13 Level $N = 4$

13.1 Odd Weights for the Fricke Group $\Gamma^*(4)$

There are six new holomorphic eta products of weight 1 for the Fricke group $\Gamma^*(4)$. They are the sign transforms of $\eta^2(z)$ and of the five eta products for $\Gamma_0(2)$ listed at the beginning of Sect. 10.1. Therefore the representations by theta series are quite similar to those in Sect. 10.1. A minor difference is that we need larger periods for the characters. It is easy to verify the following result, which allows a comfortable construction of modular forms for the Fricke group $\Gamma^*(4)$:

Lemma 13.1 *If $f(z)$ is a modular form of weight k for $\Gamma_0(2)$ (or, in particular, for the full modular group Γ_1) then its sign transform $g(z) = f\left(z+\frac{1}{2}\right)$ is a modular form of weight k for the Fricke group $\Gamma^*(4)$. If v_f denotes the multiplier system of f then the multiplier system v_g of g is given by*

$$v_g(W_4) = v_f\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \qquad v_g\begin{pmatrix} a & b \\ 4c & d \end{pmatrix} = v_f\begin{pmatrix} a+2c & b-c+\frac{d-a}{2} \\ 4c & d-2c \end{pmatrix}$$

for $\left(\begin{smallmatrix} a & b \\ 4c & d \end{smallmatrix}\right) \in \Gamma_0(4)$.

In the particular case of the Eisenstein series $E_4(z)$ we see from (10.57) that its sign transform $E_4\left(z+\frac{1}{2}\right)$ is a linear combination of eta products for $\Gamma^*(4)$.—We begin with $\left[1^{-2}, 2^6, 4^{-2}\right]$, the sign transform of $\eta^2(z)$. Not surprisingly, we find identities with theta series on three quadratic number fields as before in Example 9.1:

Example 13.2 *The residues of $2+i$, $1+6i$ and i modulo 12 can be chosen as generators of the group $(\mathcal{O}_1/(12))^\times \simeq Z_8 \times Z_2 \times Z_4$. Two characters χ_ν on $\mathcal{O}_1$ with period 12 are fixed by their values*

$$\chi_\nu(2+i) = \nu i, \qquad \chi_\nu(1+6i) = -1, \qquad \chi_\nu(i) = 1$$

G. Köhler, *Eta Products and Theta Series Identities*, Springer Monographs in Mathematics,
DOI 10.1007/978-3-642-16152-0_13,

with $\nu \in \{1, -1\}$. *The residues of* $1 + 2\omega$, $1 - 4\omega$, 5 *and* ω *modulo* $8(1+\omega)$ *can be chosen as generators of* $(\mathcal{O}_3/(8+8\omega))^\times \simeq Z_4 \times Z_2^2 \times Z_6$. *Characters* ψ_ν *on* $\mathcal{O}_3$ *with period* $8(1+\omega)$ *are defined by*

$$\psi_\nu(1+2\omega) = \nu, \qquad \psi_\nu(1-4\omega) = 1, \qquad \psi_\nu(5) = -1, \qquad \psi_\nu(\omega) = 1.$$

The residues of $2+\sqrt{3}$, $1+2\sqrt{3}$ *and* -1 *modulo* $4\sqrt{3}$ *can be chosen as generators of* $(\mathbb{Z}[\sqrt{3}]/(4\sqrt{3}))^\times \simeq Z_4 \times Z_2^2$. *A Hecke character* ξ *on* $\mathbb{Z}[\sqrt{3}]$ *modulo* $4\sqrt{3}$ *is given by*

$$\xi(\mu) = \begin{cases} \mathrm{sgn}(\mu) \\ -\mathrm{sgn}(\mu) \end{cases} \quad \textit{for} \quad \mu \equiv \begin{cases} 2+\sqrt{3} \\ -1,\ 1+2\sqrt{3} \end{cases} \bmod 4\sqrt{3}.$$

The corresponding theta series satisfy

$$\Theta_1\left(12, \xi, \tfrac{z}{12}\right) = \Theta_1\left(-4, \chi_\nu, \tfrac{z}{12}\right) = \Theta_1\left(-3, \psi_\nu, \tfrac{z}{12}\right) = \frac{\eta^6(2z)}{\eta^2(z)\eta^2(4z)}, \tag{13.1}$$

$$\Theta_5\left(-4, \chi_\nu, \tfrac{z}{12}\right) = E_4\left(z + \tfrac{1}{2}\right)\frac{\eta^6(2z)}{\eta^2(z)\eta^2(4z)} - 48\nu\left(\frac{\eta^6(2z)}{\eta^2(z)\eta^2(4z)}\right)^5, \tag{13.2}$$

$$\Theta_7\left(-3, \psi_\nu, \tfrac{z}{12}\right) = E_6\left(z + \tfrac{1}{2}\right)\frac{\eta^6(2z)}{\eta^2(z)\eta^2(4z)} - 360\nu i\sqrt{3}\left(\frac{\eta^6(2z)}{\eta^2(z)\eta^2(4z)}\right)^7. \tag{13.3}$$

The characters χ_ν, ψ_ν and the eta product $\left[1^{-2}, 2^6, 4^{-2}\right]$ will reappear in identities in Example 15.3.

Now we deal with the sign transforms of the eta products in Sect. 10.1. We obtain theta identities involving the same fields as before in that section.

Example 13.3 *The residues of* $2+i$, 3 *and* i *modulo* 8 *can be chosen as generators of the group* $(\mathcal{O}_1/(8))^\times \simeq Z_4 \times Z_2 \times Z_4$. *A pair of characters* χ_ν^* *on* $\mathcal{O}_1$ *with period* 8 *is fixed by the values*

$$\chi_\nu^*(2+i) = \nu i, \qquad \chi_\nu^*(3) = 1, \qquad \chi_\nu^*(i) = 1$$

with $\nu \in \{1, -1\}$. *The residues of* $1+\sqrt{-2}$, 3 *and* -1 *modulo* $4\sqrt{-2}$ *can be chosen as generators of* $(\mathcal{O}_2/(4\sqrt{-2}))^\times \simeq Z_4 \times Z_2^2$. *Characters* ψ_ν^* *on* $\mathcal{O}_2$ *with period* $4\sqrt{-2}$ *are defined by*

$$\psi_\nu^*(1+\sqrt{-2}) = \nu, \qquad \psi_\nu^*(3) = -1, \qquad \psi_\nu^*(-1) = 1.$$

The residues of $1+\sqrt{2}$, 3 *and* -1 *modulo* $4\sqrt{2}$ *generate the group* $(\mathbb{Z}[\sqrt{2}]/(4\sqrt{2}))^\times \simeq Z_4 \times Z_2^2$. *A Hecke character* ξ^* *on* $\mathbb{Z}[\sqrt{2}]$ *modulo* $4\sqrt{2}$ *is given by*

$$\xi^*(\mu) = \begin{cases} \mathrm{sgn}(\mu) \\ -\mathrm{sgn}(\mu) \end{cases} \quad \textit{for} \quad \mu \equiv \begin{cases} 1+\sqrt{2},\ 3 \\ -1 \end{cases} \bmod 4\sqrt{2}\,.$$

The corresponding theta series satisfy

$$\Theta_1\left(8,\,\xi^*,\,\tfrac{z}{8}\right) \;=\; \Theta_1\left(-4,\chi_\nu^*,\,\tfrac{z}{8}\right) \;=\; \Theta_1\left(-8,\psi_\nu^*,\,\tfrac{z}{8}\right) \;=\; \frac{\eta^4(2z)}{\eta(z)\eta(4z)}, \quad (13.4)$$

$$\Theta_5\left(-4,\chi_\nu^*,\,\tfrac{z}{8}\right) \;=\; E_{4,2,-1}\left(z+\tfrac{1}{2}\right)\frac{\eta^4(2z)}{\eta(z)\eta(4z)} \;-\; 48\nu\left(\frac{\eta^4(2z)}{\eta(z)\eta(4z)}\right)^5, \quad (13.5)$$

$$\Theta_3\left(-8,\psi_\nu^*,\,\tfrac{z}{8}\right) \;=\; E_{2,2,-1}\left(z+\tfrac{1}{2}\right)\frac{\eta^4(2z)}{\eta(z)\eta(4z)} \;+\; 4\nu i\sqrt{2}\left(\frac{\eta^4(2z)}{\eta(z)\eta(4z)}\right)^3, \quad (13.6)$$

$$\Theta_5\left(-8,\psi_\nu^*,\,\tfrac{z}{8}\right) \;=\; E_{4,2,1}\left(z+\tfrac{1}{2}\right)\frac{\eta^4(2z)}{\eta(z)\eta(4z)} - 8\nu i\sqrt{2}\,E_{2,2,-1}\left(z+\tfrac{1}{2}\right)\left(\frac{\eta^4(2z)}{\eta(z)\eta(4z)}\right)^3. \quad (13.7)$$

Remark. The stars in the character symbols have been introduced to avoid a clash of notation in Example 15.30 where these characters will occur together with some other characters.

Example 13.4 *The residues of* $2+i$*,* $1+6i$*,* 5 *and* i *modulo* 24 *can be chosen as generators of the group* $(\mathcal{O}_1/(24))^\times \simeq Z_8\times Z_4\times Z_2\times Z_4$*. Four characters* $\chi_{\delta,\nu}$ *on* $\mathcal{O}_1$ *with period* 24 *are fixed by their values*

$$\chi_{\delta,\nu}(2+i)=\delta, \qquad \chi_{\delta,\nu}(1+6i)=\nu i, \qquad \chi_{\delta,\nu}(5)=1, \qquad \chi_{\delta,\nu}(i)=1$$

with $\delta,\nu\in\{1,-1\}$*. The residues of* $\sqrt{3}+\sqrt{-2}$*,* $1+\sqrt{-6}$*,* 7 *and* -1 *modulo* $4\sqrt{-6}$ *can be chosen as generators of* $(\mathcal{J}_6/(4\sqrt{-6}))^\times \simeq Z_4^2\times Z_2^2$*. Four characters* $\varphi_{\delta,\nu}$ *on* $\mathcal{J}_6$ *with period* $4\sqrt{-6}$ *are defined by*

$$\varphi_{\delta,\nu}(\sqrt{3}+\sqrt{-2})=\delta, \qquad \varphi_{\delta,\nu}(1+\sqrt{-6})=\nu,$$
$$\varphi_{\delta,\nu}(7)=-1, \qquad \varphi_{\delta,\nu}(-1)=1.$$

The residues of $1+\sqrt{6}$*,* 5*,* 7 *and* -1 *modulo* $4\sqrt{6}$ *can be chosen as generators of the group* $(\mathbb{Z}[\sqrt{6}]/(4\sqrt{6}))^\times \simeq Z_4\times Z_2^3$*. Hecke characters* ξ_δ *on* $\mathbb{Z}[\sqrt{6}]$ *modulo* $4\sqrt{6}$ *are given by*

$$\xi_\delta(\mu)=\begin{cases}\delta\,\mathrm{sgn}(\mu)\\ \mathrm{sgn}(\mu)\\ -\mathrm{sgn}(\mu)\end{cases} \quad \text{for} \quad \mu\equiv\begin{cases}1+\sqrt{6}\\ 5,\,7\\ -1\end{cases} \mod 4\sqrt{6}\,.$$

The corresponding theta series of weight 1 *satisfy the identities*

$$\Theta_1\left(24,\,\xi_\delta,\,\tfrac{z}{24}\right) \;=\; \Theta_1\left(-4,\chi_{\delta,\nu},\,\tfrac{z}{24}\right) \;=\; \Theta_1\left(-24,\varphi_{\delta,\nu},\,\tfrac{z}{24}\right) \;=\; f_1(z) \;+\; 2\delta f_5(z) \quad (13.8)$$

with normalized integral Fourier series f_j with denominator 24 *and numerator classes j modulo* 24*. Both the components are eta products,*

$$f_1(z) = \frac{\eta^8(2z)}{\eta^3(z)\eta^3(4z)}, \qquad f_5(z) = \eta(z)\eta(4z). \tag{13.9}$$

Another identification with eta products for these theta series will be presented in Example 15.23, a third one in Example 24.17, and another one for f_1 in Example 25.24.

For weights 3 and 5 we have decompositions

$$\begin{aligned}
\Theta_5\left(-4, \chi_{\delta,\nu}, \tfrac{z}{24}\right) &= f_{5,1}(z) - 14\delta\, f_{5,5}(z) + 240\nu\, f_{5,13}(z) \\
&\quad + 480\delta\nu\, f_{5,17}(z), \\
\Theta_3\left(-24, \varphi_{\delta,\nu}, \tfrac{z}{24}\right) &= g_{3,1}(z) + 2\delta\, g_{3,5}(z) + 4\nu i\sqrt{6}\, g_{3,7}(z) \\
&\quad + 8\delta\nu i\sqrt{6}\, g_{3,11}(z), \\
\Theta_5\left(-24, \varphi_{\delta,\nu}, \tfrac{z}{24}\right) &= g_{5,1}(z) - 46\delta\, g_{5,5}(z) - 40\nu i\sqrt{6}\, g_{5,7}(z) \\
&\quad - 80\delta\nu i\sqrt{6}\, g_{5,11}(z)
\end{aligned}$$

where $f_{5,j}(z) = \sum_{n\equiv j \bmod 24} a_{5,j}(n)e\left(\frac{nz}{24}\right)$ and $g_{k,j}(z) = \sum_{n\equiv j \bmod 24} b_{k,j}(n)\times e\left(\frac{nz}{24}\right)$ are normalized integral or rational Fourier series and where expressions for the components $f_{5,j}$ and $g_{k,j}$ in terms of eta products are obtained by taking the sign transforms of corresponding components in Examples 10.20 and 10.24.

Closing this subsection, we state analogues for the identities in Example 10.6 for the non-cuspidal eta products of weight 1 for $\Gamma^*(4)$:

Example 13.5 *The non-cuspidal eta products of weight* 1 *for* $\Gamma^*(4)$ *are*

$$\frac{\eta^{10}(2z)}{\eta^4(z)\eta^4(4z)} = 1 + 4\sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{-1}{d}\right)\right)e(nz) = 4\,\Theta_1(-4,1,z), \tag{13.10}$$

$$\frac{\eta^2(z)\eta^2(4z)}{\eta^2(2z)} = \sum_{n=1}^{\infty}\left(\frac{2}{n}\right)\left(\sum_{d|n}\left(\frac{-1}{d}\right)\right)e\left(\frac{nz}{4}\right) = \Theta_1\left(-4,\chi,\tfrac{z}{4}\right), \tag{13.11}$$

where 1 *stands for the trivial character on $\mathcal{O}_1$ and χ denotes the character modulo* 4 *on $\mathcal{O}_1$ which is given by $\chi(\mu) = (-1)^{\frac{1}{2}xy} = \left(\frac{2}{\mu\overline{\mu}}\right)$ for $\mu = x + yi \in \mathcal{O}_1$, $x \not\equiv y \bmod 2$.*

The character χ and another identity for $\Theta_1\left(-4,\chi,\frac{z}{4}\right)$ will appear in Example 24.26. Another identity for $\Theta_1(-4,1,z)$ will be given in Example 24.31.

13.2 Even Weights for the Fricke Group $\Gamma^*(4)$

There are 12 new holomorphic eta products of weight 2 for the Fricke group $\Gamma^*(4)$. Among them, 10 are cuspidal and 2 are non-cuspidal. All of them are sign transforms of eta products of levels 1 or 2. We begin with a description of the sign transforms of $\eta^2(z)\eta^2(2z)$ and $\eta^4(z)$, and then we look for the transforms of the functions in Sect. 10.4.

Example 13.6 *The residues of* $1+2i$ *and* i *modulo* 4 *generate the group* $(\mathcal{O}_1/(4))^\times \simeq Z_2 \times Z_4$. *A character* χ *on* $\mathcal{O}_1$ *with period* 4 *is fixed by the values* $\chi(1+2i)=1$, $\chi(i)=-i$, *and explicitly given by* $\chi(x+yi)=\left(\frac{-1}{x}\right)$ *if* y *is even,* $\chi(x+yi)=-i\left(\frac{-1}{y}\right)$ *if* x *is even. The corresponding theta series of weight* 2 *satisfies*

$$\Theta_2\left(-4,\chi,\tfrac{z}{4}\right) = \frac{\eta^8(2z)}{\eta^2(z)\eta^2(4z)}. \tag{13.12}$$

Another eta identity for this theta series will be presented in Example 15.21. Identities for weights 6, 10 and 14 can be derived from corresponding identities in Example 10.7 by taking sign transforms. In this process we should be aware that possibly the numerical factors in front of the terms get twisted.— Primes $p\equiv 1 \bmod 4$ can be written as $p=x^2+y^2$ with x odd, and then the coefficient of $\eta^8(2z)/(\eta^2(z)\eta^2(4z))$ at p is given by $\left(\frac{-1}{x}\right)\cdot 2x$.

Example 13.7 *Let the generators of* $(\mathcal{O}_3/(4+4\omega))^\times \simeq Z_2^2\times Z_6$ *be chosen as in Example* 9.1, *and define a character* ψ *on* $\mathcal{O}_3$ *with period* $4(1+\omega)$ *by its values*

$$\psi(1+2\omega)=1, \qquad \psi(1-4\omega)=-1, \qquad \psi(\omega)=\overline{\omega}.$$

The corresponding theta series of weight 2 *satisfies*

$$\Theta_2\left(-3,\psi,\tfrac{z}{6}\right) = \frac{\eta^{12}(2z)}{\eta^4(z)\eta^4(4z)}. \tag{13.13}$$

There is a linear relation among eta products,

$$2\,\frac{\eta^{12}(2z)}{\eta^4(z)\eta^4(4z)} = \frac{\eta^{10}(z)}{\eta^4(\frac{z}{2})\eta^2(2z)} + \frac{\eta^4(\frac{z}{2})\eta^2(2z)}{\eta^2(z)}. \tag{13.14}$$

We will meet the character ψ and the eta product $\left[1^{-4},2^{12},4^{-4}\right]$ again in Examples 13.15, 13.24.

As before, identities for weights 8, 14 and 20 are obtained from identities in Example 9.3 by taking sign transforms. The linear relation (13.14) might look spectacular, but it can be proved by elementary arguments: We divide it by $\eta^2(2z)$ and obtain the equivalent version

$$2\left(\frac{\eta^5(2z)}{\eta^2(z)\eta^2(4z)}\right)^2 = \left(\frac{\eta^5(z)}{\eta^2(\frac{z}{2})\eta^2(2z)}\right)^2 + \left(\frac{\eta^2(\frac{z}{2})}{\eta(z)}\right)^2, \tag{13.15}$$

that is,

$$2\,\theta^2(2z) = \theta^2(z) + \theta^2(z+1).$$

This is well known ([24], p. 104, entry (26), or [36], p. 266) and shown as follows: Using (8.7) and (8.8) in Theorem 8.1, the identity (13.15) is equivalent to the relations

$$2\sum_{2(x^2+y^2)=n} 1 \;=\; \sum_{x^2+y^2=n} \left(1+(-1)^{x+y}\right)$$

for all $n \geq 0$, where in both sums the summation is on all $x, y \in \mathbb{Z}$ satisfying the indicated equation. Here, obviously, both sides are 0 if n is odd. If $n = x^2 + y^2$ is even then $(-1)^{x+y} = 1$, and $(x+yi)/(1+i) = x' + y'i$ induces a bijection of the terms on the right hand side to those on the left hand side.

Let $c(n)$ denote the coefficients in (13.13),

$$\frac{\eta^{12}(2z)}{\eta^4(z)\eta^4(4z)} = \sum_{n\equiv 1 \bmod 6} c(n) e\left(\frac{nz}{6}\right).$$

We can write this eta product as a product of two simple theta series in two different ways, $\left[1^{-4}, 2^{12}, 4^{-4}\right] = \left[1^{-3}, 2^9, 4^{-3}\right]\left[1^{-1}, 2^3, 4^{-1}\right] = \left[1^{-5}, 2^{13}, 4^{-5}\right]\left[1, 2^{-1}, 4\right]$. Now when we use (8.4), (8.6), (8.16), (8.20), we get the identities

$$c(n) = \sum_{x,y>0,\, x^2+3y^2=4n} \left(\frac{6}{x}\right)\left(\frac{-2}{y}\right) y \;=\; \sum_{x,y>0,\, x^2+3y^2=4n} \left(\frac{-6}{x}\right)\left(\frac{2}{y}\right) x.$$

Now we treat the sign transforms of the eta products in Sect. 10.4.

Example 13.8 *Let the generators of $(\mathcal{O}_2/(4\sqrt{-2}))^\times \simeq Z_4 \times Z_2^2$ be chosen as in Example 13.3, and define a pair of characters ψ_δ on $\mathcal{O}_2$ with period $4\sqrt{-2}$ by*

$$\psi_\delta(1+\sqrt{-2}) = -\delta i, \qquad \psi_\delta(3) = 1, \qquad \psi_\delta(-1) = -1$$

with $\delta \in \{1, -1\}$. The corresponding theta series of weight 2 satisfy

$$\Theta_2\left(-8, \psi_\delta, \tfrac{z}{8}\right) \;=\; f_1(z) + 2\sqrt{2}\delta\, f_3(z) \tag{13.16}$$

with normalized integral Fourier series f_j with denominator 8 and numerator classes j modulo 8. Both the components are eta products,

$$f_1(z) \;=\; \frac{\eta^{14}(2z)}{\eta^5(z)\eta^5(4z)}, \qquad f_3(z) \;=\; \eta(z)\eta^2(2z)\eta(4z). \tag{13.17}$$

Example 13.9 *Let the generators of* $(\mathcal{O}_1/(12))^\times \simeq Z_8 \times Z_2 \times Z_4$ *be chosen as in Example 13.2, and define a pair of characters* χ_δ *on* $\mathcal{O}_1$ *with period* 12 *by its values*

$$\chi_\delta(2+i) = \delta, \qquad \chi_\delta(1+6i) = 1, \qquad \chi_\delta(i) = -i$$

with $\delta \in \{1,-1\}$. *The corresponding theta series of weight* 2 *satisfy*

$$\Theta_2\left(-4, \chi_\delta, \tfrac{z}{12}\right) \;=\; f_1(z) + 4\delta\, f_5(z) \tag{13.18}$$

with normalized integral Fourier series f_j *with denominator* 12 *and numerator classes* j *modulo* 12. *Both the components are eta products,*

$$f_1(z) \;=\; \frac{\eta^{16}(2z)}{\eta^6(z)\eta^6(4z)}, \qquad f_5(z) \;=\; \eta^2(z)\eta^2(4z). \tag{13.19}$$

Example 13.10 *Let the generators of* $(\mathcal{J}_6/(4\sqrt{-6}))^\times \simeq Z_4^2 \times Z_2^2$ *be chosen as in Example 13.4, and define a quadruplet of characters* $\varphi_{\delta,\varepsilon}$ *on* $\mathcal{J}_6$ *with period* $4\sqrt{-6}$ *by*

$$\varphi_{\delta,\varepsilon}(\sqrt{3}+\sqrt{-2}) = \delta, \qquad \varphi_{\delta,\varepsilon}(1+\sqrt{-6}) = -\varepsilon i,$$
$$\varphi_{\delta,\varepsilon}(7) = 1, \qquad \varphi_{\delta,\varepsilon}(-1) = -1$$

with $\delta, \varepsilon \in \{1,-1\}$. *The corresponding theta series of weight* 2 *satisfy*

$$\Theta_2\left(-24, \varphi_{\delta,\varepsilon}, \tfrac{z}{24}\right) \;=\; f_1(z)+2\sqrt{3}\delta\, f_5(z)+2\sqrt{6}\varepsilon\, f_7(z)-4\sqrt{2}\delta\varepsilon\, f_{11}(z) \tag{13.20}$$

with normalized integral Fourier series f_j *with denominator* 24 *and numerator classes* j *modulo* 24. *All the components are eta products,*

$$f_1 = \left[\frac{2^{18}}{1^7, 4^7}\right], \qquad f_5 = \left[\frac{2^{10}}{1^3, 4^3}\right], \qquad f_7 = \left[\frac{2^6}{1, 4}\right], \qquad f_{11} = \left[\frac{1^3, 4^3}{2^2}\right]. \tag{13.21}$$

Identities for the non-cuspidal eta products of weight 2 for $\Gamma^*(4)$, $\left[1^{-8}, 2^{20}, 4^{-8}\right]$ and $\left[1^4, 2^{-4}, 4^4\right]$, have already been stated in Sect. 10.4.

13.3 Weight 1 for $\Gamma_0(4)$

In Table 13.1 we list the numbers of new holomorphic eta products of weights 1 and 2 for $\Gamma_0(4)$ which do not belong to $\Gamma^*(4)$, specified according to their denominator t and according to their property of being cuspidal or non-cuspidal.

In the present subsection we will identify the cuspidal eta products of weight 1 with (components of) Hecke theta series both on real and on imaginary quadratic fields, and we will present some identities for the non-cuspidal eta products of weight 1.

Table 13.1: Numbers of new eta products of level 4 with weights 1 and 2

denominator t	1	2	3	4	6	8	12	24
$k=1$, cuspidal	0	0	0	0	2	0	0	4
$k=1$, non-cuspidal	2	0	0	0	0	4	0	0
$k=2$, cuspidal	0	2	6	2	4	8	8	24
$k=2$, non-cuspidal	6	0	0	4	0	8	0	0

Example 13.11 *Let the generators of* $(\mathcal{O}_3/(8+8\omega))^\times \simeq Z_4 \times Z_2^2 \times Z_6$ *be chosen as in Example 13.2, and define four characters* $\psi_{\delta,\nu}$ *on* $\mathcal{O}_3$ *with period* $8(1+\omega)$ *by their values*

$$\psi_{\delta,\nu}(1+2\omega)=\delta i, \qquad \psi_{\delta,\nu}(1-4\omega)=\delta\nu, \qquad \psi_{\delta,\nu}(5)=1, \qquad \psi_{\delta,\nu}(\omega)=1$$

with $\delta,\nu\in\{1,-1\}$. *Let the generators of* $(\mathcal{J}_6/(4\sqrt{-6}))^\times \simeq Z_4^2\times Z_2^2$ *be chosen as in Example 13.4, and fix a quadruplet of characters* $\varphi_{\delta,\nu}$ *on* $\mathcal{J}_6$ *with period* $4\sqrt{-6}$ *by*

$$\varphi_{\delta,\nu}(\sqrt{3}+\sqrt{-2})=\nu, \qquad \varphi_{\delta,\nu}(1+\sqrt{-6})=\delta i,$$
$$\varphi_{\delta,\nu}(7)=-1, \qquad \varphi_{\delta,\nu}(-1)=1.$$

The residues of $1+\sqrt{2}$ *and* $3+\sqrt{2}$ *modulo* $6\sqrt{2}$ *generate the group* $(\mathbb{Z}[\sqrt{2}]/(6\sqrt{2}))^\times \simeq Z_8\times Z_4$, *where* $(3+\sqrt{2})^2\equiv -1 \bmod 6\sqrt{2}$. *Hecke characters* ξ_δ *on* $\mathbb{Z}[\sqrt{2}]$ *modulo* $6\sqrt{2}$ *are given by*

$$\xi_\delta(\mu)=\begin{cases}\operatorname{sgn}(\mu)\\ \delta i\operatorname{sgn}(\mu)\end{cases} \qquad \text{for} \qquad \mu\equiv\begin{cases}1+\sqrt{2}\\ 3+\sqrt{2}\end{cases} \bmod 6\sqrt{2}\,.$$

The corresponding theta series of weight 1 *are identical and decompose as*

$$\Theta_1\left(8,\xi_\delta,\tfrac{z}{6}\right) = \Theta_1\left(-3,\psi_{\delta,\nu},\tfrac{z}{6}\right) = \Theta_1\left(-24,\varphi_{\delta,\nu},\tfrac{z}{6}\right) = f_1(z)+2\delta i\, f_7(z) \tag{13.22}$$

where the components f_j *are normalized integral Fourier series with denominator* 6 *and numerator classes* j *modulo* 24. *They are linear combinations of two eta products which are sign transforms of each other,*

$$f_1=\tfrac{1}{2}\left(\left[1^{-2},2^5,4^{-1}\right]+\left[1^2,2^{-1},4\right]\right), \qquad f_7=\tfrac{1}{4}\left(\left[1^{-2},2^5,4^{-1}\right]-\left[1^2,2^{-1},4\right]\right). \tag{13.23}$$

The action of the Fricke involution W_4 *on* $F_\delta=f_1+2\delta i f_7$ *is given by*

$$F_\delta(W_4z)=\tfrac{\delta-i}{\sqrt{2}}\,z\left(\left[1^{-1},2^5,4^{-2}\right]-2\delta i\left[1,2^{-1},4^2\right]\right). \tag{13.24}$$

Formula (13.24) shows that $\left[1^{-1},2^5,4^{-2}\right]-2\delta i\left[1,2^{-1},4^2\right]$ is a pair of Hecke eigenforms. Taking the sign transforms gives another such pair, $\left[1,2^2,4^{-1}\right]-2\delta i\left[1^{-1},2^2,4\right]$. Their representations by Hecke theta series are given in the following example. We need characters with period $16(1+\omega)$ on $\mathcal{O}_3$.

Example 13.12 *Let the characters ξ_δ on $\mathbb{Z}[\sqrt{6}]$, $\psi_{\delta,\nu}$ on $\mathcal{O}_3$ and $\varphi_{\delta,\nu}$ on $\mathcal{J}_6$ be defined as in Example 13.11. The residues of $1+2\omega$, $1-4\omega$, 7 and ω modulo $16(1+\omega)$ can be chosen as generators of the group $(\mathcal{O}_3/(16+16\omega))^\times \simeq Z_8 \times Z_4 \times Z_2 \times Z_6$. Define characters $\widetilde{\psi}_{\delta,\nu}$ on $\mathcal{O}_3$ with period $16(1+\omega)$ by their values*

$$\widetilde{\psi}_{\delta,\nu}(1+2\omega) = \delta i, \qquad \widetilde{\psi}_{\delta,\nu}(1-4\omega) = \nu i, \qquad \widetilde{\psi}_{\delta,\nu}(7) = -1, \qquad \widetilde{\psi}_{\delta,\nu}(\omega) = 1$$

with $\delta, \nu \in \{1, -1\}$. Let the generators of $(\mathcal{J}_6/(4\sqrt{-6}))^\times \simeq Z_4^2 \times Z_2^2$ be chosen as in Example 13.4, and define characters $\rho_{\delta,\nu}$ on $\mathcal{J}_6$ with period $4\sqrt{-6}$ by

$$\rho_{\delta,\nu}(\sqrt{3}+\sqrt{-2}) = \nu i, \qquad \rho_{\delta,\nu}(1+\sqrt{-6}) = \delta i,$$
$$\rho_{\delta,\nu}(7) = -1, \qquad \rho_{\delta,\nu}(-1) = 1.$$

The residues of $1+\sqrt{2}$, $3+\sqrt{2}$, 5 and -1 modulo $12\sqrt{2}$ can be chosen as generators of $(\mathbb{Z}[\sqrt{2}]/(12\sqrt{2}))^\times \simeq Z_8 \times Z_4 \times Z_2^2$. Hecke characters $\widetilde{\xi}_\delta$ on $\mathbb{Z}[\sqrt{2}]$ modulo $12\sqrt{2}$ are given by

$$\widetilde{\xi}_\delta(\mu) = \begin{cases} \operatorname{sgn}(\mu) \\ \delta i \operatorname{sgn}(\mu) \\ -\operatorname{sgn}(\mu) \end{cases} \qquad \textit{for} \qquad \mu \equiv \begin{cases} 1+\sqrt{2} \\ 3+\sqrt{2} \\ -1 \end{cases} \mod 12\sqrt{2}\,.$$

The theta series of weight 1 for ξ_δ, $\psi_{\delta,\nu}$ and $\varphi_{\delta,\nu}$ are identical, and those for $\widetilde{\xi}_\delta$, $\widetilde{\psi}_{\delta,\nu}$ and $\rho_{\delta,\nu}$ are identical, and they decompose as

$$\begin{aligned} \Theta_1\left(8, \xi_\delta, \tfrac{z}{24}\right) &= \Theta_1\left(-3, \psi_{\delta,\nu}, \tfrac{z}{24}\right) = \Theta_1\left(-24, \varphi_{\delta,\nu}, \tfrac{z}{24}\right) \\ &= g_1(z) + 2\delta i\, g_7(z), \qquad (13.25) \\ \Theta_1\big(8, \widetilde{\xi}_\delta, \tfrac{z}{24}\big) &= \Theta_1\big(-3, \widetilde{\psi}_{\delta,\nu}, \tfrac{z}{24}\big) = \Theta_1\left(-24, \rho_{\delta,\nu}, \tfrac{z}{24}\right) \\ &= h_1(z) + 2\delta i\, h_7(z), \qquad (13.26) \end{aligned}$$

where the components h_j are normalized integral Fourier series with denominator 24 and numerator classes j modulo 24, and where $g_j(4z) = f_j(z)$ with $f_j(z)$ as declared in Example 13.11. All the components are eta products, and (g_1, h_1) and (g_7, h_7) are pairs of sign transforms. We have

$$\begin{aligned} g_1 &= \left[1^{-1}, 2^5, 4^{-2}\right], \quad h_1 = \left[1, 2^2, 4^{-1}\right], \\ g_7 &= \left[1, 2^{-1}, 4^2\right], \quad h_7 = \left[1^{-1}, 2^2, 4\right], \end{aligned} \qquad (13.27)$$

and

$$\begin{aligned} &\left[4^{-1}, 8^5, 16^{-2}\right] + 2\delta i \left[4, 8^{-1}, 16^2\right] \\ &\quad = \tfrac{1}{2}(1+\delta i)\left[1^{-2}, 2^5, 4^{-1}\right] + \tfrac{1}{2}(1-\delta i)\left[1, 2^{-1}, 4^2\right]. \qquad (13.28) \end{aligned}$$

Other versions of the identities for g_1, h_1 will be given in Example 19.3. The characters $\widetilde{\psi}_{\delta,\nu}$ and $\rho_{\delta,\nu}$ will appear in another identity in Example 15.23.

An equivalent version for (13.28) is

$$\begin{aligned}\left[4^{-1},8^5,16^{-2}\right] &= \tfrac{1}{2}\left(\left[1^{-2},2^5,4^{-1}\right]+\left[1,2^{-1},4^2\right]\right),\\ \left[4,8^{-1},16^{2}\right] &= \tfrac{1}{4}\left(\left[1^{-2},2^5,4^{-1}\right]-\left[1,2^{-1},4^2\right]\right).\end{aligned}$$

For the six non-cuspidal eta products of weight 1 we introduce the notation

$$\begin{aligned} F &= \left[1^{-2},2^7,4^{-3}\right], & \widetilde{F} &= \left[1^2,2,4^{-1}\right],\\ f_1 &= \left[1^3,2^{-2},4\right], & f_3 &= \left[1,2^{-2},4^3\right],\\ g_1 &= \left[1^{-3},2^7,4^{-2}\right], & g_3 &= \left[1^{-1},2,4^2\right].\end{aligned}$$

We observe that F and $\widetilde{F}$ have denominator 1 and numerator 0, while f_j, g_j have denominator 8 and numerator j. We have three pairs $(F,\widetilde{F})$, (f_1,g_1), (f_3,g_3) of sign transforms. The Fricke involution W_4 interchanges f_1 and f_3, and it transforms F and $\widetilde{F}$ into g_1 and g_3, respectively. Now we present six linear combinations which are Eisenstein series or Hecke theta series. They are non-cuspidal Hecke eigenforms (according to Theorem 5.1) since all the characters are induced by the norm.

Example 13.13 *For $\delta \in \{1,-1\}$, let ψ_δ be the character on $\mathcal{O}_2$ with period $2\sqrt{-2}$ which is given by*

$$\psi_\delta(\mu) = \begin{cases} (-1)^{(\mu\overline{\mu}-1)/8} \\ (\delta i)^{(\mu\overline{\mu}-3)/8} \end{cases} \quad \text{if} \quad \mu\overline{\mu} \equiv \begin{cases} 1 \bmod 8, \\ 3 \bmod 8. \end{cases}$$

Define the characters $\widetilde{\psi}_\delta$ on $\mathcal{O}_2$ by

$$\widetilde{\psi}_\delta(\mu) = \delta^{(\mu\overline{\mu}-1)/2}$$

for $2 \nmid \mu\overline{\mu}$, such that $\widetilde{\psi}_1$ is the principal character modulo $\sqrt{-2}$ and $\widetilde{\psi}_{-1}$ is the non-principal character modulo 2. Then with notations from above we have the identities

$$\tfrac{1}{4}\left(F\left(\tfrac{z}{2}\right)+\widetilde{F}\left(\tfrac{z}{2}\right)\right) = \frac{1}{2}-\sum_{n=1}^{\infty}\left((-1)^{n-1}\sum_{d|n}\left(\frac{-2}{d}\right)\right)e(nz), \tag{13.29}$$

$$\tfrac{1}{4}\left(F(z)-\widetilde{F}(z)\right) = \sum_{n=1}^{\infty}\left(\left(\frac{-1}{n}\right)\sum_{d|n}\left(\frac{-2}{d}\right)\right)e(nz), \tag{13.30}$$

$$\Theta_1\left(-8,\psi_\delta,\tfrac{z}{8}\right) = f_1(z)+2\delta i\, f_3(z), \tag{13.31}$$

$$\Theta_1\left(-8,\widetilde{\psi}_\delta,\tfrac{z}{8}\right) = g_1(z)+2\delta\, g_3(z). \tag{13.32}$$

The characters $\widetilde{\psi}_\delta$ will reappear in Examples 15.2, 15.9, 15.30, 19.8, 26.9.

13.4 Weight 2 for $\Gamma_0(4)$, Cusp Forms with Denominators $t \le 6$

The cuspidal eta products of weight 2 and denominator 2 form a pair of sign transforms $[1^{-2}, 2^5, 4]$, $[1^2, 2^{-1}, 4^3]$. The Fricke involution W_4 transforms them into eta products with denominator 8. We get the identities (13.33), (13.34), (13.53), (13.54), similar to those in (13.22), (13.23), (13.28).

Example 13.14 *The group $(\mathcal{O}_2/(2\sqrt{-2}))^\times \simeq Z_4$ is generated by the residue of $1+\sqrt{-2}$ modulo $2\sqrt{-2}$. A pair of characters χ_δ on $\mathcal{O}_2$ with period $2\sqrt{-2}$ is fixed by the value $\chi_\delta(1+\sqrt{-2}) = \delta i$ and explicitly given by*

$$\chi_\delta(x+y\sqrt{-2}) = \begin{cases} \left(\frac{-1}{x}\right) \\ \left(\frac{-1}{x}\right)\delta i \end{cases} \qquad \text{if } y \text{ is} \begin{cases} \text{even,} \\ \text{odd,} \end{cases}$$

with $\delta \in \{1, -1\}$. The corresponding theta series of weight 2 decompose as

$$\Theta_2\left(-8, \chi_\delta, \tfrac{z}{2}\right) = f_1(z) + 2\delta i\, f_3(z) \tag{13.33}$$

with normalized integral Fourier series f_j with denominator 8 and numerator classes j modulo 8. The components are linear combinations of two eta products which are sign transforms of each other,

$$\begin{aligned} f_1 &= \tfrac{1}{2}\left([1^{-2}, 2^5, 4] + [1^2, 2^{-1}, 4^3]\right), \\ f_3 &= \tfrac{1}{4}\left([1^{-2}, 2^5, 4] - [1^2, 2^{-1}, 4^3]\right). \end{aligned} \tag{13.34}$$

The action of the Fricke involution W_4 on $F_\delta = f_1 + 2\delta i f_3$ is given by

$$F_\delta(W_4 z) = -\tfrac{1+\delta i}{\sqrt{2}}\, z^2\left([1, 2^5, 4^{-2}] - 2\delta i\,[1^3, 2^{-1}, 4^2]\right). \tag{13.35}$$

The cuspidal eta products of weight 2 and denominator 3 form three pairs of sign transforms $[1^{-4}, 2^{10}, 4^{-2}]$, $[1^4, 2^{-2}, 4^2]$; $[1^{-2}, 2^7, 4^{-1}]$, $[1^2, 2, 4]$; $[1^{-2}, 2^3, 4^3]$, $[1^2, 2^{-3}, 4^5]$. The Fricke involution W_4 transforms the first pair into eta products with denominator 12, the other two pairs into eta products with denominator 24.

Example 13.15 *Let the generators of $(\mathcal{O}_3/(4+4\omega))^\times \simeq Z_2^2 \times Z_6$ be chosen as in Example 9.1, and define characters ψ_δ on $\mathcal{O}_3$ with period $4(1+\omega)$ by their values*

$$\psi_\delta(1+2\omega) = \delta, \qquad \psi_\delta(1-4\omega) = -1, \qquad \psi_\delta(\omega) = -\omega^2$$

with $\delta \in \{1, -1\}$. Put

$$F = [1^{-4}, 2^{10}, 4^{-2}], \qquad \widetilde{F} = [1^4, 2^{-2}, 4^2].$$

The corresponding theta series of weight 2 satisfy

$$\Theta_2\left(-3, \psi_1, \tfrac{z}{3}\right) = \tfrac{1}{2}\big(F(z) + \widetilde{F}(z)\big), \qquad (13.36)$$

$$\Theta_2\left(-3, \psi_{-1}, \tfrac{z}{3}\right) = \tfrac{1}{8}\big(F\left(\tfrac{z}{4}\right) - \widetilde{F}\left(\tfrac{z}{4}\right)\big). \qquad (13.37)$$

The action of W_4 on $\Psi_\delta(z) = \Theta_2\left(-3, \psi_\delta, \frac{z}{3}\right)$ is given by

$$\Psi_1(W_4 z) = -2\,z^2\big(\left[1^{-2}, 2^{10}, 4^{-4}\right] + 4\left[1^2, 2^{-2}, 4^4\right]\big), \qquad (13.38)$$

$$\Psi_{-1}(W_4 z) = -32\,z^2\big(\left[4^{-2}, 8^{10}, 16^{-4}\right] - 4\left[4^2, 8^{-2}, 16^4\right]\big). \qquad (13.39)$$

We will return to these identities in Sect. 13.5, Example 13.24 when we discuss eta products with denominator 12.

In the following example there are some subtleties in the identification of four theta series involving the remaining four eta products with denominator 3. Matter will become simpler when we study their Fricke transforms in Sect. 13.6.

Example 13.16 *The residues of $\sqrt{3}+\sqrt{-2}$, $1+\sqrt{-6}$ and -1 modulo $2\sqrt{3}$ generate the group $(\mathcal{J}_6/(2\sqrt{3}))^\times \simeq Z_2^3$. The residues of $\sqrt{3}+\sqrt{-2}$ and -1 modulo 3 generate $(\mathcal{J}_6/(3))^\times \simeq Z_6 \times Z_2$. Pairs of characters φ_δ on $\mathcal{J}_6$ with period $2\sqrt{3}$ and $\widetilde{\varphi}_\delta$ with period 3 are fixed by their values*

$$\varphi_\delta(\sqrt{3}+\sqrt{-2}) = \delta, \qquad \varphi_\delta(1+\sqrt{-6}) = -1, \qquad \varphi_\delta(-1) = -1,$$

$$\widetilde{\varphi}_\delta(\sqrt{3}+\sqrt{-2}) = \delta, \qquad \widetilde{\varphi}_\delta(-1) = -1$$

with $\delta \in \{1, -1\}$. Put

$$F_1 = \left[1^{-2}, 2^7, 4^{-1}\right], \qquad G_1 = \left[1^2, 2, 4\right],$$

$$F_2 = \left[1^{-2}, 2^3, 4^3\right], \qquad G_2 = \left[1^2, 2^{-3}, 4^5\right].$$

The corresponding theta series of weight 2 satisfy

$$\Theta_2\left(-24, \varphi_\delta, \tfrac{z}{3}\right) = \tfrac{1}{2}\left(F_1(z) + G_1(z)\right) + \tfrac{1}{\sqrt{2}}\delta i\big(F_2(z) - G_2(z)\big), \qquad (13.40)$$

$$\Theta_2\left(-24, \widetilde{\varphi}_\delta, \tfrac{z}{3}\right) = e\left(-\tfrac{1}{6}\right)\tfrac{1}{2}\left(F_2(w) + G_2(w)\right) + e\left(-\tfrac{1}{3}\right)\tfrac{1}{2\sqrt{2}}\delta i\left(F_1(w) - G_1(w)\right) \qquad (13.41)$$

where $w = \frac{1}{2}(z-1)$. The action of W_4 on $\Phi_\delta(z) = \Theta_2\left(-24, \varphi_\delta, \frac{z}{3}\right)$ is given by

$$\Phi_\delta(W_4 z) = -\tfrac{1}{2}\delta i\,z^2\left(\left[1^3, 2^3, 4^{-2}\right] - 2\left[1^5, 2^{-3}, 4^2\right] - 2\sqrt{2}\delta i\left[1^{-1}, 2^7, 4^{-2}\right] - 4\sqrt{2}\delta i\left[1, 2, 4^2\right]\right). \qquad (13.42)$$

We note that $\frac{1}{2}\left(F_2(\frac{z}{2}) + G_2(\frac{z}{2})\right)$ and $\frac{1}{4}\left(F_1(\frac{z}{2}) - G_1(\frac{z}{2})\right)$ are normalized integral Fourier series with denominators 3 and numerators 1 and 2, respectively, whose sign transforms are the components in (13.41). The eta products in (13.42) will be discussed in Example 13.26.

The cuspidal eta products with denominator 4 form a pair of sign transforms $[1^{-4}, 2^{11}, 4^{-3}]$, $[1^4, 2^{-1}, 4]$. We get a result similar to those in Examples 13.11 and 13.14.

Example 13.17 *Let the generators of* $(\mathcal{O}_1/(4+4i))^\times \simeq Z_2^2 \times Z_4$ *be chosen as in Example* 10.1*, and define a pair of characters* χ_δ *on* $\mathcal{O}_1$ *with period* $4(1+i)$ *by*

$$\chi_\delta(1+2i) = \delta, \qquad \chi_\delta(3) = 1, \qquad \chi_\delta(i) = -i$$

with $\delta \in \{1, -1\}$*. Put*

$$F = [1^{-4}, 2^{11}, 4^{-3}], \qquad \widetilde{F} = [1^4, 2^{-1}, 4].$$

The corresponding theta series of weight 2 *satisfy*

$$\Theta_2\left(-4, \chi_\delta, \tfrac{z}{4}\right) = f_1(z) + 4\delta i f_5(z) \tag{13.43}$$

with normalized integral Fourier series f_j *with denominator* 4 *and numerator classes* j *modulo* 8 *which are linear combinations of* F *and* $\widetilde{F}$*,*

$$f_1(z) = \tfrac{1}{2}\big(F(z) + \widetilde{F}(z)\big), \qquad f_5(z) = \tfrac{1}{8}\big(F(z) - \widetilde{F}(z)\big). \tag{13.44}$$

The action of W_4 *on* $H_\delta(z) = \Theta_2\left(-4, \chi_\delta, \frac{z}{4}\right)$ *is given by*

$$H_\delta(W_4 z) = -\sqrt{2}(1+\delta i)\, z^2\big([1^{-3}, 2^{11}, 4^{-4}] - 4\delta i[1, 2^{-1}, 4^4]\big). \tag{13.45}$$

The cuspidal eta products with denominator 6 form two pairs of sign transforms

$$\begin{aligned} F &= [1^{-6}, 2^{15}, 4^{-5}], & \widetilde{F} &= [1^6, 2^{-3}, 4], \\ G &= [1^{-2}, 2^9, 4^{-3}], & \widetilde{G} &= [1^2, 2^3, 4^{-1}], \end{aligned} \tag{13.46}$$

all of which have numerator 1. The Fricke involution W_4 transforms them into eta products with orders $\frac{1}{24}$, $\frac{19}{24}$, $\frac{7}{24}$, $\frac{13}{24}$, respectively, at the cusp ∞. Not surprisingly, the Fricke transforms will allow a simpler result (in Example 13.27) than the next one:

Example 13.18 *Let the generators of* $(\mathcal{O}_3/(8+8\omega))^\times \simeq Z_4 \times Z_2^2 \times Z_6$ *be chosen as in Example* 13.2*, and define four characters* $\psi_{\delta,\varepsilon}$ *on* $\mathcal{O}_3$ *with period* $8(1+\omega)$ *by their values*

$$\psi_{\delta,\varepsilon}(1+2\omega) = -\delta i, \quad \psi_{\delta,\varepsilon}(1-4\omega) = \delta\varepsilon, \quad \psi_{\delta,\varepsilon}(5) = 1, \quad \psi_{\delta,\varepsilon}(\omega) = -\omega^2$$

with $\delta, \varepsilon \in \{1, -1\}$. *The corresponding theta series of weight* 2 *satisfy*

$$\begin{aligned}\Theta_2\left(-3, \psi_{\delta,\varepsilon}, \tfrac{z}{6}\right) &= \tfrac{1}{2\sqrt{2}}\, \xi_{\delta,\varepsilon}\big(F(z) - \varepsilon i\, \widetilde{F}(z) + \delta\sqrt{3}\, G(z) + \delta\varepsilon i\sqrt{3}\, \widetilde{G}(z)\big) \\ &= f_1(z) + 2\sqrt{3}\delta f_7(z) - 4\sqrt{3}\delta\varepsilon i f_{13}(z) + 8\varepsilon i f_{19}(z)\end{aligned} \tag{13.47}$$

with primitive 24*th roots of unity*

$$\xi_{\delta,\varepsilon} = \tfrac{1}{2\sqrt{2}}\big((1 + \delta\sqrt{3}) + (1 - \delta\sqrt{3})\, \varepsilon i\big)$$

and with components f_j *which are normalized integral Fourier series with denominator* 6 *and numerator classes* j *modulo* 24, *and which are linear combinations of the eta products in* (13.46),

$$f_1 = \tfrac{1}{8}(F + \widetilde{F} + 3G + 3\widetilde{G}), \qquad f_7 = \tfrac{1}{16}(F - \widetilde{F} + G - \widetilde{G}), \tag{13.48}$$

$$f_{13} = \tfrac{1}{32}(F + \widetilde{F} - G - \widetilde{G}), \qquad f_{19} = \tfrac{1}{64}(F - \widetilde{F} - 3G + 3\widetilde{G}). \tag{13.49}$$

The action of W_4 *on* $F_{\delta,\varepsilon}(z) = \Theta_2\left(-3, \psi_{\delta,\varepsilon}, \frac{z}{6}\right)$ *is given by*

$$\begin{aligned}F_{\delta,\varepsilon}(W_4 z) &= -\,\xi_{\delta,\varepsilon}\, z^2\big(\left[1^{-5}, 2^{15}, 4^{-6}\right] + 2\sqrt{3}\delta\left[1^{-3}, 2^{9}, 4^{-2}\right] \\ &\quad + 4\sqrt{3}\delta\varepsilon i\left[1^{-1}, 2^{3}, 4^{2}\right] - 8\varepsilon i\left[1, 2^{-3}, 4^{6}\right]\big).\end{aligned} \tag{13.50}$$

We note some striking properties of the coefficients of F and G:

Corollary 13.19 *Let the expansions of the eta products in* (13.46) *be written as*

$$F(z) = \sum_{n\equiv 1 \bmod 6} a(n) e\left(\tfrac{nz}{6}\right), \qquad G(z) = \sum_{n\equiv 1 \bmod 6} b(n) e\left(\tfrac{nz}{6}\right).$$

Then the following assertions hold.

(1) *For all* $n \equiv 1 \bmod 6$ *we have*

$$a(n) = \sum_{x^2+3y^2=4n} \left(\frac{-6}{x}\right) x, \qquad b(n) = \sum_{x^2+3y^2=4n} \left(\frac{12}{x}\right)\left(\frac{-2}{y}\right) y,$$

with summation on all positive integers x, y *satisfying the indicated equation.*

(2) *We have*

$$\begin{aligned}a(n) &= \left(\tfrac{-1}{n}\right) b(n) &&\textit{for } n \equiv 1, 19 \bmod 24, \\ a(n) &= -3\left(\tfrac{-1}{n}\right) b(n) &&\textit{for } n \equiv 7, 13 \bmod 24.\end{aligned}$$

(3) *Let* $p \equiv 1 \bmod 6$ *be prime and write* $p = u^2 + 3v^2$ *with unique positive integers* u, v. *Then*

$$\begin{aligned} a(p) &= \pm 2u \quad \text{for } p \equiv 1, 19 \bmod 24, \\ a(p) &= \pm 6v \quad \text{for } p \equiv 7, 13 \bmod 24. \end{aligned}$$

Here the sign is $\left(\frac{-6}{u-3v}\right)$ *for* $p \equiv 1 \bmod 24$, *and* $-\left(\frac{-6}{u-3v}\right)$ *for* $p \equiv 13 \bmod 24$.

Proof. We can write F and G as products of two simple theta series of weights $\frac{1}{2}$,

$$F(z) = \frac{\eta^{13}(2z)}{\eta^5(z)\eta^5(4z)} \cdot \frac{\eta^2(2z)}{\eta(z)}, \qquad G(z) = \frac{\eta^9(2z)}{\eta^3(z)\eta^3(4z)} \cdot \eta(z).$$

We use (8.20), (8.5) and (8.16), (8.3). This yields assertion (1). Since $(F, \widetilde{F})$, $(G, \widetilde{G})$ are pairs of sign transforms, the identities (13.47), (13.48), (13.49) imply assertion (2).

Let a prime $p \equiv 1 \bmod 6$ be given, and write p uniquely in the form $p = u^2 + 3v^2$ with positive integers u, v. This means that $p = \mu\overline{\mu}$ where $\mu = (u-v) + 2v\omega$ and $\overline{\mu} = (u+v) - 2v\omega$. A table of values $\psi_{\delta,\varepsilon}(\mu)$ as in Figs. 9.1, 12.1 shows that $\psi_{\delta,\varepsilon}(\overline{\mu}) = \psi_{\delta,\varepsilon}(\mu)$ for $p \equiv 1, 19 \bmod 24$ and $\psi_{\delta,\varepsilon}(\overline{\mu}) = -\psi_{\delta,\varepsilon}(\mu)$ for $p \equiv 7, 13 \bmod 24$. For the coefficient $\lambda(p)$ of $\Theta_2\left(-3, \psi_{\delta,\varepsilon}, \frac{z}{6}\right)$ at the prime p this implies that $\lambda(p) = \psi_{\delta,\varepsilon}(\mu)\,(\mu + \overline{\mu}) = \psi_{\delta,\varepsilon}(\mu)\,2u$ for $p \equiv 1, 19 \bmod 24$ and $\lambda(p) = \psi_{\delta,\varepsilon}(\mu)\,(\mu - \overline{\mu}) = \psi_{\delta,\varepsilon}(\mu)\,2v(2\omega - 1) = \psi_{\delta,\varepsilon}(\mu)\,\sqrt{3}i\,2v$ for $p \equiv 7, 13 \bmod 24$. We use (13.47), (13.48), (13.49) again and obtain $a(p) = \pm 2u$ for $p \equiv 1, 19 \bmod 24$, $a(p) = \pm 6v$ for $p \equiv 7, 13 \bmod 24$.

Now we assume that $p \equiv 1 \bmod 24$. Then u is odd and v is a multiple of 4. It follows that $\psi_{\delta,\varepsilon}(\mu) = \psi_{\delta,\varepsilon}(u - 3v) = \left(\frac{-6}{u-3v}\right)$, and hence $a(p) = \left(\frac{-6}{u+3v}\right)2u$. Finally, let $p \equiv 13 \bmod 24$. Then u is odd and $v \equiv 2 \bmod 4$. We get $\psi_{\delta,\varepsilon}(\mu) = \psi_{\delta,\varepsilon}(u - 3v - 4(\omega + 1)) = \left(\frac{-6}{u-3v}\right)\delta\varepsilon$, and hence $a(p) = -\left(\frac{-6}{u+3v}\right)6v$. Thus we have proved assertion (3). It would also be possible to find rules for the sign in the remaining two cases. □

13.5 Weight 2 for $\Gamma_0(4)$, Cusp Forms with Denominators $t = 8, 12$

Now we discuss the cuspidal eta products of weight 2 and denominator 8. We start with the Fricke transforms of the functions in Example 13.14. Rescaling the eta products gives a pair of functions which are interchanged by the action of the Fricke involution.

Example 13.20 *Let χ_δ be the characters on $\mathcal{O}_2$ with period $2\sqrt{-2}$ as defined in Example 13.14. The corresponding theta series of weight 2 satisfy*

$$\Theta_2\left(-8, \chi_\delta, \tfrac{z}{8}\right) \;=\; g_1(z) + 2\delta i\, g_3(z) \tag{13.51}$$

with normalized integral Fourier series g_j with denominator 8 and numerator classes j modulo 8 which are eta products,

$$g_1 = \left[1, 2^5, 4^{-2}\right], \qquad g_3 = \left[1^3, 2^{-1}, 4^2\right]. \tag{13.52}$$

We have the identities

$$\left[4, 8^5, 16^{-2}\right] \;=\; \tfrac{1}{2}\left(\left[1^{-2}, 2^5, 4\right] + \left[1^2, 2^{-1}, 4^3\right]\right), \tag{13.53}$$

$$\left[4^3, 8^{-1}, 16^2\right] \;=\; \tfrac{1}{4}\left(\left[1^{-2}, 2^5, 4\right] - \left[1^2, 2^{-1}, 4^3\right]\right). \tag{13.54}$$

The action of W_4 on

$$F_\delta(z) = \Theta_2\left(-8, \chi_\delta, \tfrac{z}{4}\right) = \frac{\eta(2z)\eta^5(4z)}{\eta^2(8z)} + 2\delta i\, \frac{\eta^3(2z)\eta^2(8z)}{\eta(4z)}$$

is given by

$$F_\delta(W_4 z) = -2\sqrt{2}(1+\delta i) z^2 F_{-\delta}(z).$$

Corollary 13.21 *Let $a_j(n)$ for $j \in \{1, 3\}$ denote the coefficients of the functions f_j in (13.33) and, simultaneously, of the functions g_j in (13.52). Let $p \equiv 1$ or $3 \bmod 8$ be prime and write $p = x^2 + 2y^2$ with unique positive integers x, y. Then*

$$a_1(p) \;=\; \left(\tfrac{-1}{x}\right) 2x \quad \textit{for } p \equiv 1 \bmod 8,$$

$$a_3(p) \;=\; \left(\tfrac{-1}{x}\right) x \quad \textit{for } p \equiv 3 \bmod 8.$$

Proof. The character values $\chi_\delta(x + y\sqrt{-2})$ are explicitly given by a formula in Example 13.14. We write p uniquely in the form $p = \mu\overline{\mu} = x^2 + 2y^2$ with $\mu = x + y\sqrt{-2} \in \mathcal{O}_2$, $x > 0$, $y > 0$. Here y is even if $p \equiv 1 \bmod 8$, and y is odd if $p \equiv 3 \bmod 8$. Now we can compute the coefficient $\lambda(p) = \chi_\delta(\mu)\mu + \chi_\delta(\overline{\mu})\overline{\mu} = \chi_\delta(\mu)\, 2x$ of $\Theta_2(-8, \chi_\delta, \cdot)$ at p, and the assertion follows from (13.51), (13.52).

The result can also be deduced directly from the Jacobi and Gauss identities (8.5), (8.8), (8.15) when we decompose the eta products

$$\left[1, 2^5, 4^{-2}\right] = \left[1^{-2}, 2^5, 4^{-2}\right]\left[1^3\right], \qquad \left[1^3, 2^{-1}, 4^2\right] = \left[1^3\right]\left[2^{-1}, 4^2\right]$$

into products of two simple theta series. Then we get

$$a_1(n) = \sum_{x>0,\, y\in\mathbb{Z},\, x^2+8y^2=n} \left(\tfrac{-1}{x}\right)x, \qquad a_3(n) = \sum_{x,y>0,\, x^2+2y^2=n} \left(\tfrac{-1}{x}\right)x$$

for arbitrary n. □

Next we deal with the Fricke transforms of the eta products in Example 13.17. The result is quite similar to that in Example 13.20.

Example 13.22 *Let χ_δ be the characters on $\mathcal{O}_1$ with period $4(1+i)$ as defined in Example 13.17. The corresponding theta series of weight 2 satisfy*

$$\Theta_2\left(-4, \chi_\delta, \tfrac{z}{8}\right) = g_1(z) + 4\delta i\, g_5(z) \tag{13.55}$$

with normalized integral Fourier series g_j with denominator 8 and numerator classes j modulo 8 which are eta products,

$$g_1 = \left[1^{-3}, 2^{11}, 4^{-4}\right], \qquad g_5 = \left[1, 2^{-1}, 4^4\right]. \tag{13.56}$$

We have the identities

$$\left[2^{-3}, 4^{11}, 8^{-4}\right] = \tfrac{1}{2}\left(\left[1^{-4}, 2^{11}, 4^{-3}\right] + \left[1^4, 2^{-1}, 4\right]\right), \tag{13.57}$$
$$\left[2, 4^{-1}, 8^4\right] = \tfrac{1}{8}\left(\left[1^{-4}, 2^{11}, 4^{-3}\right] - \left[1^4, 2^{-1}, 4\right]\right). \tag{13.58}$$

The action of W_4 on

$$G_\delta(z) = \Theta_2\left(-4, \chi_\delta, \tfrac{z}{4\sqrt{2}}\right) = \frac{\eta^{11}(2\sqrt{2}z)}{\eta^3(\sqrt{2}z)\eta^4(4\sqrt{2}z)} + 4\delta i\, \frac{\eta(\sqrt{2}z)\eta^4(4\sqrt{2}z)}{\eta(2\sqrt{2}z)}$$

is given by

$$G_\delta(W_4 z) = -2\sqrt{2}(1+\delta i) z^2\, G_{-\delta}(z).$$

Here again, we can write

$$\left[1^{-3}, 2^{11}, 4^{-4}\right] = \left[1^{-3}, 2^9, 4^{-3}\right]\left[2^2, 4^{-1}\right], \qquad \left[1, 2^{-1}, 4^4\right] = \left[4^3\right]\left[1, 2^{-1}, 4\right]$$

as products of two simple theta series. Then we use (8.6), (8.7), (8.15), (8.16) and obtain the formulae

$$a_1(n) = \sum_{x>0,\, y\in\mathbb{Z},\, x^2+16y^2=n} (-1)^y \left(\tfrac{-2}{x}\right) x,$$
$$a_5(n) = \sum_{x,y>0,\, x^2+4y^2=n} \left(\tfrac{-1}{y}\right)\left(\tfrac{2}{x}\right) y$$

for the coefficients $a_j(n)$ of the eta products g_j in (13.56). These formulae can also be deduced from (13.56) and the definition of the characters χ_δ.

The remaining four cuspidal eta products with denominator 4 are the sign transforms of the functions discussed so far. They form two pairs of functions which are transformed into each other by W_4 and which combine to theta series:

Example 13.23 *Let the generators of* $(\mathcal{O}_1/(8))^\times \simeq Z_4 \times Z_2 \times Z_4$ *and of* $(\mathcal{O}_2/(4\sqrt{-2}))^\times \simeq Z_4 \times Z_2^2$ *be chosen as in Example 13.3. Define two pairs of characters* χ_δ *on* $\mathcal{O}_1$ *with period* 8 *and* ψ_δ *on* $\mathcal{O}_2$ *with period* $4\sqrt{-2}$ *by their values*

$$\chi_\delta(2+i)=\delta, \qquad \chi_\delta(3)=-1, \qquad \chi_\delta(i)=-1,$$
$$\psi_\delta(1+\sqrt{-2})=\delta, \qquad \psi_\delta(3)=1, \qquad \psi_\delta(-1)=-1$$

with $\delta\in\{1,-1\}$. *The corresponding theta series of weight* 2 *satisfy*

$$\Theta_2\left(-4,\chi_\delta,\tfrac{z}{8}\right) = f_1(z)+4\delta\, f_5(z), \tag{13.59}$$
$$\Theta_2\left(-8,\psi_\delta,\tfrac{z}{8}\right) = g_1(z)+2\delta\, g_3(z), \tag{13.60}$$

where f_j *and* g_j *are normalized integral Fourier series with denominator* 8 *and numerator classes* j *modulo* 8. *All of them are eta products,*

$$f_1 = \left[1^3,2^2,4^{-1}\right], \qquad f_5=\left[1^{-1},2^2,4^3\right], \tag{13.61}$$
$$g_1 = \left[1^{-1},2^8,4^{-3}\right], \qquad g_3=\left[1^{-3},2^8,4^{-1}\right]. \tag{13.62}$$

The action of W_4 *on* $F_\delta(z)=\Theta_2\left(-4,\chi_\delta,\frac{z}{8}\right)$ *and on* $G_\delta(z)=\Theta_2\left(-8,\psi_\delta,\frac{z}{8}\right)$ *is given by*

$$F_\delta(W_4 z) = -4\delta z^2\, F_\delta(z), \qquad G_\delta(W_4 z) = -4\delta z^2\, G_\delta(z). \tag{13.63}$$

As before, the eta products are products of two simple theta series, and they are the sign transforms of the eta products in Examples 13.20, 13.22. Therefore the coefficients $a_j(n)$, $b_j(n)$ of the eta products f_j, g_j in Example 13.23 are given by the formulae

$$a_1(n) = \sum_{x>0,\, y\in\mathbb{Z},\, x^2+16y^2=n} (-1)^y\left(\tfrac{-1}{x}\right)x,$$
$$a_5(n) = \sum_{x,y>0,\, x^2+4y^2=n} \left(\tfrac{-1}{y}\right)y,$$
$$b_1(n) = \sum_{x>0,\, y\in\mathbb{Z},\, x^2+8y^2=n} (-1)^y\left(\tfrac{-2}{x}\right)x,$$
$$b_3(n) = \sum_{x,y>0,\, x^2+2y^2=n} \left(\tfrac{-2}{x}\right)x$$

which are quite similar to those we got before.

Four of the cuspidal eta products with weight 2 and denominator 12 are

$$f_1=\left[\frac{2^{10}}{1^2,4^4}\right], \quad f_7=\left[\frac{1^2,4^4}{2^2}\right], \qquad g_1=\left[\frac{1^2,2^4}{4^2}\right], \quad g_7=\left[\frac{2^4,4^2}{1^2}\right]. \tag{13.64}$$

Here, (f_1, g_1) and (f_7, g_7) are pairs of sign transforms, and g_1, g_7 are interchanged by W_4. The transforms of f_1, f_7 under W_4 were discussed in Example 13.15. Two linear combinations of f_1, f_7 are, after rescaling, transformed into themselves by W_4, and one of them is the eta product for $\Gamma^*(4)$ which was discussed in Example 13.7. The four eta products combine to theta series as follows:

Example 13.24 *Let ψ_δ be the characters on $\mathcal{O}_3$ with period $4(1+\omega)$ as defined in Example 13.15. Let the generators of $(\mathcal{O}_3/(8+8\omega))^\times \simeq Z_4 \times Z_2^2 \times Z_6$ be chosen as in Example 13.2, and define a pair of characters $\widetilde{\psi}_\delta$ on $\mathcal{O}_3$ with period $8(1+\omega)$ by*

$$\widetilde{\psi}_\delta(1+2\omega) = \delta, \qquad \widetilde{\psi}_\delta(1-4\omega) = 1, \qquad \widetilde{\psi}_\delta(5) = -1, \qquad \widetilde{\psi}_\delta(\omega) = \overline{\omega}$$

with $\delta \in \{1, -1\}$. The corresponding theta series of weight 2 satisfy

$$\Theta_2\left(-3, \psi_\delta, \tfrac{z}{12}\right) = f_1(z) + 4\delta\, f_7(z), \tag{13.65}$$

$$\Theta_2\left(-3, \widetilde{\psi}_\delta, \tfrac{z}{12}\right) = g_1(z) + 4\delta\, g_7(z), \tag{13.66}$$

where the components f_j and g_j are equal to the eta products defined in (13.64). The action of W_4 on $G_\delta(z) = \Theta_2\left(-3, \widetilde{\psi}_\delta, \frac{z}{12}\right)$ is given by $G_\delta(W_4 z) = -4\delta z^2\, G_\delta(z)$. We have the eta identities

$$f_1(4z) + 4f_7(4z) = \frac{1}{2}\left(\frac{\eta^{10}(2z)}{\eta^4(z)\eta^2(4z)} + \frac{\eta^4(z)\eta^2(4z)}{\eta^2(2z)}\right) = \frac{\eta^{12}(2z)}{\eta^4(z)\eta^4(4z)}, \tag{13.67}$$

$$f_1(4z) - 4f_7(4z) = \frac{1}{8}\left(\frac{\eta^{10}(\frac{z}{2})}{\eta^4(\frac{z}{4})\eta^2(z)} - \frac{\eta^4(\frac{z}{4})\eta^2(z)}{\eta^2(\frac{z}{2})}\right). \tag{13.68}$$

The action of W_4 on $F_\delta(z) = \Theta_2\left(-3, \psi_\delta, \frac{z}{3}\right)$ is given by $F_1(W_4 z) = -4z^2 \times F_1(z)$, $F_{-1}(W_4 z) = -\frac{1}{2} z^2\, F_{-1}(z)$.

The eta products f_j, g_j in (13.64) are products of two simple theta series. So as before we get formulae which relate their coefficients $a_j(n)$, $b_j(n)$ to quadratic forms,

$$a_1(n) = \sum_{x>0,\, y\in\mathbb{Z},\, x^2+12y^2=n} \left(\tfrac{x}{3}\right)x, \qquad a_7(n) = \sum_{x,y>0,\, 3x^2+4y^2=n} \left(\tfrac{y}{3}\right)y,$$

and similar formulae for $b_j(n)$.

The other four cuspidal eta products with denominator 12 form two pairs of sign transforms

$$F = \left[1^{-4}, 2^{13}, 4^{-5}\right], \qquad \widetilde{F} = \left[1^4, 2, 4^{-1}\right],$$
$$G = \left[1^{-4}, 2^9, 4^{-1}\right], \qquad \widetilde{G} = \left[1^4, 2^{-3}, 4^3\right] \tag{13.69}$$

with numerators 1 and 5. By W_4 they are transformed into eta products with orders $\frac{5}{24}$, $\frac{17}{24}$, $\frac{1}{24}$, $\frac{13}{24}$, respectively, at the cusp ∞. The Fricke transforms will be discussed in Example 13.28. The functions (13.69) combine to four theta series as follows:

Example 13.25 *Let the generators of* $(\mathcal{O}_1/(12+12i))^\times \simeq Z_8 \times Z_2^2 \times Z_4$ *be chosen as in Example* 10.5, *and define four characters* $\varphi_{\delta,\varepsilon}$ *on* $\mathcal{O}_1$ *with period* $12(1+i)$ *by their values*

$$\varphi_{\delta,\varepsilon}(1+2i) = \varepsilon i, \qquad \varphi_{\delta,\varepsilon}(1+6i) = -\delta, \qquad \varphi_{\delta,\varepsilon}(11) = 1, \qquad \varphi_{\delta,\varepsilon}(i) = -i$$

with $\delta, \varepsilon \in \{1, -1\}$. *The corresponding theta series of weight* 2 *satisfy*

$$\Theta_2\left(-4, \varphi_{\delta,\varepsilon}, \tfrac{z}{12}\right) = f_1(z) + 2\varepsilon i\, f_5(z) + 4\delta i f_{13}(z) - 8\delta\varepsilon f_{17}(z), \tag{13.70}$$

where the components f_j *are normalized integral Fourier series with denominator* 12 *and numerator classes* j *modulo* 24 *which are linear combinations of the eta products in* (13.69),

$$f_1 = \tfrac{1}{2}(F + \widetilde{F}), \qquad f_{13} = \tfrac{1}{8}(F - \widetilde{F}), \tag{13.71}$$

$$f_5 = \tfrac{1}{2}(G + \widetilde{G}), \qquad f_{17} = \tfrac{1}{8}(G - \widetilde{G}). \tag{13.72}$$

The action of W_4 *on* $F_{\delta,\varepsilon}(z) = \Theta_2\big(-4, \varphi_{\delta,\varepsilon}, \frac{z}{12}\big)$ *is given by*

$$\begin{aligned} F_{\delta,\varepsilon}(W_4 z) = &-\varepsilon i(1+\delta i)\sqrt{2}z^2\big(\big[1^{-1}, 2^9, 4^{-4}\big] - 2\varepsilon i\big[1^{-5}, 2^{13}, 4^{-4}\big] \\ &- 4\delta i\big[1^3, 2^{-3}, 4^4\big] - 8\delta\varepsilon\big[1^{-1}, 2, 4^4\big]\big). \end{aligned} \tag{13.73}$$

There are decompositions of F and $\widetilde{F}$ into products of two simple theta series which imply coefficient formulae similar to those before. We did not find such a decomposition for G or $\widetilde{G}$.

13.6 Weight 2 for $\Gamma_0(4)$, Cusp Forms with Denominator $t = 24$

We start the discussion of the 24 cuspidal eta products of weight 2 and denominator 24 with the Fricke transforms of the eta products with denominator 3 in Example 13.16.

Example 13.26 *Let the generators of* $(\mathcal{J}_6/(2\sqrt{3}))^\times \simeq Z_2^3$ *be chosen as in Example* 13.16, *and define four characters* $\varphi_{\delta,\varepsilon}$ *on* $\mathcal{J}_6$ *with period* $2\sqrt{3}$ *by their values*

$$\varphi_{\delta,\varepsilon}(\sqrt{3}+\sqrt{-2}) = \delta\varepsilon, \qquad \varphi_{\delta,\varepsilon}(1+\sqrt{-6}) = -\varepsilon, \qquad \varphi_{\delta,\varepsilon}(-1) = -1$$

with $\delta, \varepsilon \in \{1, -1\}$. *The corresponding theta series of weight* 2 *satisfy*

$$\Theta_2\left(-24, \varphi_{\delta,\varepsilon}, \tfrac{z}{24}\right) = f_1(z) + 2\sqrt{2}\delta\varepsilon i\, f_5(z) + 2\varepsilon f_7(z) - 4\sqrt{2}\delta i f_{11}(z), \quad (13.74)$$

where the components f_j *are normalized integral Fourier series with denominator* 24 *and numerator classes* j *modulo* 24. *All of them are eta products,*

$$f_1 = \left[1^3, 2^3, 4^{-2}\right], \qquad f_5 = \left[1^{-1}, 2^7, 4^{-2}\right],$$
$$f_7 = \left[1^5, 2^{-3}, 4^2\right], \qquad f_{11} = \left[1, 2, 4^2\right]. \quad (13.75)$$

The Fricke involution W_4 *maps* $\Theta_2\big(-24, \varphi_{\delta,\varepsilon}, \frac{z}{24}\big)$ *to a multiple of*

$$\left[1^{-2}, 2^7, 4^{-1}\right] - \varepsilon\left[1^2, 2, 4\right] - \delta\varepsilon i\sqrt{2}\big(\left[1^{-2}, 2^3, 4^3\right] + \varepsilon\left[1^2, 2^{-3}, 4^5\right]\big).$$

We have the identities

$$\left[8^3, 16^3, 32^{-2}\right] - 2\left[8^5, 16^{-3}, 32^2\right] = \tfrac{1}{2}\big(\left[1^{-2}, 2^7, 4^{-1}\right] + \left[1^2, 2, 4\right]\big), \quad (13.76)$$

$$\left[8^{-1}, 16^7, 32^{-2}\right] + 2\left[8, 16, 32^2\right] = \tfrac{1}{4}\big(\left[1^{-2}, 2^3, 4^3\right] - \left[1^2, 2^{-3}, 4^5\right]\big). \quad (13.77)$$

We note that the characters $\varphi_{\delta,-1}$ in Example 13.26 coincide with the characters $\varphi_{-\delta}$ in Example 13.16. Therefore the identities (13.76), (13.77) follow from (13.40), (13.74), (13.75). We have the decompositions $f_1 = \left[1^5, 2^{-2}\right]\left[1^{-2}, 2^5, 4^{-2}\right]$, $f_5 = \left[2^5, 4^{-2}\right]\left[1^{-1}, 2^2\right]$, $f_7 = \left[1^5, 2^{-2}\right]\left[2^{-1}, 4^2\right]$, $f_{11} = \left[1^2, 2^{-1}, 4^2\right]\left[1^{-1}, 2^2\right]$ into products of simple theta series. Therefore the identities in Sect. 8 yield coefficient formulae for the eta products (13.75) similar to those in preceding cases.—The sign transforms of these eta products will appear in Example 13.29.

In the next example we treat the Fricke transforms of the eta products in Example 13.18. Rescaling the theta series in that example produces theta series whose components are eta products with denominator 24; rescaling differently, we get functions which are permuted by the Fricke involution W_4.

Example 13.27 *Let* $\psi_{\delta,\varepsilon}$ *be the characters on* $\mathcal{O}_3$ *with period* $8(1+\omega)$ *as defined in Example* 13.18. *The corresponding theta series of weight* 2 *satisfy*

$$\Theta_2\left(-3, \psi_{\delta,\varepsilon}, \tfrac{z}{24}\right) = g_1(z) + 2\sqrt{3}\delta\, g_7(z) - 4\delta\varepsilon i\sqrt{3}\, g_{13}(z) + 8\varepsilon i g_{19}(z), \quad (13.78)$$

where the components g_j *are normalized integral Fourier series with denominator* 24 *and numerator classes* j *modulo* 24. *All of them are eta products,*

$$g_1 = \left[1^{-5}, 2^{15}, 4^{-6}\right], \qquad g_7 = \left[1^{-3}, 2^9, 4^{-2}\right],$$
$$g_{13} = \left[1^{-1}, 2^3, 4^2\right], \qquad g_{19} = \left[1, 2^{-3}, 4^6\right]. \quad (13.79)$$

We have the identities $g_j(4z) = f_j(z)$ *where the* f_j *are the linear combinations* (13.48), (13.49) *of the eta products* F, $\widetilde{F}$, G, $\widetilde{G}$ *in* (13.46). *The action of* W_4 *on* $G_{\delta,\varepsilon}(z) = \Theta_2\big(-3, \psi_{\delta,\varepsilon}, \frac{z}{12}\big)$ *is given by*

$$G_{\delta,\varepsilon}(W_4 z) = -4\,\xi_{\delta,\varepsilon} z^2\, G_{\delta,-\varepsilon}(z)$$

with the 24*th roots of unity* $\xi_{\delta,\varepsilon}$ *from Example* 13.18.

There are obvious decompositions of the eta products (13.79) into products of two simple theta series. They imply coefficient formulae similar to those in preceding cases. The sign transforms $\big[1^5, 4^{-1}\big]$, $\big[1^3, 4\big]$, $\big[1, 4^3\big]$, $\big[1^{-1}, 4^5\big]$ of the functions (13.79) will be discussed in Example 13.30. Now we turn to the Fricke transforms of the eta products in Example 13.25. We get a result quite similar to that above:

Example 13.28 *Let* $\varphi_{\delta,\varepsilon}$ *be the characters on* $\mathcal{O}_1$ *with period* $12(1+i)$ *as defined in Example* 13.25. *The corresponding theta series of weight* 2 *satisfy*

$$\Theta_2\left(-4, \varphi_{\delta,\varepsilon}, \tfrac{z}{24}\right) \;=\; g_1(z) + 2\varepsilon i\, g_5(z) + 4\delta i\, g_{13}(z) - 8\delta\varepsilon g_{17}(z), \qquad (13.80)$$

where the components g_j *are normalized integral Fourier series with denominator* 24 *and numerator classes* j *modulo* 24. *All of them are eta products,*

$$\begin{aligned} g_1 &= \big[1^{-1}, 2^9, 4^{-4}\big], \qquad & g_5 &= \big[1^{-5}, 2^{13}, 4^{-4}\big], \\ g_{13} &= \big[1^3, 2^{-3}, 4^4\big], \qquad & g_{17} &= \big[1^{-1}, 2, 4^4\big]. \end{aligned} \qquad (13.81)$$

We have the identities $g_j(2z) = f_j(z)$ *where the* f_j *are the linear combinations* (13.71), (13.72) *of the eta products* F, $\widetilde{F}$, G, $\widetilde{G}$ *in* (13.69). *The action of* W_4 *on* $G_{\delta,\varepsilon}(z) = \Theta_2\big(-4, \varphi_{\delta,\varepsilon}, \frac{z}{12\sqrt{2}}\big)$ *is given by*

$$G_{\delta,\varepsilon}(W_4 z) = -2\sqrt{2}\,\varepsilon i(1+\delta i)\, z^2\, G_{-\delta,-\varepsilon}(z).$$

We can write $g_5 = \big[1^{-5}, 2^{13}, 4^{-5}\big]\big[4\big]$, $g_{17} = \big[2^{-2}, 4^5\big]\big[1^{-1}, 2^3, 4^{-1}\big]$ as products of two simple theta series, but apparently there are no such decompositions for g_1 and g_{13}. The sign transforms of the functions (13.81) will be discussed in Example 13.31. Now we describe theta series whose components are the sign transforms of the eta products in Example 13.26.

Example 13.29 *Let the generators of* $(\mathcal{J}_6/(4\sqrt{-6}))^\times \simeq Z_4^2 \times Z_2^2$ *be chosen as in Example* 13.4, *and define characters* $\psi_{\delta,\varepsilon}$ *on* $\mathcal{J}_6$ *with period* $4\sqrt{-6}$ *by*

$$\begin{aligned} &\psi_{\delta,\varepsilon}(\sqrt{3}+\sqrt{-2}) = -\delta\varepsilon i, \qquad \psi_{\delta,\varepsilon}(1+\sqrt{-6}) = \varepsilon, \\ &\psi_{\delta,\varepsilon}(7) = 1, \qquad \psi_{\delta,\varepsilon}(-1) = -1. \end{aligned}$$

The corresponding theta series of weight 2 *satisfy*

$$\Theta_2\left(-24, \psi_{\delta,\varepsilon}, \tfrac{z}{24}\right) \;=\; g_1(z) + 2\sqrt{2}\delta\varepsilon\, g_5(z) + 2\varepsilon\, g_7(z) + 4\sqrt{2}\delta\, g_{11}(z), \quad (13.82)$$

where the components g_j are normalized integral Fourier series with denominator 24 and numerator classes j modulo 24. They are equal to eta products,

$$g_1 = \left[1^{-3}, 2^{12}, 4^{-5}\right], \qquad g_5 = \left[1, 2^4, 4^{-1}\right],$$
$$g_7 = \left[1^{-5}, 2^{12}, 4^{-3}\right], \qquad g_{11} = \left[1^{-1}, 2^4, 4\right]. \tag{13.83}$$

The functions g_j are the sign transforms of the eta products f_j in Example 13.26. The action of W_4 on $G_{\delta,\varepsilon}(z) = \Theta_2\bigl(-24, \psi_{\delta,\varepsilon}, \frac{z}{24}\bigr)$ is given by

$$G_{\delta,\varepsilon}(W_4 z) = -4\varepsilon\, z^2\, G_{\delta,\varepsilon}(z).$$

The decompositions of the functions f_j in Example 13.26 into products of simple theta series imply analogous decompositions for their sign transforms g_j. In the following two examples we describe theta series whose components are the sign transforms of the eta products in Examples 13.27, 13.28.

Example 13.30 *Let the generators of $(\mathcal{O}_3/(16+16\omega))^\times \simeq Z_8 \times Z_4 \times Z_2 \times Z_6$ be chosen as in Example 13.12, and define four characters $\rho_{\delta,\varepsilon}$ on $\mathcal{O}_3$ with period $16(1+\omega)$ by*

$$\rho_{\delta,\varepsilon}(1+2\omega) = -\delta i, \qquad \rho_{\delta,\varepsilon}(1-4\omega) = \delta\varepsilon i, \qquad \rho_{\delta,\varepsilon}(7) = 1, \qquad \rho_{\delta,\varepsilon}(\omega) = \overline{\omega}$$

with $\delta, \varepsilon \in \{1, -1\}$. The corresponding theta series of weight 2 satisfy

$$\Theta_2\left(-3, \rho_{\delta,\varepsilon}, \tfrac{z}{24}\right) = h_1(z) + 2\sqrt{3}\delta\, h_7(z) + 4\sqrt{3}\delta\varepsilon\, h_{13}(z) + 8\varepsilon\, h_{19}(z), \tag{13.84}$$

where the components h_j are normalized integral Fourier series with denominator 24 and numerator classes j modulo 24. All of them are eta products,

$$h_1 = \left[1^5, 4^{-1}\right], \quad h_7 = \left[1^3, 4\right], \quad h_{13} = \left[1, 4^3\right], \quad h_{19} = \left[1^{-1}, 4^5\right]. \tag{13.85}$$

The functions h_j are the sign transforms of the eta products g_j in Example 13.27. The action of W_4 on $H_{\delta,\varepsilon}(z) = \Theta_2\bigl(-3, \rho_{\delta,\varepsilon}, \frac{z}{24}\bigr)$ is given by

$$H_{\delta,\varepsilon}(W_4 z) = -4\varepsilon\, z^2\, H_{\delta,\varepsilon}(z).$$

Example 13.31 *Let the generators of $(\mathcal{O}_1/(24))^\times \simeq Z_8 \times Z_4 \times Z_2 \times Z_4$ be chosen as in Example 13.4, and define four characters $\chi_{\delta,\varepsilon}$ on $\mathcal{O}_1$ with period 24 by their values*

$$\chi_{\delta,\varepsilon}(2+i) = -\varepsilon i, \qquad \chi_{\delta,\varepsilon}(1+6i) = -\delta i, \qquad \chi_{\delta,\varepsilon}(5) = 1, \qquad \chi_{\delta,\varepsilon}(i) = -i$$

with $\delta, \varepsilon \in \{1, -1\}$. The corresponding theta series of weight 2 satisfy

$$\Theta_2\left(-4, \chi_{\delta,\varepsilon}, \tfrac{z}{24}\right) = h_1(z) + 2\varepsilon\, h_5(z) + 4\delta\, h_{13}(z) + 8\delta\varepsilon\, h_{17}(z), \tag{13.86}$$

where the components h_j are normalized integral Fourier series with denominator 24 and numerator classes j modulo 24. All of them are eta products,

$$\begin{aligned} h_1 &= \left[1, 2^6, 4^{-3}\right], \quad h_5 = \left[1^5, 2^{-2}, 4\right], \\ h_{13} &= \left[1^{-3}, 2^6, 4\right], \quad h_{17} = \left[1, 2^{-2}, 4^5\right]. \end{aligned} \tag{13.87}$$

The functions h_j are the sign transforms of the eta products g_j in Example 13.28. The action of W_4 on $H_{\delta,\varepsilon}(z) = \Theta_2\left(-4, \chi_{\delta,\varepsilon}, \frac{z}{24}\right)$ is given by

$$H_{\delta,\varepsilon}(W_4 z) = -4\delta\, z^2\, H_{\delta,\varepsilon}(z).$$

13.7 Weight 2 for $\Gamma_0(4)$, Non-cuspidal Eta Products

The table at the beginning of Section 13.3 indicates that there are altogether 18 new non-cuspidal eta products of weight 2 for $\Gamma_0(4)$. We start with the discussion of those with denominator 8. They form four pairs of sign transforms, and they are not lacunary. There are eight linear combinations whose Fourier expansions are of Eisenstein type.

Example 13.32 *Consider the eta products*

$$\begin{aligned} f_1 &= \left[1^{-7}, 2^{17}, 4^{-6}\right], \qquad f_3 = \left[1^{-5}, 2^{11}, 4^{-2}\right], \\ f_5 &= \left[1^{-3}, 2^5, 4^2\right], \qquad f_7 = \left[1^{-1}, 2^{-1}, 4^6\right] \end{aligned} \tag{13.88}$$

with normalized integral Fourier expansions $f_j(z) = \sum_{n \equiv j \bmod 8} a_j(n) e\left(\frac{nz}{8}\right)$. Then for $\delta, \varepsilon \in \{1, -1\}$, the linear combinations

$$F_{\delta,\varepsilon}(z) = f_1(z) + 2\delta\, f_3(z) + 4\varepsilon\, f_5(z) + 8\delta\varepsilon\, f_7(z) = \sum_{n>0 \text{ odd}} \lambda_{\delta,\varepsilon}(n)\, e\left(\frac{nz}{8}\right)$$

have coefficients

$$\lambda_{\delta,\varepsilon}(n) = \sigma_{\delta,\varepsilon}(n) \sum_{d|n} \left(\frac{2}{n/d}\right) d \tag{13.89}$$

where

$$\sigma_{\delta,\varepsilon}(n) = \left(\frac{-1}{n}\right)^{\frac{\delta-1}{2}} \left(\frac{-2}{n}\right)^{\frac{\varepsilon-1}{2}} = \begin{cases} 1 \\ \delta \\ \varepsilon \\ \delta\varepsilon \end{cases} \quad \text{for} \quad n \equiv \begin{cases} 1 \\ 3 \\ 5 \\ 7 \end{cases} \bmod 8.$$

The coefficients are multiplicative and satisfy the recursion

$$\lambda_{\delta,\varepsilon}(p^{r+1}) = \lambda_{\delta,\varepsilon}(p)\lambda_{\delta,\varepsilon}(p^r) - \left(\tfrac{2}{p}\right) p\, \lambda_{\delta,\varepsilon}(p^{r-1})$$

for odd primes p. For the sign transforms $\widetilde{f}_j$ of the functions f_j,

$$\widetilde{f}_1 = \left[1^7, 2^{-4}, 4\right], \ \ \widetilde{f}_3 = \left[1^5, 2^{-4}, 4^3\right], \ \ \widetilde{f}_5 = \left[1^3, 2^{-4}, 4^5\right], \ \ \widetilde{f}_7 = \left[1, 2^{-4}, 4^7\right], \tag{13.90}$$

we have the linear combinations

$$\widetilde{F}_{\delta,\varepsilon}(z) = \widetilde{f}_1(z) + 2\delta\,\widetilde{f}_3(z) + 4\varepsilon\,\widetilde{f}_5(z) + 8\delta\varepsilon\,\widetilde{f}_7(z) = \sum_{n>0 \text{ odd}} \widetilde{\lambda}_{\delta,\varepsilon}(n) e\left(\tfrac{nz}{8}\right)$$

with coefficients $\widetilde{\lambda}_{\delta,\varepsilon}(n) = (-1)^{(n-n_0)/8}\lambda_{\delta,\varepsilon}(n)$ where n_0 is the smallest positive residue of n modulo 8. The action of W_4 on $\widetilde{F}_{\delta,\varepsilon}$ is given by

$$\widetilde{F}_{\delta,\varepsilon}(W_4 z) = -4\delta\varepsilon\, z^2\, \widetilde{F}_{\delta,\varepsilon}(z).$$

The Fricke involution W_4 sends the eta products f_j in (13.88) into eta products with denominator $t = 1$ and order 0 at the cusp ∞. We denote them by

$$\begin{aligned} g_1 &= \left[1^{-6}, 2^{17}, 4^{-7}\right], \qquad & g_3 &= \left[1^{-2}, 2^{11}, 4^{-5}\right], \\ g_5 &= \left[1^{2}, 2^{5}, 4^{-3}\right], \qquad & g_7 &= \left[1^{6}, 2^{-1}, 4^{-1}\right]. \end{aligned} \tag{13.91}$$

Then the functions $F_{\delta,\varepsilon}$ in Example 13.32 satisfy

$$F_{\delta,\varepsilon}(W_4 z) = -2\sqrt{2}z^2\left(g_1(z) + \delta g_3(z) + \varepsilon g_5(z) + \delta\varepsilon g_7(z)\right).$$

Correspondingly, we get four linear combinations of the eta products g_j which are Eisenstein series and, in particular, have multiplicative coefficients:

Example 13.33 *The eta products g_j in (13.91) form two pairs (g_1, g_7), (g_3, g_5) of sign transforms. They satisfy*

$$\begin{aligned}
\tfrac{1}{8}(g_1 + g_7 + g_3 + g_5)\left(\tfrac{z}{2}\right) &= \tfrac{1}{2} + \sum_{n=1}^{\infty}\left((-1)^{n-1}\sum_{d|n}\left(\tfrac{2}{d}\right)d\right)e(nz), \\
\tfrac{1}{32}(g_1 + g_7 - g_3 - g_5)\left(\tfrac{z}{2}\right) &= \sum_{n=1}^{\infty}\left((-1)^{n-1}\sum_{d|n}\left(\tfrac{2}{n/d}\right)d\right)e(nz), \\
\tfrac{1}{8}(g_1 - g_7 - g_3 + g_5)(z) &= \sum_{n=1}^{\infty}\left(\left(\tfrac{-2}{n}\right)\sum_{d|n}\left(\tfrac{2}{n/d}\right)d\right)e(nz) \ = \ F_{1,-1}(8z), \\
\tfrac{1}{16}(g_1 - g_7 + g_3 - g_5)(z) &= \sum_{n=1}^{\infty}\left(\left(\tfrac{-1}{n}\right)\sum_{d|n}\left(\tfrac{2}{n/d}\right)d\right)e(nz) \ = \ F_{-1,1}(8z),
\end{aligned}$$

where $F_{\delta,\varepsilon}$ is defined in Example 13.32.

We observe that $F_{1,1}(8z)$ and $F_{-1,-1}(8z)$ are the partial sums with odd-numbered coefficients in $\frac{1}{8}(g_1+g_7+g_3+g_5)\left(\frac{z}{2}\right)$ and $\frac{1}{32}(g_1+g_7-g_3-g_5)\left(\frac{z}{2}\right)$, respectively. For the coefficients in $g_j(z) = \sum_{n=0}^{\infty} b_j(n)e(nz)$ we observe

$$\begin{aligned} b_1(n) = 3b_3(n) &\qquad \text{for} \qquad n \equiv \pm 1 \bmod 8, \\ 3b_1(n) = b_3(n) &\qquad \text{for} \qquad n \equiv \pm 3 \bmod 8, \end{aligned}$$

and some more complicated rules relating $b_1(n)$, $b_3(n)$ for even n.

We have two more non-cuspidal eta products of weight 2 with order 0 at the cusp ∞. They form a pair of sign transforms, and their Fricke transforms have denominator $t = 4$. We denote these functions by

$$\begin{aligned} f &= \left[1^{-4}, 2^{14}, 4^{-6}\right], &\qquad \widetilde{f} &= \left[1^{4}, 2^{2}, 4^{-2}\right], \\ h_1 &= \left[1^{-6}, 2^{14}, 4^{-4}\right], &\qquad h_3 &= \left[1^{-2}, 2^{2}, 4^{4}\right]. \end{aligned} \tag{13.92}$$

The remaining two non-cuspidal eta products of weight 2 with denominator 4 are the sign transforms of h_1 and h_3. We denote them by

$$g_1 = \left[1^{6}, 2^{-4}, 4^{2}\right], \qquad g_3 = \left[1^{2}, 2^{-4}, 4^{6}\right]. \tag{13.93}$$

Example 13.34 *The functions in* (13.92) *combine to the Eisenstein series*

$$\begin{aligned} F(z) \;=\; \tfrac{1}{8}\big(f(z) - \widetilde{f}(z)\big) &\;=\; \sum_{n=1}^{\infty}\left(\left(\tfrac{-1}{n}\right)\sum_{d|n} d\right)e(nz), \\ \widetilde{F}(z) \;=\; -\tfrac{1}{16}\big(f\left(\tfrac{z}{4}\right) + \widetilde{f}\left(\tfrac{z}{4}\right)\big) &\;=\; -\tfrac{1}{8} + \sum_{n=1}^{\infty}\left((-1)^{n-1}\sum_{d|n,\, 4\nmid d} d\right)e(nz), \\ h_1(z) + 4h_3(z) &\;=\; \sum_{n>0 \text{ odd}} \sigma_1(n)e\left(\tfrac{nz}{4}\right), \\ h_1(z) - 4h_3(z) &\;=\; \sum_{n>0 \text{ odd}} \left(\tfrac{-1}{n}\right)\sigma_1(n)e\left(\tfrac{nz}{4}\right) \;=\; F\left(\tfrac{z}{4}\right). \end{aligned}$$

The Fricke transforms of F and $\widetilde{F}$ are

$$F(W_4 z) = -z^2\, F\left(\tfrac{z}{4}\right), \qquad \widetilde{F}(W_4 z) = -8z^2\,(h_1(4z) + 4h_3(4z)).$$

The functions in (13.93) *combine to the Eisenstein series*

$$\begin{aligned} g_1(z) + 4g_3(z) &\;=\; \sum_{n>0 \text{ odd}} \left(\tfrac{-2}{n}\right)\sigma_1(n)e\left(\tfrac{nz}{4}\right), \\ g_1(z) - 4g_3(z) &\;=\; \sum_{n>0 \text{ odd}} \left(\tfrac{2}{n}\right)\sigma_1(n)e\left(\tfrac{nz}{4}\right). \end{aligned}$$

The Fricke transform of $G_\delta(z) = g_1(z) + 4\delta g_3(z)$ is $G_\delta(W_4 z) = -4\delta z^2 G_\delta(z)$.

13.8 A Remark on Weber Functions

In [137], pp. 86, 112, Heinrich Weber introduced and used three modular functions which he denoted by f, f_1, f_2 and which we will denote, quite similarly, by $\mathfrak{f}$, $\mathfrak{f}_1$, $\mathfrak{f}_2$. The definitions are

$$\mathfrak{f}(z) = q^{-\frac{1}{48}} \prod_{n=1}^{\infty} \left(1 + q^{n-\frac{1}{2}}\right) = \frac{\eta^2(z)}{\eta(\frac{z}{2})\eta(2z)}, \qquad (13.94)$$

$$\mathfrak{f}_1(z) = q^{-\frac{1}{48}} \prod_{n=1}^{\infty} \left(1 - q^{n-\frac{1}{2}}\right) = \frac{\eta(\frac{z}{2})}{\eta(z)}, \qquad (13.95)$$

$$\mathfrak{f}_2(z) = \sqrt{2}\, q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^n) = \sqrt{2}\, \frac{\eta(2z)}{\eta(z)}, \qquad (13.96)$$

where $q = e(z)$. Immediate consequences are the relations $\mathfrak{f}\,\mathfrak{f}_1\,\mathfrak{f}_2 = \sqrt{2}$, $\mathfrak{f}(z)\,\mathfrak{f}_1(z) = \mathfrak{f}_1(2z)$, $\mathfrak{f}_1(2z)\,\mathfrak{f}_2(z) = \sqrt{2}$. We mention the Weber functions because all the pairs of sign transforms of eta products in this section are related to the modular function

$$\begin{aligned} J(z) &= \tfrac{1}{\sqrt{2}}\mathfrak{f}(2z)\mathfrak{f}_2(z) = \prod_{n=1}^{\infty}(1+q^n)(1+q^{2n-1}) \\ &= \prod_{n=1}^{\infty}(1+q^{2n})(1+q^{2n-1})^2 \\ &= \frac{\eta^3(2z)}{\eta^2(z)\eta(4z)}. \end{aligned} \qquad (13.97)$$

The eta product representation shows that the sign transform of J is the reciprocal of J itself, that is,

$$J\left(z + \tfrac{1}{2}\right) = \frac{1}{J(z)}. \qquad (13.98)$$

The product expansions show that the coefficients in

$$J(z) = \sum_{n=0}^{\infty} A(n)e(nz) \qquad (13.99)$$

allow an interpretation in terms of partitions: We have $A(n) = \sum_P 2^{r(P)}$ where the summation is on all partitions P of n in which even parts are not repeated and odd parts are repeated at most once, and where $r(P)$ is the number of those odd parts in P which are not repeated. It is easy to see that $A(n)$ is positive and even for all $n \geq 1$ and that the sequence of numbers $A(n)$ is strictly increasing with the sole exception of $A(1) = A(2) = 2$. The function J transforms according to $J(Lz) = v_J(L)J(z)$ for $L \in \Gamma_0(4)$ where

$v_J(L)$ is a certain 8th root of unity which can be computed from Theorem 1.7. We have $v_J(L) = 1$ for $L \in \Gamma_0(32)$, that is, J is a modular function for $\Gamma_0(32)$. From (13.98) one can deduce a recursion formula which expresses $A(2n)$ in terms of the products $A(j)A(2n-j)$ with $j < n$. Efficient formulae for $A(n)$ are obtained when we write J as a quotient of simple theta series which are sign transforms of each other,

$$J = \left[1^{-1}, 2^3, 4^{-1}\right]/\left[1\right] = \left[1^{-2}, 2^5, 4^{-2}\right]/\left[2^2, 4^{-1}\right] = \left[1^{-1}, 2^2\right]/\left[1, 2^{-1}, 4\right].$$

In this way we get, for example,

$$A(n) = 2\delta_n - 2 \sum_{x>0,\, 2x^2 \le n} (-1)^x A(n - 2x^2)$$

where $\delta_n = 1$ if n is a square and $\delta_n = 0$ otherwise.

Because of (13.98) it is not very surprising that the quotients of pairs of sign transforms of the eta products in this section are powers of the function J. For instance, Example 13.10 leads to

$$\left[1^{-7}, 2^{18}, 4^{-7}\right]/\left[1^7, 2^{-3}\right] = J^7, \qquad \left[1^{-3}, 2^{10}, 4^{-3}\right]/\left[1^3, 2\right] = J^3,$$

$$\left[1^{-1}, 2^6, 4^{-1}\right]/\left[1, 2^3\right] = J, \qquad \left[1^3, 2^{-2}, 4^3\right]/\left[1^{-3}, 2^7\right] = J^{-3}.$$

Example 13.15 gives

$$\left[1^{-4}, 2^{10}, 4^{-2}\right]/\left[1^4, 2^{-2}, 4^2\right] = J^4, \qquad \left[1^{-2}, 2^{10}, 4^{-4}\right]/\left[1^2, 2^4, 4^{-2}\right] = J^2,$$

and so on through all the examples in this section.