The Complexity of Three-Dimensional Critical Avalanches

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Abstract. In this work we study the complexity of the three-dimensional sandpile avalanches triggered by the addition of two critical configurations. We prove that the algorithmic problem consisting in predicting the evolution of three dimensional critical avalanches is the hardness core of the three-dimensional Abelian Sandpile Model. On the other hand we prove that three-dimensional critical avalanches are superlinear long on average. It suggests that the prediction problem is superlinear-hard on average.

Can we quickly predict the evolution of an avalanche if we are given a full description of the initial conditions? *The Abelian Sandpile Model* has been used to simulate dissipative dynamical process such as forest fires, earth quakes, extinction events, and (off course) avalanches [2]. Can we quickly predict sandpile avalanches? There is some previous work concerning the computational complexity of prediction problems related to The Abelian Sandpile Model (see for example [3], and [4]). Most of those works are focused on the analysis of The Sandpile Prediction Problem, which refers to the computation of relaxations of unstable configurations. In this work we analyze the complexity of predicting the final state of the avalanches triggered by the addition of two critical configurations, (we focus our research on three-dimensional cubic lattices). Those avalanches are called *critical avalanches.* We show that GC, the problem consisting in predicting the evolution of three-dimensional critical avalanches, is at least as hard as most of the algorithmic problems related to The Abelian Sandpile Model, that is: we show that GC is the hardness core of the predicting tasks related to the model. It is important to remark that our complexity theoretical analysis is based on the notion of NC-Turing reducibility. We have chosen to work with this notion because all the algorithmic problems considered in this paper are *Ptime* computable, and because we are interested in analyzing the *polylogarithmic time computability* of those problems.

We believe that the argued *Self-organized Criticality* [1] of The Abelian Sandpile Model is the complexity source of GC and its relatives. We show that GC is the complexity core of The Abelian Sandpile Model, and we prove that critical avalanches are *superlinear long on average.* It implies that any sequential simulation algorithm computing GC has a running time which is *superlinear on average.* Also, we prove that the criticality of the model implies some type of average-case hardness. We wanted to establish some links between the Self-Organized

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Criticality of The Abelian Sandpile Model and the algorithmic hardness of the prediction problems related to it, we believe that we have partially fulfilled this goal.

Organization of the work. This work is organized into three sections. In section one we introduce The Abelian Sandpile Model and we review some basic facts concerning this model. In section two we study the statistics of three-dimensional critical avalanches and we compute their expected length. In section three we study the algorithmic hardness of GC, we show that most algorithmic problems related to The Abelian Sandpile Model are NC^2-Turing reducible to GC.

1 The Three-Dimensional Abelian Sandpile Model

In this section we introduce the basic definitions and some of the basic results concerning The Abelian Sandpile Model.

Given $n \geq 1$, we use the symbol $\mathcal{G}_n$ to denote the *cubic lattice* of order n, whose vertex set is equal to $[n] \times [n] \times [n]$. We use the symbol $\mathcal{L}_n$ to denote the *cubic sandpile lattice* of order n, which is obtained from $\mathcal{G}_n$ by adding to it a special node $*$ called the *sink*. Furthermore, given v a node on the border of $\mathcal{G}_n$, there are $6 - \deg_{\mathcal{G}_n}(v)$ edges in $\mathcal{L}_n$ connecting v and $*$. We will use the symbol $V(\mathcal{L}_n)^*$ to denote the set $V(\mathcal{L}_n) - \{*\} = V(\mathcal{G}_n)$. Note that given $v \in V(\mathcal{L}_n)^*$ we have that $\deg(v) = 6$.

A *configuration* on $\mathcal{L}_n$ is a function $g : V(\mathcal{L}_n)^* \rightarrow \mathbb{N}$. Given g a configuration on $\mathcal{L}_n$ and given $v \in V(\mathcal{L}_n)^*$ we say that v is *g-stable* if and only if $g(v) \lneqq 6$. We say that g is an *stable configuration* if and only if for all $v \in V(\mathcal{L}_n)^*$, we have that v is g-stable.

We can attach to any sandpile lattice $\mathcal{L}_n$ a *Graph Automaton* $\mathcal{SP}(\mathcal{L}_n)$ whose underlying graph is $\mathcal{L}_n$ and whose transition rule is the *toppling rule* defined by:

Given $v \in V(\mathcal{L}_n)^*$ such that $g(v) \geq 6$, we have that $g \rightarrow g_v$ is a possible transition, where g_v is the configuration on $\mathcal{L}_n$ defined by

$$g_v(w) = \begin{cases} g(v) - 6, & \text{if } w = v, \\ g(w) + 1, & \text{if } v \text{ is a neighbor of } w \\ g(w), & \text{if } v \text{ is not a neighbor of } w \end{cases} \tag{1}$$

Any transition of $\mathcal{SP}(\mathcal{L}_n)$ is called a *firing* or a *toppling.* So, given g a configuration, the transition $g \rightarrow g_v$ is a firing, and if such transition occurs we say that node v was fired (toppled) or we say that a firing (toppling) at v has occurred. Given $\mathcal{L}_n$ a sandpile lattice and given g an initial configuration, we can choose an unstable node, fire it and obtain a new configuration. A sequence of firings $g_1 \rightarrow g_2 \rightarrow \ldots \rightarrow g_m$ is called an *avalanche* of length $m - 1$ with initial configuration g_1, and we say that it is an avalanche from g_1 to g_m. If g_m is stable we say that g_m is a *stabilization* or a *relaxation* of g_1. Given g a configuration on $\mathcal{L}_n$ we use the symbol $\mathcal{ST}(n, g)$ to denote the set of relaxations of g. Furthermore, given $\mathcal{L}_n$, g and $A = g \rightarrow g_1 \rightarrow \ldots \rightarrow g_m$ an avalanche, the *score vector* of A, which we denote with the symbol SC_A, is equal to $(t_v)_{v\in V(\mathcal{L}_n)^*}$, where for any $v \in V(\mathcal{L}_n)^*$ the entry t_v is equal to the number of times node v was fired during the occurrence of A.

Theorem 1 (The fundamental theorem of sandpiles). *Let n be a natural number and let g be a configuration on $\mathcal{L}_n$, we have:*

1. *Any avalanche beginning in g is finite.*
2. $|\mathcal{ST}(n,g)| = 1$.
3. *Given A and B two maximal avalanches beginning in g, we have that $SC_A = SC_B$.*

A proof of this theorem can be found in [2].

Remark 1. Given $n \geq 1$ we use the symbol $\mathcal{C}(n)$ to denote the set $\mathbb{N}^{V(\mathcal{L}_n)^*}$ which is equal to the set of all the configurations on $\mathcal{L}_n$. Given $g \in \mathcal{C}(n)$ we use the symbol SC_g to denote the vector SC_A, where A is any maximal avalanche beginning in g.

Let $\mathcal{ST}(n)$ be the set of all the stable configurations on $\mathcal{L}_n$. We can define a function $st_n : \mathcal{C}(n) \to \mathcal{ST}(n)$ where $st_n(g)$ is the stabilization of g.

Note that, for any n the function st_n is computable: given g a configuration on $\mathcal{L}_n$, if one wants to compute $st_n(g)$, one only has to simulate the automaton $\mathcal{SP}(\mathcal{L}_n)$ on input g.

Given $\mathcal{L}_n$ a sandpile lattice and given f_1, f_2 and f_3 three configurations, we have that

$$st_n(f_1 + f_2 + f_3) = st_n(st_n(f_1 + f_2) + f_3). \tag{2}$$

Last equation allow us to associate to any sandpile graph a sandpile monoid. To this end we define a binary operation $\oplus : \mathcal{ST}(n)^2 \to \mathcal{ST}(n)$ in the following way

$$f \oplus g = st_n(f + g) = st_n(f) \oplus st_n(g). \tag{3}$$

The pair $(st(n), \oplus)$ is a finite commutative monoid. We will use the name *Sandpile Monoid of* $\mathcal{L}_n$ to denote the pair $\mathcal{M}(n) = (\mathcal{ST}(n), \oplus)$. It is known that the *kernel* of a finite commutative monoid is an abelian group [6]. We use the symbol $\mathcal{K}(n)$ to denote the abelian group $\left(Ker(\mathcal{M}(n)), \oplus \upharpoonright_{(Ker(\mathcal{M}(n)))^2}\right)$, which we call *the critical group* of $\mathcal{L}_n$. The elements of $\mathcal{K}(n)$ will be called *critical configurations.* Intuitively, critical configurations are stable configurations of high complexity, which are very near to be unstable. This point of view is supported by the following theorem [2].

Theorem 2. *Given $\mathcal{L}_n$ a sandpile lattice and given $f \in \mathcal{M}(n)$ we have that f is a critical configuration if and only if there not exists $A \subseteq V(\mathcal{L}_n)^*$ such that for any $u \in A$ the inequality* $\deg_A(u) \gneqq f(u)$ *holds.*

Remark 2. Given G a graph, given $A \subseteq V(G)^*$ and given $v \in V(G)$ we use the symbol $\deg_A(v)$ to denote the quantity $|\{w \in A : \{w, u\} \in E(v)\}|$.

Remark 3. Given $\mathcal{M}$ a monoid, its kernel is equal to the intersection of the ideals included in $\mathcal{M}$. It implies that $Ker(\mathcal{M}(n))$ is an ideal of $\mathcal{M}(n)$ and it implies that given f a configuration and given g a critical configuration, $f \oplus g \in \mathcal{K}(n)$.

2 The Statistics of Three-Dimensional Critical Avalanches

We use the term *critical avalanches* to denote the avalanches triggered by the addition of two critical configurations. In this section we prove that the expected length of three-dimensional critical avalanches is $\Omega\left(n^4\right) = \Omega\left(|\mathcal{L}_n|^{\frac{4}{3}}\right)$.

Given $f, g \in \mathcal{K}(n)$ we will use the symbol $L(f, g)$ to denote the length of the critical avalanches triggered by $f + g$.

Definition 1. *We say that w_n is the maximal critical configuration on $\mathcal{L}_n$ if for any $v \in V(\mathcal{L}_n)^*$, $w_n(v) = 5$.*

We observe that given $f, g \in \mathcal{M}(n)$ the inequality $L(f, g) \leq L(w_n, w_n)$ holds. Also, we have that $L(w_n, w_n)$ is an upper bound on avalanche length.

Theorem 3. $L(w_n, w_n) \in \Omega\left(|\mathcal{L}_n|^{\frac{4}{3}}\right)$.

Proof. We prove that there exists a constant C such that, for any $n \geq 2$, we have

$$L(w_n, w_n) \geq Cn^4 \in \Omega\left(|\mathcal{L}_n|^{\frac{4}{3}}\right). \quad (4)$$

Given $\mathcal{L}_n$ a sandpile lattice, we use the symbol $\delta(\mathcal{L}_n)$ to denote the set

$$\left\{w \in V(\mathcal{L}_n)^* : (\{*, w\} \in E(\mathcal{L}_n))\right\}. \quad (5)$$

We use the symbol δ_n to denote the configuration defined by: given $v \in V(\mathcal{L}_n)^*$, $\delta_n(v) = 6 - \deg_{\mathcal{G}_n}(v)$.

Remember that all the avalanches triggered by $2w_n$ have the same length. Fix $n \geq 2$, we want to lowerbound the length of a very specific avalanche triggered by $2w_n$. Given $n \geq 2$, we can identify the sink of $\mathcal{L}_{n-2}$ with $\delta(\mathcal{L}_n)$ the border of $\mathcal{L}_n$. If we make such an identification, we can think of $\mathcal{L}_{n-2}$ as embedded into $\mathcal{L}_n$, and we can express the configuration w_n as $w_{n-2} + \delta_n + \gamma_n$, where γ_n is some configuration on $\mathcal{L}_n$. Note that

$$2w_n = (w_n + \delta_n) + (w_{n-2} + \gamma_n). \quad (6)$$

We know that

$$\begin{aligned} st_n(2w_n) &= st_n(st_n(w_n + \delta_n) + st_n(w_{n-2} + \gamma_n)), \\ st_n(w_n + \delta_n) &= w_n, \\ L(w_n, \delta_n) &= \left|V(\mathcal{L}_n)^*\right| = (n)^3. \end{aligned} \quad (7)$$

Thus, we have that there exists a configuration β_n such that we can pass from the configuration $2w_n$ to the configuration $2w_{n-2} + \beta_n$. Furthermore, we have that the partial avalanche carrying us from $2w_n$ to $2w_{n-2} + \beta_n$ has a length equal to n^3. This partial avalanche (it is not a maximal avalanche) is the first stage of the whole stabilization process. In the second stage we work on the subgraph $\mathcal{L}_{n-2}$

with the configuration $2w_{n-2}$. We can claim that after $(n-2)^3$ topplings we can pass from $2w_{n-2}$ to $2w_{n-4}+\beta_{n-1}$. If we continue in this way, going to the core (center) of $\mathcal{L}_n$, we have to generate $\lfloor \frac{n}{2} \rfloor - 1$ partial avalanches whose lengths are lowerbounded by $n^3, (n-2)^3, \ldots, (n-2(\lfloor \frac{n}{2} \rfloor - 2))^3$ and $(n-2(\lfloor \frac{n}{2} \rfloor - 1))^3$ (respectively). Therefore, we have that

$$L(w_n, w_n) \geq \left(\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} (n-2i)^3 \right) \in \Omega\left(n^4\right) \quad (8)$$

Definition 2. *Given $f, g \in \mathcal{C}(n)$ we use the symbol $f \leq g$ to indicate that for any $v \in V(\mathcal{L}_n)^*$ the inequality $f(v) \leq g(v)$ holds.*

We will prove that critical avalanches are superlinear long on average. First at all we have to remember the notion of accessibility: given $f, g \in \mathcal{C}(n)$ we say that g is *accessible* from f if and only if there exists a configuration $h \geq g$ and there exists a sequence of nodes such that if we begin with f and we topple the nodes in the sequence, (according to the order determined by the sequence), we obtain h. We will use the symbol $f \rightarrow g$ to indicate that g is accessible from f.

Lemma 1. *For all $f_1, \ldots, f_{70} \in \mathcal{K}(n)$ the configuration $2w_n$ is accessible from $f_1 + \cdots + f_{70}$.*

Proof. Given $f \in \mathcal{K}(n)$ and given $\{v, w\} \in E(\mathcal{L}_n)$ we have that either $f(w) \gneqq 0$ or $f(v) \gneqq 0$ (see reference [2]). Let $f_1, \ldots, f_7$ be seven critical configurations, given $v \in V(\mathcal{L}_n)^*$ we have that either there exists $i \leq 7$ such that $f_i(v) \gneqq 0$ or for any w neighbor of v and for any $i \leq 7$ we have that $f_i(w) \gneqq 0$. Suppose that for all $i \leq 7$ we have that $f_i(v) = 0$, in this case we can choose any neighbor of v, say w, and fire it. Also, we can place at least one chip on v, taking care of leaving at least one chip on w. It is clear that if we begin with the configuration $\sum_{i \leq 7} f_i$ we can choose a sequence of at most $\left|V(\mathcal{L}_n)^*\right|$ topplings to obtain a configuration h such that for any $v \in V(\mathcal{L}_n)^*$ the inequality $h(v) \geq 1$ holds. Then, given $f_1, \ldots, f_{70} \in \mathcal{K}(n)$ we have that $\sum_{i \leq 70} f_i \rightarrow 2w_n$.

Theorem 4. *(Critical configurations generate, with high probability, very long avalanches) Given $n \geq 1$ we have that*

$$\Pr_{f,g \in \mathcal{K}(n)} \left[L(f,g) \geq \frac{L(w_n, w_n)}{2^{70}} \right] \geq \frac{1}{69}. \quad (9)$$

Proof. Given $f_1, f_2, \ldots, f_{70} \in \mathcal{K}(n)$ we have that $\sum_{i \leq 70} f_i \rightarrow 2w_n$. It implies that

$$L\left(f_{70}, \sum_{i \leq 69} f_i\right) \geq L(w_n, w_n). \quad (10)$$

Also, we have that either

$$\left(L\left(f_{70}, \bigoplus_{i\leq 69} f_i\right) \geq \frac{L(w_n,w_n)}{2}\right) \text{ or } \left(L\left(f_{69}, \sum_{i\leq 68} f_i\right) \geq \frac{L(w_n,w_n)}{2}\right). \quad (11)$$

Arguing in this way we can prove that there exists $i \leq 70$ such that

$$L\left(f_i, \bigoplus_{j\leq i-1} f_j\right) \geq \frac{L(w_n,w_n)}{2^{70}}. \quad (12)$$

Thus, we have that

$$\Pr_{f_1,\ldots,f_{70}}\left[\exists i, i \leq 70 \left(L\left(f_i, \bigoplus_{j\leq i-1} f_j\right) \geq \frac{L(w_n,w_n)}{2^{70}}\right)\right] = 1. \quad (13)$$

Note that for any $f \in \mathcal{K}(n)$ and for any $i \geq 1$ we have that

$$\Pr_{f_1,\ldots,f_i}\left[\bigoplus_{j\leq i} f_j = f\right] = \frac{1}{|\mathcal{K}(n)|}. \quad (14)$$

Given $f_1,\ldots,f_\alpha$ a sequence of critical configurations on $\mathcal{L}_n$ and given $i \leq \alpha-1$, we define $g_i = \bigoplus_{j\leq i} f_j$. We have that:

1. The procedure below is a sound method to generate, uniformly at random, two elements of $\mathcal{K}(n)$.
   - Choose uniformly at random $f_1,\ldots,f_\alpha$, ($\alpha \geq 2$).
   - Choose uniformly at random $i \in \{2,\ldots,\alpha\}$.
   - Compute f_i and g_{i-1}.
2. It holds that

$$\Pr_{f_1,\ldots,f_{70}}\left[\exists i, 2 \leq i \leq 70 \left(L(f_i, g_{i-1}) \geq \frac{L(w_n,w_n)}{2^{70}}\right)\right] = 1. \quad (15)$$

From items 1 and 2 we obtain

$$\Pr_{f,g\in\mathcal{K}(n)}\left[L(f,g) \geq \frac{L(w_n,w_n)}{2^{70}}\right] =$$

$$\Pr_{2\leq i\leq 70;\ f_1,\ldots,f_{70}}\left[L(f_i,g_{i-1}) \geq \frac{L(w_n,w_n)}{2^{70}}\right] \geq \frac{1}{69} \quad (16)$$

Thus, we haven proven that

$$\Pr_{f,g\in\mathcal{K}(n)}\left[L(f,g) \geq \frac{L(w_n,w_n)}{2^{70}}\right] \geq \frac{1}{69}. \quad (17)$$

Let $X_n : \mathcal{K}(n)^2 \to \mathbb{N}$ be the random variable defined by $X_n(f,g) = L(f,g)$.

Theorem 5. *$E[X_n]$, the expected value of X_n, belongs to $\Omega(n^4)$.*

Proof. We know that there exists a positive constant K such that

$$\Pr_{f,g\in\mathcal{K}(n)}\left[X_n(f,g) \geq Kn^4\right] \geq \frac{1}{69}. \tag{18}$$

Then, we have that

$$\frac{K}{69}n^4 \leq E[X_n]. \tag{19}$$

Therefore, we have that $E[X_n] \in \Omega(n^4) = \Omega\left(|\mathcal{L}_n|^{\frac{4}{3}}\right)$.

2.1 The Algorithmic Hardness of *GC*

In this section we prove that the addition of critical configurations is, in a very specific sense, the complexity source of The Abelian Sandpile Model. Let $n \geq 1$, it is know that if we simulate the dynamics of the model on $\mathcal{L}_n$, alternating the adding of fresh chips with the relaxation process, we will arrive after a polynomial number of iterations to the set of critical (also called *recurrent*) configurations. Furthermore, once we enter $\mathcal{K}(n)$ we can not exit this set. It is the case since $\mathcal{K}(n)$ is the stationary state of The Abelian Sandpile Model on $\mathcal{L}_n$ [2]. Also, if we want to efficiently simulate the dynamics of the model we have to be able to compute the addition of any pair (f,g) of configurations, where f is critical and g is stable. We introduce a related problem below, which we denote with the symbol MC^*, and we prove that MC^* is NC^2Turing reducible to GC.

The Sandpile Prediction Problem, is the algorithmic problem defined by:

Problem 1. (SPP*, sandpile prediction*)

- *Input:* (n,g), *where* $n \in \mathbb{N}$ *and* $g \in \mathcal{C}(n)$.
- *Problem: Compute* $st_n(g)$.

Remark 4. Tardos' bound [5] implies that SPP, and each one of the algorithmic problems introduced below, can be solved in polynomial time, because of this we will analyze the relative complexity of those problems using the notion of NC-Turing reducibility.

A Second problem is MC, which corresponds to the computation of the monoid operation $\oplus$.

Problem 2. (MC*, monoid computations*)

- *Input:* (n,f,g), *where* $n \in \mathbb{N}$ *and* $f,g \in \mathcal{M}(n)$.
- *Problem: Compute* $f \oplus g$.

Now, we introduce the problem GC which is the restriction of SPP to critical avalanches.

Problem 3. (*GC, group computations*)

- *Input:* (n, f, g), *where* $n \in \mathbb{N}$ *and* $f, g \in \mathcal{K}(n)$.
- *Problem: Compute* $f \oplus g$.

Let us introduce three additional problems, which will be play an important role in our analysis.

Problem 4. (*SC, computation of score vectors*)

- *Input:* (n, f), *where* $n \in \mathbb{N}$ *and* $f \in \mathcal{C}(n)$.
- *Problem: Compute the vector* SC_f.

Problem 5. (MC^*, *mixed computations*)

- *Input:* (n, f, g), *where* $n \in \mathbb{N}$, $f \in \mathcal{K}(n)$ *and* $g \in \mathcal{M}(n)$.
- *Problem: Compute* $f \oplus g$.

Given $\mathcal{L}_n$ a three-dimensional sandpile lattice, we use the symbol $e_{\mathcal{K}(n)}$ to denote the identity of $\mathcal{K}(n)$.

Lemma 2. *Identities can be computed in constant time, if oracle access to GC is provided.*

Proof. In order to compute the identity of $\mathcal{K}(n)$, in constant time and using an oracle for GC, we can use the equations:

1. $w_n^{-1} = w_n - (w_n \oplus w_n)$.
2. $e_{\mathcal{K}(n)} = w_n \oplus w_n^{-1}$.

Lemma 3. *Inverses can be computed in time* $O(\log(n))$ *if oracle access to GC is provided.*

Proof. Let $v \in V(\mathcal{L}_n)^*$ and let $w_v = w_n - e_v$. It follows from theorem 2 that w_v is a critical configuration. Let $f \in \mathcal{K}(n)$, note that

$$f^{-1} = \left(\bigoplus_{v \in V(\mathcal{L}_n)^*} f(v)\, w_v \right) \oplus \left(\underbrace{w_n^{-1} \oplus \cdots \oplus w_n^{-1}}_{\|f\| \text{ times}} \right). \tag{20}$$

It is clear that we can compute the expression on the right side of the equation above in time $O(\log(n))$ and using a polynomial number of processors, (if oracle access to GC is provided).

Given v an element of $V(\mathcal{L}_n)^*$, we use the symbol e_v to denote the configuration

$$e_v(w) = \begin{cases} 1, & \text{if } v = w \\ 0, & \text{otherwise} \end{cases} \tag{21}$$

Let $e_n : V(\mathcal{L}_n)^* \rightarrow \mathcal{K}(n)$ be the function defined by $e_n(v) = e_{\mathcal{K}(n)} \oplus e_v$ and let $e : \mathbb{N}^3 \times \mathbb{N} \rightarrow \left(\bigcup_{i \geq 1} \mathcal{K}(n) \right) \cup \{\infty\}$ be the function defined by

$$e(v,n) = \begin{cases} e_n(v), & \text{if } v \in V(\mathcal{L}_n)^* \\ \infty, & \text{else} \end{cases}. \tag{22}$$

Problem 6. (*EC, computation of* e)

- *Input:* (n, v), *where* $n \in \mathbb{N}$ *and* $v \in V(\mathcal{L}_n)^*$.
- *Problem: Compute* $e(v)$.

Next theorem is the main theorem of this section.

Theorem 6. *(The relative hardness of sandpile prediction problems)*

1. *SPP and SC are* NC^2*-Turing equivalent.*
2. *SPP is* NC^2*-reducible to MC.*
3. MC^* *can be computed in time* $O(\log^2(n))$ *if oracle access to EC and GC is provided.*
4. *EC is NC-Turing reducible to GC.*
5. *The problems* MC^* *and GC are* NC^2*-Turing equivalent.*

Proof. The proof of item 1 can be found in [3]. The proof of item 2 is very easy, also we prove items 3 and 4, item 5 follows from items 3 and 4.

1. (Proof of item 3) Let (n, f, g) be an instance of MC^*. We observe that
$$f \oplus g = f \oplus g \oplus \underbrace{e_{\mathcal{K}(n)} \oplus \cdots \oplus e_{\mathcal{K}(n)}}_{\|g\|\text{-times}}. \tag{23}$$
If we express g as $\sum_{v \in V(\mathcal{L}_n)^*} m_v e_v$ we get
$$f \oplus g = f \oplus \left(\bigoplus_{v \in V(\mathcal{L}_n)^*} m_v e_n(v) \right). \tag{24}$$
Also, we can use n^3 processors to compute $\{m_v e_n(v)\}_{v \in V(\mathcal{L}_n)^*}$, this computation takes $O(\log^2(n + \|g\|))$ time units, since we are supposing that we have oracle access to EC. We can use the same n^3 processors to compute $f \oplus \left(\bigoplus_{v \in V(\mathcal{L}_n)^*} m_v e_n(v) \right)$ in time $O(\log^2(n + \|f\| + \|g\|))$, since we are supposing that we have oracle access to GC.
2. (Proof of item 4) Observe that
$$e_n(v) = e_v \oplus e_{\mathcal{K}(n)} = e_v \oplus (w_v \oplus w_v^{-1}) = w_n \oplus w_v^{-1}. \tag{25}$$
Thus, if one wants to compute $e_n(v)$, one only has to compute $w_n \oplus w_v^{-1}$ (note that $w_n, w_v^{-1} \in \mathcal{K}(n)$). We can compute w_v^{-1} in time $O(\log(n))$ if oracle access to GC is provided. Then, we can solve EC in time $O(\log(n))$ using an oracle for GC.

Next theorem follows easily from the results obtained in section 2, it brings together the results concerning the algorithmic hardness of GC and the results concerning the statistics of critical avalanches. Let $\mathcal{SA}$ be the naive (sequential) sandpile automata simulation algorithm, and let $\mathcal{B}$ be the parallel sandpile automata simulation algorithm (we topple all the unstable nodes at once). We will use the symbol $t_{\mathcal{SA}}(n, f, g)$ to denote the running time of $\mathcal{SA}$ on input (n, f, g), (we define $t_{\mathcal{B}}(n, f, g)$ accordingly).

Theorem 7. *Let $n \geq 1$ be a natural number*

1. *There exists a positive constant K such that*

$$\Pr_{f,g \in \mathcal{K}(n)} \left[t_{\mathcal{SA}}(n, f, g) \geq K n^4 \right] \geq \frac{1}{69}. \tag{26}$$

2. *There exists a positive constant R such that*

$$\Pr_{f,g \in \mathcal{K}(n)} \left[t_{\mathcal{B}}(n, f, g) \geq R n \right] \geq \frac{1}{69}. \tag{27}$$

Theorem 7 suggests that the problem GC is $n^{\frac{1}{3}}$-hard on average, which means that given an algorithm $\mathcal{M}$ computing the problem GC, there exists two positive constants K, D such that

$$\Pr_{f,g \in \mathcal{K}(n)} \left[t_{\mathcal{M}}(n, f, g) \geq K n \right] \geq D. \tag{28}$$

Let us finish this work stating the following conjecture.

Conjecture 1. The problem GC is $n^{\frac{1}{3}}$-hard on average.

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