Chapter 11
A Mixed Linear-Logarithmic Specification for Lotka-Volterra Models with Endogenously Generated SDLS-Variables

In Arbia and Paelinck (2003a, b), a Lotka-Volterra model (LVM) is applied to the convergence-divergence problem of European regions in terms of incomes per capita. As the latter have to be non-negative, a double logarithmic version may be substituted for the original specification, a modification that removes at least part of the non-linearity of LVMs; this chapter introduces this non-linearity again. Discussion begins with a general section on LVMs, to go on with a mixed linear-logarithmic specification, of which the positivity of the (possible) equilibrium solution is proved, and for which a (sufficient) stability condition is derived. Section 11.3 presents an application of the model to the classical four macro-regions in which the Netherlands is subdivided.

11.1 Lotka-Volterra Models

In this section, generalized Lotka-Volterra models are introduced, and examples given of some applications, including estimation aspects of the latter.

11.1.1 A General Specification

A generalized LVM can be written in matrix-vector notation as

$$\dot{\mathbf{u}} = \hat{\mathbf{u}} \ (\mathbf{A}\,\mathbf{u} \ + \ \mathbf{a})\,, \tag{11.1}$$

where $\mathbf{u}$ is a column-vector of (endogenous) variables, $\hat{\mathbf{u}}$ its diagonal matrix version**,** $\mathbf{A}$ a square matrix, and $\mathbf{a}$ is a column-vector of fixed coefficients; the •-notation denotes the time derivative, $\partial/\partial t$.

Given equation (11.1), the variables $\mathbf{u}$ describe a time path that can take all the characteristics of general continuous dynamic processes (e.g., convergence, divergence, limit circles; see Braun, 1975, §4.9; Gandolfo, 1996, in particular §24.4; Peschel and Mende, 1986). What can be said about equation (11.1) to converge to its focus, $-\mathbf{A}^{-1}\,\mathbf{a}$? Constructing a Lyapunov-function (Hahn, 1963)

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$$v = (\mathbf{u} + \mathbf{A}^{-1}\mathbf{a})'(\mathbf{u} + \mathbf{A}^{-1}\mathbf{a}) \tag{11.2}$$

gives

$$v = 2(\mathbf{u} + \mathbf{A}^{-1}\mathbf{a})'\hat{\mathbf{u}}\,\mathbf{A}(\hat{\mathbf{u}} + \mathbf{A}^{-1}\mathbf{a}) \tag{11.3a}$$

$$= (\mathbf{u} + \mathbf{A}^{-1}\mathbf{a})'\,(\hat{\mathbf{u}}\mathbf{A} + \mathbf{A}\,\hat{\mathbf{u}})\,(\mathbf{u} + \mathbf{A}^{-1}\mathbf{a}). \tag{11.3b}$$

In the purely linear case, if the real parts of **A**'s eigenvalues are negative, *v* is negative definite (Hahn, 1965, p. 26; La Salle and Lefschetz, 1961, p. 48), and the sufficient conditions for asymptotic stability are satisfied. In the LVM case the problem is more involved; the proof of the above sufficiency conditions still being satisfied is given in Paelinck (1992, pp. 142–143).

11.1.2 Applications

Originally, special versions of the LVM have been applied to the field of bio-mathematics, i.a., to build so-called "predator-prey" models. An example is the following model (all parameters strictly positive):

$$\dot{x} = x\,(a - b\,y), \tag{11.4a}$$

$$\dot{y} = y(-c + dx). \tag{11.4b}$$

Here *x* is the prey, developing at a constant rate *a*, but preyed upon by the predators *y*; the latter, in the absence of prey animals, fade out at a rate *c*, but are kept alive by *x*.

The resulting state diagram in the *x-y* plane shows a "pseudo-elliptic" closed curve, and the time-explicit graph shows sinusoidal lagged curves of different amplitudes.

The model just described was proposed by Samuelson (1971) as a candidate for dynamic economic analysis, and applied by Dendrinos and Mullaly (1981) to the evolution of urban populations, although no explicit econometric estimation was performed. Before presenting some econometric results, an appropriate estimation method is unfolded here.

The flexibility of the LVM specification is shown by the various time-paths and singular points resulting from various parameter combinations (presence or absence, signs); Braun (1975, pp. 590–599) gives examples of this. For instance, if a term $-ex$ is added to equation (11.4a), and a term $-fy$ to equation (11.4b), both terms representing competition for limited resources, within the prey and the predator group, the solution becomes $[x^o = a/e;\ y^o = 0]$ for $c/d > a/e$.

11.1.3 Simultaneous Dynamic Least Squares (SDLS) Estimation

Consider a (e.g., sectorally, spatially, dynamically) interdependent econometric model (Paelinck, 1996b, §2.1)

$$\mathbf{A}\,\mathbf{u} + \mathbf{B}\,\mathbf{x} = \boldsymbol{\varepsilon}, \tag{11.5}$$

where **u** is a column-vector of endogenous variables, **x** a column-vector of exogenous ones, $\boldsymbol{\varepsilon}$ being the usual column-vector of random elements. Equation (11.5) always can be rewritten as

$$\mathbf{y} = \mathbf{Z}\,\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \tag{11.6}$$

where **Z** comprises at the same time endogenous and exogenous variables. The basic idea of SDLS is to minimize the sum of squared deviations between the observed and the *endogenously computed* (shown by caps) values of the endogenous variables, **u;** this leads to

$$\mathbf{u} - \hat{\mathbf{u}} = [\mathbf{u} - (\mathbf{Z} - \hat{\mathbf{Z}}\boldsymbol{\beta}] - \mathbf{Z}\boldsymbol{\beta})], \tag{11.7}$$

and minimizing as said before gives

$$\hat{\boldsymbol{\beta}} = (\mathbf{Z}'\hat{\mathbf{Z}})^{-1}\mathbf{Z}\,\mathbf{u}, \tag{11.8}$$

where $\hat{\mathbf{Z}}$ includes the computed values of the endogenous variables; a possible computing process is an iterative one, but Sect. 11.4 presents a specification with endogenously computed **u** values.

The following properties hold (Paelinck, 1990b, p. 7–8):

– $\boldsymbol{\beta}$ is a generalized reduced form estimator;
– if $\boldsymbol{\varepsilon} \sim \mathbf{N}(\mathbf{0}, \boldsymbol{\sigma}^2\mathbf{I})$, then $\boldsymbol{\beta}$ is a maximum likelihood estimator; and,
– $\boldsymbol{\beta}$ is a consistent estimator, and plim $\boldsymbol{\beta}\boldsymbol{\beta}' = \boldsymbol{\sigma}^2(\mathbf{Z}'\mathbf{Z})^{-1}$, for homoscedastic $\boldsymbol{\varepsilon}$.

The method has been applied to a two-equation full-parameter LVM process for the city of Rotterdam, The Netherlands, and for the time period 1946–1978 (Paelinck, 1996a, §3), the equations being

$$\Delta'\ln x_t = a + b\,x_{t-1} + c\,y_{t-1}, \tag{11.9a}$$

$$\Delta'\ln y_t = d + e\,x_{t-1} + f\,y_{t-1}, \tag{11.9b}$$

Table 11.1 Parameter values of the Rotterdam application

Parameter	Value	Student's t
a	−0.8798	−7.68
b	0.0711	7.33
c	0.3988	4.22
d	1.0870	9.49
e	−0.0825	−8.51
f	−0.5355	−5.67
a*	0.0362	1.64
d*	0.0538	2.43

where *x* represents population and *y* per capita income. Table 11.1 (taken from Paelinck, 1990a) presents the estimation results, which can be given plausible interpretations.

Note also the presence of the two parameters a^* and d^*, which allow the initial values to be shifted optimally by the computed process with respect to the observed initial values. Moreover, the resulting relevant eigenvalues lie between *–1* and *0*, so—abstracting from the discretionarity problem (see Gandolfo, 1996, pp. 411–412)—the process would be asymptotically convergent in terms of population and per capita income. According to the divergence criterion (Gandolfo, 1996, p. 456) the system is possibly anti-dissipative along certain stretches of its time-path, although conservative in its non-trivial singular point.

11.2 Mixed Specification

In this section, a combined linear-logarithmic specification is presented.

11.2.1 Equations

Instead of equation (11.1), now consider

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$$\hat{\mathbf{u}}^{-1}\mathbf{u} = \mathbf{A}\,(\mathbf{u} + \ln \mathbf{u}) + \mathbf{a}, \tag{11.10}$$

for which the equilibrium solution (if it exists) is

$$\mathbf{u^o} + \ln \mathbf{u^o} = -\mathbf{A}^{-1}\mathbf{a}. \tag{11.11}$$

For each variable u_i, the equilibrium solution can be written as

$$u_i{}^o = b_i - \ln(u_i{}^o), \tag{11.12}$$

where b_i is generated by the i-th row of $-\mathbf{A}^{-1}$ times $\mathbf{a}$. Now while u_i^o increases linearly starting from zero, $-ln(u_i^o)$ decreases monotonically from $+\infty$ to $-\infty$. Thus, equation (11.12) should be satisfied for some strictly positive value of u_i^o.

11.2.2 Stability

Instead of equations (11.3), consider

$$\mathrm{v} = (\mathbf{u} + \mathbf{ln\,u} + \mathbf{A}^{-1}\mathbf{a})'(\mathbf{u} + \mathbf{ln\,u} + \mathbf{A}^{-1}\mathbf{a}), \tag{11.13}$$

from which one can derive

$$\dot{\mathrm{v}} = 2(\mathbf{u} + \mathbf{ln\,u} + \mathbf{A}^{-1}\mathbf{a})'(\mathbf{I} + \hat{\mathbf{u}})^{\mathbf{1/2}}[(\mathbf{I} + \hat{\mathbf{u}})^{\mathbf{1/2}}\mathbf{A}\,(\mathbf{I} + \hat{\mathbf{u}})^{-\mathbf{1/2}}](\mathbf{I} + \hat{\mathbf{u}})^{\mathbf{1/2}} (\mathbf{u} + \mathbf{ln\,u} + \mathbf{A}^{-1}\mathbf{a}). \tag{11.14}$$

Matrix **A** has undergone a similarity transformation which keeps the eigenvalues unchanged (Allen, 1956, p. 468). Again Hahn's argument quoted at the end of Sect. 11.1.1 can be invoked here, which completes the proof of the fact that a sufficient condition for the mixed LVM to be stable is the negativity of the real parts of **A**'s eigenvalues.

11.3 Application

Model (11.10) has been applied to the relative GDPs of the four classical Dutch macro-regions; Table 11.2 lists the numbers (N = North; S = South; E = East; W = West, the latter region being known as the "Rimcity"); see Fig. 11.1. The numbers are percentages and relate to the years 1988–2000.

SDLS estimates using the numbers in Table 11.2 are obtained by introducing the following equations (discrete versions of equations (11.10); tildes relate to the computed SDLS endogenous variables)

$$\tilde{\mathbf{u}}_{\mathbf{t}} = (\mathbf{I} + \mathbf{A})\,\tilde{\mathbf{u}}_{\mathbf{t-1}} + \mathbf{A}\,\mathbf{ln}\,\tilde{\mathbf{u}}_{\mathbf{t-1}} + \mathbf{a}, \tag{11.15}$$

into a mathematical programming model, minimizing the squared residuals of (11.5) or (11.6). As only the fourth observation (relating to 1991) produces a sum diverging significantly from 100 (see also Figs. 11.1,[1] 11.2, 11.3, and 11.4), no extra constraint was introduced. Furthermore, optimal starting points were computed by optimizing simultaneously over the starting vector of computed SDLS variables.

[1]We thank Martijn Smit, Vrije Universiteit Amsterdam (VU University Amsterdam), for furnishing us with the digital map necessary to construct this figure.

Table 11.2 Numbers used in the Dutch application

N	S	E	W
10.80984	20.03057	17.32632	51.83327
10.43814	20.19805	17.24065	52.12317
10.41169	20.4517	17.43168	51.70493
10.60662	20.45458	17.50469	51.43412
10.96084	20.45568	17.46016	51.12332
10.62076	20.5621	17.66965	51.14749
10.58301	20.39573	17.88161	51.13965
10.22753	20.68775	17.87189	51.21283
10.06042	20.96244	17.78718	51.18996
10.31112	20.93548	17.71927	51.03413
10.15061	20.71473	17.69594	51.43872
9.74983	20.93522	17.6603	51.65465
9.33526	21.08442	17.85378	51.72653
9.54633	20.99156	17.73859	51.72352

Fig. 11.1 The four Dutch regions

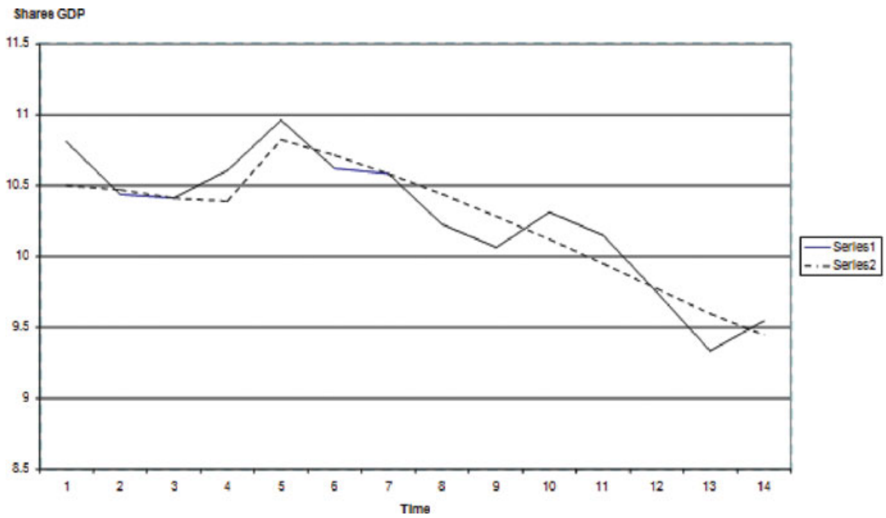

Fig. 11.2 Shares GDP time-series plot: northern Netherlands

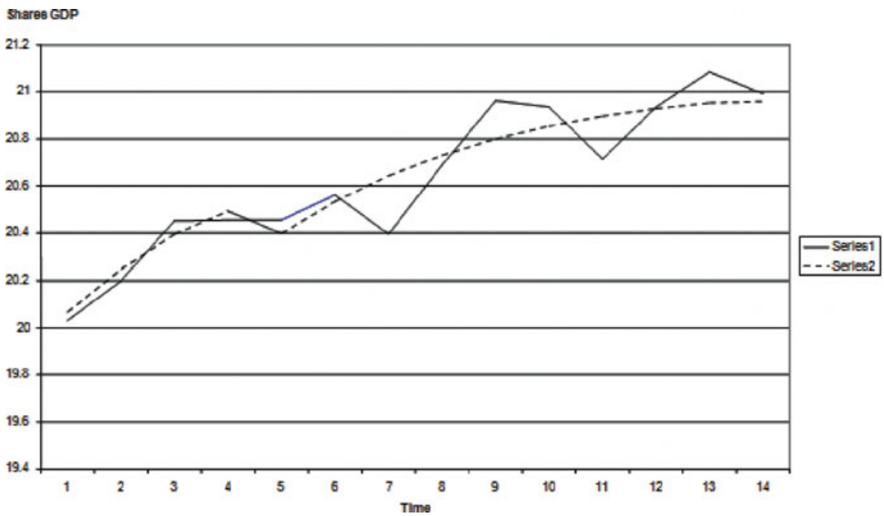

Fig. 11.3 Shares GDP time-series plot: southern Netherlands

Table 11.3 presents the main econometric results.

Every region has been assigned only three parameters: its own influence (a_1), that of the other three regions (a_2), and a constant (a_3).

A *4x4 matrix* can be constructed, dividing each a_2 by *3*. From the trace (*5.5192*) this matrix appears to be non-negative definite (two out of the four a_is are positive), so no mathematical convergence toward the right hand side of equation (11.11) is present.

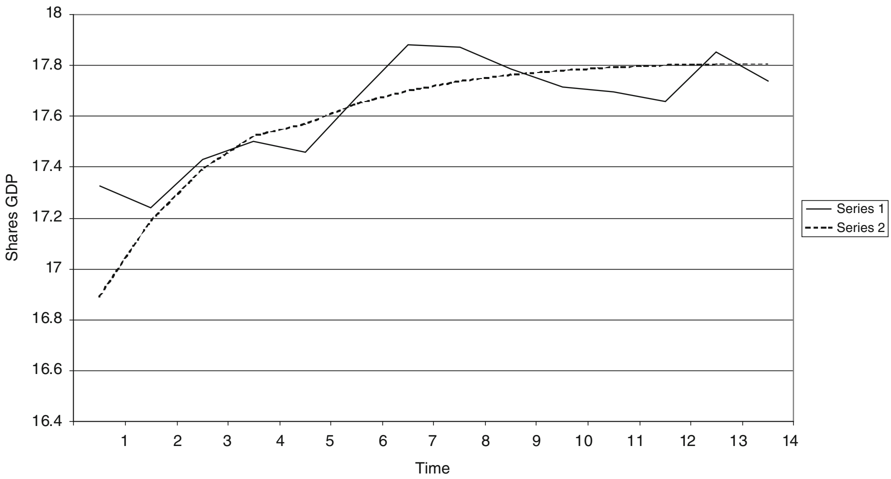

Fig. 11.4 GDP time-series plot: eastern Netherlands

Table 11.3 Econometric results of the Dutch application

Parameters	N	S	E	W
a_1	−1.2368	0.2450	−0.2108	6.7218
a_2	−1.2618	0.4712	0.1072	6.6220
a_3	141.1399	−47.4387	−5.4545	−748.6387
Pseudo-R^2	0.8870	0.8816	0.8074	0.8894

Figs. 11.2, 11.3, 11.4, and 11.5 present the observed (series 1) and the SDLS-computed (series 2) series.

Table 11.4 presents simulation results over 20 periods, starting from the value of the year 2000. One notices a progressive decline in the share of the West ("Rimcity") in favor of all other regions; whether this should be taken at its face value is a problem related to what will be said in the conclusions.

11.4 Conclusion

The method has proven itself to be workable and could be combined with an appropriate estimation method (SDLS); it has moreover the nice property that, if convergence is present, it will lead to economically acceptable (positive) equilibrium values. This means that discrete LVMs, adapted in the way described, could be an ever more useful tool for future research in multiregional dynamics.

Inspection of Figs. 11.2, 11.3, 11.4, and 11.5 shows some local discrepancies between observed and SDLS computed values. Though the specification chosen is

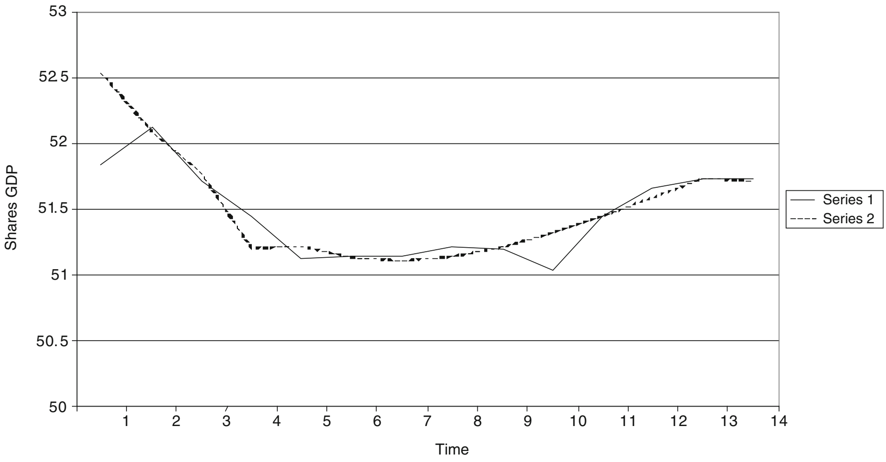

Fig. 11.5 GDP time-series plot: western Netherlands

Table 11.4 Simulation results of the Dutch application

N	S	E	W
9.454351	20.97449	17.81837	51.75279
9.488464	21.08845	17.93343	51.48965
9.513482	21.18737	18.01698	51.28217
9.531703	21.27253	18.07830	51.11747
9.544808	21.34540	18.12382	50.98597
9.554045	21.40749	18.15801	50.88045
9.560352	21.46022	18.18403	50.79539
9.564440	21.50492	18.20409	50.72655
9.566852	21.54275	18.21977	50.67062
9.568005	21.57475	18.23219	50.62505
9.568217	21.60183	18.24217	50.58779
9.567738	21.62474	18.25027	50.55725
9.566758	21.64415	18.25695	50.53215
9.565425	21.66061	18.26250	50.51146
9.563852	21.67460	18.26718	50.49437
9.562128	21.68650	18.27115	50.48022
9.560319	21.69666	18.27455	50.46848
9.558476	21.70534	18.27748	50.45871
9.556637	21.71277	18.28003	50.45056
9.554830	21.71917	18.28226	50.44374
9.553075	21.72468	18.28422	50.43803

already very flexible, it is not the only one. Other candidates (min-algebraic specifications, finite automata; see Chap. 13) are available, and should be tested against the present specification. Tools are also available (see Chap. 12) and will be used in future research.