12 Prime Levels $N = p \geq 5$

12.1 Odd Weights for the Fricke Groups $\Gamma^*(p)$, $p = 5, 7, 11, 23$

For primes $p \geq 5$ the only holomorphic eta product of weight 1 and level p is $\eta_p(z) = \eta(z)\eta(pz)$. It belongs to the Fricke group. If the order at ∞ satisfies $\frac{p+1}{24} \leq 1$ then we can find complementary components such that a linear combination with $\eta_p(z)$ becomes a Hecke theta series. For $p \in \{5, 7, 11, 23\}$ the numerator of the eta product is one, $\frac{p+1}{24} = \frac{1}{t}$. Then $\eta_p(z)$ itself is a Hecke theta series. These cases are known from [31] and [65]. The result for $p = 23$ was discussed even earlier by van der Blij [12] and Schoeneberg [123]. For $p = 5$ and $p = 7$ theta series identities involving real quadratic fields are known from [63], [56].

Example 12.1 *Let $\mathcal{J}_5$ be the system of ideal numbers for $\mathbb{Q}(\sqrt{-5})$ as defined in Example 7.1. The residue of $(1+\sqrt{-5})/\sqrt{2}$ modulo 2 generates the group $(\mathcal{J}_5/(2))^\times \simeq Z_4$. A pair of characters ψ_ν on $\mathcal{J}_5$ with period 2 is fixed by*

$$\psi_\nu\left(\tfrac{1+\sqrt{-5}}{\sqrt{2}}\right) = \nu i$$

with $\nu \in \{1, -1\}$. The residues of $2+i$ and i modulo $2(2-i)$ can be chosen as generators for the group $(\mathcal{O}_1/(4-2i))^\times \simeq Z_2 \times Z_4$. A character χ on $\mathcal{O}_1$ with period $2(2-i)$ is fixed by its values

$$\chi(2+i) = -1, \qquad \chi(i) = 1.$$

Let $\widehat{\chi}$ be the character with period $2(2+i)$ which is defined by $\widehat{\chi}(\mu) = \chi(\overline{\mu})$ for $\mu \in \mathcal{O}_1$. The residues of $\frac{1}{2}(1+\sqrt{5})$ and -1 modulo 4 generate the group $\left(\mathcal{O}_{\mathbb{Q}(\sqrt{5})}/(4)\right)^\times \simeq Z_6 \times Z_2$. A Hecke character ξ on $\mathcal{O}_{\mathbb{Q}(\sqrt{5})}$ is given by

$$\xi(\mu) = \begin{cases} \mathrm{sgn}(\mu) \\ -\mathrm{sgn}(\mu) \end{cases} \quad \text{for} \quad \mu \equiv \begin{cases} \frac{1}{2}(1+\sqrt{5}) \\ -1 \end{cases} \bmod 4.$$

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DOI 10.1007/978-3-642-16152-0_12,

The theta series of weight 1 *for the characters* ξ, ψ_ν, χ *and* $\widehat{\chi}$ *are identical; we have*

$$\Theta_1\left(5, \xi, \tfrac{z}{4}\right) = \Theta_1\left(-20, \psi_\nu, \tfrac{z}{4}\right) = \Theta_1\left(-4, \chi, \tfrac{z}{4}\right) = \Theta_1\left(-4, \widehat{\chi}, \tfrac{z}{4}\right) = \eta(z)\eta(5z). \tag{12.1}$$

With $\eta_5(z) = \eta(z)\eta(5z)$*, the theta series of weights* 3 *and* 5 *satisfy*

$$\Theta_3\left(-20, \psi_\nu, \tfrac{z}{4}\right) = E_{2,5,-1}(z)\eta_5(z) - 2\nu\sqrt{5}\,\eta_5^3(z), \tag{12.2}$$

$$\Theta_5\left(-20, \psi_\nu, \tfrac{z}{4}\right) = \left(E_{2,5,-1}^2(z)\eta_5(z) - 36\eta_5^5(z)\right) + 8\nu\sqrt{5}\,E_{2,5,-1}(z)\eta_5^3(z), \tag{12.3}$$

$$\Theta_5\left(-4, \chi, \tfrac{z}{4}\right) = \left(\tfrac{4}{25}(4+3i)E_{4,5,1}(z) + \tfrac{3}{25}(3-4i)E_{2,5,-1}^2(z)\right)\eta_5(z) + \tfrac{84}{25}(3-4i)\eta_5^5(z), \tag{12.4}$$

$$\Theta_5\left(-4, \widehat{\chi}, \tfrac{z}{4}\right) = \left(\tfrac{4}{25}(4-3i)E_{4,5,1}(z) + \tfrac{3}{25}(3+4i)E_{2,5,-1}^2(z)\right)\eta_5(z) + \tfrac{84}{25}(3+4i)\eta_5^5(z). \tag{12.5}$$

In Examples 24.25 and 24.29 we will identify $\eta(z)\eta(5z)$ and $\eta(5z)\eta(20z)$ with differences of non-cuspidal eta products of level 20.

The identity (12.2) shows that $\eta^3(z)\eta^3(5z)$ is a linear combination of two Hecke theta series, and hence is lacunary. This is also clear since this function is a product of two superlacunary series, $\eta^3(z)$ and $\eta^3(5z)$. Because of (12.3), (12.4) and (12.5), $\eta^5(z)\eta^5(5z)$ is a linear combination of four Hecke theta series, and therefore it is lacunary. This was shown in [25], §3.2.

The quadratic form $x^2 + 5y^2$ represents the primes $p \equiv 1$ and $9 \bmod 20$. The characters ψ_ν in Example 12.1 satisfy $\psi_\nu(x + y\sqrt{-5}) = (-1)^y$. Therefore the identity (12.1) gives a rule whether p is represented by $x^2 + 20y^2$:

Corollary 12.2 *A prime* $p \equiv 1$ *or* $9 \bmod 20$ *is represented by the quadratic form* $x^2 + 20y^2$ *if and only if the coefficient in* $\eta(z)\eta(5z) = \sum_{n\equiv 1 \bmod 4} a(n) \times e\left(\frac{nz}{4}\right)$ *at the prime* p *satisfies* $a(p) = 2$*. If* p *is not represented by that form then* $a(p) = -2$*.*

Now we deal with level $N = 7$. Similarly as before in Example 12.1, the eta product $\eta(z)\eta(7z)$ is identified with theta series on a real quadratic field and on two imaginary quadratic fields. For one of these fields we have conjugate complex non-real periods of the characters.

In the following figure we show the values inside and close to period meshes for the characters on $\mathcal{O}_1$ and $\mathcal{O}_3$ in Examples 12.1, 12.3 which are both denoted by χ. (See also Fig. 12.1.)

Example 12.3 *The group* $(\mathcal{O}_7/(3))^\times \simeq Z_8$ *is generated by the remainder of* $\frac{1}{2}(1 + \sqrt{-7})$ *modulo* 3*. A pair of characters* ψ_ν *on* $\mathcal{O}_7$ *with period* 3 *is given*

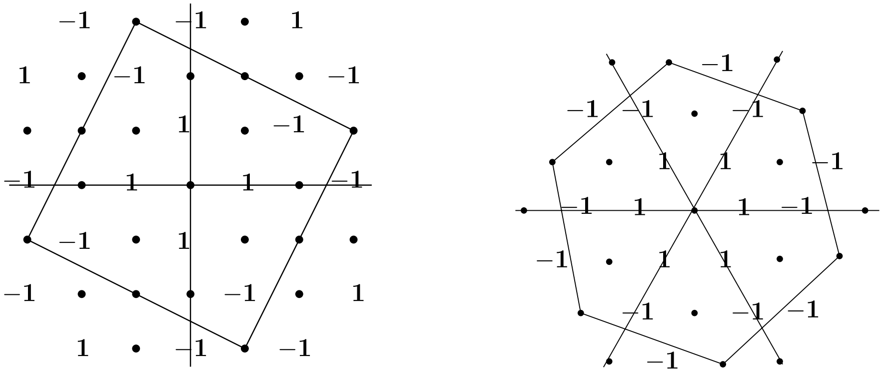

Figure 12.1: Values of the characters χ in Examples 12.1, 12.3 in period meshes

by

$$\psi_\nu\left(\tfrac{1}{2}(1+\sqrt{-7})\right) = \nu i$$

with $\nu \in \{1, -1\}$. *The remainders of* 2 *and* -1 *modulo* $4+\omega$ *can be chosen as generators for the group* $(\mathcal{O}_3/(4+\omega))^\times \simeq Z_6 \times Z_2$. *A character* χ *on* $\mathcal{O}_3$ *with period* $4+\omega$ *is fixed by the values*

$$\chi(2) = -1, \qquad \chi(-1) = 1.$$

Let $\widehat{\chi}$ *be the character with period* $5-\omega$ *which is defined by* $\widehat{\chi}(\mu) = \chi(\overline{\mu})$ *for* $\mu \in \mathcal{O}_3$. *The coprime residues modulo* $M = \frac{1}{2}(3+\sqrt{21})$ *in* $\mathcal{O}_{\mathbb{Q}(\sqrt{21})}$ *form a group of order* 2, *and a Hecke character* ξ *modulo* M *on* $\mathcal{O}_{\mathbb{Q}(\sqrt{21})}$ *is given by* $\xi(\mu) = -\mathrm{sgn}(\mu)$ *for* $\mu \equiv -1 \bmod M$. *The theta series of weight* 1 *for the characters* ξ, ψ_ν, χ *and* $\widehat{\chi}$ *are identical*; *we have*

$$\Theta_1\left(21, \xi, \tfrac{z}{3}\right) = \Theta_1\left(-7, \psi_\nu, \tfrac{z}{3}\right) = \Theta_1\left(-3, \chi, \tfrac{z}{3}\right) = \Theta_1\left(-3, \widehat{\chi}, \tfrac{z}{3}\right) = \eta(z)\eta(7z). \tag{12.6}$$

Put $\eta_7(z) = \eta(z)\eta(7z)$, *and let*

$$\Theta(z) = 2\,\Theta_1(-7, 1, z) = \sum_{\mu \in \mathcal{O}_7} e(\mu\overline{\mu}z)$$

be the theta series of weight 1 *for the trivial character on* $\mathcal{O}_7$. *Then the theta series of weights* 3 *and* 5 *for* ψ_ν *satisfy*

$$\Theta_3\left(-7, \psi_\nu, \tfrac{z}{3}\right) = E_{2,7,-1}(z)\eta_7(z) - \nu\sqrt{7}\,\Theta(z)\eta_7^2(z), \tag{12.7}$$

$$\begin{aligned}\Theta_5\left(-7, \psi_\nu, \tfrac{z}{3}\right) &= \left(E_{4,7,1}(z) + \tfrac{216}{5}\Theta(z)\eta_7^3(z)\right)\eta_7(z) \\ &\quad + 3\nu\sqrt{7}\left(\Theta^3(z) - 4\eta_7^3(z)\right)\eta_7^2(z).\end{aligned} \tag{12.8}$$

The next weight with non-vanishing theta series for χ and $\widehat{\chi}$ would be $k = 7$.

In subsequent examples we will write χ_1 and χ_{-1} for characters like χ and $\widehat{\chi}$. The advantage is a single entry $\Theta_1(D, \chi_\nu, \frac{z}{t})$ instead of two entries in formulae like (12.1), (12.6).

We note some further identities among Eisenstein series, eta products and theta series for the trivial character 1 on $\mathcal{O}_7$. They can be used to reshape (12.7) and (12.8):

Example 12.4 *Let* 1 *denote the trivial character on* $\mathcal{O}_7$. *For* $\Theta(z) = 2\,\Theta_1(-7, 1, z)$ *and weights* 3 *and* 5 *we have the identities*

$$E_{2,7,-1}(z) = \Theta^2(z), \tag{12.9}$$

$$\Theta_3(-7, 1, z) = \eta^3(z)\eta^3(7z), \tag{12.10}$$

$$\Theta_5(-7, 1, z) = E_{2,7,-1}(z)\Theta_3(-7, 1, z). \tag{12.11}$$

The identities (12.7) and (12.8) show that the modular forms $\Theta(z)\eta_7^2(z)$ and $\eta_7^3(z)$ have lacunary Fourier expansions. For $\eta_7^3(z)$ this is clear since it is a product of two superlacunary series. (Levels $N \geq 6$ are not treated in [25].)— We apply (12.6) to determine the coefficients of $\eta(z)\eta(7z)$ at primes p which satisfy $\left(\frac{p}{3}\right) = \left(\frac{p}{7}\right) = 1$. Then $p = \mu\overline{\mu} = x^2 + 7y^2$ for some $\mu = x + y\sqrt{-7} \in \mathcal{O}_7$ which is unique when we require that $x > 0$, $y > 0$. Because of $p \equiv 1 \bmod 3$ we have $xy \equiv 0 \bmod 3$. The characters ψ_ν on $\mathcal{O}_7$ satisfy

$$\psi_\nu(\mu) = \psi_\nu(\overline{\mu}) = \begin{cases} 1 \\ -1 \end{cases} \quad \text{if} \quad \begin{cases} 3|y, \\ 3|x. \end{cases}$$

Therefore we obtain the first result in the following corollary. For the second result we consider the coefficients $b(n)$ of $\Theta_3\left(\psi_\nu, \frac{z}{3}\right)$. If p is as before and $p = \mu\overline{\mu} = x^2 + 63y^2$ with $\mu = x + 3y\sqrt{-7}$, then we obtain $b(p) = \mu^2 + \overline{\mu}^2 = 2(x^2 - 63y^2)$. It follows that $b(p) = 2$ if and only if $x^2 - 63y^2 = 1$. So there is another opportunity to apply Theorem 10.4. Now the fundamental solution of our Pell equation is $x_1 = 8$, $y_1 = 1$, and for $p_m = 2x_m^2 - 1$ we find the primes $p_1 = 127$, $p_2 = 32257$,

$$p_{16} = 1500\,38171\,39490\,50304\,32003\,28185\,43397\,10977,$$

while $p_4 = 193 \cdot 107\,82529$ and $p_8 = 598\,98367 \cdot 14\,46008\,68351$ are composite.

Corollary 12.5 *Define* $a(n)$ *and* $b(n)$ *by the expansions*

$$\eta(z)\eta(7z) = \sum_{n \equiv 1 \bmod 3} a(n) e\left(\tfrac{nz}{3}\right),$$

$$E_{2,7,-1}(z)\eta(z)\eta(7z) = \sum_{n \equiv 1 \bmod 3} b(n) e\left(\tfrac{nz}{3}\right).$$

Then for primes p with $\left(\frac{p}{3}\right) = \left(\frac{p}{7}\right) = 1$ we have

$$a(p) = \begin{cases} 2 \\ -2 \end{cases} \quad \text{if} \quad \begin{cases} p = x^2 + 63y^2 \\ p = 9x^2 + 7y^2 \end{cases}$$

for some $x, y \in \mathbb{N}$. Moreover, $b(p) = 2$ if and only if $p = x^2 + 63y^2$ and $x^2 - 63y^2 = 1$ for some $x, y \in \mathbb{N}$.

For level $N = 11$, the eta product $\eta(z)\eta(11z)$ is a theta series for just one imaginary quadratic field:

Example 12.6 *The remainder of $\frac{1}{2}(1+\sqrt{-11})$ modulo 2 generates the cyclic group $(\mathcal{O}_{11}/(2))^\times \simeq Z_3$. A pair of characters ψ_ν on $\mathcal{O}_{11}$ with period 2 is given by*

$$\psi_\nu\left(\tfrac{1}{2}(1+\sqrt{-11})\right) = \omega^{2\nu} = \tfrac{1}{2}(-1+\nu\sqrt{-3})$$

with $\nu \in \{1, -1\}$. The theta series of weight 1 for ψ_ν satisfy

$$\Theta_1\left(-11, \psi_\nu, \tfrac{z}{2}\right) = \eta(z)\eta(11z). \tag{12.12}$$

Put $\eta_{11}(z) = \eta(z)\eta(11z)$, and let

$$\Theta(z) = \Theta_1(-11, 1, z) = \frac{1}{2}\sum_{\mu\in\mathcal{O}_{11}} e(\mu\overline{\mu}z)$$

be the theta series of weight 1 for the trivial character on $\mathcal{O}_{11}$. Then for weights 3 and 5 we have the identities

$$\Theta_3\left(-11, \psi_\nu, \tfrac{z}{2}\right) = \Theta^2(z)\eta_{11}(z) - \tfrac{1}{2}\left(1+\nu\sqrt{33}\right)\eta_{11}^3(z), \tag{12.13}$$

$$\begin{aligned}\Theta_5\left(-11, \psi_\nu, \tfrac{z}{2}\right) &= \Theta^4(z)\eta_{11}(z) - \tfrac{1}{2}\left(-21+5\nu\sqrt{33}\right)\Theta^2(z)\eta_{11}^3(z) \\ &\quad + 4\big(5-\nu\sqrt{33}\big)\eta_{11}^5(z).\end{aligned} \tag{12.14}$$

Corollary 12.7 *Let $\Theta(z)$ be given as in Example 12.6. Define $a_1(n)$, $a_3(n)$ and $c(n)$ by the expansions*

$$\begin{aligned}\eta(z)\eta(11z) &= \sum_{n\equiv 1 \bmod 2} a_1(n)e\big(\tfrac{nz}{2}\big), \\ \eta^3(z)\eta^3(11z) &= \sum_{n\equiv 1 \bmod 2} a_3(n)e\big(\tfrac{nz}{2}\big), \\ \Theta^2(z)\eta(z)\eta(11z) &= \sum_{n\equiv 1 \bmod 2} c(n)e\big(\tfrac{nz}{2}\big).\end{aligned}$$

Then for primes p with $\left(\frac{p}{11}\right) = 1$ the following assertions hold:

(1) *We have*

$$a_1(p) = \begin{cases} -1 \\ \;\;\,2 \end{cases} \quad \textit{if} \quad \begin{cases} p = \frac{1}{4}(x^2 + 11y^2) \;\; \textit{with} \;\; x, y \;\; \textit{odd}, \\ p = x^2 + 11y^2. \end{cases}$$

(2) *If* $a_1(p) = 2$, $p = x^2 + 11y^2$ *then* $a_3(p) = 0$ *and* $c(p) = 2(x^2 - 11y^2)$. *We have* $c(p) = 2$ *if and only if the prime* p *belongs to the sequence of numbers* P_m *defined by* $P_1 = 199$, $P_{m+1} = 2P_m^2 - 1$.

Proof. Let p be a prime with $\left(\frac{p}{11}\right) = 1$. Then p is split in $\mathcal{O}_{11}$, hence $p = \mu\overline{\mu} = \frac{1}{4}(x^2 + 11y^2)$ with $x \equiv y \bmod 2$, and $\mu = \frac{1}{2}(x + y\sqrt{-11})$ is unique when we require that $x > 0$, $y > 0$. If x, y are odd then $\psi_\nu(\mu) = \omega^2$, $\psi_\nu(\overline{\mu}) = \omega^{-2}$ or vice versa, and then (12.12) implies $a_1(p) = \omega^2 + \omega^{-2} = -1$. If x, y are even we write $2x, 2y$ instead of x, y. Then $\psi_\nu(\mu) = \psi_\nu(\overline{\mu}) = 1$, and (12.12) implies $a_1(p) = 2$. This proves (1).

From Jacobi's identity (1.7) we infer

$$a_3(n) = \sum_{u,v>0,\, u^2+11v^2=4n} \left(\frac{-1}{uv}\right) uv.$$

We suppose that $a_1(p) = 2$. Then $p = x^2 + 11y^2$ has a unique solution in positive integers x, y. Since the prime 2 is inert in $\mathcal{O}_{11}$ it follows that $4p = u^2 + 11v^2$ has no solution in integers. Therefore the sum for $a_3(p)$ is empty, hence $a_3(p) = 0$. Now from (12.13) it follows that $c(p)$ is the coefficient of $\Theta_3\left(\psi_\nu, \frac{z}{2}\right)$ at p, i.e.,

$$c(p) = \mu^2 + \overline{\mu}^2 = 2(x^2 - 11y^2).$$

Finally, we have $c(p) = 2$ if and only if $x^2 - 11y^2 = 1$. The fundamental solution of this Pell equation is $x_1 = 10$, $y_1 = 3$. Hence from Theorem 10.4 we obtain the last assertion in (2). In this example, $P_1 = 199$ and $P_2 = 79201$ are prime, while $P_3 = 31 \cdot 4046\,96671$ and P_4, P_5 are composite. □

Now we discuss the prime level $N = 23$. The eta product $\eta(z)\eta(23z)$ has denominator $t = 1$. It is a theta series for $\mathbb{Q}(\sqrt{-23})$ whose characters have period 1, i.e., they are characters of the ideal class group of this field.

Example 12.8 *Let* $\Lambda = \Lambda_{23} = \sqrt[3]{(3 + \sqrt{-23})/2}$ *and* $\mathcal{J}_{23} = \mathcal{O}_{23} \cup \mathcal{A}_2 \cup \mathcal{A}_3$ *be given as in Example* 7.13. *Let* ψ_ν *be the non-trivial characters of the ideal class group of* $\mathbb{Q}(\sqrt{-23})$, *defined on* $\mathcal{J}_{23}$ *by* $\psi_\nu(\mu) = 1$ *for* $\mu \in \mathcal{O}_{23}$, $\psi_\nu(\mu) = \omega^{2\nu}$ *for* $\mu \in \mathcal{A}_2$, $\psi_\nu(\mu) = \omega^{-2\nu}$ *for* $\mu \in \mathcal{A}_3$, *with* $\nu \in \{1, -1\}$. *Then we have*

$$\Theta_1(-23, \psi_\nu, z) = \eta(z)\eta(23z). \tag{12.15}$$

From (12.15) we deduce some of the results of van der Blij [12] and Schoeneberg [123]. (See also Zagier's article in [16].) We define $a(n)$ by the expansion

$$\eta(z)\eta(23z) = \sum_{n=1}^{\infty} a(n)e(nz).$$

We recall that the three subsets of $\mathcal{J}_{23}$ correspond to the ideal classes A_1, A_2, A_3 in $\mathbb{Q}(\sqrt{-23})$ (with A_1 the principal class), which in turn correspond to the classes of binary quadratic forms of discriminant $D = -23$, represented by $\frac{1}{4}((2x+y)^2+23y^2)$ and $\frac{1}{8}((4x\pm y)^2+23y^2)$. If A is one of the ideal classes, let $a(n, A)$ denote the number of ideals in A whose norm is n.

Let p be a prime with $\left(\frac{p}{23}\right) = 1$. Then we have $p = \mu\overline{\mu}$ where either $\mu, \overline{\mu} \in \mathcal{O}_{23}$ or $\mu \in \mathcal{A}_2$, $\overline{\mu} \in \mathcal{A}_3$. In the first case (12.15) yields $a(p) = 2$, and necessarily p is of the form $p = x^2 + 23y^2$ with $x, y \in \mathbb{N}$, $6|xy$. In the second case we get $a(p) = \omega^2 + \overline{\omega}^2 = -1$, and there is a representation $8p = x^2 + 23y^2$ with $2 \nmid xy$, $3|xy$. Thus we have

$$a(p) = \begin{cases} 2 = a(p, A_1) \\ -1 = -a(p, A_2) = -a(p, A_3) \end{cases} \quad \text{if} \quad \begin{cases} p = x^2 + 23y^2, \\ 8p = x^2 + 23y^2. \end{cases} \tag{12.16}$$

It follows that $a(n) = a(n, A_1) - a(n, A_2)$ for all n.

From the definition of the characters in Example 12.8 we obtain

$$2\,\Theta_k(-23, \psi_\nu, z) = \sum_{\mu\in\mathcal{O}_{23}} \mu^{k-1}e(\mu\overline{\mu}z) + \sum_{\mu\in\mathcal{A}_2} \left(\omega^{2\nu}\mu^{k-1} + \overline{\omega}^{2\nu}\overline{\mu}^{k-1}\right) e(\mu\overline{\mu}z)$$

for any odd $k \geq 1$. On the other hand, for the trivial character 1 on $\mathcal{J}_{23}$ we get

$$2\,\Theta_k(-23, 1, z) = \sum_{\mu\in\mathcal{O}_{23}} \mu^{k-1}e(\mu\overline{\mu}z) + \sum_{\mu\in\mathcal{A}_2} \left(\mu^{k-1} + \overline{\mu}^{k-1}\right) e(\mu\overline{\mu}z).$$

Adding the relations, and using that $\omega^2 + \overline{\omega}^2 + 1 = 0$, we obtain

$$2\left(\Theta_k(-23, \psi_1, z) + \Theta_k(-23, \psi_{-1}, z) + \Theta_k(-23, 1, z)\right) = 3 \sum_{\mu\in\mathcal{O}_{23}} \mu^{k-1}e(\mu\overline{\mu}z).$$

Similarly we can represent $\sum_{\mu\in\mathcal{A}_2} \mu^{k-1}e(\mu\overline{\mu}z)$ as a linear combination of three theta series. Thus we get two linearly independent modular forms

$$\sum_{\mu\in\mathcal{O}_{23}} \mu^{k-1}e(\mu\overline{\mu}z) \qquad \text{and} \qquad \sum_{\mu\in\mathcal{A}_2} \mu^{k-1}e(\mu\overline{\mu}z)$$

which are cusp forms for weight $k \geq 3$ and non-cuspidal for weight $k = 1$. The procedure is a symmetrization by means of the characters of the ideal class group and was, of course, known to Hecke. The result is also contained as a special case in Kahl's Theorem 5.2. Schoeneberg [123] observed that the relation $a(n) = a(n, A_1) - a(n, A_2)$ holds more generally for the coefficients of $\eta(z)\eta(|D|z)$ for any discriminant $D < 0$, $D \equiv 1 \bmod 24$, and suitable ideal classes A_1, A_2 of $\mathbb{Q}(\sqrt{D})$. A similar, though more complicated result for $D = -184$ will be obtained in Example 21.3.

12.2 Weight 1 for the Fricke Groups $\Gamma^*(p)$, $p = 13, 17, 19$

The eta product $\eta(z)\eta(13z)$ is a component in theta series for the fields $\mathbb{Q}(\sqrt{-13})$, $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{39})$. Gordon and Hughes [42] identified the other component with a linear combination of eta products of level 156. When we checked their formula we had to change two numerical factors and to replace two of the functions by their sign transforms; note the discrepancies between our formula for the component f_1 below and that in [42], p. 429.

Example 12.9 *Let $\mathcal{J}_{13}$ be the system of ideal numbers for $\mathbb{Q}(\sqrt{-13})$ as defined in Example 7.1. The residues of $2+\sqrt{-13}$ and $(3+\sqrt{-13})/\sqrt{2}$ modulo 6 can be chosen as generators of the group $(\mathcal{J}_{13}/(6))^\times \simeq Z_8 \times Z_4$, where $(2+\sqrt{-13})^4 \equiv -1 \bmod 6$. Four characters $\chi_{\delta,\nu}$ on $\mathcal{J}_{13}$ with period 6 are fixed by their values*

$$\chi_{\delta,\nu}(2+\sqrt{-13}) = \delta\nu i, \qquad \chi_{\delta,\nu}\left(\tfrac{1}{\sqrt{2}}(3+\sqrt{-13})\right) = -\nu i$$

with $\delta, \nu \in \{1, -1\}$. The residues of $3+\omega$, $1+6\omega$, $9+4\omega$ and ω modulo $4(5+2\omega)$ can be chosen as generators of the group $(\mathcal{O}_3/(20+8\omega))^\times \simeq Z_{12} \times Z_2^2 \times Z_6$. Two characters $\psi_{\delta,1}$ on $\mathcal{O}_3$ with period $4(5+2\omega)$ are given by

$$\psi_{\delta,1}(3+\omega) = 1, \qquad \psi_{\delta,1}(1+6\omega) = -\delta, \qquad \psi_{\delta,1}(9+4\omega) = -1, \qquad \psi_{\delta,1}(\omega) = 1.$$

Let $\psi_{\delta,-1}$ be the characters with period $4(5+2\overline{\omega})$ which are defined by $\psi_{\delta,-1}(\mu) = \psi_{\delta,1}(\overline{\mu})$ for $\mu \in \mathcal{O}_3$. Let the ideal numbers $\mathcal{J}_{\mathbb{Q}(\sqrt{39})}$ be chosen as in Example 7.17. The residues of $\frac{1}{\sqrt{2}}(7+\sqrt{39})$ and -1 modulo $M = 2(6+\sqrt{39})$ are generators of $\left(\mathcal{J}_{\mathbb{Q}(\sqrt{39})}/(M)\right)^\times \simeq Z_4 \times Z_2$. Hecke characters ξ_δ on $\mathcal{J}_{\mathbb{Q}(\sqrt{39})}$ with period M are given by

$$\xi_\delta(\mu) = \begin{cases} \delta\,\mathrm{sgn}(\mu) \\ -\mathrm{sgn}(\mu) \end{cases} \quad \textit{for} \quad \begin{cases} \frac{1}{\sqrt{2}}(7+\sqrt{39}) \\ -1 \end{cases} \bmod M.$$

The theta series of weight 1 for the characters ξ_δ, $\chi_{\delta,\nu}$, $\psi_{\delta,\nu}$ satisfy the identities

$$\Theta_1\left(156, \xi_\delta, \tfrac{z}{12}\right) = \Theta_1\left(-52, \chi_{\delta,\nu}, \tfrac{z}{12}\right) = \Theta_1\left(-3, \psi_{\delta,\nu}, \tfrac{z}{12}\right) = f_1(z) + 2\delta f_7(z) \tag{12.17}$$

with normalized integral Fourier series f_j with denominator 12 and numerator classes j modulo 12. The component f_7 is an eta product,

$$f_7(z) = \eta(z)\eta(13z). \tag{12.18}$$

The component f_1 is a linear combination of eta products of level 156,

$$\begin{aligned} f_1 \;&=\; \left[2^{-1}, 4, 6^2, 12^{-1}, 39^{-2}, 78^5, 156^{-2}\right] \\ &\quad + \left[3^{-2}, 6^5, 12^{-2}, 26^{-1}, 52, 78^2, 156^{-1}\right] \\ &\quad - 2\left[6^{-1}, 12^2, 13^{-1}, 26^2, 39, 52^{-1}, 78^{-1}, 156\right] \\ &\quad - 2\left[1^{-1}, 2^2, 3, 4^{-1}, 6^{-1}, 12, 78^{-1}, 156^2\right]. \end{aligned}$$

A corresponding result for the sign transforms is stated in Example 22.5.

For level $N = 17$ we find the expected component $\eta(z)\eta(17z)$ and another component which is a combination of eta products of level 68:

Example 12.10 *Let $\mathcal{J}_{17}$ be the system of ideal numbers for $\mathbb{Q}(\sqrt{-17})$ as defined in Example 7.9. The residue of $\Lambda = \Lambda_{17} = \sqrt{\frac{1}{\sqrt{2}}(1+\sqrt{-17})}$ modulo 2 generates the group $(\mathcal{J}_{17}/(2))^\times \simeq Z_8$. Four characters $\chi_{\delta,\nu}$ on $\mathcal{J}_{17}$ with period 2 are fixed by their value*

$$\chi_{\delta,\nu}(\Lambda) = \xi = \tfrac{1}{\sqrt{2}}(\delta + \nu i),$$

a primitive 8th root of unity, with $\delta, \nu \in \{1, -1\}$. The theta series of weight 1 for $\chi_{\delta,\nu}$ satisfy

$$\Theta_1\left(-68, \chi_{\delta,\nu}, \tfrac{z}{4}\right) = f_1(z) + \delta\sqrt{2}\, f_3(z) \tag{12.19}$$

with normalized integral Fourier series f_j with denominator 4 and numerator classes j modulo 4. The components are eta products or linear combinations thereof,

$$f_1 = \left[\frac{4^2, 34^5}{2, 17^2, 68^2}\right] - \left[\frac{2^5, 68^2}{1^2, 4^2, 34}\right], \qquad f_3 = [1, 17]. \tag{12.20}$$

The characters in Example 12.10 are not induced by the norm, and therefore (by Theorem 5.1) the components f_1, f_7 are cusp forms. Remarkably, in (12.20) the cusp form f_1 is written as a difference of two non-cuspidal eta products of level 68 which do not belong to the Fricke group. The sign transforms of f_1, f_7 will appear in Example 22.1 when we discuss level 68. From the definition of $\chi_{\delta,\nu}$, or from (12.20), (8.5), (8.8) we see that the coefficient of $f_1(z)$ at an integer $n \equiv 1 \bmod 4$ is given by

$$\sum_{x>0,\, y\in\mathbb{Z},\, x^2+68y^2=n} 1 \;-\; \sum_{x>0,\, y\in\mathbb{Z},\, 4x^2+17y^2=n} 1.$$

In particular, if $p \equiv 1 \bmod 4$ is prime and $\left(\frac{p}{17}\right) = 1$, then this coefficient is 2 or -2 if p is represented by the quadratic form $x^2 + 68y^2$ or $4x^2 + 17y^2$, respectively.

The theta series in Example 12.10 will appear once more in Example 22.14.

For level $N = 19$ there is a theta series with component $\eta(z)\eta(19z)$. In [42] the other component is identified with a linear combination of eta products with level 456:

Example 12.11 *The residue of* $\frac{1}{2}(1+\sqrt{-19})$ *modulo* 6 *generates the group* $(\mathcal{O}_{19}/(6))^\times \simeq Z_{24}$. *A quadruplet of characters* $\chi_{\delta,\nu}$ *on* $\mathcal{O}_{19}$ *with period* 6 *is given by*

$$\chi_{\delta,\nu}\left(\tfrac{1}{2}(1+\sqrt{-19})\right) = \xi = \tfrac{1}{2}(\delta\sqrt{3}+\nu i),$$

a primitive 12*th root of unity, with* $\delta, \nu \in \{1, -1\}$. *The theta series of weight* 1 *for* $\chi_{\delta,\nu}$ *decomposes as*

$$\Theta_1\left(-19, \chi_{\delta,\nu}, \tfrac{z}{6}\right) = f_1(z) + \delta\sqrt{3}\, f_5(z) \tag{12.21}$$

with normalized integral Fourier series f_j *with denominator* 6 *and numerator classes* j *modulo* 6. *The component* f_5 *is an eta product,*

$$f_5(z) = \eta(z)\eta(19z). \tag{12.22}$$

The component f_1 *is a linear combination of six eta products of level* 456,

$$\begin{aligned} f_1 \;=\;& \left[4^{-1}, 8, 12^2, 24^{-1}, 114^{-2}, 228^5, 456^{-2}\right] \\ &+ \left[3^{-1}, 6^2, 19^{-1}, 38, 57^2, 114^{-1}\right] \\ &- \left[6^{-2}, 12^5, 24^{-2}, 76^{-1}, 152, 228^2, 456^{-1}\right] \\ &- \left[1^{-1}, 2, 3^2, 6^{-1}, 57^{-1}, 114^2\right] \\ &- 2\left[12^{-1}, 24^2, 38^{-1}, 76^2, 114, 152^{-1}, 228^{-1}, 456\right] \\ &+ 2\left[2^{-1}, 4^2, 6, 8^{-1}, 12^{-1}, 24, 228^{-1}, 456^2\right]. \end{aligned}$$

In Example 21.2, in a similar result for the sign transform of $\eta(z)\eta(19z)$, we will need characters on $\mathcal{O}_{19}$ with period 12.

12.3 Weight 2 for $\Gamma_0(p)$

The only new eta product of weight 2 for the Fricke group $\Gamma^*(p)$ is $\eta^2(z)\eta^2(pz)$. For $p = 5$ and $p = 11$ it is a Hecke eigenform. But it is not lacunary, so there cannot be an identity of the kind listed in this monograph. The function $\eta^2(z)\eta^2(11z)$ is a prominent example of a weight 2 cuspidal eigenform: Its associated Dirichlet series is the zeta function of the elliptic curve

$Y^2 - Y = X^3 - X^2$ with conductor 11 ([55], p. 321, [136], p. 365). This is the simplest example for the celebrated relation between elliptic curves and weight 2 modular forms. Martin and Ono [93] determined all eta products which are weight 2 newforms and listed the corresponding elliptic curves. The cusp form $\eta^2(z)\eta^2(11z)$ can be identified with a linear combination of two non-cusp forms; with $\Theta(z)$ as in Example 12.6 we have

$$\eta^2(z)\eta^2(11z) = \tfrac{5}{8}\left(\Theta^2(z) - E_{2,11,-1}(z)\right). \tag{12.23}$$

For all primes p there are the new weight 2 eta products $\left[1^3, p\right]$ and $\left[1, p^3\right]$ for $\Gamma_0(p)$. These are the only ones if $p \geq 7$. They are lacunary since they are products of two superlacunary series. For $\Gamma_0(5)$ there are, in addition, two new non-cuspidal eta products $\left[1^5, 5^{-1}\right]$ and $\left[1^{-1}, 5^5\right]$ of weight 2.

In [42], linear combinations of $\left[1^3, 5\right]$ and $\left[5^3, 1\right]$ are identified with Hecke theta series:

Example 12.12 *Let $\mathcal{J}_{15}$ be the system of ideal numbers for $\mathbb{Q}(\sqrt{-15})$ as defined in Example* 7.3. *The residues of $\frac{1}{2}\left(\sqrt{3}+\sqrt{-5}\right)$ and -1 modulo $\sqrt{3}$ generate the group $(\mathcal{J}_{15}/(\sqrt{3}))^\times \simeq Z_2^2$. A pair of characters ψ_δ on $\mathcal{J}_{15}$ with period $\sqrt{3}$ is given by*

$$\psi_\delta\left(\tfrac{1}{2}\left(\sqrt{3}+\sqrt{-5}\right)\right) = \delta, \qquad \psi_\delta(-1) = -1$$

with $\delta \in \{1, -1\}$. The corresponding theta series of weight 2 *decompose as*

$$\Theta_2\left(-15, \psi_\delta, \tfrac{z}{3}\right) = f_1(z) + \delta i\sqrt{5}\, f_2(z) \tag{12.24}$$

with normalized integral Fourier series f_j with denominator 3 *and numerator classes j modulo* 3. *Both the components are eta products,*

$$f_1(z) = \eta^3(z)\eta(5z), \qquad f_2(z) = \eta(z)\eta^3(5z). \tag{12.25}$$

For $p = 7$ and $p = 11$ one finds complementary components such that linear combinations with $\left[1^3, p\right]$ and $\left[1, p^3\right]$ are Hecke theta series. The result for $p = 7$ is known from [42]:

Example 12.13 *Let $\mathcal{J}_{21}$ be the system of ideal numbers for $\mathbb{Q}(\sqrt{-21})$ as defined in Example* 7.6. *The residues of $\frac{1}{\sqrt{2}}(\sqrt{3}+\sqrt{-7})$ and $\sqrt{-7}$ modulo $2\sqrt{3}$ can be chosen as generators of $(\mathcal{J}_{21}/(2\sqrt{3}))^\times \simeq Z_4^2$. Four characters $\psi_{\delta,\varepsilon}$ on $\mathcal{J}_{21}$ with period $2\sqrt{3}$ are fixed by their values*

$$\psi_{\delta,\varepsilon}\left(\tfrac{1}{\sqrt{2}}\left(\sqrt{3}+\sqrt{-7}\right)\right) = \delta i, \qquad \psi_{\delta,\varepsilon}(\sqrt{-7}) = \varepsilon i$$

with $\delta, \varepsilon \in \{1, -1\}$. The corresponding theta series of weight 2 *decompose as*

$$\Theta_2\left(-84, \psi_{\delta,\varepsilon}, \tfrac{z}{12}\right) = f_1(z) + \delta i\sqrt{6}\, f_5(z) - \varepsilon\sqrt{7}\, f_7(z) + \delta\varepsilon i\sqrt{42}\, f_{11}(z) \tag{12.26}$$

with normalized integral Fourier series f_j with denominator 12 and numerator classes j modulo 12. The components f_5 and f_{11} are eta products,

$$f_5(z) = \eta^3(z)\eta(7z), \qquad f_{11}(z) = \eta(z)\eta^3(7z). \tag{12.27}$$

The components f_1 and f_7 are linear combinations of eta products of level 28,

$$f_1 = \left[2^5, 4^{-2}, 7^{-2}, 14^5, 28^{-2}\right] + 4\left[1^2, 2^{-1}, 4^2, 14^{-1}, 28^2\right], \tag{12.28}$$
$$f_7 = \left[1^{-2}, 2^5, 4^{-2}, 14^5, 28^{-2}\right] + 4\left[2^{-1}, 4^2, 7^2, 14^{-1}, 28^2\right]. \tag{12.29}$$

Example 12.14 *Let $\mathcal{J}_{33}$ be the system of ideal numbers for $\mathbb{Q}(\sqrt{-33})$ as defined in Example 7.6. The residues of $\frac{1}{\sqrt{2}}(\sqrt{3}+\sqrt{-11})$, $\sqrt{-11}$ and -1 modulo $2\sqrt{3}$ can be chosen as generators of $(\mathcal{J}_{33}/(2\sqrt{3}))^\times \simeq Z_4 \times Z_2^2$. Four characters $\psi_{\delta,\varepsilon}$ on $\mathcal{J}_{33}$ with period $2\sqrt{3}$ are given by*

$$\psi_{\delta,\varepsilon}\left(\tfrac{1}{\sqrt{2}}\left(\sqrt{3}+\sqrt{-11}\right)\right) = \varepsilon, \qquad \psi_{\delta,\varepsilon}(\sqrt{-11}) = \delta\varepsilon, \qquad \psi_{\delta,\varepsilon}(-1) = -1$$

with $\delta, \varepsilon \in \{1, -1\}$. The theta series of weight 2 for $\psi_{\delta,\varepsilon}$ decompose as

$$\Theta_2\left(-132, \psi_{\delta,\varepsilon}, \tfrac{z}{12}\right) = f_1(z) + \delta i\sqrt{66}\, f_5(z) + \varepsilon\sqrt{6}\, f_7(z) + \delta\varepsilon i\sqrt{11}\, f_{11}(z) \tag{12.30}$$

with normalized integral Fourier series f_j with denominator 12 and numerator classes j modulo 12. The components f_5 and f_7 are eta products,

$$f_5(z) = \eta(z)\eta^3(11z), \qquad f_7(z) = \eta^3(z)\eta(11z). \tag{12.31}$$

For $p = 13$ there are theta series on $\mathbb{Q}(\sqrt{-39})$ whose "second" components are linear combinations of the eta products $\left[1^3, 13\right]$ and $\left[1, 13^3\right]$ with denominator $t = 3$. We use the system of ideal numbers $\mathcal{J}_{39}$ from Example 7.8, where $\Lambda = \Lambda_{39}$ is a root of the polynomial $X^8 - 5X^4 + 16$. The eight roots are $\pm c \pm di$, $\pm d \pm ci$ with $c = \frac{1}{2}\sqrt{4+\sqrt{13}} > d = \frac{1}{2}\sqrt{4-\sqrt{13}} > 0$. Theorem 5.1 asks for characters with period $\sqrt{-3}$. The group $(\mathcal{J}_{39}/(\sqrt{-3}))^\times \simeq Z_4 \times Z_2$ is generated by the residues of Λ and -1, and we have $\overline{\Lambda} \equiv -\Lambda^3 \bmod \sqrt{-3}$. For weight 2 we need characters χ with $\chi(-1) = -1$. The four choices for the value at Λ yield four different theta series. For different choices of the root Λ the four theta series are merely permuted. We obtain the following result:

Example 12.15 *Let $\mathcal{J}_{39}$ be the system of ideal numbers for $\mathbb{Q}(\sqrt{-39})$ as defined in Example 7.8, where $\Lambda = \Lambda_{39}$ is a root of the polynomial $X^8 - 5X^4 + 16$. The residues of Λ and -1 modulo $\sqrt{-3}$ can be chosen as generators of $(\mathcal{J}_{39}/(\sqrt{-3}))^\times \simeq Z_4 \times Z_2$. Two pairs of characters χ_δ and ψ_δ on $\mathcal{J}_{39}$ with period $\sqrt{-3}$ are given by*

$$\chi_\delta(\Lambda) = \delta, \quad \chi_\delta(-1) = -1, \qquad \psi_\delta(\Lambda) = \delta i, \quad \psi_\delta(-1) = -1$$

with $\delta \in \{1, -1\}$. *If we choose* $\Lambda = \frac{1}{2}(\sqrt{4+\sqrt{13}} + i\sqrt{4-\sqrt{13}})$ *then the theta series of weight* 2 *for* χ_δ *and* ψ_δ *decompose as*

$$\begin{aligned} \Theta_2\left(-39, \chi_\delta, \tfrac{z}{3}\right) &= f_{1,\delta}^{(-1)}(z) + f_{2,\delta}^{(-1)}(z), \\ \Theta_2\left(-39, \psi_\delta, \tfrac{z}{3}\right) &= f_{1,\delta}^{(1)}(z) + f_{2,\delta}^{(1)}(z) \end{aligned} \tag{12.32}$$

with Fourier series $f_{j,\delta}^{(\nu)}(z) = \sum_{n\equiv j \bmod 3} \alpha_{j,\delta}^{(\nu)}(n) e\left(\frac{nz}{3}\right)$ *whose coefficients are algebraic integers. The components* $f_{2,\delta}^{(-1)}$ *and* $f_{2,\delta}^{(1)}$ *are linear combinations of eta products,*

$$f_{2,\delta}^{(\nu)}(z) = \delta i \sqrt{4+\nu\sqrt{13}}\left(\eta^3(z)\eta(13z) + \nu\sqrt{13}\eta(z)\eta^3(13z)\right). \tag{12.33}$$

A different choice for Λ *results in a permutation of the theta series.*

We close this subsection with a description of the non-cuspidal eta products of weight 2 for $\Gamma_0(5)$. They constitute an example of Hecke's Eisenstein series in Theorem 1.9:

Example 12.16 *We have the identities*

$$\frac{\eta^5(z)}{\eta(5z)} = 1 - 5\sum_{n=1}^{\infty}\left(\sum_{d>0,\, d|n}\left(\frac{d}{5}\right) d\right) e(nz), \tag{12.34}$$

$$\frac{\eta^5(5z)}{\eta(z)} = \sum_{n=1}^{\infty}\left(\sum_{d>0,\, d|n}\left(\frac{n/d}{5}\right) d\right) e(nz). \tag{12.35}$$

The formula (12.35) is equivalent with a famous formula of Ramanujan; see [9], p. 107.

12.4 Weights 3 and 5 for $\Gamma_0(5)$

In this subsection we present the results of Cooper, Gun and Ramakrishnan [25] on lacunary eta products of level 5 with weights $k > 2$ which do not belong to the Fricke group. There are four of them with weight 3 and two with weight 5. Each of the eta products $[1^7, 5^{-1}]$, $[1, 5^5]$, $[1^5, 5]$ and $[1^{-1}, 5^7]$ is a linear combination of four theta series on the Gaussian field $\mathbb{Q}(\sqrt{-1})$, and hence is lacunary:

Example 12.17 *The residues of* $2-i$, $2+3i$ *and* i *modulo* $6(2+i)$ *can be chosen as generators of* $(\mathcal{O}_1/(12+6i))^\times \simeq Z_8 \times Z_2 \times Z_4$. *Characters* $\varphi_{\delta,1}$ *on* $\mathcal{O}_1$ *with period* $6(2+i)$ *are defined by*

$$\varphi_{\delta,1}(2-i) = \delta i, \qquad \varphi_{\delta,1}(2+3i) = -1, \qquad \varphi_{\delta,1}(i) = -1$$

with $\delta \in \{1, -1\}$. *Define the characters* $\varphi_{\delta,-1}$ *on* $\mathcal{O}_1$ *with period* $6(2-i)$ *by* $\varphi_{\delta,-1}(\mu) = \varphi_{\delta,1}(\overline{\mu})$ *for* $\mu \in \mathcal{O}_1$. *For* $\delta, \varepsilon \in \{1, -1\}$, *the theta series of weight* 3 *for these characters satisfy*

$$\Theta_3\left(-4, \varphi_{\delta,\varepsilon}, \tfrac{z}{12}\right) = \frac{\eta^7(z)}{\eta(5z)} + (7 - 24\varepsilon i)\,\eta(z)\eta^5(5z) + \delta\varepsilon\left((4+3\varepsilon i)\,\eta^5(z)\eta(5z) + 5(4-3\varepsilon i)\,\frac{\eta^7(5z)}{\eta(z)}\right). \tag{12.36}$$

The last example from [25] shows that $\left[1^{11}, 5^{-1}\right]$ and $\left[1^{-1}, 5^{11}\right]$ are lacunary. They are linear combinations of the theta series of weight 5 from Example 12.1:

Example 12.18 *Let* χ, $\widehat{\chi}$ *and* ψ_ν *be the characters on* $\mathcal{O}_1$ *and on* $\mathcal{J}_5$, *respectively, as defined in Example* 12.1. *The corresponding theta series of weight* 5 *satisfy*

$$\frac{\eta^{11}(z)}{\eta(5z)} + 55\,\frac{\eta^{11}(5z)}{\eta(z)} = \tfrac{9}{32}\left(\Theta_5\left(-20, \psi_1, \tfrac{z}{4}\right) + \Theta_5\left(-20, \psi_{-1}, \tfrac{z}{4}\right)\right) + \tfrac{7}{32}\left(\Theta_5\left(-4, \chi, \tfrac{z}{4}\right) + \Theta_5\left(-4, \widehat{\chi}, \tfrac{z}{4}\right)\right), \tag{12.37}$$

$$\frac{\eta^{11}(z)}{\eta(5z)} + 195\,\frac{\eta^{11}(5z)}{\eta(z)} = \tfrac{1}{2}\left(\Theta_5\left(-20, \psi_1, \tfrac{z}{4}\right) + \Theta_5\left(-20, \psi_{-1}, \tfrac{z}{4}\right)\right) + \tfrac{7i}{24}\left(\Theta_5\left(-4, \chi, \tfrac{z}{4}\right) - \Theta_5\left(-4, \widehat{\chi}, \tfrac{z}{4}\right)\right). \tag{12.38}$$

Concerning prime levels, we finally mention a recent paper by Clader, Kemper and Wage [23]. The authors raise the problem to find all lacunary eta products of the special form $\eta^b(az)/\eta(z)$ with b odd, and they end up with a complete list of 19 such functions. Of course, for $a = 1$ they recover Serre's list of seven lacunary powers $\eta^{b-1}(z)$ with integral weight $\frac{1}{2}(b-1)$. Then for $a = 2, 3, 4, 5$ they recover ten of the lacunary eta products known from Gordon and Robins [43] and Cooper, Gun and Ramakrishnan [25]. The list is completed by two eta products of level 7 with weights 4 and 7. Theta series identities for 16 out of these special eta products (all of them with the exception of $\left[1^{-1}, 4^7\right]$, $\left[1^{-1}, 7^9\right]$, $\left[1^{-1}, 7^{15}\right]$) are to be found in Sects. 9, 10, 11, 12, 13 of our monograph.