Iterative Learning Control Using Stochastic Approximation Theory with Application to a Mechatronic System

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Abstract. In this paper it is shown how Stochastic Approximation theory can be used to derive and analyse well-known Iterative Learning Control algorithms for linear systems. The Stochastic Approximation theory gives conditions that, when satisfied, ensure almost sure convergence of the algorithms to the optimal input in the presence of stochastic disturbances. The practical issues of monotonic convergence and robustness to model uncertainty are considered. Specific choices of the learning matrix are studied, as well as a model-free choice. Moreover, the model-free method is applied to a linear motor system, leading to greatly improved tracking.

1 Introduction

Iterative Learning Control (ILC) is a technique used to enhance the tracking performance of systems that perform repetitive operations. In this approach, information 'learnt' from previous repetitions is used to improve the performance of the system during the next repetition/iteration i.e. reduce the tracking error. ILC has been shown to be very effective for systems that are predominately affected by deterministic, repetitive disturbances, which are learnt from one iteration to the next. However, when the system is affected by stochastic disturbances the tracking performance is greatly diminished [9, 5]. It is, therefore, important to develop ILC algorithms that have reduced sensitivity to this type of disturbance.

Although the deterministic aspects of ILC have received more attention, certain researchers have already proposed algorithms that are robust to the presence of stochastic disturbances.

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1) The use of a forgetting factor in ILC was first proposed in [12] for a D-type ILC law. It was then proposed in [2] for P-type ILC. It is shown that by introducing the forgetting factor the system's output converges to a neighbourhood of the desired one, despite the presence of norm-bounded initialisation errors, fluctuations of the dynamics and random disturbances. However, in [19] and [5], it is shown that the use of a forgetting factor can increase the expected value and variance of the error signal compared to standard ILC algorithms.

2) The filtering of the ILC command has been proposed in certain papers as a way of reducing the influence of noise on the error [15]. However, whilst it reduces the error variance, it causes a nonzero converged mean error.

3) Kalman filtering-type techniques have also been applied to ILC to estimate the controlled output, in the presence of disturbances [22, 20, 21, 14, 8, 1]. In the case of perfect knowledge of the disturbance covariance matrices and system parameters, convergence to the optimal input can be shown. However, perfect knowledge is unrealistic.

4) In [22] another ILC algorithm is proposed using a learning gain that decreases inversely proportionally to the iteration number and has the form of a Stochastic Approximation (SA) algorithm. No detailed analysis is, however, carried out. An algorithm with a similar iteration decreasing learning gain is also developed in [17] for repetitive disturbance rejection in the presence of measurement noise. This algorithm is derived in a similar way to recursive least squares identification algorithms, without mention to SA. The application of SA theory to ILC is most directly considered in [6] and [7] for the linear and nonlinear cases respectively. It is shown that the proposed ILC law converges almost surely to the optimal input and the output error is minimised in the mean square sense as the number of iterations tends to infinity. The algorithm requires only that the optimal input is realisable. Knowledge of neither the disturbance covariance matrix nor the system matrices is required because a simultaneous perturbation type algorithm is employed, which uses random perturbations to estimate the gradient. The disadvantage of this approach is slow convergence.

The main contribution of this paper is to show how ILC for linear systems affected by stochastic disturbances fits into the SA theory framework. Using SA theory it is possible to derive necessary conditions for well-known ILC algorithms to converge almost surely to the optimal input signal in the presence of stochastic disturbances. In addition, the important practical issues of monotonic convergence of the error signal and robustness to system uncertainty are addressed. Also two choices of learning matrix based on an uncertain model are studied, as well as a model-free choice. These choices are compared in a simulation example in [4].

In [6] the input is randomly perturbed and applied to the system in a second experiment at each iteration in order to estimate the gradient of the proposed cost function. In contrast, here either an uncertain system model or a second special experiment is considered. These choices will typically lead to faster convergence.

Steepest descent algorithms have been applied to ILC for the discrete-time case in [11]. Although certain similarities exist between the algorithms considered here and steepest descent algorithms, the major difference is the conditions SA sets on the step sizes between iterations. These conditions are necessary to ensure almost sure convergence to the optimal input in the presence of stochastic disturbances.

This paper is organised as follows. In Section 2 the notational framework is defined and the assumptions are stated. In Section 3 ILC is considered from an SA perspective. Then in Section 4 possible choices of the learning matrix are considered. In Section 5 experimental results obtained on a linear motor system are presented. Finally in Section 6 some conclusions are made.

2 Notation

We consider the linear time-invariant (LTI), discrete-time, stable SISO system $G(q)$, shown in Fig. 1, that carries out a finite-time, repetitive tracking task and whose controlled output $z_k(t)$, at time t and repetition k, is given by:

$$z_k(t) = G(q)u_k(t) + d_k(t), \tag{1}$$

where $u_k(t)$ is the input to the system, $d_k(t)$ is the load disturbance and q is the forward-shift time domain operator. The system's measured output, $y_k(t)$, is:

$$y_k(t) = z_k(t) + n_k(t), \tag{2}$$

where $n_k(t)$ is the measurement disturbance. It should be mentionned that if $G(q)$ represents a closed-loop transfer function then $d_k(t)$ and $n_k(t)$ will be the signals resulting from the filtering of external disturbances by the corresponding closed-loop transfer functions.

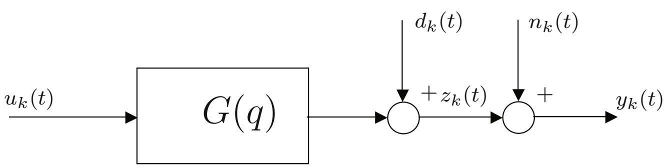

Fig. 1. System affected by stochastic disturbances

The controlled tracking error signal is defined as:

$$\epsilon_k(t) = y_d(t) - z_k(t), \tag{3}$$

where $y_d(t)$ is the bounded desired system output, which is defined over a finite repetition duration for $t = 0, \dots, N-1$, and the measured error signal is given by:

$$e_k(t) = y_d(t) - y_k(t). \tag{4}$$

As the signals are defined over a finite duration, it is possible to express the system's input-output relationship by a matrix representation. Taking advantage of the non-causal filtering possibilities of ILC, the lifted-system representation is used. For a system with a relative degree m we define the vectors:

$$\mathbf{u}_k = [u_k(0), u_k(1), \ldots, u_k(N-m-1)]^T \tag{5}$$
$$\mathbf{z}_k = [z_k(m), z_k(m+1), \ldots, z_k(N-1)]^T. \tag{6}$$

The vectors $\mathbf{y}_k$, $\mathbf{d}_k$, $\mathbf{n}_k$ and $\mathbf{y_d}$ are defined similarly to $\mathbf{z}_k$. Using these vectors, the measured output of the system is:

$$\mathbf{y}_k = \mathbf{G}\mathbf{u}_k + \mathbf{d}_k + \mathbf{n}_k, \tag{7}$$

where $\mathbf{G}$ is:

$$\mathbf{G} = \begin{bmatrix} g_m & 0 & \ldots & 0 \\ g_{m+1} & g_m & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_{N-1} & g_{N-2} & \cdots & g_m \end{bmatrix}, \tag{8}$$

g_i being the ith Markov parameter of $G(q)$. The controlled error vector is:

$$\boldsymbol{\epsilon}_k(\mathbf{u}_k) = \mathbf{y_d} - \mathbf{z}_k = \mathbf{y_d} - \mathbf{G}\mathbf{u}_k - \mathbf{d}_k \tag{9}$$

and the measured error vector:

$$\mathbf{e}_k(\mathbf{u}_k) = \mathbf{y_d} - \mathbf{y}_k = \boldsymbol{\epsilon}_k(\mathbf{u}_k) - \mathbf{n}_k, \tag{10}$$

where the errors' dependence on $\mathbf{u}_k$ is explicitly stated.

Furthermore, we have that the real system can be represented as:

$$G(q) = \hat{G}(q)[1 + \Delta(q)] \tag{11}$$

where $\hat{G}(q)$ is a model of the system and $\Delta(q)$ represents the multiplicative uncertainty. This representation is given in lifted-system form as:

$$\mathbf{G} = \hat{\mathbf{G}}[\mathbf{I} + \boldsymbol{\Delta}] \tag{12}$$

where $\mathbf{I}$ is the identity matrix, and $\hat{\mathbf{G}}$ and $\mathbf{I} + \boldsymbol{\Delta}$ are Toeplitz matrices formed similarly to (8) from the Markov parameters of $q^{\hat{m}}\hat{G}(q)$ and $q^{m-\hat{m}}[1 + \Delta(q)]$, respectively. $\hat{m}$ is the relative degree of $\hat{G}(q)$.

Definition: A real, square matrix $\mathbf{M}$ (not necessarily symmetric) is called positive definite $\mathbf{M} > 0$ if and only if all the eigenvalues of its symmetric part $(\mathbf{M} + \mathbf{M}^T)/2$ are positive.

2.1 Assumptions

(A1) The ideal input: $\mathbf{u}^* = \mathbf{G}^{-1}\mathbf{y_d}$ is realisable.
(A2) The system uncertainty satisfies: $\mathbf{I} + \mathbf{\Delta} > 0$.
(A3) The disturbances $\mathbf{d}_k$ and $\mathbf{n}_k$ are zero-mean, weakly stationary random vectors with unknown covariance matrices $\mathbf{R_d}$ and $\mathbf{R_n}$, respectively. Additionally, they have bounded, unknown cross-covariance matrices $\mathbf{R_{dn}}$ and $\mathbf{R_{nd}}$. Moreover, different realisations of $\mathbf{d}_k$ and $\mathbf{n}_k$ between iterations are mutually independent.
(A4) The mean input is bounded for all iterations: $E\{\mathbf{u}_k\} < \infty \quad \forall k$.

Remarks:

1) It is shown in [10] that a sufficient condition for Assumption **(A2)** is that the filter $q^{m-\hat{m}}[1+\Delta(q^{-1})]$ is strictly positive real (SPR). So when $m = \hat{m}$, Assumption **(A2)** is satisfied when $\|\Delta\|_\infty < 1$. This condition occurs frequently in the model uncertainty representation and so is a reasonable assumption.
2) The validity of Assumption **(A4)** will be discussed later in the chapter.

3 ILC from a SA Viewpoint

The ideal aim of tracking control is to achieve zero controlled error. When stochastic disturbances affect a system this objective is not possible. A reasonable aim is then to set the mean controlled error equal to zero. We can state a goal of the ILC algorithm, thus, as to iteratively calculate the optimal input signal $\mathbf{u}^*$ such that:

$$E\{\mathbf{L}\boldsymbol{\epsilon}_k(\mathbf{u}^*)\} = E\{\mathbf{L}\mathbf{e}_k(\mathbf{u}^*)\} = \mathbf{0}, \tag{13}$$

where $E\{\cdot\}$ denotes the mathematical expectation and $\mathbf{L}$ is a non-singular matrix.

It is straightforward to see that the solution to criterion (13) is $\mathbf{u}^*$ in Assumption **(A1)**. However, in order to calculate the ideal input $\mathbf{u}^*$ directly exact knowledge of $\mathbf{G}$ is needed, which is not available. Nevertheless, $\mathbf{u}^*$ can be found using an iterative stochastic approximation (SA) procedure, such as the Robbins-Monro algorithm [18], which does not require exact system knowledge. This algorithm calculates the input iteratively as:

$$\mathbf{u}_{k+1} = \mathbf{u}_k + \gamma_k \mathbf{L}\mathbf{e}_k(\mathbf{u}_k). \tag{14}$$

This algorithm clearly has the form of a standard P-type ILC law with an iteration varying learning gain γ_k. In the next subsection conditions will be given that, according to SA theory, ensure almost sure convergence of the algorithm to the ideal input.

3.1 Almost Sure Convergence

Theorem 1. *Under the Assumptions* **(A1)**, **(A3)** *and* **(A4)**, *the iterative update algorithm (14) converges almost surely to the solution* $\mathbf{u}^*$ *of (13) when* $k \to \infty$ *if:*

(C1) *The sequence* γ_k *of positive steps satisfies:*

$$\sum_{k=0}^{\infty} \gamma_k = \infty \quad \textit{and} \quad \sum_{k=0}^{\infty} \gamma_k^2 < \infty. \tag{15}$$

(C2) $E\{\mathbf{Le}_k(\mathbf{u}_k)\}$ *is monotonically decreasing:*

$$\mathbf{Q}(\mathbf{u}_k) = \frac{d}{d\mathbf{u}_k} E\{\mathbf{Le}_k(\mathbf{u}_k)\} < 0. \tag{16}$$

Proof. The proof is similar to that of the Robbins-Monro stochastic approximation algorithm.

Condition **(C1)** should be fulfilled by an appropriate choice of the sequence γ_k. $\mathbf{Q}(\mathbf{u}_k)$, in Condition **(C2)**, can be rewritten as:

$$\mathbf{Q}(\mathbf{u}_k) = \frac{d}{d\mathbf{u}_k} E\{\mathbf{Le}_k(\mathbf{u}_k)\} = \frac{d}{d\mathbf{u}_k}\ E\{\mathbf{Ly_d} - \mathbf{LGu}_k + \mathbf{Ld}_k + \mathbf{Lv}_k\}$$

$$= -\mathbf{LG} = -\mathbf{L}\hat{\mathbf{G}}[\mathbf{I} + \boldsymbol{\Delta}] \tag{17}$$

and so Condition **(C2)** becomes:

$$\mathbf{L}\hat{\mathbf{G}}[\mathbf{I} + \boldsymbol{\Delta}] > 0. \tag{18}$$

Remark: By combining equations (9), (10), (14) and **A1** we can obtain the input error evolution as:

$$\mathbf{e}_{k+1}^u = \mathbf{u}^* - \mathbf{u}_{k+1} = (\mathbf{I} - \gamma_k \mathbf{LG})\mathbf{e}_k^u + \gamma_k \mathbf{L}(\mathbf{y_d} - \mathbf{d_k} - \mathbf{n_k}). \tag{19}$$

A necessary, but not sufficient, condition for asymptotic convergence of the input error, in the absence of disturbances, is:

$$|\lambda_i(\mathbf{I} - \gamma_k \mathbf{LG})| < 1 \quad \forall k, \forall i \tag{20}$$

where $\lambda_i(\cdot)$ is the i^{th} eigenvalue. If $\mathbf{L}$ represents a causal operator and is therefore a real, lower triangular matrix, a link between this condition and those given by SA theory can be made, as detailed below. Since $\mathbf{I} - \gamma_k \mathbf{LG}$ will be a real, lower triangular matrix, its eigenvalues will be real. (20) therefore implies:

$$\overline{\lambda}(\mathbf{I} - \gamma_k \mathbf{LG}) < 1 \iff 1 - \gamma_k \underline{\lambda}(\mathbf{LG}) < 1 \iff \gamma_k \underline{\lambda}(\mathbf{LG}) > 0 \tag{21}$$

and

$$\underline{\lambda}(\mathbf{I} - \gamma_k \mathbf{LG}) > -1 \iff 1 - \overline{\lambda}(\gamma_k \mathbf{LG}) > -1 \iff \gamma_k \overline{\lambda}(\mathbf{LG}) < 2, \qquad (22)$$

where $\underline{\lambda}(\cdot)$ and $\overline{\lambda}(\cdot)$ are the minimum and maximum eigenvalues, respectively. Moreover we have $\mathbf{LG}\mathbf{x}_i = \lambda_i \mathbf{x}_i$, where $\mathbf{x}_i$ is the real eigenvector corresponding to λ_i. Taking the transpose of the both sides gives:

$$\mathbf{x}_i^T (\mathbf{LG})^T = \lambda_i \mathbf{x}_i^T. \qquad (23)$$

Right multiplying (23) by $\mathbf{x}_i$ and adding it with its transpose gives:

$$\mathbf{x}_i^T \mathbf{LG}\mathbf{x}_i + \mathbf{x}_i^T (\mathbf{LG})^T \mathbf{x}_i = 2\lambda_i \mathbf{x}_i^T \mathbf{x}_i \iff \mathbf{x}_i^T \left(\frac{\mathbf{LG} + (\mathbf{LG})^T}{2} \right) \mathbf{x}_i = \lambda_i \mathbf{x}_i^T \mathbf{x}_i. \qquad (24)$$

So, if $\mathbf{LG}$ is positive definite, (21) is satisfied. (22) can be satisfied by an appropriate choice of γ_k.

3.2 Monotonic Convergence

Whilst almost sure convergence of the input sequence to the solution $\mathbf{u}^*$ when $k \to \infty$ is, obviously, of utmost importance, practically it is not the only type of convergence of interest. The monotonic convergence, from one iteration to the next, of a norm of the controlled error is also of concern.

To proceed, we will need the following lemma:

Lemma 1. *If a real, square matrix* $\mathbf{M}$ *(not necessarily symmetric) is positive definite, there exists an* $\alpha > 0$ *such that:*

$$\overline{\sigma}(\mathbf{I} - \alpha \mathbf{M}) < 1, \qquad (25)$$

where $\overline{\sigma}(\cdot)$ *is the maximum singular value.*

Proof. Condition (25) is true iff:

$$\begin{aligned} &\lambda_i \left(I - \alpha(\mathbf{M}^T + \mathbf{M}) + \alpha^2 \mathbf{M}^T \mathbf{M} \right) < 1 \quad \forall i \\ \iff &1 - \lambda_i \left(\alpha(\mathbf{M}^T + \mathbf{M}) - \alpha^2 \mathbf{M}^T \mathbf{M} \right) < 1 \quad \forall i \\ \iff &\lambda_i \left(\mathbf{M}^T + \mathbf{M} - \alpha \mathbf{M}^T \mathbf{M} \right) > 0 \quad \forall i. \end{aligned} \qquad (26)$$

Furthermore the eigenvalues satisfy:

$$[\mathbf{M}^T + \mathbf{M} - \alpha \mathbf{M}^T \mathbf{M}]\mathbf{x}_i = \lambda_i \mathbf{x}_i. \qquad (27)$$

Left multiplying (27) by $\mathbf{x}_i^T$ *we get:*

$$\mathbf{x}_i^T \left(\mathbf{M}^T + \mathbf{M} \right) \mathbf{x}_i - \alpha \mathbf{x}_i^T \mathbf{M}^T \mathbf{M} \mathbf{x}_i = \lambda_i \mathbf{x}_i^T \mathbf{x}_i. \qquad (28)$$

So if $\mathbf{M} > 0$*, (26), and thus Condition (25), are satisfied when:*

$$0 < \alpha < \min_i \frac{\mathbf{x}_i^T \left(\mathbf{M}^T + \mathbf{M}\right) \mathbf{x}_i}{\mathbf{x}_i^T \mathbf{M}^T \mathbf{M} \mathbf{x}_i}. \tag{29}$$

Theorem 2. *If* $\hat{\mathbf{G}}[\mathbf{I} + \mathbf{\Delta}]\mathbf{L} > 0$*, there exists a sequence of positive step sizes* γ_k*, satisfying Condition* **(C1)***, such that monotonic convergence of the 2-norm of the mean controlled error is achieved.*

Proof. By combining equations (9), (10), (12) and (14) we can obtain the controlled error evolution equation as:

$$\boldsymbol{\epsilon}_{k+1}(\mathbf{u}_{k+1}) = (\mathbf{I} - \gamma_k \hat{\mathbf{G}}[\mathbf{I} + \mathbf{\Delta}]\mathbf{L})\boldsymbol{\epsilon}_k(\mathbf{u}_k) + \mathbf{d}_k - \mathbf{d}_{k+1} + \gamma_k \hat{\mathbf{G}}[\mathbf{I} + \mathbf{\Delta}]\mathbf{L}\mathbf{n}_k. \tag{30}$$

The mean value of equation (30) is:

$$E\{\boldsymbol{\epsilon}_{k+1}(\mathbf{u}_{k+1})\} = (\mathbf{I} - \gamma_k \hat{\mathbf{G}}[\mathbf{I} + \mathbf{\Delta}]\mathbf{L})E\{\boldsymbol{\epsilon}_k(\mathbf{u}_k)\}. \tag{31}$$

Monotonic convergence of the 2-norm of the mean controlled error is obtained if the following condition is satisfied (see e.g. Theorem 2, [16]):

$$\overline{\sigma}(\mathbf{I} - \gamma_k \hat{\mathbf{G}}[\mathbf{I} + \mathbf{\Delta}]\mathbf{L}) < 1 \quad \forall k. \tag{32}$$

If a given sequence γ_k*, satisfying Condition* **(C1)***, does not satisfy (32), a new, scaled sequence* $\gamma_k \triangleq \beta\gamma_k$*,* $\beta > 0$ *can always be defined that does, as follows from Lemma 1.*

Remarks:

1) Theorem 2's requirement that $\hat{\mathbf{G}}[\mathbf{I} + \mathbf{\Delta}]\mathbf{L}$ be positive definite is satisfied when $\mathbf{L}$ and $\hat{\mathbf{G}}[\mathbf{I} + \mathbf{\Delta}]$ commute, i.e. when $L(q)$ is causal, and condition (18) is satisfied.
2) Since the system $G(q)$ is assumed stable, its output and internal states will be bounded if its input is bounded. Combining equations (9), (10), (12) and (14) gives the input evolution equation as:

$$\mathbf{u}_{k+1} = (\mathbf{I} - \gamma_k \mathbf{L}\hat{\mathbf{G}}[\mathbf{I} + \mathbf{\Delta}])\mathbf{u}_k + \gamma_k \mathbf{L}(\mathbf{y_d} - \mathbf{d_k} - \mathbf{n_k}). \tag{33}$$

According to Theorem 5 of [16] the input will remain bounded from one iteration to the next if a) (33) is a uniformly exponentially stable iterative system, b) for a finite constant β, $\|\gamma_k \mathbf{L}\| < \beta\ \forall k$, and c) $\mathbf{y_d}$, $\mathbf{d_k}$ and $\mathbf{n_k}$ are bounded. As stated in Corollary 1 of [16], (33) is a uniformly exponentially stable iterative system if $\overline{\sigma}(\mathbf{I} - \gamma_k \mathbf{L}\hat{\mathbf{G}}[\mathbf{I} + \mathbf{\Delta}]) < 1 \quad \forall k$. This condition is considered in Lemma 1, implying that, when $\mathbf{L}\hat{\mathbf{G}}[\mathbf{I} + \mathbf{\Delta}] > 0$, a sequence γ_k exists that achieves uniform exponential stability. Furthermore, since $\|\gamma_k \mathbf{L}\| = |\gamma_k|\|\mathbf{L}\|$, there exists a sequence γ_k that satisfies the condition $\|\gamma_k \mathbf{L}\| < \beta\ \forall k$. So the boundedness of the system's signals requires the

disturbances to be bounded, which can usually be assumed to be the case in practice.
It should be noted that the mean input $E\{\mathbf{u}_k\}$ will be bounded if only the means of the disturbances are bounded, rather than the disturbances themselves.

3.3 Asymptotic Distribution of the Input Estimation Error

The asymptotic distribution of the input estimation error is given by the following theorem:

Theorem 3. *Assume that:*

i) Algorithm (14) converges almost surely to the solution $\mathbf{u}^*$ *as* $k \to \infty$.
ii) The sequence of step sizes is chosen as $\gamma_k = \frac{\alpha}{k+1}$.
iii) All the eigenvalues of the matrix $\mathbf{D} = \mathbf{I}/2 + \alpha\mathbf{Q}(\mathbf{u}^*)$ *have negative real parts.*

Then the sequence $\sqrt{k}(\mathbf{u}_k - \mathbf{u}^*) \in As\,\mathcal{N}(\mathbf{0}, \mathbf{V})$ *i.e it converges asymptotically in distribution to a zero-mean normal distribution with covariance:*

$$\mathbf{V} = \alpha^2 \int_0^\infty \exp(\mathbf{D}\mathbf{x})\mathbf{P}\exp(\mathbf{D}^T\mathbf{x})dx \tag{34}$$

where $\mathbf{P}$ *is the covariance matrix of* $\mathbf{L}\mathbf{e}(\mathbf{u}^*)$*:*

$$\mathbf{P} = E\{\mathbf{L}\mathbf{e}_k(\mathbf{u}^*)(\mathbf{L}\mathbf{e}_k(\mathbf{u}^*))^T\}. \tag{35}$$

Proof. The proof can be found in [13] (Theorem 6.1 p.147).
Using Theorem 3 we have that:

$$\begin{aligned}\mathbf{P} &= E\{\mathbf{L}\mathbf{e}_k(\mathbf{u}^*)(\mathbf{L}\mathbf{e}_k(\mathbf{u}^*))^T\} = E\{(-\mathbf{L}(\mathbf{d}_k + \mathbf{n}_k))(-\mathbf{L}(\mathbf{d}_k + \mathbf{n}_k))^T\} \\ &= \mathbf{L}(\mathbf{R_d} + \mathbf{R_{dn}} + \mathbf{R_{nd}} + \mathbf{R_n})\mathbf{L}^T.\end{aligned} \tag{36}$$

Additionally, as $\mathbf{Q}(\mathbf{u}^*) = \frac{d}{d\mathbf{u}_k}E\{\mathbf{L}\mathbf{e}_k(\mathbf{u}_k)\}|_{\mathbf{u}(k)=\mathbf{u}_0} = -\mathbf{L}\mathbf{G}$, we have that:

$$\mathbf{D} = (\mathbf{I}/2 - \alpha\mathbf{L}\mathbf{G}). \tag{37}$$

The covariance matrix $\mathbf{V}$ is then the unique symmetric solution of the following Lyapunov equation:

$$2\alpha^2\mathbf{L}(\mathbf{R_d} + \mathbf{R_{dn}} + \mathbf{R_{nd}} + \mathbf{R_n})\mathbf{L}^T + (\mathbf{I} - 2\alpha\mathbf{L}\mathbf{G})\mathbf{V} + \mathbf{V}(\mathbf{I} - 2\alpha\mathbf{L}\mathbf{G})^T = \mathbf{0}. \tag{38}$$

It is shown in [3] (Proposition 4, p.112) that if, instead of using a scalar learning gain α, we use a non-singular learning matrix $\mathbf{K}$, then the optimal matrix $\mathbf{K}^*$ to mimimise the trace of $\mathbf{V}$ is given by:

$$\mathbf{K}^* = -\mathbf{Q}(\mathbf{u}^*)^{-1} = (\mathbf{L}\mathbf{G})^{-1}. \tag{39}$$

Using this gain matrix results in the learning law:

$$\mathbf{u}_{k+1} = \mathbf{u}_k + \frac{\mathbf{G}^{-1}}{k+1}\mathbf{e}_k(\mathbf{u}_k), \tag{40}$$

and the optimal asymptotic covariance matrix:

$$\mathbf{V}^* = \mathbf{G}^{-1}(\mathbf{R_d} + \mathbf{R_{dn}} + \mathbf{R_{nd}} + \mathbf{R_n})\mathbf{G}^{-T}, \tag{41}$$

which means that the sequence $\sqrt{k}(\mathbf{u}_k - \mathbf{u}^*) \in \text{As}\,\mathcal{N}(\mathbf{0}, \mathbf{V}^*)$.

Moreover we have that $\boldsymbol{\epsilon}_k(\mathbf{u}_k) = -\mathbf{G}(\mathbf{u}_k - \mathbf{u}^*) - \mathbf{d}_k$ so the covariance matrix of $\boldsymbol{\epsilon}_k(\mathbf{u}_k)$ is then given by:

$$\text{cov}(\boldsymbol{\epsilon}_k(\mathbf{u}_k)) = E\{\boldsymbol{\epsilon}_k(\mathbf{u}_k)\boldsymbol{\epsilon}_k^T(\mathbf{u}_k)\} = \mathbf{G}E\{(\mathbf{u}_k - \mathbf{u}^*)(\mathbf{u}_k - \mathbf{u}^*)^T\}\mathbf{G}^T + \mathbf{R_d}. \tag{42}$$

Using the optimal gain matrix $\mathbf{K}^*$ means that the sequence $\boldsymbol{\epsilon}_k(\mathbf{u}_k)$ will have a converged covariance matrix given by

$$\text{cov}(\boldsymbol{\epsilon}_k(\mathbf{u}_k)) = \frac{1}{k}(\mathbf{R_d} + \mathbf{R_{dn}} + \mathbf{R_{nd}} + \mathbf{R_n}) + \mathbf{R_d}$$

and in the limit we have: $\lim_{k\to\infty} \text{cov}(\boldsymbol{\epsilon}_k(\mathbf{u}_k)) = \mathbf{R_d}$. However, $\mathbf{K}^*$ is not implementable because exact knowledge of $\mathbf{G}$ is not achievable. Nonetheless it gives an ideal law to aim for in the design of a stochastic ILC algorithm.

4 Specific Choices of L

In this section specific choices of the learning matrix $\mathbf{L}$ will be considered.

4.1 Use of the Uncertain System Inverse

We consider here the choice of $\mathbf{L} = \hat{\mathbf{G}}^{-1}$ i.e. the inverse of the uncertain system model. This choice is motivated by the fact that $\mathbf{L} = \hat{\mathbf{G}}^{-1}$ is an approximation of the optimal learning gain used in (40).

Theorem 4. *Under Assumption (**A2**) and when $\mathbf{L} = \hat{\mathbf{G}}^{-1}$, there exists a sequence of positive step sizes γ_k, satisfying Condition* **(C1)***, that ensures that the ILC algorithm (14) converges almost surely to $\mathbf{u}^*$ and that the 2-norm of the mean controlled error convergences monotonically.*

*Proof. Condition (18) is automatically satisfied when $\mathbf{L} = \hat{\mathbf{G}}^{-1}$, under Assumption (**A2**). Therefore, when the sequence of positive step sizes γ_k satisfies Condition* **(C1)***, the ILC algorithm (14) converges almost surely to $\mathbf{u}^*$, as stated by Theorem 1. Moreover, because $\mathbf{I} + \boldsymbol{\Delta}$ is a lower triangular Toeplitz matrix, $\mathbf{I}+\boldsymbol{\Delta}$ commutes with $\hat{\mathbf{G}}$ and, under Assumption (**A2**), $\hat{\mathbf{G}}[\mathbf{I}+\boldsymbol{\Delta}]\mathbf{L} > 0$. This result means Theorem 2 applies, implying the existence of a sequence, satisfying Condition* **(C1)***, that ensures monotonic convergence.*

4.2 Use of the Uncertain System Transpose

Another choice is $\mathbf{L} = \hat{\mathbf{G}}^T$. This choice is motivated by the fact that it can be used when $\hat{\mathbf{G}}$ is ill conditioned, as may be the case when $\hat{G}(q)$ has unstable zeros. The previously considered choice of $\mathbf{L}$, on the other hand, may not be usable because the input signal generated by the ILC algorithm can grow unacceptably large before converging to the ideal input.

Theorem 5. *Under Assumption (***A2***) and when* $\mathbf{L} = \hat{\mathbf{G}}^T$*, there exists a sequence of positive step sizes* γ_k*, satisfying Condition* **(C1)***, that ensures that the ILC algorithm (14) converges almost surely to* $\mathbf{u}^*$ *and that the 2-norm of the mean controlled error convergences monotonically.*

Proof. Since $\mathbf{I}+\mathbf{\Delta}$ *is a lower triangular Toeplitz matrix,* $\mathbf{I}+\mathbf{\Delta}$ *commutes with* $\hat{\mathbf{G}}$ *and condition (18) can be written as* $\hat{\mathbf{G}}^T[\mathbf{I}+\mathbf{\Delta}]\hat{\mathbf{G}} > 0$*, when* $\mathbf{L} = \hat{\mathbf{G}}^T$*. This condition is fulfilled when* $\hat{\mathbf{G}}$ *is non-singular and* $\mathbf{I} + \mathbf{\Delta} > 0$*. The former is true because* N *is finite and the latter is Assumption (***A2***). Therefore, when the sequence of positive step sizes* γ_k *satisfies Condition* **(C1)***, the ILC algorithm (14) converges almost surely to* $\mathbf{u}^*$*, as stated by Theorem 1. Moreover, Theorem 2 applies, implying the existence of a sequence, satisfying Condition* **(C1)***, that ensures monotonic convergence.*

4.3 Use of an Experiment

So far the use of a model to give an $\mathbf{L}$ that can then be used in (14) to evaluate $\mathbf{L}\mathbf{e}_k(\mathbf{u}_k)$ has been considered. For the specific choice of $\mathbf{L} = \mathbf{G}^T$, it is, however, possible to use an extra experiment per iteration to evaluate $\mathbf{L}\mathbf{e}_k(\mathbf{u}_k)$. Condition (18) is automatically satisfied with this choice, and Theorem 2 also applies.

The fact that a special experiment can be used is seen by noting that $\mathbf{e2} = \mathbf{G}^T\mathbf{e}_k(\mathbf{u}_k)$ is equal to the following filtering operations:

$$e1(t) = G(q)e_k(N - t, u_k(t)) \tag{43}$$

$$e2(t) = e1(N - t). \tag{44}$$

We see that, in the disturbance free case, $\mathbf{e2}$ can be found using an experiment on the true system, where the time reversed error signal is fed into the system as its input, the system output is measured and then time reversed itself. In reality the special experiment will have its own disturbances $d2(t)$ and $v2(t)$ associated with it. Nonetheless, an unbiased estimate of $\mathbf{e2}$ can still be found since:

$$\begin{aligned} E\{\mathbf{e2}\} &= E\{\mathbf{G}^T\mathbf{e}_k(\mathbf{u}_k) + \mathbf{d2} + \mathbf{v2}\} = E\{\mathbf{G}^T\mathbf{e}_k(\mathbf{u}_k)\} + E\{\mathbf{d2}\} + E\{\mathbf{n2}\} \\ &= \mathbf{G}^T E\{\boldsymbol{\epsilon}_k(\mathbf{u}_k)\} + \mathbf{0} + \mathbf{0}. \end{aligned} \tag{45}$$

This method of evaluating **e2** is attractive as it avoids the problems of model uncertainty. It does, however, require an additional, non-standard, experiment at each iteration, which, depending on the application, may not always be possible. One case where it may be useful is when ILC is used to tune the input to improve the system's performance before the system is used in its intended application.

Remarks:

1. So far the motivation of the ILC algorithms considered has been to find the input that solves the root-finding type criterion (13), which aims to set the mean controlled error to zero. The model-free algorithm can be motivated differently. Instead of criterion (13), a logical alternative objective is the minimisation of the trace of the controlled error covariance matrix i.e.:

$$\min_{\mathbf{u}_k} J_k(\mathbf{u}_k) = \min_{\mathbf{u}_k} \frac{1}{2} \mathrm{tr}\left(E\{\boldsymbol{\epsilon}_k(\mathbf{u}_k)\boldsymbol{\epsilon}_k(\mathbf{u}_k)^T\}\right). \tag{46}$$

   The minimum of this criterion occurs when:

$$\left.\frac{dJ_k(\mathbf{u}_k)}{d\mathbf{u}_k}\right|_{\mathbf{u}_k=\mathbf{u}^*} = E\left\{ \left(\left.\frac{\partial \boldsymbol{\epsilon}_k(\mathbf{u}_k)}{\partial \mathbf{u}_k}\right|_{\mathbf{u}_k=\mathbf{u}^*} \right)^T \boldsymbol{\epsilon}_k(\mathbf{u}^*) \right\} = -\mathbf{G}^T E\{\boldsymbol{\epsilon}_k(\mathbf{u}^*)\} = \mathbf{0}. \tag{47}$$

   $E\{\boldsymbol{\epsilon}_k(\mathbf{u}_k)\}$ is not directly measurable. Nonetheless, because equation (47) can be written as:

$$\left.\frac{dJ_k(\mathbf{u}_k)}{d\mathbf{u}_k}\right|_{\mathbf{u}_k=\mathbf{u}^*} = -\mathbf{G}^T E\{\boldsymbol{\epsilon}_k(\mathbf{u}^*)\} = -\mathbf{G}^T E\{\mathbf{e}_k(\mathbf{u}^*)\} = \mathbf{0} \tag{48}$$

   it is possible to find the minimiser of the criterion, again, using the Robbins-Monro algorithm:

$$\mathbf{u}_{k+1} = \mathbf{u}_k + \gamma_k \mathbf{G}^T \mathbf{e}_k(\mathbf{u}_k), \tag{49}$$

   i.e. (14) with $\mathbf{L} = \mathbf{G}^T$.
2. The model-free algorithm has similarities to that proposed in [23] where reversed time inputs are used to cancel the system phase and produce monotonic convergence. Stochastic aspects are not considered, however.
3. It also has similarities to [11], which uses the steepest descent method, and calls $\mathbf{G}^T$ the adjoint of $\mathbf{G}$. It shows that by using this 'adjoint' with an iteration-varying gain, monotonic convergence occurs. The gain sequence is calculated via an optimisation, which does not consider stochastic disturbances. The gain at iteration k is given by:

$$\gamma_k = \frac{\|\mathbf{G}^T \mathbf{e}_{k-1}\|^2}{w + \|\mathbf{G}\mathbf{G}^T \mathbf{e}_{k-1}\|^2}, \tag{50}$$

where w is a weight on γ_k in the cost function. Since the measured error signal is used to calculate the gain, it will be affected by stochastic disturbances. This means $\lim_{k\to\infty} \|\mathbf{G}^T \mathbf{e}_{k-1}\|^2 \neq 0$ and so $\lim_{k\to\infty} \gamma_k \neq 0$. This implies that the second series of condition **C1** cannot be satisfied. Therefore, whilst the algorithm developed can lead to fast deterministic convergence to the optimal input, this cannot be proved when stochastic disturbances are present.

5 Experimental Results

The model-free algorithm was applied to the tracking control of a linear, permanent magnet, synchronous motor (LPMSM), which forms the upper axis of an x-y positioning table. LPMSMs are very stiff and have no mechanical transmission components. They, therefore, do not suffer from backlash and so allow very high positioning accuracy to be achieved. Additionally they are capable of high velocities and accelerations. These properties make them a very appealing, and thus common, choice for use in industries where rapid, high precision movements are required.

A standard two-degree-of-freedom position controller is used to control the motor's position. It operates at a sampling frequency of 2kHz. An analog position encoder using sinusoidal signals with periods of 2μm, which are then interpolated with 8192 intervals/period to obtain a resolution of 0.24nm, is used to measure the motor's position. However, the accuracy of this type of encoders is limited to 20nm.

The input, $\mathbf{u}_k$, computed by the ILC algorithm, is used as the position reference signal of the closed-loop system.

The desired output position, $y_d(t)$, was a series of three low-pass filtered steps, each of amplitude 25mm in the positive direction, followed by a similar series of filtered steps in the negative direction. This movement represents a typical industrial positioning motion. It has $N = 8192$. This value corresponds to the maximum number of points in the look-up table into which the new reference signal is fed at each iteration.

The sequence $\gamma_k = \frac{\alpha}{k+1}$ is used with $\alpha = 0.85$, which was chosen to achieve monotonic convergence.

For the experiment $\mathbf{u}_0 = \mathbf{y}_d$ was used and 100 iterations were carried out. Figures 2 and 3 show the convergence of $\boldsymbol{\epsilon}_k^T(\mathbf{u}_k)\boldsymbol{\epsilon}_k(\mathbf{u}_k)$ and the initial and final tracking achieved, respectively.

As can be seen from the figures and the values $\boldsymbol{\epsilon}_0^T(\mathbf{u}_0)\epsilon_0(\mathbf{u}_0) = 6.0143 \times 10^{-8}\text{m}^2$ and $\boldsymbol{\epsilon}_{100}^T(\mathbf{u}_{100})\boldsymbol{\epsilon}_{100}(\mathbf{u}_{100}) = 1.9913 \times 10^{-14}\text{m}^2$ the algorithm considerably improves the tracking.

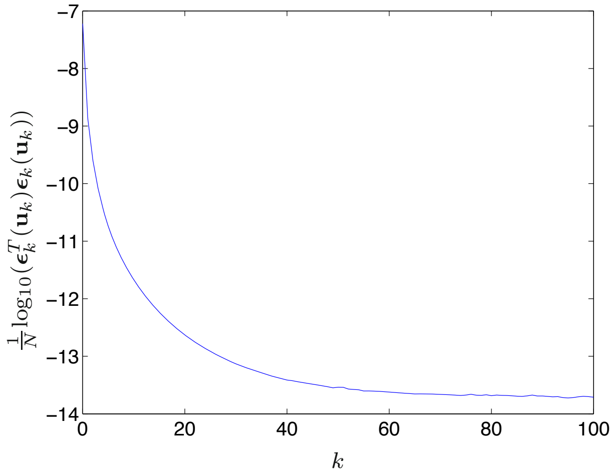

Fig. 2. $\boldsymbol{\epsilon}_k^T(\mathbf{u}_k)\boldsymbol{\epsilon}_k(\mathbf{u}_k)$ obtained using the model-free method

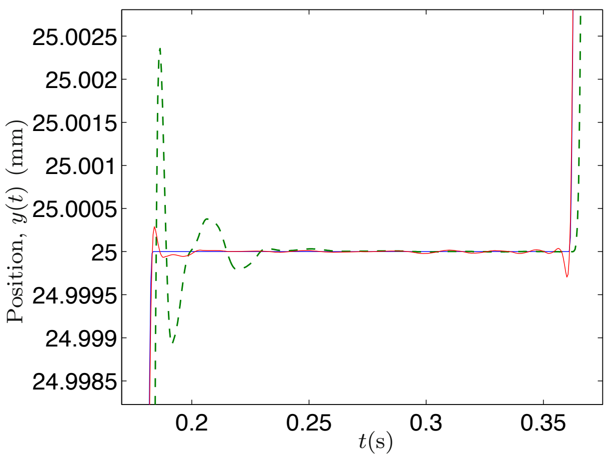

Fig. 3. Tracking at iteration $k = 0$ (green-dashed) and $k = 100$ (red) using the model-free method

6 Conclusions

The main contribution of this paper is to show how stochastic approximation theory can be used to derive and analyse Iterative Learning Control algorithms for linear time-invariant systems that are robust to non-repetitive disturbances. SA theory has provided general conditions that ensure almost sure convergence of the algorithm to the optimal input in the presence of stochastic disturbances.

ILC for LTI systems has been considered in this paper. The majority of the results apply, however, to linear time-varying (LTV) systems as well. In this case, however, the matrix $\mathbf{G}$ will not be lower triangular Toeplitz but a general lower triangular matrix instead. This implies that $\mathbf{L}$, $\hat{\mathbf{G}}$ and $\mathbf{I} + \mathbf{\Delta}$ will not, in general, commute.

The conditions imposed by SA require the learning gain to tend to zero as the iterations tend to infinity. This requirement is essential for stochastic learning algorithms. Practically it means that the learning ceases after a large number of iterations and if the desired output or repetitive disturbances change the algorithm will not react and the tracking will deteriorate. It is thus necessary to have a surveillance program that restarts the learning when the errors rise above a certain threshold.

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