, in contrast to (16.29), due to the fact that the coefficients at multiples of the prime 3 vanish:

Example 16.19 *The residues of* $3-\omega$, -5 *and* ω *modulo* $6(2+\omega)$ *can be chosen as generators of the group* $(\mathcal{O}_3/(12+6\omega))^\times \simeq Z_6 \times Z_3 \times Z_6$*. A character* χ_1 *on* $\mathcal{O}_3$ *with period* $6(2+\omega)$ *is fixed by its values*

$$\chi_1(3-\omega) = \omega, \qquad \chi_1(-5) = 1, \qquad \chi_1(\omega) = \overline{\omega}.$$

Let χ_{-1} *denote the character on* $\mathcal{O}_3$ *with period* $6(2+\overline{\omega})$ *which is given by* $\chi_{-1}(\mu) = \overline{\chi_1(\overline{\mu})}$*. Then for* $\delta \in \{1,-1\}$ *we have the identity*

$$\Theta_2\left(-3, \chi_\delta, \tfrac{z}{6}\right) = \frac{\eta^4(3z)\eta^2(7z)}{\eta(z)\eta(21z)} + (1+\delta i\sqrt{3})\,\eta(z)\eta^2(3z)\eta(21z) - (1+3\delta i\sqrt{3})\,\eta(3z)\eta(7z)\eta^2(21z) + (2-\delta i\sqrt{3})\,\frac{\eta^2(z)\eta^4(21z)}{\eta(3z)\eta(7z)}. \quad (16.32)$$

There is a linear combination of the cuspidal eta products with denominator $t = 12$ and numerators $s \equiv 1 \bmod 12$ which has multiplicative coefficients and which is closely related to the theta series in Example 11.17. We get an identity relating eta products of weight 2 of levels 3 and 21:

Example 16.20 *We have the eta identity*

$$9\left[\frac{3^6, 7}{1^2, 21}\right] + 5\left[\frac{1^2, 7^3}{21}\right] - 13\left[\frac{3, 7^6}{1, 21^2}\right] - 9\left[\frac{3^3, 21^2}{1}\right] = \left[\frac{1^5}{3}\right] + 27\left[\frac{21^5}{7}\right], \quad (16.33)$$

and this function is equal to

$$\tfrac{1}{2}\left(\Theta_2\left(-3,\psi_1,\tfrac{z}{12}\right)+\Theta_2\left(-3,\psi_{-1},\tfrac{z}{12}\right)\right) \\ -\ \tfrac{3i\sqrt{3}}{2}\left(\Theta_2\left(-3,\psi_1,\tfrac{7z}{12}\right)-\Theta_2\left(-3,\psi_{-1},\tfrac{7z}{12}\right)\right),$$

where ψ_δ are the characters on $\mathcal{O}_3$ with period 12 *from Example* 11.17*.*

There are four linear combinations of eight cuspidal eta products with denominator 12 which are Hecke theta series:

Example 16.21 *The residues of $3-\omega$, $3+\omega$, 13 and ω modulo $12(2+\omega)$ are generators of $(\mathcal{O}_3/(24+12\omega))^\times \simeq Z_6^2\times Z_2\times Z_6$. Characters $\psi_{1,\varepsilon}$ on $\mathcal{O}_3$ with period $12(2+\omega)$ are given by*

$$\psi_{1,\varepsilon}(3-\omega)=\varepsilon\overline{\omega},\qquad \psi_{1,\varepsilon}(3+\omega)=\overline{\omega},\qquad \psi_{1,\varepsilon}(13)=-1,\qquad \psi_{1,\varepsilon}(\omega)=\overline{\omega}.$$

Define characters $\psi_{-1,\epsilon}$ on $\mathcal{O}_3$ with period $12(2+\overline{\omega})$ by $\psi_{-1,\varepsilon}(\mu)=\overline{\psi_{1,\varepsilon}(\overline{\mu})}$. Then for $\delta,\varepsilon\in\{1,-1\}$ we have the identity

$$\begin{aligned}\Theta_2\left(-3,\psi_{\delta,\varepsilon},\tfrac{z}{12}\right) &= f_1(z)+C_\delta f_{13}(z)\\ &\quad+(C_\delta-1)\widetilde{f}_{13}(z)+(C_\delta+6)f_{25}(z)\\ &\quad+\varepsilon\big(C_\delta f_7(z)+f_{19}(z)+(C_\delta+6)\widetilde{f}_{19}(z)\\ &\quad+(C_\delta-1)f_{31}(z)\big),\end{aligned} \tag{16.34}$$

where $C_\delta=\frac{1}{2}(1-3\sqrt{3}\delta i)$, and where the components are eta products which make up four pairs of Fricke transforms, with the subscripts indicating the numerators,

$$f_1=\left[\frac{3^3,7^2}{21}\right],\quad f_{13}=\left[1^2,3,21\right],\quad \widetilde{f}_{13}=\left[\frac{3^2,7^3}{1}\right],\quad f_{25}=\left[1,7,21^2\right], \tag{16.35}$$

$$f_7=\left[1,3^2,7\right],\quad f_{19}=\left[\frac{1^3,21^2}{7}\right],\quad \widetilde{f}_{19}=\left[3,7^2,21\right],\quad f_{31}=\left[\frac{1^2,21^3}{3}\right]. \tag{16.36}$$