Nonholonomic Mechanics, Dissipation and Quantization*

Anthony M. Bloch

Abstract. In this review paper we consider some of the basics of nonholonomic systems, considering in particular how it is possible do derive nonholonomic equations of motion as a limit of a Lagrangian system subject to dissipation. This in then extended to show how dissipation may be induced from a Hamiltonian field with a view to quantization of the system.

Keywords: Nonholonomic Systems, Dissipation, Quantization.

1 Introduction

In this (mainly) review paper we consider some of the basics of nonholonomic systems, considering in particular how it is possible do derive nonholonomic equations of motion as a limit of Lagrangian system subject to dissipation. This is then extended to show how dissipation may be induced from a Hamiltonian field, thus keeping the full system Hamiltonian, with a view to quantization of the system. Some of the basic ideas in nonholonomic systems theory may be found in [Bloch, Krishnaprasad, Marsden, and Murray(1996)] and [Bloch(2003)] (see also e.g. [Bullo and Lewis(2005)]) which thus give further background on the ideas described here. Below we give various references which link systems with nonholonomic constraints to the limit of infinite

A. Bloch
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109
Tel.: +734-647-4980
Fax: +734-763-0937
e-mail: abloch@umich.edu

* Support from NSF grants DMS-0604307 and DMS-0907949 is gratefully acknowledged.

J. Lévine & P. Müllhaupt (Eds.): Adv. in Theory of Control, Signals, LNCIS 407, pp. 141–152.
springerlink.com

friction. The key idea goes back to Caratheodory and there have been various interesting contributions since then including those by Fufaev and Kozlov among others.

2 Vertical Disk

We being by discussing a key example which is useful for illustrating many of the key idea in nonholonomic mechanics and control, the vertical disk (see [Bloch(2003)]). In this example the configuration space: $Q = \mathbb{R}^2 \times S^1 \times S^1$, parameterized by coordinates $q = (x, y, \theta, \varphi)$.

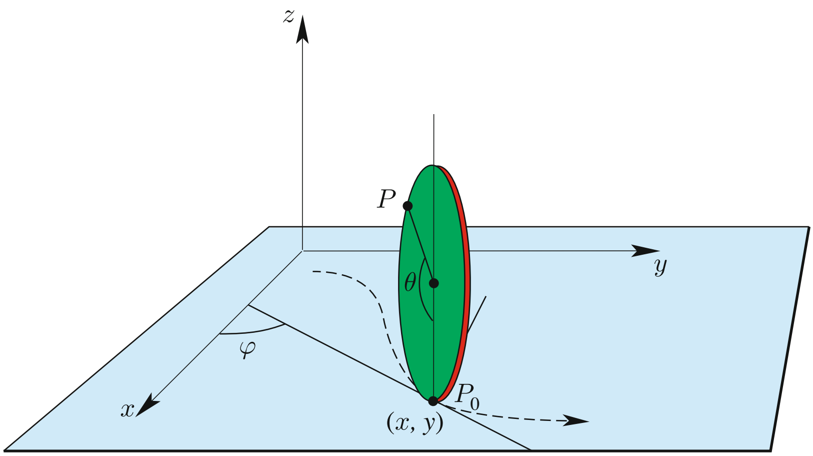

Fig. 1. The geometry for the rolling disk

The Lagrangian for the system is simply the kinetic energy

$$L(x, y, \theta, \phi, \dot{x}, \dot{y}, \dot{\theta}, \dot{\phi}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2.$$

If R is the radius of the disk, the nonholonomic constraints of rolling without slipping are

$$\dot{x} = R(\cos\varphi)\dot{\theta}$$
$$\dot{y} = R(\sin\varphi)\dot{\theta}, .$$

Dynamics of the Controlled Disk. We consider the case where we have two controls, one that can steer the disk and another that determines the roll torque. We obtain the Lagrange d'Alembert equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = u_1 X_1 + u_2 X_2 + \lambda_1 W_1 + \lambda_2 W_2,$$

where

$$\frac{\partial L}{\partial \dot{q}} = (m\dot{x}, m\dot{y}, I\dot{\theta}, J\dot{\varphi})^T,$$

$$X_1 = (0,0,1,0)^T, X_2 = (0,0,0,1)^T,$$

and

$$W_1^T = (1, 0, -R\cos\varphi, 0), \quad W_2^T = (0, 1, -R\sin\varphi, 0)^T,$$

together with the constraint equations.

Here u_1, u_2 are natural controls. We call the variables θ and ϕ "base" or "controlled" variables and the variables x and y "fiber" variables. While θ and φ are controlled directly, the variables x and y are controlled indirectly via the constraints. This a special case of a general construction on bundles (see [Bloch(2003)]).

It is clear here that the base variables are controllable in any sense we can imagine. Moreover the full system is controllable also by virtue of the nonholonomic (nonintegrable) nature of the constraints.

Also of interest is the so called ***Kinematic Controlled Disk***. In this case we imagine we have direct control over velocities rather than forces and, accordingly, we consider the most general first order system satisfying the constraints or lying in the "constraint distribution".

In this case the system is

$$\dot{q} = u_1 \overline{X}_1 + u_2 \overline{X}_2$$

where $\overline{X}_1 = (\cos\varphi, \sin\varphi, 1, 0)^T$ and $\overline{X}_2 = (0,0,0,1)^T$.

Interesting problems related to this system including motion planning and stabilization. Aspects of this are discussed in [Bloch(2003)] and references therein.

Nonholonomic Equations of Motion. We now discuss the nonholonomic equations of motion in general: see e.g [Bloch(2003)].

The Lagrange-d'Alembert Principle. Consider a system with a configuration space Q, local coordinates q^i and m nonintegrable constraints

$$\dot{s}^a + A^a_\alpha(r,s)\dot{r}^\alpha = 0$$

where $q = (r,s) \in \mathbb{R}^{n-p} \times \mathbb{R}^p$, which we write as $q^i = (r^\alpha, s^a)$, where $1 \leq \alpha \leq n-p$ and $1 \leq a \leq p$.

We also assume we have a Lagrangian $L(q^i, \dot{q}^i)$. The equations of motion given by Lagrange-d'Alembert principle.

Definition 1. The ***Lagrange-d'Alembert equations of motion*** for the system are those determined by

$$\delta \int_a^b L(q^i, \dot{q}^i)\, dt = 0,$$

where we choose variations $\delta q(t)$ of the curve $q(t)$ that satisfy $\delta q(a) = \delta q(b) = 0$ and $\delta q(t)$ satisfies the constraints for each t where $a \le t \le b$.

This principle is supplemented by the condition that the curve itself satisfies the constraints. Note that we take the variation *before* imposing the constraints; that is, we do not impose the constraints on the family of curves defining the variation.

This leads to the equations of motion

$$\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{r}^\alpha} - \frac{\partial L}{\partial r^\alpha} \right) = A^a_\alpha \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{s}^a} - \frac{\partial L}{\partial s^a} \right), \qquad \alpha = 1, \dots, n - m. \tag{1}$$

The equations (1) combined with the constraint equations

$$\dot{s}^a = -A^a_\alpha \dot{r}^\alpha, \qquad a = 1, \dots, m, \tag{2}$$

give a complete description of the *equations of motion* of the system. Notice that they consist of $n-m$ second-order equations and m first-order equations.

We remark that this is in contrast to so called variational nonholonomic systems (sometimes called vakonomic systems) where we constrain the class of curves over which we take variations. Such constrained variational problems may be solved by appending the constraints to the Lagrangian via Lagrange multipliers (for details and background see [Bloch(2003)].

3 Chaplygin Sleigh

One of the striking feature of nonholonomic systems is that while they conserve energy they need not conserve volume in the phase space (or momentum, even in the presence of symmetries). For more on this see [Bloch(2003)], [Zenkov and Bloch(2003)] and [Bloch, Marsden, and Zenkov(2009)].

Here we describe the Chaplygin sleigh, perhaps the simplest mechanical system which illustrates the possible dissipative nature of energy preserving nonholonomic systems.

If v denotes the velocity of the system along the direction of the blade and ω its angular velocity one can show that the equations of motion reduce to

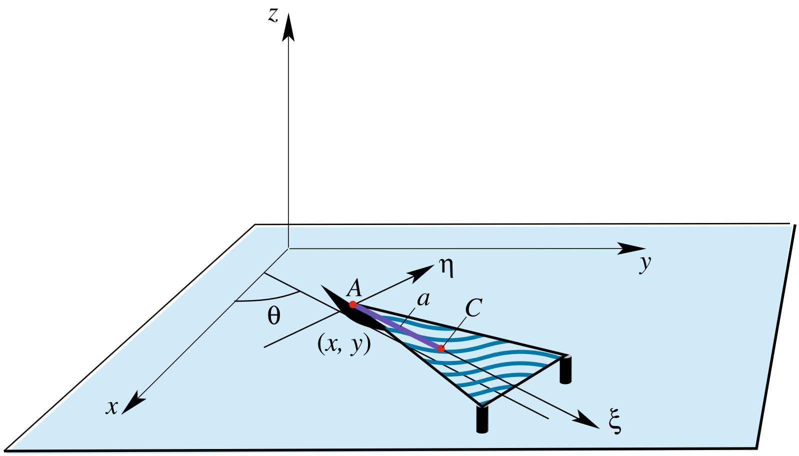

Fig. 2. The Chaplygin sleigh is a rigid body moving on two sliding posts and one knife edge

$$\dot{v} = a\omega^2$$
$$\dot{\omega} = -\frac{ma}{I + ma^2} v\omega$$

The equations have a family of relative equilibria given by $(v, \omega)|v =$ const, $\quad \omega = 0$.

Linearizing about any of these equilibria one finds one zero eigenvalue and one negative eigenvalue. In fact the solution curves are ellipses in $v - \omega$ plane with the positive v-axis attracting all solutions.

This is a special case of the so-called *Euler-Poincaré-Suslov* equations, an important special case of the reduced nonholonomic equations.

Another example is the *Euler-Poincaré-Suslov Problem* on $SO(3)$. In this case the problem can be formulated as the standard Euler equations

$$I\dot{\omega} = I\omega \times \omega$$

where $\omega = (\omega_1, \omega_2, \omega_3)$ are the system angular velocities in a frame where the inertia matrix is of the form $I = \mathrm{diag}(I_1, I_2, I_3)$ and the system is subject to the constraint

$$a \cdot \omega = 0$$

where $a = (a_1, a_2, a_3)$. The nonholonomic equations of motion are then given by

$$I\dot{\omega} = I\omega \times \omega + \lambda a$$

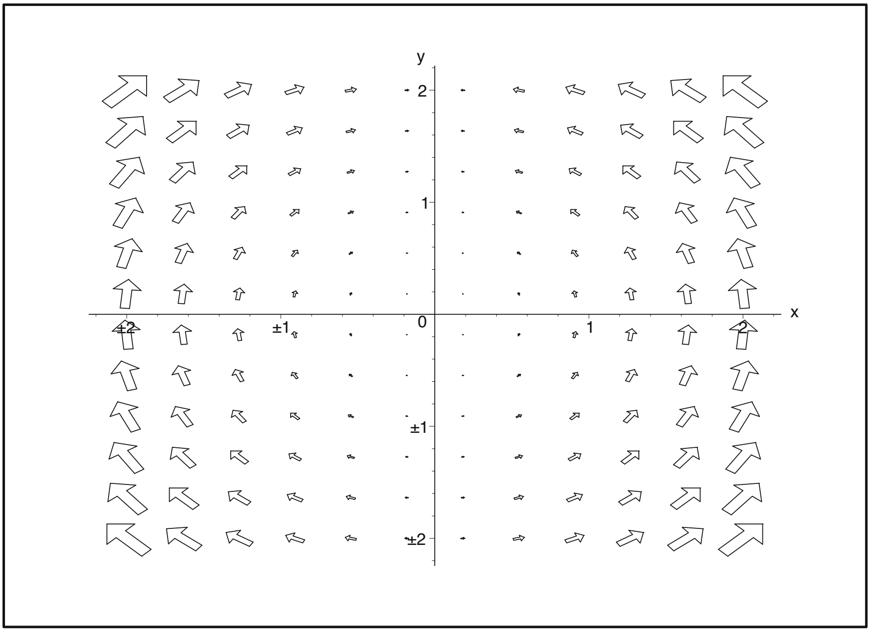

Fig. 3. Chaplygin Sleigh phase portrait

subject to the constraint. Solving for λ we get

$$\lambda = -\frac{I^{-1}a \cdot (I\omega \times \omega)}{I^{-1}a \cdot a}\,.$$

If a is an eigenvector of the moment of inertia tensor the flow is measure preserving.

We can extend to the general Euler-Poincaré-Suslov equations on a Lie algebra $\mathfrak{g}$ where the system is characterized by the Lagrangian $L = \frac{1}{2}\mathbb{I}_{AB}\Omega^A\Omega^B$ and the left-invariant constraint

$$\langle a, \Omega \rangle = a_A \Omega^A = 0. \tag{3}$$

Here $a = a_A e^A \in \mathfrak{g}^*$ and $\Omega = \Omega^A e_A$, where e_A, $A = 1, \ldots, k$, is a basis for $\mathfrak{g}$ and e^A is its dual basis. Multiple constraints may be imposed as well. The classical examples of such systems are the systems just discussed: the *Chaplygin Sleigh* and the *Suslov problem* introduced by Chaplygin in 1895 and Suslov in 1902, respectively.

4 Lamb Model of Damping

Our goal here is to implement the constraints in the sleigh model by an external field which in turn imposes dissipative motion on the sleigh. The model of dissipation that we use goes back to Lamb in 1900 (see [Lamb(1900)] and was discussed in detail in [Bloch, Hagerty and Weinstein (2004)] The original Lamb model is an oscillator physically coupled to a string. The vibrations of the oscillator transmit waves into the string and are carried off to infinity. Hence the oscillator loses energy and is effectively damped by the string.

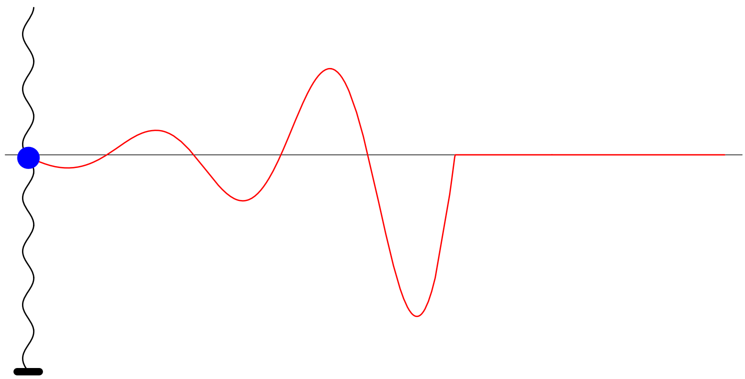

Fig. 4. Lamb model of an oscillator coupled to a string

Let $w(x,t)$ denote displacement of a string. with mass density ρ, tension T. Assuming a singular mass density at $x=0$, we couple to this an oscillator with position q and mass M (see figure 4) yielding the dynamics:

$$\begin{aligned}\frac{\partial^2 w}{\partial t^2} &= c^2\frac{\partial^2 w}{\partial x^2}\\ M\ddot{q}+Vq &= T[w_x]_{x=0}\\ q(t) &= w(0,t).\end{aligned}$$

$[w_x]_{x=0}=w_x(0+,t)-w_x(0-,t)$ is the jump discontinuity of the slope of the string. Note that this is a Hamiltonian system.

We can now solve for w and reduce (via elementary Fourier analysis) to obtain a reduced form of the dynamics describing the explicit motion of the oscillator subsystem,

$$M\ddot{q}+\frac{2T}{c}\dot{q}+Vq=0.$$

The coupling term arises explicitly as a Rayleigh dissipation term $\frac{2T}{c}\dot{q}$ in the dynamics of the oscillator.

5 Nonholonomic Systems as Limit

There is an interesting history behind the question of whether the Lagrange–d'Alembert equations can be obtained by starting with an unconstrained system subject to appropriately chosen dissipative forces, and then letting these forces go to infinity in an appropriate manner.

Nonholonomic constraints can be regarded in some sense as due to "infinite" friction. Several authors have asked if this can be quantified. Interestingly this goes back at least to the work of Caratheodory who asked if the limiting case of such friction could explain the motion of Chaplygin's sleigh. Caratheodory claimed this could not be done, but Fufaev in [Fufaev(1964)] showed that this was indeed possible. The general case was considered by Kozlov, [Kozlov(1983)] and Karapetyan [Karapetyan(1983)].

Kozlov ([Kozlov(1992)] showed also that variational nonholonomic equations (i.e. solutions of a constrained variational problem such as an optimal control problem, see [Bloch(2003)]) can be obtained as the result of another limiting process: He added a parameter-dependent "inertial term" to the Lagrangian of the constrained system, and then showed that the unconstrained equations approach the variational equations as the parameter approaches infinity.

The key idea in the nonholonomic setting is to take a nonlinear Rayleigh dissipation function of the form

$$F = -\frac{1}{2}k\sum_{j=1}^{m}\left(\sum_{i=1}^{n} a_i^{(j)}(\mathbf{q})\dot{q}_i\right)^2 \tag{4}$$

where $\sum_{i=1}^{n} a_i^{(j)}(\mathbf{q})\dot{q}_i = 0$, $i = 1 \ldots m$ are the constraints and $k >$ is a positive constant. Taking the limit as k goes to zero and using Tikhonov's theorem yields the nonholonomic dynamics.

However, the system in this setting is still not Hamiltonian. The goal here is to keep the system in the class of Hamiltonian systems by emulating the dissipation by coupling to an external field. We shall consider this issue in the next section.

Now consider again the Chaplygin sleigh which illustrates in very nice fashion the approach to limiting friction.

This mechanical system has three coordinates, two for the center of mass (x_C, y_C) and one "internal" angular variable θ for the rotation with respect to the knife edge located at $(x, y) = (x_C + a\cos\theta, y_C + a\sin\theta)$. The system can rotate freely around (x, y) but is only allowed to translate in the direction $(\cos\theta, \sin\theta)$: if we choose our coordinates as $\mathbf{q} = (x, y, \theta)$ there is a single constraint given by

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0, \tag{5}$$

or, $\mathbf{a}^{(1)} = (\sin\theta, -\cos\theta, 0)$.

The equations of motion can be also obtained using the virtual force method starting with the unconstrained Lagrangian

$$L_0 = \frac{m}{2}\left[\left(\dot{x} - a\dot{\theta}\sin\theta\right)^2 + \left(\dot{y} + a\dot{\theta}\cos\theta\right)^2\right] + \frac{I}{2}\dot{\theta}^2, \tag{6}$$

and using a Lagrange multiplier in the equations of motion:

$$\begin{aligned} m\frac{d}{dt}\left(\dot{x} - a\dot{\theta}\sin\theta\right) &= -\lambda\sin\theta, \\ m\frac{d}{dt}\left(\dot{y} + a\dot{\theta}\cos\theta\right) &= \lambda\cos\theta, \\ (I + ma^2)\ddot{\theta} + ma\dot{\theta}(\dot{x}\cos\theta + \dot{y}\sin\theta) &= 0. \end{aligned} \tag{7}$$

Carathedory and Fufaev added a viscous friction force of the from

$$R = -Nu \tag{8}$$

to the sleigh equations.where u is the velocity in the direction perpendicular to the blade. (Note that we interchange u and v compared to the original paper of Fufaev.)

Setting

$$k^2 = \frac{m}{I + ma^2}, \quad \epsilon = \frac{I}{Na^2} \tag{9}$$

the equations with dissipation become

$$u = \epsilon a\dot{\omega} \tag{10}$$

$$\dot{v} = a\omega^2 + \epsilon a\omega\dot{\omega} \tag{11}$$

$$ak^2\dot{\omega} + v\omega = -\epsilon a\ddot{\omega} \tag{12}$$

It is clear that as ϵ goes to zero one recovers the original equations. Cartheodory incorrectly argued however that since no matter how small ϵ is these equations yield trajectories which differ from that of the original system, dissipation cannot yield the nonholonomic constraints.

Fufaev realized this is not correct since the system degenerates from a system of three to two equations and thus there is a singularity. Setting $\mu = \epsilon a$ and $\sigma = \dot{\omega}$ we then get

$$\dot{\omega} = \sigma \tag{13}$$

$$\dot{v} = a\omega^2 + \mu\omega\sigma \tag{14}$$

$$\mu\dot{\sigma} = -ak^2\sigma - v\omega\,. \tag{15}$$

Then as $\mu \to 0$ we get rapid motion except for the surface

$$ak^2\sigma + \mu\omega = 0. \tag{16}$$

The slow motion of this surface onto the v-ω plane then gives the correct equations of motion.

6 Dissipation and Quantization

One can show ([Bloch and Rojo (2008)]) that the sleigh equations can be obtained from a variational principle as reduced equations of motion after the system is coupled to an environment described by an $U(1)$ infinite field of the form $\mathbf{a}(\mathbf{z},t) \equiv [\cos\alpha(\mathbf{z},t), \sin\alpha(\mathbf{z},t)]$. For the Lagrangian of the free field we choose

$$L_{\mathrm{F}} = \frac{K}{2}\int d^2\mathbf{z}\,\dot{\mathbf{a}}^2, \tag{17}$$

and we couple the sleigh and the field with a term of the form

$$L_1 = \int d^2\mathbf{z}\,\delta(\mathbf{z}-\mathbf{x})\,\left[\gamma\dot{\mathbf{x}}\cdot\mathbf{a} + \mu\cos\left(\alpha(\mathbf{z},t)-\theta\right)\right]. \tag{18}$$

The first term in square brackets corresponds to a minimal coupling that favors $\dot{\mathbf{x}}$ in the direction of $\mathbf{a}$; the second has the form of a potential coupling that favors an alignment of the internal variable θ with the local direction of $\mathbf{a}$.

The total action is $S = \int dt(L_0 + L_F + L_1)$ where L_0 is the Lagrangian of the free sleigh

$$L_0 = \frac{m}{2}\left[\left(\dot{x} - a\dot{\theta}\sin\theta\right)^2 + \left(\dot{y} + a\dot{\theta}\cos\theta\right)^2\right] + \frac{I}{2}\dot{\theta}^2, \tag{19}$$

and can be regarded as a full "microscopic" theory of the sleigh coupled to an environment.

The equations of motion of the combined system are now obtained from a variational principle, $\delta S = 0$.

Now take the limit $\mu \to \infty$ and use singular perturbation theory. For very large μ we can show that we have a very slow dynamics on the right hand side of the equations of motion., which amounts to saying that in the $\mu \to \infty$ limit the variables $\alpha(\mathbf{x},t)$ and θ are pinned to the same value. We also obtain

$$\dot{x}\sin\alpha(\mathbf{x},t) - \dot{y}\cos\alpha(\mathbf{x},t) = \dot{x}\sin\theta - \dot{y}\cos\theta = 0, \tag{20}$$

which means that the constraint is satisfied and one can show the full equations are given also.

One can now consider quantization of the system. in the case $a = 0$.

The Hamiltonian in this limit has the form

$$H = \frac{1}{2m}\left[p_x - \lambda\cos\alpha(\mathbf{x})\right]^2 + \frac{1}{2m}\left[p_y - \lambda\sin\alpha(\mathbf{x})\right]^2 + \frac{1}{2I}p_\theta^2 \tag{21}$$

$$+\frac{1}{2K}\int d\mathbf{z}\,\Pi^2\left(\alpha(\mathbf{z})\right) + \mu\cos\left[\theta - \alpha(\mathbf{x})\right]. \tag{22}$$

For the quantization of H we proceed with the usual replacements

$$\mathbf{p} = -i\hbar(\partial_x, \partial_y), \quad p_\theta = -i\hbar\partial_\theta, \quad \Pi\left(\alpha(\mathbf{z})\right) = -i\hbar\partial_{\alpha(\mathbf{z})}. \tag{23}$$

We can then analyze the corresponding Schroedinger equation. In the quasiclassical limit the fluctuations of the angle are small and centered around given eigenstates $\theta = \theta_{\mathbf{k}}$. This means that, up to small quantum fluctuations, the knife edge is pointing in the direction defined by the classical constraint. Details may be found in [Bloch and Rojo (2008)].

We note also that an alternate approach to quantization can be obtained using the inverse problem of the calculus of variations (see [Bloch, Fernandez and Mestdag (2009)]). In this setting one obtains an associated system which give the nonholonomic equations on invariant manifolds. This system can shown to be variational using the inverse problem and can then be quantized.

We note finally that control of such nonholonomic systems with internal dissipation is of interest and the dissipation leads to interesting controlled dynamics. We are currently pursuing work in this area with Luis Narnanjo and Dmitry Zenkov. See also [Osborne and Zenkov(2005)].

References

[Arnold, Kozlov, and Neishtadt(1988)] Arnold, V.I., Kozlov, V.V., Neishtadt, A.I.: Dynamical Systems III. Encyclopedia of Math., vol. 3. Springer, Heidelberg (1988)

[Bloch(2003)] Bloch, A.M.: Nonholonomic Mechanics and Control. Interdisciplinary Applied Mathematics. Springer, Heidelberg (2003)

[Bloch, Fernandez and Mestdag (2009)] Bloch, A.M., Fernandez, O., Mestdag, T.: Hamiltonization of Nonholonomic Systems and the Inverse Problem of the Calculus of Variations. Reports on Mathematical Physics 63, 225–249 (2009)

[Bloch and Rojo (2008)] Bloch, A.M., Rojo, A.: Quantization of a nonholonomic system. Phys. Rev. Letters 101, 030404 (2008)

[Bloch, Krishnaprasad, Marsden, and Murray(1996)] Bloch, A.M., Krishnaprasad, P.S., Marsden, J.E., Murray, R.: Nonholonomic mechanical systems with symmetry. Arch. Rat. Mech. An. 136, 21–99 (1996)

[Bloch, Krishnaprasad, Marsden, and Ratiu(1996)] Bloch, A.M., Krishnaprasad, P.S., Marsden, J.E., Ratiu, T.S.: The Euler–Poincaré equations and double bracket dissipation. Comm. Math. Phys. 175, 1–42 (1996)

[Bloch, Hagerty and Weinstein (2004)] Bloch, A.M., Hagerty, P., Weinstein, M.: Radiation induced instability. SIAM J. Applied Math. 64, 484–524 (2004)

[Bloch, Marsden, and Zenkov(2009)] Bloch, A.M., Marsden, J.E., Zenkov, D.E.: Quasivelocities and symmetries in nonholonomic systems. Dynamical Systems 24, 187–222 (2009)

[Bullo and Lewis(2005)] Bullo, F., Lewis, A.D.: Geometric control of mechanical systems. Texts in Applied Mathematics, vol. 49. Springer, New York (2005) (Modeling, analysis, and design for simple mechanical control systems)

[Carathéodory(1933)] Carathéodory, C.: Der Schlitten. Z. Angew. Math. und Mech. 13, 71–76 (1933)

[Cendra, Marsden, and Ratiu(2001b)] Cendra, H., Marsden, J.E., Ratiu, T.S.: Geometric mechanics, Lagrangian reduction and nonholonomic systems. In: Enquist, B., Schmid, W. (eds.) Mathematics Unlimited-2001 and Beyond, pp. 221–273. Springer, Heidelberg (2001)

[Fufaev(1964)] Fufaev, N.A.: On the possibility of realizing a nonholonomic constraint by means of viscous friction forces. Prikl. Math. Mech 28, 513–515 (1964)

[Karapetyan(1983)] Karapetyan, A.V.: On realizing nonholonomic constraints by viscous friction forces and Celtic stones stability. Prikl. Matem. Mekhan. USSR 45, 30–36 (1983)

[Kozlov(1983)] Kozlov, V.V.: Realization of nonintegrable constraints in classical mechanics. Sov. Phys. Dokl. 28, 735–737 (1983)

[Kozlov(1992)] Kozlov, V.V.: The problem of realizing constraints in dynamics. J. Appl. Math. Mech. 56(4), 594–600 (1992)

[Lamb(1900)] Lamb, H.: On the peculiarity of wave-system due to the free vibrations of a nucleus in an extended medium. Proc. London Math, Society 32, 208–211 (1900)

[Marsden and Ratiu(1999)] Marsden, J.E., Ratiu, T.S.: Introduction to Mechanics and Symmetry. Texts in Applied Mathematics, vol. 17. Springer, Heidelberg (1999), First Edition (1994), Second Edition (1999)

[Neimark and Fufaev(1972)] Neimark, J.I., Fufaev, N.A.: Dynamics of Nonholonomic Systems. Translations of Mathematical Monographs, vol. 33. AMS, Providence (1972)

[Osborne and Zenkov(2005)] Osborne, J.M., Zenkov, D.V.: Steering the Chaplygin sleigh by a moving mass. In: Proc. IEEE CDC, vol. 44, pp. 1114–1118 (2005)

[Zenkov and Bloch(2003)] Zenkov, D.V., Bloch, A.M.: Invariant measures of nonholonomic flows with internal degrees of freedom. Nonlinearity 16, 1793–1807 (2003)