14 Levels $N = p^2$ with Primes $p \geq 3$

14.1 Weight 1 for Level $N = 9$

For primes $p \geq 5$ there are exactly 6 holomorphic eta products of weight 1 and level $N = p^2$. The only new one among them is $\eta(z)\eta(p^2 z)$. Since its order at ∞ is $\frac{1+p^2}{24} > 1$, there is little chance to find complementary eta products for the construction of eigenforms which might be represented by Hecke theta series,—at least when we stick to level p^2. The chances are improved when we consider $\eta(z)\eta(p^2 z)$ as an old eta product of level $2p^2$, and indeed the function $\eta(z)\eta(25z)$ will play its rôle in Sect. 20.3.

Thus for weight 1 we are confined to the level $N = 3^2 = 9$. In this case there are exactly 13 holomorphic eta products, among which only 4 are new. Two of them are the cuspidal eta products $\left[1, 9\right]$ and $\left[1^{-1}, 3^4, 9^{-1}\right]$ for the Fricke group $\Gamma^*(9)$. The other two are non-cuspidal, $\left[1^2, 3^{-1}, 9\right]$ and $\left[1, 3^{-1}, 9^2\right]$, with orders $\frac{1}{3}$ and $\frac{2}{3}$ at ∞. In the first example of this section we describe theta series whose components are the two cuspidal eta products:

Example 14.1 *The residues of* $\alpha = 2 + i$ *and* $\beta = 2 + 3i$ *modulo* 18 *can be chosen as generators of* $(\mathcal{O}_1/(18))^\times \simeq Z_{24} \times Z_6$. *We have* $\alpha^6\beta^3 \equiv i \bmod 18$, $\alpha^{12} \equiv -1 \bmod 18$. *Four characters* $\chi_{\delta,\nu}$ *on* $\mathcal{O}_1$ *with period* 18 *are fixed by their values*

$$\chi_{\delta,\nu}(2+i) = \xi, \qquad \chi_{\delta,\nu}(2+3i) = \xi^2, \qquad \text{with} \qquad \xi = \xi_{\delta,\nu} = \tfrac{1}{2}(\delta\sqrt{3} + \nu i)$$

a primitive 12*th root of unity, and* $\delta, \nu \in \{1, -1\}$. *The corresponding theta series of weight* 1 *satisfy*

$$\Theta_1\left(-4, \chi_{\delta,\nu}, \tfrac{z}{12}\right) = f_1(z) + \delta\sqrt{3}\, f_5(z) \tag{14.1}$$

where the components f_j *are normalized integral Fourier series with denominator* 12 *and numerator classes* j *modulo* 12, *and both of them are eta products,*

$$f_1(z) = \frac{\eta^4(3z)}{\eta(z)\eta(9z)}, \qquad f_5(z) = \eta(z)\eta(9z). \tag{14.2}$$

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The Fricke involution W_9 *maps* $F_\delta(z) = \Theta_1\big(-4, \chi_{\delta,\nu}, \frac{z}{12}\big)$ *to* $F_\delta(W_9 z) = -3iz\, F_\delta(z)$.

For the non-cuspidal eta products which we mentioned above we introduce the notation

$$\begin{aligned} g_1 &= \big[1^2, 3^{-1}, 9\big], \qquad g_2 = \big[1, 3^{-1}, 9^2\big], \\ g_j(z) &= \sum_{n \equiv j \bmod 3} b_j(n) e\left(\tfrac{nz}{3}\right). \end{aligned} \tag{14.3}$$

We find two linear combinations which are Eisenstein series whose divisor sums involve non-real characters:

Example 14.2 *For* $\delta \in \{1, -1\}$, *let* χ_δ *be the Dirichlet character modulo* 9 *on* $\mathbb{Z}$ *which is fixed by the value* $\chi_\delta(2) = \omega^\delta$ *for the primitive root* 2 *modulo* 9. *Then the eta products* g_1, g_2 *in* (14.3) *satisfy*

$$g_1(z) + \delta i\sqrt{3}\, g_2(z) = \sum_{n=1}^{\infty} \Big(\chi_\delta(n) \sum_{d|n} \chi_\delta(d)\Big) e\Big(\frac{nz}{3}\Big). \tag{14.4}$$

The Fricke involution W_9 *transforms* $G_\delta = g_1 + \delta i\sqrt{3}\, g_2$ *into* $G_\delta(W_9 z) = 3\delta z\, G_{-\delta}(z)$.

The identity (14.4) implies that the coefficients $b_j(p)$ of g_j at primes p are

$$b_1(p) = \begin{cases} -1 \\ 2 \end{cases} \text{for } p \equiv \begin{cases} 7, 13 \\ 1 \end{cases} \bmod 18,$$

$$b_2(p) = \begin{cases} -1 \\ 1 \\ 0 \end{cases} \text{for } p \equiv \begin{cases} 5 \\ 11 \\ 17 \end{cases} \bmod 18.$$

For the coefficients $\lambda(p^r)$ of G_δ at prime powers p^r we get the recursions $\lambda(p^{r+1}) = \lambda(p)\lambda(p^r) - \big(\frac{p}{3}\big)\lambda(p^{r-1})$.

14.2 Weight 2 for the Fricke Group $\Gamma^*(9)$

There are 6 new holomorphic eta products of weight 2 for $\Gamma^*(9)$, four of them cuspidal and two non-cuspidal. Besides, there are 14 new cuspidal and 6 new non-cuspidal eta products of level 9 which do not belong to the Fricke group.

We get 4 linear combinations of the cuspidal eta products for $\Gamma^*(9)$ which are Hecke eigenforms. Two eta products with orders $\frac{1}{3}$ and $\frac{2}{3}$ at ∞ combine to the eigenforms

$$\frac{\eta^6(3z)}{\eta(z)\eta(9z)} + \delta\sqrt{3}\,\eta(z)\eta^2(3z)\eta(9z),$$

and the other two eta products with orders $\frac{1}{6}$ and $\frac{5}{6}$ at ∞ combine to the eigenforms

$$\frac{\eta^8(3z)}{\eta^2(z)\eta^2(9z)} + 3\delta\,\eta^2(z)\eta^2(9z),$$

with $\delta \in \{1, -1\}$. These functions are, however, not lacunary, and hence cannot be identified with theta series. We remark that a few of their coefficients at primes vanish, in accordance with Serre's theorem [128].

The non-cuspidal eta products of weight 2 for $\Gamma^*(9)$ have orders 0 and 1 at ∞. One of them is the Eisenstein series $E_{2,9,-1}$ from Proposition 1.8, the other one is an Eisenstein series similar to those in Theorem 1.9:

Example 14.3 *We have the identities*

$$\frac{\eta^{10}(3z)}{\eta^3(z)\eta^3(9z)} = 1 + \sum_{n=1}^{\infty}\Bigg(\sum_{d|n,\, 9\nmid d} d\Bigg)e(nz) = E_{2,9,-1}(z)\,, \tag{14.5}$$

$$\frac{\eta^3(z)\eta^3(9z)}{\eta^2(3z)} = \sum_{n=1}^{\infty}\Bigg(\left(\frac{n}{3}\right)\sum_{d|n} d\Bigg)e(nz)\,. \tag{14.6}$$

14.3 Weight 2 for $\Gamma_0(9)$

One of the cuspidal eta products of weight 2 for $\Gamma_0(9)$ is $\eta^3(z)\eta(9z)$ with order $\frac{1}{2}$ at ∞. It is an eigenform of the Hecke operators T_p for all primes $p \neq 3$. It is completed to an eigenform by the oldform $\eta^4(9z)$, yielding a theta series which is well known from Example 11.7. Fricke transformation leads to the eta product $\eta(z)\eta^3(9z)$ with order $\frac{7}{6}$ at ∞. We list the following results:

Example 14.4 *Let $\rho = \rho_1$ be the character on $\mathcal{O}_3$ with period 6 as defined in Example 11.7. Then we have the identities*

$$\Theta_2\left(-3, \rho, \tfrac{z}{2}\right) = \eta^3(z)\eta(9z) + 3\,\eta^4(9z)\,, \tag{14.7}$$

$$\Theta_2\left(-3, \rho, \tfrac{z}{6}\right) = \eta^4(z) + 9\,\eta(z)\eta^3(9z)\,, \tag{14.8}$$

and, with $\Theta(z)$ as defined in (11.1),

$$\Theta(3z)\eta(3z)\eta(9z) = \eta^4(3z) + 9\,\eta(3z)\eta^3(27z) = \eta^3(z)\eta(9z) + 3\,\eta^4(9z). \tag{14.9}$$

The Fricke involution W_9 transforms $F = \left[1^3, 9\right] + 3\left[9^4\right]$ into $F(W_9 z) = -3z^2 F\left(\frac{z}{3}\right)$.

The only new cuspidal eta product of level 9, weight 2 and denominator 4 is $[1^3, 3^{-1}, 9^2]$, with order $\frac{3}{4}$ at ∞. We could not find an eigenform involving this eta product as a constituent in one of its components. The same must be said for its Fricke transform $[1^2, 3^{-1}, 9^3]$ with order $\frac{13}{12}$ at ∞. However, there are theta series whose components are old eta products with denominator 4: These are the functions

$$[1^3, 3] + 3(1 + \sqrt{2}\,\delta i)[3, 9^3]$$

which were discussed in Example 11.18.

There are exactly two new cuspidal eta products of weight 2 with denominator 6. They are Fricke transforms of each other, and their linear combinations

$$F_\delta(z) = \frac{\eta(z)\eta^4(3z)}{\eta(9z)} + 3\delta\,\frac{\eta^4(3z)\eta(9z)}{\eta(z)}$$

are Hecke eigenforms. These functions are not lacunary, and hence there is no theta series identity. We have $F_\delta(W_9 z) = -9z^2 F_\delta(z)$.

There are 9 new cuspidal eta products of weight 2 with denominator 12 for $\Gamma_0(9)$. One of them is $[1^2, 3^{-1}, 9^3]$ which was mentioned before. For the others we introduce the notations

$$f_1 = [1^{-1}, 3^7, 9^{-2}], \qquad f_5 = [1^{-2}, 3^7, 9^{-1}], \qquad f_7 = [1^2, 3, 9], \qquad f_{11} = [1, 3, 9^2], \tag{14.10}$$

$$g_1 = [1^2, 3^3, 9^{-1}], \qquad g_{13} = [1^{-1}, 3^3, 9^2], \qquad g_5 = [1^4, 3^{-1}, 9], \qquad g_{17} = [1, 3^{-1}, 9^4]. \tag{14.11}$$

Here the subscripts are equal to the numerators of the eta products, and (f_1, f_5), (f_7, f_{11}), (g_1, g_{13}), (g_5, g_{17}) are pairs of Fricke transforms. We get four Hecke eigenforms

$$F_{\delta,\varepsilon} = f_1 + \sqrt{3}\delta\varepsilon i f_5 + 3\varepsilon i f_7 + 3\sqrt{3}\delta f_{11},$$

with $\delta, \varepsilon \in \{1, -1\}$, which are not lacunary. They satisfy $F_{\delta,\varepsilon}(W_9 z) = -9\delta\varepsilon i F_{\delta,-\varepsilon}(z)$. There are four linear combinations of the eta products g_j which are represented by Hecke theta series:

Example 14.5 *Let the generators of $(\mathcal{O}_1/(18))^\times \simeq Z_{24} \times Z_6$ be chosen as in Example 14.1, and define four characters $\psi_{\delta,\varepsilon}$ on $\mathcal{O}_1$ with period 18 by*

$$\psi_{\delta,\varepsilon}(2 + i) = \xi, \quad \psi_{\delta,\varepsilon}(2 + 3i) = -i\xi^2, \quad \text{with } \xi = \tfrac{1}{2\sqrt{2}}\varepsilon\left((1 + \delta\sqrt{3}) + (1 - \delta\sqrt{3})i\right)$$

a primitive 24th root of unity, and with $\delta, \varepsilon \in \{1, -1\}$. Then we have the identity

$$\Theta_2\left(-4, \psi_{\delta,\varepsilon}, \tfrac{z}{12}\right) = g_1(z) + 3\sqrt{3}\delta\, g_{13}(z) + \tfrac{1}{2}\varepsilon i\sqrt{6}(\sqrt{3} - \delta)\left(g_5(z) - 3\sqrt{3}\delta\, g_{17}(z)\right), \tag{14.12}$$

where the components g_j are the eta products in (14.11). The Fricke involution W_9 transforms $G_{\delta,\varepsilon} = \Theta_2\left(-4, \psi_{\delta,\varepsilon}, \frac{z}{12}\right)$ into $G_{\delta,\varepsilon}(W_9 z) = -9\delta z^2 G_{\delta,-\varepsilon}(z)$.

Primitive 24th roots of unity appeared in Example 13.18, with a similar notation. Four of eight roots are involved here. We remark that $\xi^2 = \frac{1}{2}(\delta\sqrt{3}-i)$, $\xi^3 = \varepsilon\,\frac{1-i}{\sqrt{2}}$, $\xi^6 = -i$. Since $i \equiv (2+i)^6(2+3i)^3 \bmod 18$, we obtain $\psi_{\delta,\varepsilon}(i) = \xi^6(-i\xi^2)^3 = i\xi^{12} = -i$ as it should be for weight $k=2$. Formulae such as $\mu\xi - \overline{\mu}\overline{\xi} = \frac{\varepsilon i}{\sqrt{2}}\left((x+y) - \delta\sqrt{3}(x-y)\right)$ for $\mu = x+yi$ are useful for the evaluation of coefficients of $\Theta_2\left(-4, \psi_{\delta,\varepsilon}, \frac{z}{12}\right)$.

Now we discuss the non-cuspidal eta products of weight 2 and level 9. Two of them, $\left[1^3, 3^2, 9^{-1}\right]$ and $\left[1^{-1}, 3^2, 9^3\right]$, with denominators 1 and 3, form a pair of Fricke transforms which we could not identify with constituents of Hecke eigenforms. For the others we introduce the notation

$$\begin{aligned} h_1 &= \left[1^5, 3^{-2}, 9\right], \qquad h_4 = \left[1^2, 3^{-2}, 9^4\right], \\ h_2 &= \left[1^4, 3^{-2}, 9^2\right], \qquad h_5 = \left[1, 3^{-2}, 9^5\right]. \end{aligned} \tag{14.13}$$

All of them have denominator 3, the subscripts are equal to the numerators, and (h_1, h_5), (h_2, h_4) are pairs of Fricke transforms. There are four linear combinations which are eigenforms and can be represented by Eisenstein series:

Example 14.6 *For $\delta, \varepsilon \in \{1, -1\}$, let $\rho_{\delta,\varepsilon}$ and χ_ε be the Dirichlet characters modulo 9 on $\mathbb{Z}$ which are fixed by the values*

$$\rho_{\delta,\varepsilon}(2) = \delta\omega^{-\varepsilon}, \qquad \chi_\varepsilon(2) = \omega^{2\varepsilon}$$

for the primitive root 2 modulo 9. Then the eta products h_j in (14.13) satisfy

$$\begin{aligned} &\left(h_1(z) + 3A_\varepsilon\, h_4(z)\right) + \delta\left(A_\varepsilon\, h_2(z) + 9h_5(z)\right) \\ &\quad = \sum_{n=1}^{\infty}\left(\rho_{\delta,\varepsilon}(n)\sum_{d|n}\chi_\varepsilon(d)d\right)e\left(\frac{nz}{3}\right), \end{aligned} \tag{14.14}$$

where $A_\varepsilon = 1+\omega^\varepsilon$. Let $H_{\delta,\varepsilon}$ denote the functions in (14.14). Then the action of W_9 on $H_{\delta,\varepsilon}$ is given by $H_{\delta,\varepsilon}(W_9 z) = -9\delta z^2\, H_{\delta,\varepsilon}(z)$.

From (14.14) one can deduce explicit formulae for the coefficients of the eta products (14.13). At primes they read as follows:

Corollary 14.7 *Let the Fourier expansions of the eta products (14.13) be written as*

$$h_j(z) = \sum_{n \equiv j \bmod 3} c_j(n) e\left(\tfrac{nz}{3}\right).$$

Then for primes p we have

$$c_1(p) = \begin{cases} p+1 \\ 1-2p \\ p-2 \end{cases}, \qquad c_4(p) = \begin{cases} 0 \\ \frac{p-1}{3} \\ -\frac{p-1}{3} \end{cases} \qquad \textit{for} \qquad p \equiv \begin{cases} 1 \\ 4 \\ 7 \end{cases} \mod 9,$$

$$c_2(p) = \begin{cases} p-1 \\ -(p-1) \\ 0 \end{cases}, \qquad c_5(p) = \begin{cases} -\frac{p-2}{9} \\ \frac{2p-1}{9} \\ -\frac{p+1}{9} \end{cases} \qquad \textit{for} \qquad p \equiv \begin{cases} 2 \\ 5 \\ 8 \end{cases} \mod 9.$$

14.4 Weight 2 for Levels $N = p^2$, $p \geq 5$

For each prime $p \geq 7$ there are exactly 6 new holomorphic eta products of weight 2 and level p^2. There are 5 more of them for level 25. (In accordance with Propositions 3.3, 3.5 we get more or at least the same number of eta products for smaller primes.)

We start to discuss level 25. Among the 11 new eta products, 3 belong to the Fricke group $\Gamma^*(25)$. One of them is $\left[1, 5^2, 25\right]$ with denominator 2 and order $\frac{3}{2}$ at ∞. It can be combined with old eta products to get an eigenform. At the beginning of Sect. 12.3 we mentioned that $\left[1^2, 5^2\right]$ is an eigenform. Now we find that

$$F = \left[1^2, 5^2\right] + 4\left[1, 5^2, 25\right] + 5\left[5^2, 25^2\right]$$

is an eigenform and satisfies $F(W_{25}z) = -25z^2F(z)$. Its coefficients at multiples of 5 vanish. With $\left[1^2, 5^2\right]$ it shares the property that it is not lacunary.

The other two eta products for $\Gamma^*(25)$ have denominator 6 and orders $\frac{1}{6}$ and $\frac{13}{6}$ at ∞. They combine to an eigenform which is a theta series for the field $\mathbb{Q}(\sqrt{-3})$:

Example 14.8 *The residues of* $2+\omega$ *and* ω *modulo* $10(1+\omega)$ *can be chosen as generators of the group* $(\mathcal{O}_3/(10+10\omega))^\times \simeq Z_{24} \times Z_6$. *A character* ψ *on* $\mathcal{O}_3$ *with period* $10(1+\omega)$ *is given by*

$$\psi(2+\omega) = \omega, \qquad \psi(\omega) = \overline{\omega}.$$

The corresponding theta series of weight 2 *satisfies*

$$\Theta_2\left(-3, \psi, \tfrac{z}{6}\right) = \frac{\eta^6(5z)}{\eta(z)\eta(25z)} + 4\,\eta^2(z)\eta^2(25z). \tag{14.15}$$

For the construction of another eigenform involving $\left[1^{-1}, 5^6, 25^{-1}\right]$ and $\left[1^2, 25^2\right]$ one would need a third constituent; we did not find one.

The cuspidal eta products of weight 2 for $\Gamma_0(25)$ are $[1^3, 25]$, $[1, 25^3]$, $[1^2, 5, 25]$, $[1, 5, 25^2]$ with orders $\frac{7}{6}$, $\frac{19}{6}$, $\frac{4}{3}$, $\frac{7}{3}$ at ∞. The orders are > 1, and the numerators s and denominators t satisfy $s \equiv 1 \bmod t$. These are the obstacles why we did not find eigenforms involving these eta products as constituents.

A more favorable situation prevails for the non-cuspidal eta products of level 25 and weight 2. All of them have denominator 1. We denote them by $f_1, \ldots, f_4$ where s is the order of f_s at ∞.

Example 14.9 *Put*

$$\begin{aligned} f_1 &= [1^4, 5^{-1}, 25], & f_2 &= [1^3, 5^{-1}, 25^2], \\ f_3 &= [1^2, 5^{-1}, 25^3], & f_4 &= [1, 5^{-1}, 25^4]. \end{aligned} \tag{14.16}$$

For $\delta \in \{1, -1\}$, let χ_δ be the Dirichlet character modulo 5 on $\mathbb{Z}$ which is fixed by the value $\chi_\delta(2) = \delta i$ for the primitive root 2 modulo 5. Then we have the identities

$$f_1(z) + (4 + 3\delta i) f_2(z) + 5(2 + \delta i) f_3(z) + 5(1 + 2\delta i) f_4(z) = \sum_{n=1}^{\infty} (\chi_\delta(n)\sigma_1(n))\, e(nz), \tag{14.17}$$

$$f_1(z) + 3f_2(z) + 5f_3(z) + 5f_4(z) = \sum_{n=1}^{\infty} \left(\left(\frac{n}{5}\right) \sum_{d|n} \left(\frac{n/d}{5}\right) d \right) e(nz), \tag{14.18}$$

$$f_1(z) + 5 f_2(z) + 15 f_3(z) + 25 f_4(z) = \sum_{n=1}^{\infty} \left(\sum_{d|n} \left(\frac{n/d}{5}\right) d \right) e(nz) - 5 \sum_{5|n} \left(\sum_{d|n} \left(\frac{n/d}{5}\right) d \right) e(nz). \tag{14.19}$$

The holomorphic eta products of weight 2 and levels $N = p^2$ with primes $p \geq 7$ are $[1, (p)^2, (p^2)]$, $[1^2, (p^2)^2]$, $[1, p, (p^2)^2]$, $[1, (p^2)^3]$, $[1^2, p, (p^2)]$, $[1^3, (p^2)]$. All of them have orders > 1 at ∞. We did not find eigenforms involving any of these eta products.