All Ternary Permutation Constraint Satisfaction Problems Parameterized above Average Have Kernels with Quadratic Numbers of Variables*

Gregory Gutin[1], Leo van Iersel[2], Matthias Mnich[3], and Anders Yeo[1]

[1] Royal Holloway, University of London, United Kingdom
{gutin,anders}@cs.rhul.ac.uk
[2] University of Canterbury, Christchurch, New Zealand
l.j.j.v.iersel@gmail.com
[3] Technische Universiteit Eindhoven, Eindhoven, The Netherlands
m.mnich@tue.nl

Abstract. A ternary Permutation-CSP is specified by a subset Π of the symmetric group $\mathcal{S}_3$. An instance of such a problem consists of a set of variables V and a multiset of constraints, which are ordered triples of distinct variables of V. The objective is to find a linear ordering α of V that maximizes the number of triples whose rearrangement (under α) follows a permutation in Π. We prove that all ternary Permutation-CSPs parameterized above average have kernels with quadratic numbers of variables.

1 Introduction

Parameterized complexity theory is a multivariate framework for a refined analysis of hard (NP-hard) problems. A *parameterized problem* is a subset $L \subseteq \Sigma^* \times \mathbb{N}$ over a finite alphabet Σ. L is *fixed-parameter tractable* if the membership of an instance (I, k) in $\Sigma^* \times \mathbb{N}$ can be decided in time $f(k) \cdot |I|^{O(1)}$ where f is a computable function of the *parameter* k only [9,10,26]. (We would like $f(k)$ to grow as slowly as possible.)

Given a pair L, L' of parameterized problems, a *bikernelization from L to L'* is a polynomial-time algorithm that maps an instance (x, k) to an instance (x', k') (the *bikernel*) such that (i) $(x, k) \in L$ if and only if $(x', k') \in L'$, (ii) $k' \leq h(k)$, and (iii) $|x'| \leq g(k)$ for some functions h and g. The function $g(k)$ is called the *size* of the bikernel. A *kernelization* of a parameterized problem L is simply a bikernelization from L to itself, i.e., a *kernel* is a bikernel when $L = L'$.

The notion of a bikernelization was introduced by Alon et al. [1], who observed that a decidable parameterized problem L is fixed-parameter tractable if and only if it admits a bikernelization to a decidable parameterized problem L'. Not every fixed-parameter tractable problem has a kernel of polynomial size unless

* Part of this research has been supported by the EPSRC, grant EP/E034985/1, the Netherlands Organisation for Scientific Research (NWO), grant 639.033.403, and the Allan Wilson Centre for Molecular Ecology and Evolution.

M. de Berg and U. Meyer (Eds.): ESA 2010, Part I, LNCS 6346, pp. 326–337, 2010.

coNP ⊆ NP/poly [2,3]; low degree polynomial size kernels are of main interest due to applications [15].

For maximization problems whose lower bound on the solution value is a monotonically increasing unbounded function of the instance size, the standard parameterization by solution value is trivially fixed-parameter tractable. Mahajan and Raman [24] were the first to recognize both practical and theoretical importance of parameterizing maximization problems differently: above tight lower bounds. They considered MAX SAT with the tight lower bound $m/2$, where m is the number of clauses, and the problem is to decide whether we can satisfy at least $m/2 + k$ clauses, where k is the parameter. Mahajan and Raman proved that this parameterization of MAX SAT is fixed-parameter tractable by obtaining a kernel with $O(k)$ variables. Despite clear importance of parameterizations above tight lower bounds, until recently only a few sporadic non-trivial results on the topic were obtained [17,20,21,24,28].

Massive interest in parameterizations above tight lower bound came with the paper of Mahajan et al. [25], who stated several questions on fixed-parameter tractability of maximization problems parameterized above tight lower bounds, some of which are still open. Several of those questions were answered by newly-developed methods [1,7,8,18,19], using algebraic, probabilistic and harmonic analysis tools. In particular, an advanced probabilistic approach allowed Gutin et al. [18] to prove the existence of a quadratic kernel for the parameterized BETWEENNESS ABOVE AVERAGE (BETWEENNESS-AA) problem, thus, answering an open question of Benny Chor [26].

BETWEENNESS is just one representative of a rich family of *ternary Permutation Constant Satisfaction Problems (CSPs)*. A ternary Permutation-CSP is specified by a subset Π of the symmetric group $\mathcal{S}_3$. An instance of such a problem consists of a set of variables V and a multiset of constraints, which are ordered triples of distinct variables of V. The objective is to find a linear ordering α of V that maximizes the number of triples whose rearrangement (under α) follows a permutation in Π. Important special cases are BETWEENNESS [5,12,18,27] and CIRCULAR ORDERING [11,13], which find applications in circuit design and computational biology [6,27], and in qualitative spatial reasoning [23], respectively.

In this paper, we prove that all ternary Permutation-CSPs have kernels with quadratic numbers of variables, when parameterized above average (AA), which is a tight lower bound. This result is obtained by first reducing all the problems to just one, LINEAR ORDERING-AA, then showing that LINEAR ORDERING-AA has a kernel with quadratic numbers of variables and constraints and, thus, concluding that there is a bikernel with a quadratic number of variables from each of the problems AA to LINEAR ORDERING-AA. Using the last result, we prove that there are bikernels with a quadratic number of variables from all ternary Permutation-CSPs to most ternary Permutation-CSPs. This implies the existence of kernels with a quadratic number of variables for most ternary Permutation-CSPs. The remaining ternary Permutation-CSPs are proved to be equivalent to ACYCLIC SUBDIGRAPH-AA (a *binary Permutation-CSP* defined in Section 4) and since ACYCLIC SUBDIGRAPH-AA, as shown in [19], has a kernel with a quadratic number of variables, the remaining ternary Permutation-CSPs have a kernel with a quadratic number of variables.

The most difficult part of this set of arguments is the proof that Linear Ordering-AA has a kernel with quadratic numbers of variables and constraints. We can show that if we want to prove this in a similar way as for Betweenness-AA (that is, eliminate all instances of Linear Ordering-AA whose optimal solution coincides with the lower bound) we need an infinite number of reduction rules. See [16] for further details. So, determining fixed-parameter tractability of Linear Ordering-AA turns out to be much harder than for Betweenness-AA. Fortunately, we found a nontrivial way of reducing Linear Ordering-AA to a combination of Betweenness-AA and Acyclic Subdigraph-AA. Using further probabilistic and deterministic arguments for the mixed problem, we prove that Linear Ordering-AA has a kernel with quadratic numbers of variables and constraints.

The rest of the paper is organized as follows. In Section 2, we define and discuss ternary Permutation-CSPs; we also reduce all nontrival ternary Permutation-CSPs AA to Linear Ordering-AA. In Section 3, we describe probabilistic and harmonic analysis tools used in the paper. In Section 4, we obtain some results on Betweenness-AA and Acyclic Subdigraph-AA needed in the following section, where we prove that Linear Ordering-AA has a quadratic kernel. In Section 5, we also prove our main result, Theorem 4, that all ternary Permutation-CSPs parameterized above average have kernels with a quadratic number of variables. Due to the space limit, many proofs are omitted; they can be found in [16].

2 Permutation CSPs Parameterized above Average

Let V be a set of n variables. A *linear ordering of* V is a bijection $\alpha : V \to [n]$, where $[n] = \{1, 2, \dots, n\}$. The symmetric group on three elements is $\mathcal{S}_3 = \{(123), (132), (213), (231), (312), (321)\}$. A *constraint set over* V is a multiset $\mathcal{C}$ of *constraints*, which are permutations of three distinct elements of V. For each subset $\Pi \subseteq \mathcal{S}_3$ and a linear ordering α of V, a constraint $(v_1, v_2, v_3) \in \mathcal{C}$ is *Π-satisfied by* α if there is a permutation $\pi \in \Pi$ such that $\alpha(v_{\pi(1)}) < \alpha(v_{\pi(2)}) < \alpha(v_{\pi(3)})$. If Π is fixed, we will simply say that $(v_1, v_2, v_3) \in \mathcal{C}$ is *satisfied by* α.

For each subset $\Pi \subseteq \mathcal{S}_3$, the problem Π-CSP is to decide whether for a given pair $(V, \mathcal{C})$ of variables and constraints there is a linear ordering α of V that Π-satisfies all constraints in $\mathcal{C}$. A complete dichotomy of the Π-CSP problems with respect to their computational complexity was given by Guttmann and Maucher [22]. For that, they reduced $2^{|\mathcal{S}_3|} = 64$ problems by two types of symmetry. First, two problems differing just by a consistent renaming of the elements of their permutations are of the same complexity. Second, two problems differing just by reversing their permutations are of the same complexity. The symmetric reductions leave 13 problems Π_i-CSP, $i = 0, 1, \dots, 12$, whose time complexity is polynomial for $\Pi_{11} = \emptyset$ and $\Pi_{12} = \mathcal{S}_3$ and was otherwise established by Guttmann and Maucher [22], see Table 1.

Table 1. Ternary Permutation-CSPs (after symmetry considerations)

$\Pi \subseteq \mathcal{S}_3$	Common Problem Name	Complexity to Saisfy All Constraints
$\Pi_0 = \{(123)\}$	LINEAR ORDERING	polynomial
$\Pi_1 = \{(123), (132)\}$		polynomial
$\Pi_2 = \{(123), (213), (231)\}$		polynomial
$\Pi_3 = \{(132), (231), (312), (321)\}$		polynomial
$\Pi_4 = \{(123), (231)\}$		NP-complete
$\Pi_5 = \{(123), (321)\}$	BETWEENNESS	NP-complete
$\Pi_6 = \{(123), (132), (231)\}$		NP-complete
$\Pi_7 = \{(123), (231), (312)\}$	CIRCULAR ORDERING	NP-complete
$\Pi_8 = \mathcal{S}_3 \setminus \{(123), (231)\}$		NP-complete
$\Pi_9 = \mathcal{S}_3 \setminus \{(123), (321)\}$	NON-BETWEENNESS	NP-complete
$\Pi_{10} = \mathcal{S}_3 \setminus \{(123)\}$		NP-complete

The maximization version of Π_i-CSP is the problem MAX-Π_i-CSP of finding a linear ordering α of V that Π_i-satisfies a maximum number of constraints in $\mathcal{C}$. Clearly, for $i = 4, \ldots, 10$ the problem MAX-Π_i-CSP is NP-hard. In [16] we prove that MAX-Π_i-CSP is NP-hard also for $i = 0, 1, 2, 3$.

Now observe that given a variable set V and a constraint multiset $\mathcal{C}$ over V, for a random linear ordering α of V, the probability of a constraint in $\mathcal{C}$ being Π-satisfied by α equals $\frac{|\Pi|}{6}$. Hence, the expected number of satisfied constraints from $\mathcal{C}$ is $\frac{|\Pi|}{6}|\mathcal{C}|$, and thus there is a linear ordering α of V satisfying at least $\frac{|\Pi|}{6}|\mathcal{C}|$ constraints (and this bound is tight). A derandomization argument leads to $\frac{|\Pi_i|}{6}$-approximation algorithms for the problems MAX-Π_i-CSP [5]. No better constant factor approximation is possible assuming the Unique Games Conjecture [5].

We study the parameterization of MAX-Π_i-CSP above tight lower bound:

Π-ABOVE AVERAGE (Π-AA)
Input: A finite set V of variables, a multiset $\mathcal{C}$ of ordered triples of distinct variables from V and an integer $k \geq 0$.
Parameter: k.
Question: Is there a linear ordering α of V such that at least $\frac{|\Pi|}{6}|\mathcal{C}| + k$ constraints of $\mathcal{C}$ are Π-satisfied by α?

For example, choose $\Pi = \{(123), (321)\}$ for BETWEENNESS-AA. Π_0-AA is called the LINEAR ORDERING-AA problem.

Let Π be a subset of $\mathcal{S}_3$. Clearly, if Π is the empty set or equal to $\mathcal{S}_3$ then the corresponding problem Π-AA can be solved in polynomial time. The following simple result allows us to study the Π-AA problems using Π_0-AA.

Proposition 1. *Let Π be a subset of $\mathcal{S}_3$ such that $\Pi \notin \{\emptyset, \mathcal{S}_3\}$. There is a polynomial time transformation f from Π-AA to Π_0-AA such that an instance*

$(V, \mathcal{C}, k)$ of Π-AA is a "yes"-instance if and only if $(V, \mathcal{C}_0, k) = f(V, \mathcal{C}, k)$ is a "yes"-instance of Π_0-AA.

Proof. From an instance $(V, \mathcal{C}, k)$ of Π-AA, construct an instance $(V, \mathcal{C}_0, k)$ of Π_0-AA as follows. For each triple $(v_1, v_2, v_3) \in \mathcal{C}$, add $|\Pi|$ triples $(v_{\pi(1)}, v_{\pi(2)}, v_{\pi(3)})$, $\pi \in \Pi$, to $\mathcal{C}_0$.

Observe that a triple $(v_1, v_2, v_3) \in \mathcal{C}$ is Π-satisfied if and only if exactly one of the triples $(v_{\pi(1)}, v_{\pi(2)}, v_{\pi(3)})$, $\pi \in \Pi$, is Π_0-satisfied. Thus, $\frac{|\Pi|}{6}|\mathcal{C}| + k$ constraints from $\mathcal{C}$ are Π-satisfied if and only if the same number of constraints from $\mathcal{C}_0$ are Π_0-satisfied. It remains to observe that $\frac{|\Pi|}{6}|\mathcal{C}| + k = \frac{1}{6}|\mathcal{C}_0| + k$ as $|\mathcal{C}_0| = |\Pi| \cdot |\mathcal{C}|$. □

For a variable set V, a constraint multiset $\mathcal{C}$ over V and a linear ordering α of V, the *α-deviation of* $(V, \mathcal{C})$ is the number $\mathsf{dev}(V, \mathcal{C}, \alpha)$ of constraints of $\mathcal{C}$ that are Π-satisfied by α minus $\frac{|\Pi|}{6}|\mathcal{C}|$. The *maximum deviation of* $(V, \mathcal{C})$, denoted $\mathsf{dev}(V, \mathcal{C})$, is the maximum of $\mathsf{dev}(V, \mathcal{C}, \alpha)$ over all linear orderings α of V. Now the problem Π-AA can be reformulated as the problem of deciding whether $\mathsf{dev}(V, \mathcal{C}) \geq k$.

3 Probabilistic and Harmonic Analysis Tools

We build on the probabilistic *Strictly Above Expectation* method by Gutin et al. [19] to prove non-trivial lower bounds on the minimum fraction of satisfiable constraints in instances belonging to a restricted subclass. For such an instance with parameter k, we introduce a random variable X such that the instance is a "yes"-instance if and only if X takes with positive probability a value greater than or equal to k. If X happens to be a symmetric random variable with finite second moment then $\mathbb{P}(X \geq \sqrt{\mathbb{E}[X^2]}) > 0$; it hence suffices to prove $\mathbb{E}[X^2] = h(k)$ for some monotonically increasing unbounded function h. (Here, $\mathbb{P}(\cdot)$ and $\mathbb{E}[\cdot]$ denote probability and expectation, respectively.) If X is not symmetric then the following lemma can be used instead.

Lemma 1 (Alon et al. [1]). *Let X be a real random variable and suppose that its first, second and forth moments satisfy $\mathbb{E}[X] = 0$, $\mathbb{E}[X^2] = \sigma^2 > 0$ and $\mathbb{E}[X^4] \leq c\sigma^4$, respectively, for some constant c. Then $\mathbb{P}(X > \frac{\sigma}{2\sqrt{c}}) > 0$.*

We combine this result with the following result from harmonic analysis.

Lemma 2 (Hypercontractive Inequality [4,14]). *Let $f = f(x_1, \ldots, x_n)$ be a polynomial of degree r in n variables $x_1, \ldots, x_n$ with domain $\{-1, 1\}$. Define a random variable X by choosing a vector $(\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n$ uniformly at random and setting $X = f(\epsilon_1, \ldots, \epsilon_n)$. Then $\mathbb{E}[X^4] \leq 9^r \mathbb{E}[X^2]^2$.*

4 Facts on the Betweenness and Acyclic Subdigraph Problems

Let u, v, w be variables. We denote a betweenness constraint "v is between u and w" by $(v, \{u, w\})$, and call a 3-set S of betweenness constraints over $\{u, v, w\}$

complete if $S = \{(u, \{v, w\}), (v, \{u, w\}), (w, \{u, v\})\}$. Since every linear ordering of $\{u, v, w\}$ satisfies exactly one constraint in S, we obtain the following reduction.

Lemma 3. *Let* $(V, \mathcal{B})$ *be an instance of* Betweenness *and let* α *be a linear ordering of* V*. Let* $\mathcal{B}'$ *be the set of constraints obtained from* $\mathcal{B}$ *by deleting all complete subsets. Then* $\mathsf{dev}(V, \mathcal{B}, \alpha) = \mathsf{dev}(V, \mathcal{B}', \alpha)$*.*

An instance of Betweenness without complete subsets of constraints is called *reduced*.

Let $(V, \mathcal{B})$ be an instance of Betweenness, with $\mathcal{B} = \{B_1, \ldots, B_m\}$, and let ϕ be a fixed function from V to $\{0, 1, 2, 3\}$. A linear ordering α of V is called *ϕ-compatible* if for each pair $u, v \in V$ with $\alpha(u) < \alpha(v)$ it holds $\phi(u) \leq \phi(v)$. For a random ϕ-compatible linear ordering π of V, define a binary random variable y_p that takes value one if and only if $B_p \in \mathcal{B}$ is satisfied by π (if B_p is falsified by π, then $y_p = 0$). Let $Y_p = \mathbb{E}[y_p] - 1/3$ for each $p \in [m]$, and let $Y = \sum_{p=1}^{m} Y_p$.

Now let ϕ be a random function from V to $\{0, 1, 2, 3\}$. Then $Y, Y_1, \ldots, Y_m$ are random variables. For a constraint $B_p = (v, \{u, w\})$, the distribution of Y_p as it is given in Table 2 implies that $\mathbb{E}[Y_p] = 0$. Thus, by linearity of expectation, $\mathbb{E}[Y] = 0$.

Table 2. Distribution of Y_p for constraint $B_p = (v, \{u, w\})$

$\lvert\{\phi(u), \phi(v), \phi(w)\}\rvert$	Relation	Value of Y_p	Prob.
1	$\phi(u) = \phi(v) = \phi(w)$	0	1/16
2	$\phi(v) \neq \phi(u) = \phi(w)$	$-1/3$	3/16
2	$\phi(v) \in \{\phi(u), \phi(w)\}$	1/6	6/16
3	$\phi(v)$ is between $\phi(u)$ and $\phi(w)$	2/3	2/16
3	$\phi(v)$ is not between $\phi(u)$ and $\phi(w)$	$-1/3$	4/16

The following lemma was proved by Gutin et al. [18] for Betweenness in which $\mathcal{B}$ is a set, not a multiset, but a simple modification of its proof gives us the following (see [16] for details):

Lemma 4. *For a reduced instance* $(V, \mathcal{B})$ *of* Betweenness*,* $\mathbb{E}[Y^2] \geq \frac{11}{768} m$*.*

In the Acyclic Subdigraph problem we are given a directed multigraph $D = (U, A)$, with parallel arcs allowed, and ask for a linear ordering π of V which maximizes the number of satisfied arcs, where an arc $(u, v) \in A$ is *satisfied by* π if $\pi(u) < \pi(v)$. If π is a uniformly-at-random linear ordering of V then the probability of an arc of D being satisfied is $1/2$. Thus, there is a linear ordering π of V in which the number of satisfied arcs is at least $|A|/2$. We therefore define, for a digraph $D = (U, A)$ and a linear ordering π of U, the *π-deviation* of D as the number of arcs satisfied by π minus $|A|/2$, and denote it by $\mathsf{dev}(V, A, \pi)$. In the Acyclic Subdigraph-AA problem we are given a directed multigraph

$D = (U, A)$ and asked to decide whether there is a linear ordering π of U with π-*deviation* at least k, where k is a parameter.

As every linear ordering of U satisfies exactly one of two mutually opposite arcs (u, v) and (v, u), we obtain the following reduction.

Lemma 5. *Let $D = (U, A)$ be a directed multigraph and let π be a linear ordering of V. Let A' be the set of arcs obtained from A by deleting all pairs of mutually opposite arcs. Then* $\mathsf{dev}(V, A, \pi) = \mathsf{dev}(V, A', \pi)$.

A directed multigraph without mutually opposite arcs is called *reduced*.

Let $D = (U, A)$ be a directed multigraph with $A = \{a_1, \ldots, a_m\}$ as multiset of arcs, and let ϕ be a fixed function from U to $\{0, 1, 2, 3\}$. For a random ϕ-compatible linear ordering π of U, define a binary random variable x_p that takes value one if and only if a_p is satisfied by π. Let $X_p = \mathbb{E}[x_p] - 1/2$ for each $p \in [m]$ and let $X = \sum_{p=1}^{m} X_p$.

Now let ϕ be a random function from U to $\{0, 1, 2, 3\}$. Then $X, X_1, \ldots, X_m$ are random variables. For an arc (u, v), the distribution of X_p as it is given in Table 3 implies that $\mathbb{E}[X_p] = 0$. Thus, by linearity of expectation, $\mathbb{E}[X] = 0$.

Table 3. Distribution of X_p for an arc (u, v)

Relation between $\phi(u)$ and $\phi(v)$	Value of X_p	Prob.
$\phi(u) = \phi(v)$	0	1/4
$\phi(u) < \phi(v)$	1/2	3/8
$\phi(u) > \phi(v)$	−1/2	3/8

We have the following analogue of Lemma 4 proved in [16].

Lemma 6. *For reduced directed multigraphs D it holds that $\mathbb{E}[X^2] \geq \frac{1}{32}m$.*

The following theorem was proved in [19].

Theorem 1. Acyclic Subdigraph-AA *has a kernel with a quadratic number of vertices and arcs.*

5 Kernels for *Π*-AA Problems

We start from the following key construction of this paper. With an instance $(V, \mathcal{C})$ of Linear Ordering, we associate an instance $(V, \mathcal{B})$ of Betweenness and two instances (V, A') and (V, A'') of Acyclic Subdigraph as follows: If $C_p = (u, v, w) \in \mathcal{C}$, then $B_p = (v, \{u, w\}) \in \mathcal{B}$, $a'_p = (u, v) \in A'$, and $a''_p = (v, w) \in A''$. The following lemma is proved in [16].

Lemma 7. *Let (V, C, k) be an instance of* Linear Ordering-AA *and let α be a linear ordering of V. Then*

$$\mathsf{dev}(V, \mathcal{C}, \alpha) = \frac{1}{2}\left[\mathsf{dev}(V, A', \alpha) + \mathsf{dev}(V, A'', \alpha) + \mathsf{dev}(V, \mathcal{B}, \alpha)\right].$$

Let $(V, \mathcal{C}, k)$ be an instance of Linear Ordering-AA, and let ϕ be a function from V to $\{0,1,2,3\}$. For a random ϕ-compatible linear ordering π of V, define a binary random variable z_p that takes value one if and only if C_p is satisfied by π. Let $Z_p = \mathbb{E}[z_p] - 1/6$ for each $p \in [m]$, and let $Z = \sum_{p=1}^{m} Z_p$.

Lemma 8. *If $Z \geq k$ then $(V, \mathcal{C}, k)$ is a "yes"-instance of* Linear Ordering-AA*.*

Proof. By linearity of expectation, $Z \geq k$ implies $\mathbb{E}[\sum_{p=1}^{m} z_p] \geq m/6 + k$. Thus, if $Z \geq k$ then there is a ϕ-compatible permutation π that satisfies at least $m/6 + k$ constraints. □

Fix a function $\phi: V \rightarrow \{0,1,2,3\}$ and assign variables Y_p, X'_p, X''_p, respectively, to the three instances of Betweenness and Acyclic Subdigraph above.

Lemma 9. *For each $p \in [m]$, we have $Z_p = \frac{1}{2}\left[X'_p + X''_p + Y_p\right]$.*

Proof. Let $C_p = (u, v, w) \in \mathcal{C}$. Table 4 shows the values of X'_p, X''_p, Y_p, Z_p for some relations between $\phi(u)$, $\phi(v)$ and $\phi(w)$. The values of X'_p, X''_p and Y_p can be computed using Tables 2 and 3. In all cases of Table 4 it holds $Z_p = \frac{1}{2}(X'_p + X''_p + Y_p)$. Thus, $Z_p = \frac{1}{2}[X'_p + X''_p + Y_p]$ for each possible relation between $\phi(u)$, $\phi(v)$ and $\phi(w)$. □

Let $X = \sum_{p=1}^{m}[X'_p + X''_p]$, let $Y = \sum_{p=1}^{m} Y_p$ and let ϕ be a random function from V to $\{0,1,2,3\}$. Then $X, X'_1, \ldots, X'_m, X''_1, \ldots, X''_m, Y, Y_1, \ldots, Y_m, Z, Z_1, \ldots, Z_m$ are random variables. From $\mathbb{E}[X'] = \mathbb{E}[X''] = \mathbb{E}[Y] = 0$ it follows that $\mathbb{E}[Z] = 0$.

Table 4. Values of X'_p, X''_p, Y_p, Z_p

Relation between $\phi(u)$, $\phi(v)$ and $\phi(w)$	X'_p	X''_p	Y_p	Z_p
$\phi(u) = \phi(v) = \phi(w)$	0	0	0	0
$\phi(v) < \phi(u) = \phi(w)$	-1/2	1/2	-1/3	-1/6
$\phi(v) > \phi(u) = \phi(w)$	1/2	-1/2	-1/3	-1/6
$\phi(v) = \phi(u) < \phi(w)$	0	1/2	1/6	1/3
$\phi(v) = \phi(u) > \phi(w)$	0	-1/2	1/6	-1/6
$\phi(u) < \phi(v) = \phi(w)$	1/2	0	1/6	1/3
$\phi(u) > \phi(v) = \phi(w)$	-1/2	0	1/6	-1/6
$\phi(u) < \phi(v) < \phi(w)$	1/2	1/2	2/3	5/6
$\phi(u) < \phi(w) < \phi(v)$	1/2	-1/2	-1/3	-1/6
$\phi(v) < \phi(u) < \phi(w)$	-1/2	1/2	-1/3	-1/6
$\phi(v) < \phi(w) < \phi(u)$	-1/2	1/2	-1/3	-1/6
$\phi(w) < \phi(u) < \phi(v)$	1/2	-1/2	-1/3	-1/6
$\phi(w) < \phi(v) < \phi(u)$	-1/2	-1/2	2/3	-1/6

We will be able to use Lemma 2 in the proof of Lemma 12 due to the following:

Lemma 10. *The random variable Z can be expressed as a polynomial of degree 6 in independent uniformly distributed random variables with values -1 and 1.*

Proof. Consider $C_p = (u, v, w) \in \mathcal{C}$. Let $\epsilon_1^u = -1$ if $\phi(u) = 0$ or 1 and $\epsilon_1^u = 1$, otherwise. Let $\epsilon_2^u = -1$ if $\phi(u) = 0$ or 2 and $\epsilon_2^u = 1$, otherwise. Similarly, we can define $\epsilon_1^v, \epsilon_2^v, \epsilon_1^w, \epsilon_2^w$. Now $\epsilon_1^u\epsilon_2^u$ can be seen as a binary representation of a number from the set $\{0, 1, 2, 3\}$ and $\epsilon_1^u\epsilon_2^u\epsilon_1^v\epsilon_2^v\epsilon_1^w\epsilon_2^w$ can be viewed as a binary representation of a number from the set $\{0, 1, \dots, 63\}$, where -1 plays the role of 0. Then we can write Z_p as the polynomial

$$\frac{1}{64}\sum_{q=0}^{63}(-1)^{s_q}W_q \cdot (\epsilon_1^u + c_1^{uq})(\epsilon_2^u + c_2^{uq})(\epsilon_1^v + c_1^{vq})(\epsilon_2^v + c_2^{vq})(\epsilon_1^w + c_1^{wq})(\epsilon_2^w + c_2^{wq}),$$

where $c_1^{uq}c_2^{uq}c_1^{vq}c_2^{vq}c_1^{wq}c_2^{wq}$ is the binary representation of q, s_q is the number of digits equal -1 in this representation, and W_q equals the value of Z_p for the case when the binary representations of $\phi(u), \phi(v)$ and $\phi(w)$ are $c_1^{uq}c_2^{uq}$, $c_1^{vq}c_2^{vq}$ and $c_1^{wq}c_2^{wq}$, respectively. The actual values for Z_p for each case are given in the proof of Lemma 9. The above polynomial is of degree 6. It remains to recall that $Z = \sum_{p=1}^{m} Z_p$. □

Let us consider the following natural transformation of our key construction introduced in the beginning of this section. Let $(V, \mathcal{C})$ be an instance of Linear Ordering and $(V, \mathcal{B})$, (V, A') and (V, A'') be the associated instances of Betweenness and Acyclic Subdigraph. Let b be the number of pairs of mutually opposite arcs in the directed multigraph $D = (V, A' \cup A'')$ that are deleted by our reduction rule, and let $r = 2(m - b)$. Let t be the number of complete 3-sets of constraints in $\mathcal{B}$ whose deletion from $\mathcal{B}$ eliminates all complete 3-sets of constraints in $\mathcal{B}$ and let $s = m - 3t$.

Lemma 11. *We have $\mathbb{E}[Z^2] \geq \frac{11}{3072}(r + s)$.*

Proof. Let $A = A' \cup A'' = \{a_1, \dots, a_{2m}\}$ and $D = (V, A)$. Fix a function $\phi: V \rightarrow \{0, 1, 2, 3\}$. For a random ϕ-compatible linear ordering π of V, define a binary random variable x_i that takes value one if and only if a_i is satisfied by π. Analogously, define a binary random variable y_i that takes value one if and only if B_i is satisfied by π. Let $X_i = \mathbb{E}[x_i] - 1/2$ for all $i = 1, \dots, 2m$, let $Y_j = \mathbb{E}[y_j] - 1/3$ for all $j = 1, \dots, m$ and let $X = \sum_{i=1}^{2m} X_i$, $Y = \sum_{i=1}^{m} Y_i$. Recall that b is the number of deleted pairs of mutually opposite arcs from D, and t is the number of complete 3-sets deleted from $\mathcal{B}$. Assume, without loss of generality, that the remaining arcs are $a_1, \dots, a_r$ and the remaining betweenness constraints are $B_1, \dots, B_s$. Then $X = \sum_{i=1}^{2m} X_i = \sum_{i=1}^{r} X_i$, $Y = \sum_{i=1}^{m} Y_i = \sum_{i=1}^{s} Y_i$ and, by

Lemma 9, $Z = X + Y/2$. Now let ϕ be a random function from V to $\{0, 1, 2, 3\}$. We have the following:

$$\mathbb{E}[Z^2] = \mathbb{E}[X^2 + XY + Y^2/4] = \mathbb{E}[X^2] + \mathbb{E}[Y^2]/4 + \mathbb{E}\left[\left(\sum_{i=1}^{r} X_i\right)\left(\sum_{j=1}^{s} Y_j\right)\right]$$
$$= \mathbb{E}[X^2] + \mathbb{E}[Y^2]/4 + \sum_{i=1}^{r}\sum_{j=1}^{s} \mathbb{E}[X_i Y_j].$$

We will show that $\mathbb{E}[X_i Y_j] = 0$ for any pair (i, j). Let $\phi' : V \rightarrow \{0, 1, 2, 3\}$ be defined as $\phi'(x) = 3 - \phi(x)$ for all x. Let $X_i(\phi)$ be the value of X_i when considering ϕ-compatible orderings and define $X_i(\phi')$, $Y_i(\phi)$ and $Y_i(\phi')$ analogously. From Table 2 we note that $Y_j(\phi) = Y_i(\phi')$, and from Table 3 we note that $X_j(\phi) = -X_i(\phi')$. From $\mathbb{E}[X_i Y_j] = \frac{1}{4^{|V|}} \sum_{\phi} X_i(\phi) Y_j(\phi)$ it follows that

$$2\mathbb{E}[X_i Y_j] = 2\left[\frac{1}{4^{|V|}} \sum_{\phi} X_i(\phi) Y_j(\phi)\right] = \frac{1}{4^{|V|}} \sum_{\phi} [X_i(\phi) Y_j(\phi) + X_i(\phi') Y_j(\phi')] = 0.$$

Therefore, $\mathbb{E}[Z^2] = \mathbb{E}[X^2] + \mathbb{E}[Y^2]/4$. It follows from Lemmas 4 and 6 that $\mathbb{E}[X^2] \geq r/32$ and $\mathbb{E}[Y^2] \geq \frac{11}{768} s$. We conclude that $\mathbb{E}[Z^2] \geq \frac{11}{3072}(r + s)$. □

Lemma 12. *There is a constant $c > 0$ such that if $r + s \geq ck^2$, then $(V, \mathcal{C}, k)$ is a "yes"-instance of* Linear Ordering-AA.

Proof. By Lemmas 10 and 2, we have $\mathbb{E}[Z^4] \leq 9^6 (\mathbb{E}[Z^2])^2$. As $\mathbb{E}[Z] = 0$, it follows from Lemma 1 that $\mathbb{P}\left(Z > \frac{\sqrt{\mathbb{E}[Z^2]}}{2 \cdot 9^3}\right) > 0$. By Lemma 11, $\mathbb{E}[Z^2] \geq \frac{11}{3072}(r + s)$. Hence, $\mathbb{P}\left(Z > \frac{\sqrt{\frac{11}{3072}(r+s)}}{2 \cdot 9^3}\right) > 0$. Therefore if $r + s \geq ck^2$, where $c = 4 \cdot 9^6 \cdot 3072/11$, then by Lemma 8 $(V, \mathcal{C}, k)$ is a "yes"-instance of Linear Ordering-AA. □

After we have deleted mutually opposite arcs from D and complete 3-sets of constraints from $\mathcal{B}$ we may assume, by Lemma 12, that D has an arc multiset $A = \{a_1, \ldots, a_r\}$ left, with $r = O(k^2)$, and $\mathcal{B}$ now contains $s = O(k^2)$ constraints $B_1, \ldots, B_s$. By Lemma 7, $\mathrm{dev}(V, \mathcal{C}) = \max_{\pi}[(\mathrm{dev}(V, A, \pi) + \mathrm{dev}(V, B, \pi))/2]$, where the maximum is taken over all linear orderings π of V.

We now create a new instance $(V', \mathcal{C}', k)$ of Linear Ordering-AA as follows. Let ω be a new variable not in V. For every $a_i = (u_i, v_i)$ add the constraints (ω, u_i, v_i), (u_i, ω, v_i) and (u_i, v_i, ω) to $\mathcal{C}'$. For every $B_i = (a_i, \{b_i, c_i\})$ add the constraints (b_i, a_i, c_i) and (c_i, a_i, b_i) to $\mathcal{C}'$. Let V' be the set of variables that appear in some constraint in $\mathcal{C}'$. Then (V', C') is an instance of Linear Ordering with $O(k^2)$ variables and constraints. Now the number of constraints in $\mathcal{C}'$ satisfied by any linear ordering α of V' equals the number of arcs in D satisfied by α plus the number of constraints in $\mathcal{B}$ satisfied by α. As the average number of constraints satisfied in $(V', \mathcal{C}')$ equals $(3r + 2s)/6 = r/2 + s/3$, it follows that $\mathsf{dev}(V, \mathcal{C}) = \max_\pi[(\mathsf{dev}(V, A, \pi) + \mathsf{dev}(V, \mathcal{B}, \pi))/2] = \mathsf{dev}(V', C')/2$. Hence, (V', C', k) is a kernel of LINEAR ORDERING-AA with $O(k^2)$ variables and constraints. We have established the following theorem.

Theorem 2. LINEAR ORDERING-AA *has a kernel with* $O(k^2)$ *variables and constraints.*

Using Proposition 1 and Theorem 2 we prove the following in [16].

Theorem 3. *There is a bikernel with* $O(k^2)$ *variables from* Π_i-AA *to* Π_j-AA *for each pair* (i, j) *such that* $0 \le i \le 10$ *and* $0 \le j \le 10$ *but* $j \notin \{2, 7\}$.

Using Theorems 1 and 3 we prove the following in [16].

Theorem 4. *All ternary Permutation-CSPs parameterized above average have kernels with* $O(k^2)$ *variables.*

Acknowledgements. The authors thank Mark Jones for carefully reading the paper and finding several minor mistakes that we have corrected.

References

1. Alon, N., Gutin, G., Kim, E.J., Szeider, S., Yeo, A.: Solving MAX-r-SAT above a tight lower bound. Tech. Report arXiv:0907.4573, `http://arxiv.org/abs/0907.4573` ; A preliminary version was published in Proc. SODA, pp. 511–517 (2010)
2. Bodlaender, H.L., Downey, R.G., Fellows, M.R., Hermelin, D.: On problems without polynomial kernels. J. Comput. Syst. Sci. 75(8), 423–434 (2009)
3. Bodlaender, H.L., Thomassé, S., Yeo, A.: Kernel bounds for disjoint cycles and disjoint paths. In: Fiat, A., Sanders, P. (eds.) ESA 2009. LNCS, vol. 5757, pp. 635–646. Springer, Heidelberg (2009)
4. Bonami, A.: Étude des coefficients de Fourier des fonctions de $L^p(G)$. Ann. Inst. Fourier 20(2), 335–402 (1970)
5. Charikar, M., Guruswami, V., Manokaran, R.: Every permutation CSP of arity 3 is approximation resistant. In: Proc. of CCC 2009, pp. 62–73 (2009)
6. Chor, B., Sudan, M.: A geometric approach to betweenness. SIAM J. Discrete Mathematics 11(4), 511–523 (1998)
7. Crowston, R., Gutin, G., Jones, M.: Note on Max Lin-2 above average. Inform. Proc. Lett. 110, 451–454 (2010)
8. Crowston, R., Gutin, G., Jones, M., Kim, E.J., Ruzsa, I.Z.: Systems of linear equations over $\mathbb{F}_2$ and problems parameterized above average. To appear in Proc. SWAT 2010 (2010)
9. Downey, R.G., Fellows, M.R.: Parameterized Complexity. Springer, Heidelberg (1999)
10. Flum, J., Grohe, M.: Parameterized Complexity Theory. Springer, Heidelberg (2006)
11. Galil, Z., Megiddo, N.: Cyclic ordering is NP-complete. Theor. Comput. Sci. 5(2), 179–182 (1977)

12. Goerdt, A.: On random betweenness constraints. In: Gębala, M. (ed.) FCT 2009. LNCS, vol. 5699, pp. 157–168. Springer, Heidelberg (2009)
13. Goerdt, A.: On random ordering constraints. In: Frid, A., Morozov, A., Rybalchenko, A., Wagner, K.W. (eds.) CSR 2009. LNCS, vol. 5675, pp. 105–116. Springer, Heidelberg (2009)
14. Gross, L.: Logarithmic Sobolev inequalities. Amer. J. Math. 97, 1061–1083 (1975)
15. Guo, J., Niedermeier, R.: Invitation to data reduction and problem kernelization. ACM SIGACT News 38, 31–45 (2007)
16. Gutin, G., van Iersel, L., Mnich, M., Yeo, A.: All Ternary Permutation Constraint Satisfaction Problems Parameterized Above Average Have Kernels with Quadratic Number of Variables. Technical Report arXiv:1004.1956v2 [cs.DS]
17. Gutin, G., Kim, E.J., Lampis, M., Mitsou, V.: Vertex Cover Problem Parameterized Above and Below Tight Bounds. Theory Comput. Syst. (in press)
18. Gutin, G., Kim, E.J., Mnich, M., Yeo, A.: Betweenness Parameterized Above Tight Lower Bound. J. Comput. Sys. Sci. (in press)
19. Gutin, G., Kim, E.J., Szeider, S., Yeo, A.: A probabilistic approach to problems parameterized above or below tight bounds. J. Comput. Sys. Sci. (in press); A preliminary version is in Proc. IWPEC 2009. LNCS. 5917, pp. 234–245 (2009)
20. Gutin, G., Rafiey, A., Szeider, S., Yeo, A.: The linear arrangement problem parameterized above guaranteed value. Theory Comput. Syst. 41, 521–538 (2007)
21. Gutin, G., Szeider, S., Yeo, A.: Fixed-parameter complexity of minimum profile problems. Algorithmica 52(2), 133–152 (2008)
22. Guttmann, W., Maucher, M.: Variations on an ordering theme with constraints. In: Navarro, G., Bertossi, L., Kohayakwa, Y. (eds.) Proc. 4th IFIP International Conference on Theoretical Computer Science-TCS 2006, pp. 77–90. Springer, Heidelberg (2006)
23. Isli, A., Cohn, A.G.: A new approach to cyclic ordering of 2D orientations using ternary relation algebras. Artificial Intelligence 122(1-2), 137–187 (2000)
24. Mahajan, M., Raman, V.: Parameterizing above guaranteed values: MaxSat and MaxCut. J. Algorithms 31(2), 335–354 (1999)
25. Mahajan, M., Raman, V., Sikdar, S.: Parameterizing above or below guaranteed values. J. Comput. Syst. Sci. 75(2), 137–153 (2009)
26. Niedermeier, R.: Invitation to fixed-parameter algorithms. Oxford Univ. Press, Oxford (2006)
27. Opatrný, J.: Total ordering problem. SIAM J. Comput. 8(1), 111–114 (1979)
28. Villanger, Y., Heggernes, P., Paul, C., Telle, J.A.: Interval completion is fixed parameter tractable. SIAM J. Comput. 38(5), 2007–2020 (2009)