# 11 The Prime Level $N = 3$

## 11.1 Weight 1 and Other Weights $k \equiv 1 \bmod 6$ for $\Gamma^*(3)$ and $\Gamma_0(3)$

For $\Gamma_0(3)$ and weight $k = 1$ there are three holomorphic eta products,

$$[1,3], \quad \left[1^3, 3^{-1}\right], \quad \left[1^{-1}, 3^3\right].$$

The first one is cuspidal and belongs to the Fricke group $\Gamma^*(3)$, the others are non-cuspidal. Here we have an illustration for Theorem 3.9 (3): The lattice points on the boundary of the simplex $S(2,1)$ do not belong to $S(3,1)$, and two of the interior lattice points in $S(2,1)$ are on the boundary of $S(3,1)$. At this point it becomes clear that $\eta(z)\eta(pz)$ is the only holomorphic eta product of level $p$ and weight 1 for primes $p \geq 5$. The eta product $\eta(z)\eta(3z)$ is identified with a Hecke theta series for $\mathbb{Q}(\sqrt{-3})$; the result (11.2) is known from [31], [75]. In the identities for higher weights we need the trivial character 1 on $\mathcal{O}_3$ and the corresponding theta series

$$\Theta(z) = 6\,\Theta_1(-3,1,z) = \sum_{\mu \in \mathcal{O}_3} e(\mu\overline{\mu}z) = 1 + 6\sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{-3}{d}\right)\right)e(nz) \quad (11.1)$$

of weight 1 and level 3. It is an instance for a non-cusp form in Theorem 5.1 and appeared in Hecke [51] as an example of a modular form of "Nebentypus". It satisfies

$$\Theta(W_3 z) = \Theta\left(-\frac{1}{3z}\right) = -i\sqrt{3}z\,\Theta(z);$$

hence it belongs to the Fricke group $\Gamma^*(3)$. The identities

$$E_{2,3,-1}(z) = \Theta_1^2(-3,1,z), \qquad E_{4,3,1}(z) = E_{2,3,-1}^2(z) = \Theta_1^4(-3,1,z)$$

and several others are known from [74]; they are easily deduced from the fact that certain spaces of modular forms are one-dimensional.

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**Example 11.1** *The residues of* $2+\omega$ *and* $\omega$ *modulo* 6 *can be chosen as generators for the group* $(\mathcal{O}_3/(6))^\times \simeq Z_3 \times Z_6$. *Two characters* $\psi_1 = \psi$ *and* $\psi_{-1} = \overline{\psi}$ *on* $\mathcal{O}_3$ *with period* 6 *are fixed by their values*

$$\psi_\nu(2+\omega) = \omega^{2\nu} = e\left(\tfrac{\nu}{3}\right) = \tfrac{1}{2}\left(-1+\nu\sqrt{-3}\right), \qquad \psi_\nu(\omega) = 1.$$

*The corresponding theta series are not identically* 0 *for weights* $k \equiv 1 \bmod 6$ *and satisfy*

$$\Theta_1\left(\psi, \tfrac{z}{6}\right) = \Theta_1\left(\overline{\psi}, \tfrac{z}{6}\right) = \eta(z)\eta(3z) \tag{11.2}$$

*and, with* $\Theta(z)$ *and* $\eta_3(z) = \eta(z)\eta(3z)$ *from* (11.1),

$$\Theta_7\left(\psi, \tfrac{z}{6}\right) = \left(\Theta^6(z) - 432\,\eta_3^6(z)\right)\eta_3(z), \tag{11.3}$$

$$\Theta_7\left(\overline{\psi}, \tfrac{z}{6}\right) = \left(\Theta^6(z) + 648\,\eta_3^6(z)\right)\eta_3(z), \tag{11.4}$$

$$\Theta_{13}\left(\psi, \tfrac{z}{6}\right) = \left(\Theta^{12}(z) + 231120\,\Theta^6(z)\eta_3^6(z) - 93312\,\eta_3^{12}(z)\right)\eta_3(z), \tag{11.5}$$

$$\Theta_{13}\left(\overline{\psi}, \tfrac{z}{6}\right) = \left(\Theta^{12}(z) - 77760\,\Theta^6(z)\eta_3^6(z) + 5038848\,\eta_3^{12}(z)\right)\eta_3(z). \tag{11.6}$$

The identities (11.3) and (11.4) imply that $\eta^7(z)\eta^7(3z)$ is a linear combination of two Hecke theta series, and hence its Fourier expansion is lacunary. According to the exhaustive list in [25], Theorem 1.3, this is the highest weight eta product of level 3 which is lacunary.

We consider the coefficients in

$$\eta(z)\eta(3z) = \sum_{n\equiv 1 \bmod 6} c_1(n)\, e\left(\tfrac{nz}{6}\right), \qquad c_1(n) = \sum_{x,y>0,\, x^2+3y^2=4n} \left(\frac{12}{xy}\right). \tag{11.7}$$

For primes $p \equiv 1 \bmod 6$ we have $p = \mu\overline{\mu}$ for some $\mu \in \mathcal{O}_3$. From (11.2) and the definition of the character $\psi$ we obtain $\psi(\mu) = 1$ and $c_1(p) = 2$ if and only if one of the conjugates of $\mu$ has residue 1 modulo 6. Then we may assume that $\mu = 1+6a+6b\omega \equiv 1 \bmod 6$, and thus $p = (1+6a+3b)^2 + 27a^2$ is represented by the quadratic form $x^2+27y^2$. Otherwise we get $c_1(p) = \omega^2 + \overline{\omega}^2 = -1$. We have proved statement (1) in the following Corollary:

**Corollary 11.2** *Let* $\eta_3(z) = \eta(z)\eta(3z)$ *and* $\Theta(z) = 6\,\Theta_1(-3,1,z)$ *be given as in Example* 11.1. *Then for primes* $p \equiv 1 \bmod 6$ *the following assertions hold*:

(1) *The coefficient of* $\eta_3(z)$ *at* $p$ *is* $c_1(p) = 2$ *if* $p$ *is represented by the quadratic form* $x^2+27y^2$, *and* $c_1(p) = -1$ *otherwise.*

(2) *Let* $c_7(n)$ *denote the Fourier coefficients of* $\eta_3^7(z)$, *and let* $p = x^2+xy+y^2$. *Then*

$$c_7(p) = \begin{cases} 0 \\ \pm\,\frac{1}{120}\,xy(x+y)(x-y)(2x+y)(x+2y) \end{cases} \quad \text{if} \quad c_1(p) = \begin{cases} 2, \\ -1. \end{cases}$$

*Proof* of assertion (2). From (11.3) and (11.4) we infer

$$\eta_3^7(z) = \tfrac{1}{1080}\left(\Theta_7\left(\overline{\psi}, \tfrac{z}{6}\right) - \Theta_7\left(\psi, \tfrac{z}{6}\right)\right).$$

We have $p = \mu\overline{\mu} = x^2 + xy + y^2$ for some $\mu = x + y\omega \in \mathcal{O}_3$ which is unique up to associates and conjugates. This implies

$$c_7(p) = \tfrac{1}{1080}\left(\overline{\psi}(\mu)\mu^6 + \overline{\psi}(\overline{\mu})\overline{\mu}^6 - \psi(\mu)\mu^6 - \psi(\overline{\mu})\overline{\mu}^6\right).$$

If $c_1(p) = 2$ then $\psi(\mu) = \psi(\overline{\mu}) = 1$, and hence we get $c_7(p) = 0$. Otherwise we have $\psi(\mu) = \omega^2$, $\psi(\overline{\mu}) = \overline{\omega}^2$ or vice versa, and we get

$$c_7(p) = \pm\,\frac{\omega^2 - \overline{\omega}^2}{1080}\,(\mu^6 - \overline{\mu}^6) \;=\; \pm\,\frac{\sqrt{-3}}{1080}\,(\mu^2 - \overline{\mu}^2)(\mu^2 + \omega\overline{\mu}^2)(\mu^2 + \overline{\omega}\,\overline{\mu}^2).$$

Evaluating the factors yields the desired result. □

There is a famous theorem of Gauss (Werke, vol. 8, p. 5) on the representation of primes by the quadratic form $x^2 + 27y^2$. It follows from a law of cubic reciprocity; a proof is given in [59], Proposition 9.6.2:

**Theorem (Gauss)** *Let $p \equiv 1 \bmod 6$ be prime. The polynomial $X^3 - 2$ splits completely into linear factors over the $p$-element field $\mathbb{F}_p$ if and only if $p$ is represented by the quadratic form $x^2 + 27y^2$.*

For primes $p \equiv -1 \bmod 6$ the order $p - 1$ of the cyclic group $\mathbb{F}_p^\times$ and the exponent 3 are relatively prime, and therefore $X^3 - 2$ splits into a linear and an irreducible quadratic factor over $\mathbb{F}_p$. Hiramatsu [56] says that a *reciprocity law* for an irreducible polynomial $f(X)$ over $\mathbb{Z}$ is a rule how $f(X)$ decomposes over the $p$-element fields $\mathbb{F}_p$ for primes $p$. One of his examples (Theorem 1.1 in [56]) is the result on $c_1(p)$ in Corollary 11.2 (1). We state several equivalent criteria for the splitting of $X^3 - 2$. The equivalence with (b) in the following list is borrowed from Satgé [119]. The list will be prolonged in Corollary 11.10:

**Corollary 11.3** *For primes $p \equiv 1 \bmod 6$ the following statements are equivalent*:

(a) *The polynomial $X^3 - 2$ splits into three linear factors over the field $\mathbb{F}_p$.*
(b) *The prime $p$ splits completely in the field $\mathbb{Q}(\omega, \sqrt[3]{2})$.*
(c) *The prime $p$ is represented by the quadratic form $x^2 + 27y^2$.*
(d) *The Fourier coefficient of the weight 1 eta product $\eta(z)\eta(3z)$ at $p$ is equal to 2.*
(e) *The Fourier coefficient of the weight 7 eta product $\eta^7(z)\eta^7(3z)$ at $p$ is equal to 0.*

We briefly deal with the non-cuspidal eta products of weight 1 for $\Gamma_0(3)$. An inspection of their Fourier expansions yields the following identities:

**Example 11.4** *We have the identities*

$$\begin{aligned}\frac{\eta^3(z)}{\eta(3z)} &= 1 - 3\sum_{n=1}^{\infty}\Bigl(\sum_{d|n}\bigl(\tfrac{d}{3}\bigr)\Bigr)e(nz) + 9\sum_{n=1}^{\infty}\Bigl(\sum_{d|n}\bigl(\tfrac{d}{3}\bigr)\Bigr)e(3nz)\\ &= -3\,\Theta_1(-4,1,z) + 9\,\Theta_1(-4,1,3z), \qquad (11.8)\\ \frac{\eta^3(3z)}{\eta(z)} &= \sum_{n>0,\,3\nmid n}\Bigl(\sum_{d|n}\bigl(\tfrac{d}{3}\bigr)\Bigr)e\bigl(\tfrac{nz}{3}\bigr) = \Theta_1\bigl(-3,\psi_0,\tfrac{z}{3}\bigr), \qquad (11.9)\end{aligned}$$

*where* $1$ *stands for the trivial character on* $\mathcal{O}_3$ *and* $\psi_0$ *is the principal character modulo* $1+\omega$ *on* $\mathcal{O}_3$.

The coefficients at $n$ in the series (11.8) and (11.9) vanish whenever there is an odd power of a prime $p \equiv 5 \bmod 6$ in the factorization of $n$. Therefore, both these series are lacunary. According to [25], Theorem 1.4, $\eta^3(z)/\eta(3z)$ and $\eta^3(3z)/\eta(z)$ are the only non-cuspidal eta products of the form $[1^a, N^b]$ with level $N \geq 3$ which are lacunary.

## 11.2 Even Weights for the Fricke Group $\Gamma^*(3)$

The only holomorphic eta product of weight 2 for $\Gamma^*(3)$ is $\eta_3^2(z)$ with $\eta_3(z) = \eta(z)\eta(3z)$. Another modular form of weight 2 for this group is $\Theta(z)\eta_3(z)$ where $\Theta(z)$ is defined in (11.1). Both functions are Hecke eigenforms and can be identified with Hecke theta series for $\mathbb{Q}(\sqrt{-3})$. Theorem 5.1 predicts period 3 for a character $\psi$ to represent $\eta_3^2(z)$. For weight 2 we must have $\psi(\omega) = \overline{\omega}$, and $\psi$ is uniquely determined by this value.

**Example 11.5** *Let* $\eta_3(z) = \eta(z)\eta(3z)$ *and* $\Theta(z) = 6\,\Theta_1(-3,1,z)$ *as in Sect.* 11.1. *The residue of* $\omega$ *modulo* 3 *generates the group* $(\mathcal{O}_3/(3))^\times \simeq Z_6$. *Let* $\psi$ *be the character with period* 3 *on* $\mathcal{O}_3$ *which is fixed by the value* $\psi(\omega) = \overline{\omega}$, *and let* $\overline{\psi}$ *be the conjugate complex character. The theta series for* $\psi$ *are not identically* 0 *for weights* $k \equiv 2 \bmod 6$ *and satisfy*

$$\begin{aligned}\Theta_2\bigl(\psi,\tfrac{z}{3}\bigr) &= \eta_3^2(z), \qquad (11.10)\\ \Theta_8\bigl(\psi,\tfrac{z}{3}\bigr) &= \bigl(\Theta^6(z) - 162\,\eta_3^6(z)\bigr)\,\eta_3^2(z), \qquad (11.11)\\ \Theta_{14}\bigl(\psi,\tfrac{z}{3}\bigr) &= \bigl(\Theta^{12}(z) - 8262\,\Theta^6(z)\eta_3^6(z) - 157464\,\eta_3^{12}(z)\bigr)\,\eta_3^2(z). \qquad (11.12)\end{aligned}$$

*The theta series for* $\overline{\psi}$ *are not identically* 0 *for weights* $k \equiv 0 \bmod 6$ *and* *satisfy*

$$\begin{aligned}
\Theta_6\left(\overline{\psi}, \tfrac{z}{3}\right) &= E_{4,3,-1}(z)\eta_3^2(z), && (11.13)\\
\Theta_{12}\left(\overline{\psi}, \tfrac{z}{3}\right) &= \left(\Theta^6(z) - 2052\,\eta_3^6(z)\right) E_{4,3,-1}(z)\eta_3^2(z), && (11.14)\\
\Theta_{18}\left(\overline{\psi}, \tfrac{z}{3}\right) &= \big(\Theta^{12}(z) - 131112\,\Theta^6(z)\eta_3^6(z) \\
&\quad + 2496096\,\eta_3^{12}(z)\big) E_{4,3,-1}(z)\eta_3^2(z). && (11.15)
\end{aligned}$$

**Corollary 11.6** *Let $c_1(n)$ be the coefficients of $\eta_3(z) = \eta(z)\eta(3z)$ as in (11.7), and define $c_2(n)$ by the expansion*

$$\eta_3^2(z) = \sum_{n \equiv 1 \bmod 3} c_2(n) e\left(\tfrac{nz}{3}\right).$$

*Then for primes $p \equiv 1 \bmod 6$ the following assertions hold*:

(1) *We have $p \nmid c_2(p)$ and*

$$c_2(p) \equiv \begin{cases} p+1 \bmod 36, \\ p-8 \bmod 18, \end{cases}$$

$$c_2(p) \equiv \begin{cases} 2 \\ -1 \end{cases} \bmod 6 \quad \text{if} \quad c_1(p) = \begin{cases} 2, \\ -1. \end{cases}$$

(2) *Let $a_4(n)$ denote the coefficients of $\eta^4(z)$. Then $c_2(p) = a_4(p)$ holds if and only if $c_1(p) = 2$.*

(3) *We have $c_2(p) = -1$ if and only if $4p = 27v^2+1$, and we have $c_2(p) = 2$ if and only if $p = 108v^2 + 1$ for some $v \in \mathbb{N}$.*

(4) *We have $|c_2(p)| \leq \sqrt{4p - 27}$ with equality if and only if $4p = m^2 + 27$ for some $m \in \mathbb{N}$.*

*Proof.* We have $p = \mu\overline{\mu} = x^2 + xy + y^2$ where we can choose $\mu = x + y\omega$ among its associates and conjugates such that $x \equiv 1 \bmod 3$, $y \equiv 0 \bmod 3$. Then $\mu \equiv \overline{\mu} \equiv 1 \bmod 3$, $\psi(\mu) = \psi(\overline{\mu}) = 1$, and (11.10) implies

$$c_2(p) = \mu + \overline{\mu} = 2x + y.$$

Hence $c_2(p)$ is even if and only if $y$ is even. But then $x$ is odd, $\mu \equiv 1 \bmod 6$, whence $c_1(p) = 2$ by Corollary 11.2. Otherwise, if $y$ is odd, we have $c_1(p) = -1$. This proves the congruences modulo 6 in (1).

We write $x = 1 + 3u$, $y = 3v$. Then we obtain $c_2(p) = 2 + 6u + 3v$ and $p+1 = 1+(1+3u)^2+3v(1+3u)+9v^2 = c_2(p)+9(u^2+uv+v^2) \equiv c_2(p) \bmod 9$. If $y$ is even then we get $u^2 + uv + v^2 \equiv 0 \bmod 4$ and $p + 1 \equiv c_2(p) \bmod 36$. If $v$ is odd then $u^2 + uv + v^2$ is odd, hence $p + 1 \equiv 9 + c_2(p) \bmod 18$.—Since

$(\mu)$ and $(\overline{\mu})$ are distinct prime ideals in $\mathcal{O}_3$, $c_2(p) = \mu + \overline{\mu}$ is not a multiple of either of them. This establishes the assertions in (1).

The character $\psi'$ in the representation (9.10) of $\eta^4(z)$ as a theta series (denoted by $\psi$ in Example 9.3) and the character presently denoted by $\psi$ satisfy $\psi'(\mu) = \psi(\mu) = 1$ if $\mu \equiv 1 \bmod 2(1+\omega)$, and in this case we get $c_2(p) = a_4(p)$, $c_1(p) = 2$. Otherwise, different values of $\psi'(\mu)$ and $\psi(\mu)$ yield different values of $a_4(p)$ and $c_2(p)$. This proves (2).

We write $y = 3v$. Then $c_2(p) = 2x + y = -1$ is equivalent to $p = \frac{1}{4}((2x + y)^2 + 3y^2) = \frac{1}{4}(1 + 27v^2)$. Similarly, $c_2(p) = 2x + y = 2$ is equivalent to $p = \frac{1}{4}(4 + 27v^2)$. Necessarily, $v$ is a multiple of 4. We write $4v$ instead of $v$ and obtain $p = 108v^2 + 1$. This proves (3).

We get large values of $|c_2(p)|/\sqrt{p}$ when $\mu$ is close to the real axis. Hence we get an upper bound if we take $y = 3$. In this case, $c_2(p) = 2x + 3$ and $4p = (2x + 3)^2 + 3 \cdot 3^2 = c_2(p)^2 + 27$. This implies (4). □

For the representation of $\Theta(z)\eta_3(z)$ as a theta series on $\mathcal{O}_3$ we need a character with period 6. From Example 11.1 we know that the residues of $2 + \omega$ and $\omega$ modulo 6 generate the group $(\mathcal{O}_3/(6))^\times$.

**Example 11.7** *A pair of characters $\rho_1 = \rho$ and $\rho_{-1} = \overline{\rho}$ on $\mathcal{O}_3$ with period 6 is given by*

$$\rho_\nu(2 + \omega) = 1, \qquad \rho_\nu(\omega) = \omega^{-\nu} = e\left(\tfrac{-\nu}{6}\right) = \tfrac{1}{2}(1 - \nu\sqrt{-3}).$$

*Let $\eta_3(z)$ and $\Theta(z)$ be defined as in Example 11.5. The theta series for $\rho$ are not identically 0 for weights $k \equiv 2 \bmod 6$ and satisfy*

$$\Theta_2\left(\rho, \tfrac{z}{6}\right) = \Theta(z)\eta_3(z), \tag{11.16}$$

$$\Theta_8\left(\rho, \tfrac{z}{6}\right) = \left(\Theta^6(z) - 1296\,\eta_3^6(z)\right)\Theta(z)\eta_3(z), \tag{11.17}$$

$$\Theta_{14}\left(\rho, \tfrac{z}{6}\right) = \left(\Theta^{12}(z) - 229392\,\Theta^6(z)\eta_3^6(z) + 42830208\,\eta_3^{12}(z)\right)\Theta(z)\eta_3(z). \tag{11.18}$$

*The theta series for $\overline{\rho}$ are not identically 0 for weights $k \equiv 0 \bmod 6$ and satisfy*

$$\Theta_6\left(\overline{\rho}, \tfrac{z}{6}\right) = E_{4,3,-1}(z)\Theta(z)\eta_3(z), \tag{11.19}$$

$$\Theta_{12}\left(\overline{\rho}, \tfrac{z}{6}\right) = \left(\Theta^6(z) - 76896\,\eta_3^6(z)\right)E_{4,3,-1}(z)\Theta(z)\eta_3(z), \tag{11.20}$$

$$\Theta_{18}\left(\overline{\rho}, \tfrac{z}{6}\right) = \left(\Theta^{12}(z) + 24930288\,\Theta^6(z)\eta_3^6(z) + 3142188288\,\eta_3^{12}(z)\right)E_{4,3,-1}(z)\Theta(z)\eta_3(z). \tag{11.21}$$

The character $\rho = \rho_1$ will reappear in Example 14.4.

**Corollary 11.8** *Let $\eta_3(z)$, $\Theta(z)$, $c_1(n)$, $c_2(n)$ and $a_4(n)$ be given as in Example 11.5 and Corollary 11.6. For primes $p \equiv 1 \bmod 6$ the coefficients $\gamma_2(p)$ in the expansion*

$$\Theta(z)\eta_3(z) = \sum_{n\equiv 1 \bmod 6} \gamma_2(n) e\left(\tfrac{nz}{6}\right)$$

*have the following properties:*

(1) *We have $p \nmid \gamma_2(p)$,*

$$\gamma_2(p) \equiv \begin{cases} p+1 \bmod 36, \\ p-2 \bmod 18, \end{cases}$$

$$\gamma_2(p) \equiv \begin{cases} 2 \\ -1 \end{cases} \bmod 6 \quad \text{if} \quad c_1(p) = \begin{cases} 2, \\ -1, \end{cases}$$

*and $\gamma_2(p) = c_2(p)$ if and only if $c_1(p) = 2$.*

(2) *We have $\gamma_2(p) = -1$ if and only if $4p = 3m^2 + 1$ with $m \equiv \pm 5 \bmod 12$.*

(3) *We have $|\gamma_2(p)| \le \sqrt{4p-3}$ where equality holds if and only if $4p = m^2 + 3$ with $m \equiv \pm 5 \bmod 12$.*

*Proof.* In $p = \mu\overline{\mu} = x^2 + xy + y^2$ we choose $\mu = x + y\omega \equiv 1 \bmod 3$ as in the proof of Corollary 11.6. If $y$ is even then $\mu \equiv 1 \bmod 6$, $\rho(\mu) = 1$, and we get $\gamma_2(p) = 2x + y = c_2(p) = a_4(p)$, $c_1(p) = 2$. Otherwise, when $y$ is odd, an inspection of the values $\rho(\mu)$ yields $\gamma_2(p) = -x - 2y$ if $x$ is odd, $\gamma_2(p) = y - x$ if $x$ is even. Now we argue as in the proof of Corollary 11.6 and obtain the assertions in (1).

It follows that $\gamma_2(p) = -1$ if and only if $y = 3v$ is odd and $x + 2y = 1$ or $x - y = 1$. This is equivalent to $4p = (2x+y)^2 + 3y^2 = 3(2y \pm 1)^2 + 1 = 3m^2 + 1$ with $m = 6v \pm 1 \equiv \pm 5 \bmod 12$. Thus we have proved (2).

For even $y$ the upper bound in Corollary 11.6, (4) is valid for $\gamma_2(p) = c_2(p)$. For odd $y = 3v$ we get maximal values of $|\gamma_2(p)|/\sqrt{p}$ when $\omega\overline{\mu} = y + x\omega$ or $\omega\mu = -y + (x+y)\omega$ is close to the real axis. This means that $\gamma_2(p) = -x - 2y$ with $x = 1$ or $\gamma_2(p) = y - x$ with $x + y = 1$, and gives the asserted estimate $|\gamma_2(p)| \le \sqrt{4p-3}$ with equality for $4p = m^2 + 3$, $m = 6v \pm 1 \equiv \pm 5 \bmod 12$. □

In the following discussion of weights $k \equiv 4 \bmod 6$ we need the Eisenstein series of weight 3 for $\Gamma^*(3)$ which were introduced in Sect. 1.6. One verifies the identities

$$E_{3,3,i}(z) = \Theta^3(z), \qquad E_{3,3,-i}(z)\,\Theta(z) = E_{4,3,-1}(z).$$

The products $E_{3,3,i}(z)\eta_3(z) = \Theta^3(z)\eta_3(z)$ and $E_{3,3,-i}(z)\eta_3(z)$ are cusp forms of weight 4 for $\Gamma^*(3)$. Lists of coefficients display multiplicative properties and gaps which suggest that they are both eigenforms and Hecke theta series for $\mathbb{Q}(\sqrt{-3})$. Again, we need characters with period 6.

**Example 11.9** *Let $\eta_3(z)$ and $\Theta(z)$ be defined as in Example 11.5. A pair of characters $\chi = \chi_1$ and $\overline{\chi} = \chi_{-1}$ on $\mathcal{O}_3$ with period 6 is given by*

$$\chi_\nu(2+\omega) = \omega^{-2\nu} = e\left(\tfrac{-\nu}{3}\right), \qquad \chi_\nu(\omega) = -1.$$

*The corresponding theta series are not identically 0 for weights $k \equiv 4 \bmod 6$ and satisfy*

$$\Theta_4\left(\chi, \tfrac{z}{6}\right) = \Theta^3(z)\eta_3(z), \tag{11.22}$$
$$\Theta_{10}\left(\chi, \tfrac{z}{6}\right) = \left(\Theta^6(z) + 7776\,\eta_3^6(z)\right)\Theta^3(z)\eta_3(z), \tag{11.23}$$
$$\Theta_{16}\left(\chi, \tfrac{z}{6}\right) = \left(\Theta^{12}(z) - 4239216\,\Theta^6(z)\eta_3^6(z) - 186437376\,\eta_3^{12}(z)\right)\Theta^3(z)\eta_3(z), \tag{11.24}$$
$$\Theta_4\left(\overline{\chi}, \tfrac{z}{6}\right) = E_{3,3,-i}(z)\eta_3(z), \tag{11.25}$$
$$\Theta_{10}\left(\overline{\chi}, \tfrac{z}{6}\right) = \left(\Theta^6(z) + 4752\,\eta_3^6(z)\right)E_{3,3,-i}(z)\eta_3(z), \tag{11.26}$$
$$\Theta_{16}\left(\overline{\chi}, \tfrac{z}{6}\right) = \left(\Theta^{12}(z) + 2994192\,\Theta^6(z)\eta_3^6(z) + 8864640\,\eta_3^{12}(z)\right)E_{3,3,-i}(z)\eta_3(z). \tag{11.27}$$

For the coefficients in

$$\Theta^3(z)\eta_3(z) = \sum_{n\equiv 1 \bmod 6} \gamma_4(n)e\left(\tfrac{nz}{6}\right),$$
$$E_{3,3,-i}(z)\eta_3(z) = \sum_{n\equiv 1 \bmod 6} \gamma_4'(n)e\left(\tfrac{nz}{6}\right)$$

at primes $p \equiv 1 \bmod 6$ one can deduce similar properties as in the preceding cases. We omit the proofs, but note the results

$$\gamma_4(p) \equiv \gamma_4'(p) \equiv \left\{\begin{array}{c} 2 \\ -1 \end{array}\right. \bmod 18 \qquad \text{if} \qquad c_1(p) = \left\{\begin{array}{r} 2, \\ -1, \end{array}\right. \tag{11.28}$$

$$\left\{\begin{array}{l} \gamma_4(p) = \gamma_4'(p) = a_8(p) \\ \gamma_4(p) + \gamma_4'(p) + a_8(p) = 0 \end{array}\right. \qquad \text{if} \qquad c_1(p) = \left\{\begin{array}{r} 2, \\ -1, \end{array}\right. \tag{11.29}$$

where $a_8(n)$ denote the coefficients of $\eta^8(z)$ in Sect. 9.4,

$$\gamma_4(p) \le -(6p-8) \quad \text{or} \quad \gamma_4(p) \ge 3p-1 \qquad \text{if} \qquad c_1(p) = 2,$$

$$|\gamma_4(p)| \le (p-3)\sqrt{4p-3} \qquad \text{if} \qquad c_1(p) = 2.$$

We continue the list of equivalent statements in Corollary 11.3, using Corollaries 11.6, 11.8 and (11.28), (11.29):

**Corollary 11.10** *Let $\eta_3(z)$ and $\Theta(z)$ be defined as in Example 11.5. For primes $p \equiv 1 \bmod 6$, the statements in Corollary 11.3 and the following statements are equivalent to each other*:

(f) *The coefficient* $c_2(p)$ *of* $\eta_3^2(z)$ *at* $p$ *satisfies* $c_2(p) \equiv 2 \bmod 6$.

(g) *The coefficients of* $\eta_3^2(z)$ *and* $\eta^4(z)$ *at the prime* $p$ *are equal to each other.*

(h) *The coefficient* $\gamma_2(p)$ *of* $\Theta(z)\eta_3^2(z)$ *at* $p$ *satisfies* $\gamma_2(p) \equiv 2 \bmod 6$.

(i) *The coefficients of* $\Theta(z)\eta_3(z)$ *and* $\eta^4(z)$ *at the prime* $p$ *are equal to each other.*

(j) *The coefficient* $\gamma_4(p)$ *of* $\Theta^3(z)\eta_3^2(z)$ *at* $p$ *satisfies* $\gamma_4(p) \equiv 2 \bmod 18$.

(k) *The coefficients of* $\Theta^3(z)\eta_3(z)$ *and* $\eta^8(z)$ *at* $p$ *are equal to each other.*

(l) *The coefficients of* $\Theta^3(z)\eta_3(z)$ *and* $E_{3,3,-i}(z)\eta_3(z)$ *at* $p$ *are equal to each other.*

## 11.3 Weights $k \equiv 3,\, 5 \bmod 6$ for the Fricke Group $\Gamma^*(3)$

We continue to use the notations $\eta_3(z) = \eta(z)\eta(3z)$ and $\Theta(z) = 6\,\Theta_1(-3, 1, z)$ from Sect. 11.1. There are three cusp forms of weight 3 for $\Gamma^*(3)$, with expansions

$$\eta_3^3(z) = \sum_{n \equiv 1 \bmod 2} c_3(n) e\left(\tfrac{nz}{2}\right), \tag{11.30}$$

$$\Theta(z)\eta_3^2(z) = \sum_{n \equiv 1 \bmod 3} \gamma_3(n) e\left(\tfrac{nz}{3}\right), \qquad \Theta^2(z)\eta_3(z) = \sum_{n \equiv 1 \bmod 6} \lambda_3(n) e\left(\tfrac{nz}{6}\right). \tag{11.31}$$

Lists of coefficients suggest that all of them are Hecke theta series for $\mathbb{Q}(\sqrt{-3})$. According to Theorem 5.1, we need characters with periods 2, 3 and 6, respectively. The group $(\mathcal{O}_3/(2))^\times$ is cyclic of order 3 with the residue of $\omega$ modulo 2 as a generator.

**Example 11.11** *A character* $\psi_2$ *on* $\mathcal{O}_3$ *with period* 2 *is fixed by the value* $\psi_2(\omega) = -\omega$. *The corresponding theta series are not identically* 0 *for weights* $k \equiv 3 \bmod 6$ *and satisfy*

$$\Theta_3\left(\psi_2, \tfrac{z}{2}\right) = \eta_3^3(z), \tag{11.32}$$

$$\Theta_9\left(\psi_2, \tfrac{z}{2}\right) = \left(\Theta^6(z) + 48\,\eta_3^6(z)\right)\eta_3^3(z), \tag{11.33}$$

$$\Theta_{15}\left(\psi_2, \tfrac{z}{2}\right) = \left(\Theta^{12}(z) - 2256\,\Theta^6(z)\eta_3^6(z) + 58752\,\eta_3^{12}(z)\right)\eta_3^3(z). \tag{11.34}$$

*The theta series for the conjugate complex character* $\overline{\psi}_2$ *are not identically* 0 *for weights* $k \equiv 5 \bmod 6$ *and satisfy*

$$\Theta_5\left(\overline{\psi}_2, \tfrac{z}{2}\right) = \Theta^2(z)\eta_3^3(z), \tag{11.35}$$

$$\Theta_{11}\left(\overline{\psi}_2, \tfrac{z}{2}\right) = \left(\Theta^6(z) - 288\,\eta_3^6(z)\right)\Theta^2(z)\eta_3^3(z), \tag{11.36}$$

$$\Theta_{17}\left(\overline{\psi}_2, \tfrac{z}{2}\right) = \left(\Theta^{12}(z) + 6480\,\Theta^6(z)\eta_3^6(z) - 255744\,\eta_3^{12}(z)\right)\Theta^2(z)\eta_3^3(z). \tag{11.37}$$

**Corollary 11.12** *For primes* $p \equiv 1 \bmod 6$ *the coefficients* $c_3(p)$ *of* $\eta_3^3(z)$ *have the following properties*:

(1) *We have* $c_3(p) \equiv 2p \bmod 12$ *and* $c_3(p) \equiv 2 \bmod 24$.

(2) *Every odd prime divisor* $q$ *of* $c_3(p)$ *satisfies* $q \equiv \pm 1 \bmod 12$ *and* $\left(\frac{p}{q}\right) = \left(\frac{2}{q}\right)$.

(3) *We have* $c_3(p) = 2$ *if and only if* $p = 2u^2 - 1$ *and* $u^2 - 3v^2 = 1$ *for some* $u, v \in \mathbb{N}$.

(4) *We have* $-2(p-2) \le c_3(p) \le 2(p-6)$. *Equality* $c_3(p) = -2(p-2)$ *holds if and only if* $p = 12m^2 + 1$, *and equality* $c_3(p) = 2(p-6)$ *holds if and only if* $p = 4m^2 + 3$ *for some* $m \in \mathbb{N}$.

(5) *The coefficient* $\lambda_5(p)$ *of* $\Theta^2(z)\eta_3^3(z)$ *at* $p$ *satisfies* $\lambda_5(p) = (c_3(p))^2 - 2p^2$.

*Proof.* We have $p = \mu\overline{\mu} = x^2 + xy + y^2$ where we can choose $\mu = x + y\omega \in \mathcal{O}_3$ with $\mu \equiv 1 \bmod 2$. Then $\psi_2(\mu) = \psi_2(\overline{\mu}) = 1$, and (11.32) implies

$$c_3(p) = \mu^2 + \overline{\mu}^2 = 2p - 3y^2 = (2x+y)^2 - 2p. \qquad (11.38)$$

Since $y$ is even, we get $c_3(p) = 2p - 3y^2 \equiv 2p \bmod 12$. Since $p \equiv 1$ or $7 \bmod 12$ according to $y \equiv 0$ or $2 \bmod 4$, we also get $c_3(p) \equiv 2 \bmod 24$. Thus (1) is established.

Let $q$ be an odd prime divisor of $c_3(p)$. Then $6p - (3y)^2 \equiv 2p - (2x+y)^2 \equiv 0 \bmod q$, hence $\left(\frac{6p}{q}\right) = \left(\frac{2p}{q}\right) = 1$. Therefore we get $\left(\frac{p}{q}\right) = \left(\frac{2}{q}\right)$ and $\left(\frac{3}{q}\right) = 1$, i.e., $q \equiv \pm 1 \bmod 12$. Thus we proved (2).

From (1) it is clear that $c_3(p) \ge 2$ or $c_3(p) \le -22$. The case $c_3(p) = 2$ means that $2p - 3y^2 = (2x+y)^2 - 2p = 2$. Here, $y = 2v$ and $2x + y = 2u$ are even, and we obtain $p = 2u^2 - 1$, $u^2 - 3v^2 = 1$. This proves (3).

From $|c_3(p)| < 2p$ and (1) it is clear that $-2(p-2) \le c_3(p) \le 2(p-6)$. From (11.38) we see that $c_3(p) = 2p - 12$ holds if and only if $y^2 = 4$, and this means that $p = (x+1)^2 + 3 = 4m^2 + 3$ for some $m \in \mathbb{Z}$. Also, we see that $c_3(p) = -2p + 4$ holds if and only if $(2x+y)^2 = 4$, and this means that $p = 1 + \frac{3}{4}y^2 = 1 + 12m^2$ for some $m \in \mathbb{Z}$. This proves (4).

With $\mu$ chosen as before, (11.35) implies $\lambda_5(p) = \mu^4 + \overline{\mu}^4 = (\mu^2 + \overline{\mu})^2 - 2\mu^2\overline{\mu}^2 = (c_3(p))^2 - 2p^2$, which is (5). □

**Remark.** All positive solutions $u_m, v_m$ of Pell's equation $u^2 - 3v^2 = 1$ in Corollary 11.12 (3) are given by $u_m + v_m\sqrt{3} = (2+\sqrt{3})^m$. If $p_m = 2u_m^2 - 1$ is a prime then $m = 2^a$ is a power of 2, according to Theorem 10.4. Thus $p_1 = 7$, $p_2 = 97$, $p_8 = 708158977$ are the only primes below $10^{35}$ with $c_3(p) = 2$, since $p_4 = 31 \cdot 607$ and $p_{16} = 127 \cdot 7897466719774591$ are composite. We recall relation (10.40) for the coefficients of $\eta^8(z)\eta^{-2}(2z)$ which lead us to the "even" solutions $u_{2m}$, $v_{2m}$ of $u^2 - 3v^2 = 1$.

**Example 11.13** *Let $\psi^2$ and $\overline{\psi}^2$ be the characters with period 3 on $\mathcal{O}_3$ which are fixed by the values $\psi^2(\omega) = -\omega$, $\overline{\psi}^2(\omega) = \omega^2$ and which are the squares of the characters $\psi$, $\overline{\psi}$ in Example 11.5. The theta series for $\psi^2$ are not identically 0 for weights $k \equiv 3 \bmod 6$ and satisfy*

$$\Theta_3\left(\psi^2, \tfrac{z}{3}\right) = \Theta(z)\eta_3^2(z), \tag{11.39}$$

$$\Theta_9\left(\psi^2, \tfrac{z}{3}\right) = \left(\Theta^6(z) + 216\,\eta_3^6(z)\right)\Theta(z)\eta_3^2(z), \tag{11.40}$$

$$\Theta_{15}\left(\psi^2, \tfrac{z}{3}\right) = \left(\Theta^{12}(z) + 16308\,\Theta^6(z)\eta_3^6(z) + 903960\,\eta_3^{12}(z)\right)\Theta(z)\eta_3^2(z). \tag{11.41}$$

*The theta series for $\overline{\psi}^2$ are not identically 0 for weights $k \equiv 5 \bmod 6$ and satisfy*

$$\Theta_5\left(\overline{\psi}^2, \tfrac{z}{3}\right) = \Theta^3(z)\eta_3^2(z), \tag{11.42}$$

$$\Theta_{11}\left(\overline{\psi}^2, \tfrac{z}{3}\right) = \left(\Theta^6(z) + 972\,\eta_3^6(z)\right)\Theta^3(z)\eta_3^2(z), \tag{11.43}$$

$$\Theta_{17}\left(\overline{\psi}^2, \tfrac{z}{3}\right) = \left(\Theta^{12}(z) + 65448\,\Theta^6(z)\eta_3^6(z) - 14486688\,\eta_3^{12}(z)\right)\Theta^3(z)\eta_3^2(z). \tag{11.44}$$

From (11.39) one derives properties of the coefficients of $\Theta(z)\eta_3^2(z)$. We omit the proofs, which are similar to preceding cases, except for part (3):

**Corollary 11.14** *For primes $p \equiv 1 \bmod 6$ the coefficients $\gamma_3(p)$ of $\Theta(z)\eta_3^2(z)$ have the following properties*:

(1) *We have $\gamma_3(p) \equiv 2p \bmod 27$ and*

$$\gamma_3(p) \equiv \left\{ \begin{array}{c} 2 \\ -1 \end{array} \right. \bmod 12 \qquad \text{if} \qquad c_1(p) = \left\{ \begin{array}{c} 2, \\ -1. \end{array} \right.$$

*Moreover, $\gamma_3(p) = c_3(p)$ if and only if $c_1(p) = 2$.*

(2) *Every odd prime divisor $q$ of $\gamma_3(p)$ satisfies $q \equiv \pm 1 \bmod 12$ and $\left(\frac{p}{q}\right) = \left(\frac{2}{q}\right)$.*

(3) *There is no prime with $\gamma_3(p) = 2$. The only prime with $\gamma_3(p) = -1$ is $p = 13$.*

(4) *We have $-2p + 1 \le \gamma_3(p) \le 2p - 27$. Equality $\gamma_3(p) = -2p + 1$ holds if and only if $4p = 27m^2 + 1$, and equality $\gamma_3(p) = 2p - 27$ holds if and only if $p = m^2 + 3m + 9$ for some $m \in \mathbb{N}$.*

*Proof* of part (3). As in the proof of Corollary 11.12 (3), one finds that $\gamma_3(p) = 2$ if and only if $p = 2U^2 - 1$ and $U^2 - 27V^2 = 1$ for some $U, V \in \mathbb{N}$. The positive solutions $U_m, V_m$ of Pell's equation $U^2 - 27V^2 = 1$ are $U_m = u_{3m}$, $V_m = v_{3m}$ where $u_m, v_m$ is defined in the remark after Corollary 11.12. Now

Theorem 10.4 says that all numbers $P_m = 2U_m^2 - 1 = p_{3m}$ are composite.— We have $\gamma_3(p) = -1$ if and only if $2p = u^2 + 1$, $u^2 - 27v^2 = -2$ for some $u, v \in \mathbb{N}$. The positive solutions $u_m, v_m$ of $u^2 - 27v^2 = -2$ are given by $u_m + v_m\sqrt{27} = (5 + \sqrt{27})(26 + 5\sqrt{27})^{m-1}$. An easy induction shows that all numbers $p'_m = \frac{1}{2}(u_m^2 + 1)$ are multiples of the prime $p'_1 = 13$. □

**Example 11.15** *Let $\rho^2$ be the character with period 6 on $\mathcal{O}_3$ which is fixed by the values $\rho^2(2+\omega) = 1$, $\rho^2(\omega) = -\omega$ and which is the square of the character $\rho$ in Example 11.7. The corresponding theta series are not identically 0 for weights $k \equiv 3 \bmod 6$ and satisfy*

$$\Theta_3\left(\rho^2, \tfrac{z}{6}\right) = \Theta^2(z)\eta_3(z), \tag{11.45}$$

$$\Theta_9\left(\rho^2, \tfrac{z}{6}\right) = \left(\Theta^6(z) - 4320\,\eta_3^6(z)\right)\Theta^2(z)\eta_3(z), \tag{11.46}$$

$$\Theta_{15}\left(\rho^2, \tfrac{z}{6}\right) = \left(\Theta^{12}(z) - 72144\,\Theta^6(z)\eta_3^6(z) - 118506240\,\eta_3^{12}(z)\right)\Theta^2(z)\eta_3(z). \tag{11.47}$$

*The theta series for the conjugate complex character $\overline{\rho}^2$ are not identically 0 for weights $k \equiv 5 \bmod 6$ and satisfy*

$$\Theta_5\left(\overline{\rho}^2, \tfrac{z}{6}\right) = \Theta^4(z)\eta_3(z), \tag{11.48}$$

$$\Theta_{11}\left(\overline{\rho}^2, \tfrac{z}{6}\right) = \left(\Theta^6(z) - 33048\,\eta_3^6(z)\right)\Theta^4(z)\eta_3(z), \tag{11.49}$$

$$\Theta_{17}\left(\overline{\rho}^2, \tfrac{z}{6}\right) = \left(\Theta^{12}(z) + 6728832\,\Theta^6(z)\eta_3^6(z) - 1562042880\,\eta_3^{12}(z)\right)\Theta^4(z)\eta_3(z). \tag{11.50}$$

From (11.45) one obtains properties of the coefficients of $\Theta^2(z)\eta_3(z)$. We omit the proofs.

**Corollary 11.16** *For primes $p \equiv 1 \bmod 6$ the coefficients $\lambda_3(p)$ of $\Theta^2(z) \times \eta_3(z)$ have the following properties*:

(1) *We have*

$$\lambda_3(p) \equiv \begin{cases} 2p & \bmod 108, \\ 2p - 3 & \bmod 72, \end{cases}$$

$$\lambda_3(p) \equiv \begin{cases} 2 \\ -1 \end{cases} \bmod 12 \quad \text{if} \quad c_1(p) = \begin{cases} 2, \\ -1. \end{cases}$$

*Moreover, $\lambda_3(p) = c_3(p)$ if and only if $c_1(p) = 2$.*

(2) *Every odd prime divisor $q$ of $\lambda_3(p)$ satisfies $q \equiv \pm 1 \bmod 12$ and $\left(\frac{p}{q}\right) = \left(\frac{2}{q}\right)$.*

(3) *We have $-2p+1 \le \lambda_3(p) \le 2p-3$. Equality $\lambda_3(p) = -2p+1$ holds if and only if $4p = 3y^2 + 1$ for some $y \equiv 1 \bmod 6$, and equality $\lambda_3(p) = 2p - 3$ holds if and only if $p = x^2 + x + 1$ for some $x \equiv 2$ or $3 \bmod 6$.*

It can be shown that $\lambda_3(p) \neq -1$ for all primes $p$. If $\lambda_3(p) = 2$ then we also have $\gamma_3(p) = 2$, and the statements in Corollary 11.14 and the following remark apply.

With Corollaries 11.12, 11.14, 11.16 it is easily possible to prolong the list of equivalent statements in Example 11.5 and Corollary 11.10. We refrain from stating the results.

## 11.4 Weight $k = 2$ for $\Gamma_0(3)$

For level $N = 3$ and weight $k = 2$ there are seven new holomorphic eta products. In Example 11.5 the function $\eta^2(z)\eta^2(3z)$ was identified with a theta series. Of the remaining 6 functions, there are 4 cuspidal and 2 non-cuspidal eta products. For two of the cusp forms with denominator $t = 12$ there is a neat representation by Hecke series:

**Example 11.17** *The residues of* $2+\omega$, $5$ *and* $\omega$ *modulo* $12$ *can be chosen as generators of the group* $(\mathcal{O}_3/(12))^\times \simeq Z_6 \times Z_2 \times Z_6$. *A pair of characters* $\psi_\delta$ *on* $\mathcal{O}_3$ *with period* $12$ *is given by*

$$\psi_\delta(2+\omega) = \delta\omega, \qquad \psi_\delta(5) = 1, \qquad \psi_\delta(\omega) = \overline{\omega}$$

*with* $\delta \in \{1, -1\}$. *The corresponding theta series of weight* $2$ *have a decomposition*

$$\Theta_2\left(-3, \psi_\delta, \tfrac{z}{12}\right) = f_1(z) + 3\sqrt{3}\delta i\, f_7(z) \tag{11.51}$$

*with normalized integral Fourier series* $f_j$ *with denominator* $12$ *and numerator classes* $j$ *modulo* $12$ *which are eta products,*

$$f_1(z) = \frac{\eta^5(z)}{\eta(3z)}, \qquad f_7(z) = \frac{\eta^5(3z)}{\eta(z)}. \tag{11.52}$$

The cuspidal eta product $\eta^3(z)\eta(3z)$ has denominator 4 and numerator 1. Its coefficients enjoy partially multiplicative properties, but it is not a Hecke eigenform. We do not get an eigenform by adding a complementary Fourier series for the remainder 3 modulo 4. But in the following example we will obtain a theta series by adding an old eta product of level 9 with order $\frac{5}{4}$ at $\infty$. The eta product $\eta(z)\eta^3(3z)$ is cuspidal with order $\frac{5}{12}$ at $\infty$, and its coefficients also have some partially multiplicative properties. Here one can construct an eigenform, which also is a theta series, by adding a complementary component for the remainder 1 modulo 12. It turns out that the missing component is obtained by rescaling the variable in $\eta^3(z)\eta(3z)$, thus passing from level $N = 3$ to the higher level 9. So we get identities which are more complicated than those in the examples so far in this section.

**Example 11.18** *Let the generators of* $(\mathcal{O}_1/(6))^\times \simeq Z_8 \times Z_2$ *be chosen as in Example* 9.1, *and define a pair of characters* $\chi_\delta$ *on* $\mathcal{O}_1$ *with period* 6 *by the assignment*

$$\chi_\delta(2+i) = \tfrac{\delta}{\sqrt{2}}(1+i), \qquad \chi_\delta(2+3i) = 1$$

*with* $\delta \in \{1, -1\}$. *The corresponding theta series of weight* 2 *have a decomposition*

$$\Theta_2\left(-4, \chi_\delta, \tfrac{z}{12}\right) = f_1(z) + 3\sqrt{2}\,\delta i\, f_5(z) \tag{11.53}$$

*with normalized integral Fourier series* $f_j(z)$ *with denominator* 12 *and numerator classes* $j$ *modulo* 12. *The component* $f_5$ *is an eta product, and* $f_1$ *is a linear combination of eta products. We have*

$$f_1(z) = \eta^3\left(\tfrac{z}{3}\right)\eta(z) + 3\eta(z)\eta^3(3z), \qquad f_5(z) = \eta(z)\eta^3(3z), \tag{11.54}$$

$$\Theta_2\left(-4, \chi_\delta, \tfrac{z}{4}\right) = \eta^3(z)\eta(3z) \;+\; 3(1+\sqrt{2}\,\delta i)\,\eta(3z)\eta^3(9z).$$

*The Fricke involution* $W_9$ *acts on* $F_\delta(z) = \Theta_2\left(-4, \chi_\delta, \frac{z}{4}\right)$ *by* $F_\delta(W_9 z) = -3\sqrt{3}(1+\sqrt{2}\delta i)z^2\, F_{-\delta}(z)$.

We introduce coefficients for the functions in the last example by setting

$$\begin{aligned} \eta^3(z)\eta(3z) &= \sum_{n\equiv 1,5 \bmod 12} a(n) e\left(\frac{nz}{4}\right), \\ \eta(3z)\eta^3(9z) &= \sum_{n\equiv 5 \bmod 12} b(n) e\left(\frac{nz}{4}\right), \end{aligned}$$

and $F_\delta(z) = \sum_{n\equiv 1,5 \bmod 12} \lambda_\delta(n) e\left(\frac{nz}{4}\right)$. Then we use (8.3), (8.15) to relate $a(n)$, $b(n)$ to the positive solutions of $x^2 + y^2 = 2n$ and $9x^2 + y^2 = 2n$ with $x$ odd and $\gcd(y, 6) = 1$. It follows that $b(n) = -3a(n)$ for $n \equiv 5 \bmod 12$ and

$$\lambda_\delta(n) = \begin{cases} a(n) \\ -\sqrt{2}\delta i\, a(n) \end{cases} \quad \text{for} \quad n \equiv \begin{cases} 1 \\ 5 \end{cases} \bmod 12.$$

The non-cuspidal eta products of weight 2 and level 3 are the squares of the functions in Example 11.4, $\eta^6(z)/\eta^2(3z)$ and $\eta^6(3z)/\eta^2(z)$. Below, for the first one we present a complicated identity with Eisenstein series. The second one has denominator 3 and numerator 2, and one needs a complementary component with numerator 1 to construct eigenforms:

**Example 11.19** *We have the identities*

$$\frac{\eta^6(z)}{\eta^2(3z)} = 1 \;+\; 3\sum_{n=1}^{\infty}\left(\sum_{9\nmid d\,|\,n} d\right) e(nz) \;-\; 9 \sum_{n\equiv 1 \bmod 3} \sigma_1(n) e(nz), \tag{11.55}$$

$$f_1(z) + 3\,\frac{\eta^6(3z)}{\eta^2(z)} = \sum_{3\nmid n} \sigma_1(n) e\left(\tfrac{nz}{3}\right), \tag{11.56}$$

$$f_1(z) - 3\,\frac{\eta^6(3z)}{\eta^2(z)} = \sum_{n=1}^{\infty} \left(\tfrac{n}{3}\right) \sigma_1(n) e\left(\tfrac{nz}{3}\right) \tag{11.57}$$

*with* $f_1(z) = \sum_{n\equiv 1 \bmod 3} \sigma_1(n) e\left(\frac{nz}{3}\right)$.

## 11.5 Lacunary Eta Products with Weights $k > 2$ for $\Gamma_0(3)$

Cooper, Gun and Ramakrishnan [25] determined all lacunary eta products of levels $N = 3, 4$ and 5. The preceding examples in this section comprise all those of level 3 which have weights $k \le 2$ or belong to the Fricke group. Besides, there are 8 more with weights $k > 2$, and all of them have weight 4. The representations of these eta products by theta series have already been established by Gordon and Hughes [42] and Ahlgren [2]. The first example from [2] shows that $\left[1^{10}, 3^{-2}\right]$ and $\left[1^{-2}, 3^{10}\right]$ are lacunary:

**Example 11.20** *Let the generators of* $(\mathcal{O}_3/(6))^\times \simeq Z_3 \times Z_6$ *be chosen as in Example* 11.1, *fix a character* $\psi_1$ *on* $\mathcal{O}_3$ *with period* 6 *by its values*

$$\psi_1(2+\omega) = -\omega = e\left(-\tfrac{1}{3}\right), \qquad \psi_1(\omega) = -1,$$

*and let* $\psi_{-1} = \overline{\psi_1}$ *be the conjugate complex character. Then for* $\delta \in \{1, -1\}$ *the corresponding theta series of weight* 4 *satisfy*

$$\Theta_4\left(-3, \psi_\delta, \tfrac{z}{6}\right) = \frac{\eta^{10}(z)}{\eta^2(3z)} + 27\delta\,\frac{\eta^{10}(3z)}{\eta^2(z)}\,. \tag{11.58}$$

The next example from [2] shows that $\left[1^9, 3^{-1}\right]$ and $\left[1^{-3}, 3^{11}\right]$ are lacunary since they are linear combinations of Hecke theta series:

**Example 11.21** *Let the generators of* $(\mathcal{O}_1/(6))^\times \simeq Z_8 \times Z_2$ *be chosen as in Example* 9.1. *For* $\delta \in \{1, -1\}$, *define a pair of characters* $\varphi_\delta$ *on* $\mathcal{O}_1$ *with period* 6 *by*

$$\varphi_\delta(2+i) = \delta\xi = \delta\,\tfrac{1-i}{\sqrt{2}}, \qquad \varphi_\delta(2+3i) = 1.$$

*The residues of* $1 + 2\omega$ *and* $\omega$ *modulo* 4 *generate the group* $(\mathcal{O}_3/(4))^\times \simeq Z_2 \times Z_6$. *A pair of characters* $\psi_\delta$ *on* $\mathcal{O}_3$ *with period* 4 *is given by*

$$\psi_\delta(1+2\omega) = \delta, \qquad \psi_\delta(\omega) = -1.$$

*Let $\psi'_\delta$ be the imprimitive character on $\mathcal{O}_3$ with period $4(1+\omega)$ which is induced by $\psi_\delta$. Then the corresponding theta series of weight 4 satisfy*

$$\frac{\eta^9(z)}{\eta(3z)} - 9\,\frac{\eta^{11}(3z)}{\eta^3(z)} = \tfrac{1}{2}(1+i\sqrt{2})\,\Theta_4\left(-4,\varphi_1,\tfrac{z}{4}\right) + \tfrac{1}{2}(1-i\sqrt{2})\,\Theta_4\big(-4,\varphi_{-1},\tfrac{z}{4}\big), \tag{11.59}$$

$$\frac{\eta^9(z)}{\eta(3z)} + 9\,\frac{\eta^{11}(3z)}{\eta^3(z)} = \tfrac{3}{2}\left(\Theta_4\left(-3,\psi'_1,\tfrac{z}{4}\right) + \Theta_4\left(-3,\psi'_{-1},\tfrac{z}{4}\right)\right) + \big(\Theta_4\left(-3,\psi_1,\tfrac{z}{4}\right) + \Theta_4\left(-3,\psi_{-1},\tfrac{z}{4}\right)\big). \tag{11.60}$$

The identities (11.59), (11.60) imply relations among the coefficients of the two eta products: Let us write

$$\frac{\eta^9(z)}{\eta(3z)} = \sum_{n\equiv 1 \bmod 4} a(n)e\left(\tfrac{nz}{4}\right), \qquad \frac{\eta^{11}(3z)}{\eta^3(z)} = \sum_{n\equiv 1 \bmod 4} b(n)e\left(\tfrac{nz}{4}\right).$$

Then we have

$$b(n) = \begin{cases} 9a(n) \\ -9a(n) \end{cases} \quad \text{for} \quad n \equiv \begin{cases} 5 \\ 9 \end{cases} \bmod 12.$$

For primes $p \equiv 1 \bmod 12$ we have $p = \mu\overline{\mu} = x^2+y^2$ where we can choose $\mu = x+yi \in \mathcal{O}_1$ with $3|y$, $x \equiv 1$ or $2 \bmod 6$, and we have $p = \lambda\overline{\lambda} = u^2+12t^2$ with $\lambda = u+2t\sqrt{-3} \in \mathcal{O}_3$. Then the characters in Example 11.21 satisfy $\varphi_\delta(\mu) = \varphi_\delta(\overline{\mu}) = 1$, $\psi_\delta(\lambda) = \psi_\delta(\overline{\lambda}) = \pm 1$, and we get

$$a(p) - 9b(p) = \mu^3 + \overline{\mu}^3 = 2x(x^2-3y^2),$$
$$a(p) + 9b(p) = \pm(\lambda^3 + \overline{\lambda}^3) = \pm 2u(u-6t)(u+6t).$$

Under the Fricke involution $W_3$, the functions $\left[1^9, 3^{-1}\right] \pm 9\left[1^{-3}, 3^{11}\right]$ in Example 11.21 are transformed into multiples of $\left[1^{11}, 3^{-3}\right] \pm 81\left[1^{-1}, 3^9\right]$. The representations of these functions by Hecke theta series looks somewhat simpler than the preceding identities. Moreover, we identify $\left[1^3, 3^5\right]$ and $\left[1^5, 3^3\right]$ with components of theta series, thus proving their lacunarity as in [42]:

**Example 11.22** *Let $\varphi_\delta$, $\psi_\delta$ and $\psi'_\delta$ be defined as in Example 11.21. Then we have*

$$\frac{\eta^{11}(z)}{\eta^3(3z)} - 81\,\frac{\eta^9(3z)}{\eta(z)} = \tfrac{1}{2}\big(\Theta_4\left(-4,\varphi_1,\tfrac{z}{12}\right) + \Theta_4\left(-4,\varphi_{-1},\tfrac{z}{12}\right)\big), \tag{11.61}$$

$$\frac{\eta^{11}(z)}{\eta^3(3z)} + 81\,\frac{\eta^9(3z)}{\eta(z)} = \tfrac{1}{2}\big(\Theta_4\left(-3,\psi'_1,\tfrac{z}{12}\right) + \Theta_4\left(-3,\psi'_{-1},\tfrac{z}{12}\right)\big). \tag{11.62}$$

*Moreover, we have decompositions*

$$\Theta_4\left(-3, \psi'_\delta, \tfrac{z}{12}\right) = f_1(z) + 18\delta i\sqrt{3}\, f_7(z), \tag{11.63}$$

$$3\,\Theta_4\left(-3, \psi'_\delta, \tfrac{z}{4}\right) \;-\; 2\,\Theta_4\left(-3, \psi_\delta, \tfrac{z}{4}\right) = g_1(z) - 6\delta i\sqrt{3}\, g_3(z), \tag{11.64}$$

*where the components $f_j$ and $g_j$ are normalized integral Fourier series with denominators* 12 *and* 4, *respectively, and numerator classes $j$ modulo their denominators, and all of them are eta products or linear combinations thereof,*

$$f_1(z) \;=\; \frac{\eta^{11}(z)}{\eta^3(3z)} + 81\,\frac{\eta^9(3z)}{\eta(z)}, \qquad f_7(z) = \eta^5(z)\eta^3(3z), \tag{11.65}$$

$$g_1(z) \;=\; \frac{\eta^9(z)}{\eta(3z)} + 9\,\frac{\eta^{11}(3z)}{\eta^3(z)}, \qquad g_3(z) = \eta^3(z)\eta^5(3z). \tag{11.66}$$