Algorithms for Dominating Set in Disk Graphs: Breaking the $\log n$ Barrier*
(Extended Abstract)**

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Abstract. We consider the problem of finding a lowest cost dominating set in a given disk graph containing n disks. The problem has been extensively studied on subclasses of disk graphs, yet the best known approximation for disk graphs has remained $O(\log n)$ – a bound that is asymptotically no better than the general case. We improve the status quo in two ways: for the unweighted case, we show how to obtain a PTAS using the framework recently proposed (independently) by Mustafa and Ray [16] and by Chan and Har-Peled [4]; for the weighted case where each input disk has an associated rational weight with the objective of finding a minimum cost dominating set, we give a randomized algorithm that obtains a dominating set whose weight is within a factor $2^{O(\log^* n)}$ of a minimum cost solution, with high probability – the technique follows the framework proposed recently by Varadarajan [19].

1 Introduction

For a set $\mathcal{D}$ of n disks in the Euclidean plane, define an intersection graph, $G = (V, E)$, thus: $V = \mathcal{D}$; $\{u, v\} \in E \Leftrightarrow \textbf{disk}\,(u) \cap \textbf{disk}\,(v) \neq \emptyset$. G is called a *disk graph*; it is a *unit disk graph* when the disk radii are identical.

Given a graph the *minimum dominating set* (MDS) problem is to find a smallest subset $\mathcal{D}' \subseteq V$ such that every vertex is either in $\mathcal{D}'$ or is adjacent to a vertex in $\mathcal{D}'$. On general graphs, the problem is $(1-\varepsilon)\ln n$ hard to approximate for any $\varepsilon > 0$ under standard complexity theoretic assumptions [10,5], while a greedy algorithm yields an $O(\log n)$ approximation [20].

Nevertheless, better approximations are possible for restricted domains. For example, the problem admits a *polynomial-time approximation scheme* (PTAS)

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** Several details are left out due to space constraints. For the full version of the paper, please see [11].

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for unit disk graphs and *growth-bounded graphs* [12,17]. The problem is NP-hard on these domains [6]. However, for the disk graph case, $o(\log n)$ approximations have remained elusive – perhaps, in part, because known techniques for unit disk graphs and solutions to other problems on disk graphs have either relied on packing properties [12,17,8,3], or when packing property does not hold, as in the *minimum weighted dominating set* on unit disk graphs, the fact that disk radii are uniform [1,18]. Erlebach and van Leeuwen recently studied the dominating set problem on *fat objects*, e.g., disk graphs, [9]. They note that existing techniques for disk graphs do not seem sufficient to solve MDS [9]; they also give an $O(1)$-approximation for fat objects of *bounded ply.* Recently, Kammer and Tholey [14] give an $O(1)$-approximation algorithm for MDS when the input is a disk graph with some special properties as well as for intersection graphs with r-regular polygons and other fat objects.

In their recent break-through papers, Chan and Har-Peled [4], and Mustafa and Ray [16] independently showed how a simple *local search* algorithm on certain geometric graphs yields a PTAS for some problems; Chan and Har-Peled [4] show local search yields a PTAS for maximum independent set problem on admissible objects, while Mustafa and Ray [16] show local search yields a PTAS for the minimum hitting set problem given a collection of points and half-spaces in $\mathbb{R}^3$, and also for points and admissible regions in $\mathbb{R}^2$. They both use the *planar separator theorem* to relate the cost of the local search solution with the optimum solution. In the framework, at the crux lies the analysis of a certain graph whose vertices are objects found by local search and ones that belong to an optimum solution, and whose edges (which are only between the two kinds of vertices) satisfy a property relating the two solutions. They show that there exists such a graph which is also planar. Mustafa and Ray [16] refer to the existence of such a planar graph as the *locality condition.*

Results: Our first result is a PTAS for the minimum dominating set problem for disk graphs via a local search algorithm, as in [4,16]. Our analysis also uses the framework introduced by these two papers. Our main new contribution is to show the existence of a planar graph satisfying the locality condition. This graph turns out to be the dual of a weighted Voronoi diagram in the plane.

The minimum dominating set problem for disk graphs can be reduced to the problem of hitting half-spaces in $\mathbb{R}^4$ with the smallest number of a given set of points. That is, given the set $\mathcal{D}$ of disks that form the input to the MDS problem, we can easily compute a map π from $\mathcal{D}$ to a set of points in $\mathbb{R}^4$, and a map h from $\mathcal{D}$ to a set of half-spaces in $\mathbb{R}^4$, with the following property: Two disks d_1 and d_2 from $\mathcal{D}$ intersect if and only if $\pi(d_1)$ lies in $h(d_2)$. Thus we can efficiently reduce the MDS problem for disks to a hitting set problem for points and half-spaces in $\mathbb{R}^4$. While there is a PTAS for the hitting set problem in $\mathbb{R}^3$, as shown by [16], there is none known for $\mathbb{R}^4$. It is not hard to see that a local search such as the one in [16] does not yield a PTAS in $\mathbb{R}^4$.

Rather than reduce to a hitting set problem, we are able to establish the locality condition by staying in the plane itself. In fact, the graph for the locality condition is the dual of the weighted Voronoi diagram of the centers of the disks in the local search solution and the optimal solution, where the weights are the radii of the disk. This can be seen as generalizing the situation considered by [16] for the hitting set problem with points and disks in the plane. In that case, the graph for the locality condition is the Delaunay triangulation, which is the dual of the unweighted Voronoi diagram.

For the case when the disks are weighted, we give the first $o(\log n)$ approximation algorithm; we give a $2^{O(\log^* n)}$ approximation algorithm[1]. This result is based on the framework recently introduced by Varadarajan for the weighted geometric set cover problem [19]. Our contribution here is to observe that the framework is applicable to our dominating set problem as well; the weighted Voronoi diagram is the key to this result also.

We assume that the inputs for both problems satisfy non-degeneracy assumptions – no three disk centers on a line and no four disks tangent to a circle. This is without loss of generality, as these conditions can be enforced by simple perturbations. In Section 2, we present our PTAS for the unweighted dominating set problem, and in Section 3 our algorithm for weighted dominating set.

2 The Unweighted Case: PTAS via Local Search

In this section, we give our PTAS for minimum dominating set for disk graphs. Here, we are given a disk graph with a set $\mathcal{D}$ of n disks in the Euclidean plane, and we are interested in computing a minimum cardinality dominating set of the disk graph. The algorithm is given in Section 2.1 and the analysis of the approximation ratio is given in Section 2.2.

2.1 The Algorithm

Local Search. Call a subset of disks, $B \subseteq \mathcal{D}$, b-locally optimal if one cannot obtain a smaller dominating set by removing a subset $X \subseteq B$ of size at most b from B and replacing that with a subset of size at most $|X|-1$ from $\mathcal{D} \setminus B$. Our algorithm will compute a b-locally optimal set of disks for $b = \frac{c}{\epsilon^2}$ where $c > 0$ is a large enough constant. Our algorithm begins with an arbitrary feasible set of disks and proceeds by making small local exchanges of size $b = O(\frac{1}{\epsilon^2})$, for a given $\epsilon > 0$. We stop when no further local improvements are possible.

Suppose that the solution returned is B. Finally, for reasons apparent in the analysis, we check to see if for any disk $u \in B$ there is a disk $v \in \mathcal{D}$ such that u is completely contained in $v \in \mathcal{D} \setminus B$. If such a disk exists, then simply replace u with the largest such disk v. We return this as our final solution and call it B. Our replacement step ensures that there is no disk in B that is properly contained in some other disk in $\mathcal{D}$.

2.2 Approximation Ratio

We will show that our algorithm is a PTAS, thus proving the following theorem:

[1] $\log^* n$ is the fewest number of iterated “logarithms” applied to n to yield a constant.

Theorem 1. *For any $\epsilon > 0$, there exists a polynomial time algorithm for the minimum dominating set problem on disk graphs that returns a solution whose cost is at most $(1 + \epsilon)OPT$ where OPT is the cost of an optimal solution.*

Let R be the disks in an optimal solution; we may assume no disk in R is properly contained in any other disk in $\mathcal{D}$. Thus, no disk in $R \cup B$ is properly contained in any other disk of $R \cup B$. Note that by the definition of PTAS, we need to show that $|B| \leq (1 + \epsilon) \cdot |R|$. We will refer to R as the set of red disks and B as the set of blue disks. Without loss of generality, we will assume that $R \cap B = \emptyset$, i.e. there is no disk that is both red and blue. For a disk $u \in \mathcal{D}$, we say a disk $v \in R \cup B$ is a *dominator of* u if u and v intersect. Similarly, we also say that v *dominates* u.

We must show the existence of an appropriate planar graph which relates the disks in R with the disks in B. Here, we state the *locality condition* as per Mustafa and Ray [16]:

Lemma 1 (Locality Condition). *There exists a planar graph with vertex set $R \cup B$, such that for every $d \in \mathcal{D}$, there is a disk u from amongst the red dominators of d and a disk v amongst the blue dominators of d such that $\{u, v\}$ is an edge in the graph.*

Section 2.3 is devoted to a proof of Lemma 1. In this extended abstract, we skip the argument (from [4,16]) that has now become standard which uses the lemma to show that $|B| < (1 + \epsilon)|R|$; we also skip the running time analysis.

2.3 Establishing the Locality Condition

This section is devoted to the proof of Lemma 1, that is, the construction of an appropriate planar graph which satisfies the locality condition.

Weighted Voronoi Diagram. We will be using a generalization of Voronoi diagrams called a *weighted Voronoi Diagram* (WVD). Instead of defining cells with respect to a set of points, we will be defining cells with respect to red and blue disks. In order to do this generalization for disks, we must define the distance between a point in the plane and a disk.

Let u be a disk and let x be a point in the plane. We define $\mathbf{wvd}(x, u) = d(x, c_u) - r_u$ where c_u is the center of u, r_u is the radius of u, and $d(x, c_u)$ is the Euclidean distance between x and c_u. Intuitively, for a point x, $\mathbf{wvd}(x, u)$ is the Euclidean distance from x to the boundary of u; the distance to a disk is negative for points that are strictly inside the disk. Alternatively, if $x \notin u$, then $\mathbf{wvd}(x, u)$ is the amount we would need to increase the radius of u so that x lies on the boundary of u; if $x \in u$, then $\mathbf{wvd}(x, u)$ is the negative of the amount we would need to decrease the radius of u so that x lies on the boundary of u. See Figure 1 for an illustration.

For a disk u in any collection of disks, let $\mathbf{cell}(u)$ be the set of points x in the plane such that $\mathbf{wvd}(x, u) \leq \mathbf{wvd}(x, v), u \neq v$. The cells of all the disks in the collection induce a decomposition of the plane, and this is the WVD. This is just the

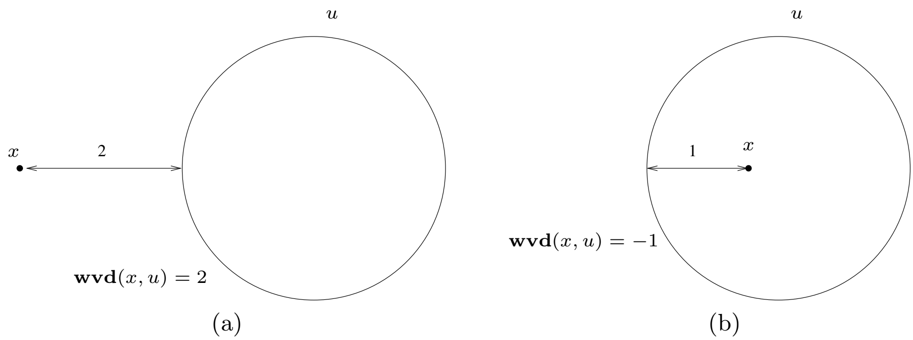

Fig. 1. An illustration for the distances used in our WVD. (a) $\mathbf{wvd}(x, u)$ when x is not in u. (b) $\mathbf{wvd}(x, u)$ when x is in u.

standard weighted Voronoi diagram of the centers of the disks, where the weight of the center of a disk is simply the radius of the disk [2].

Consider the WVD of the disks in $R \cup B$. First, we will show that for every $u \in R \cup B$, u has a non-empty cell in the WVD. That is, there is some point in the plane that is closer to u than it is to any other red or blue disk.

Lemma 2. *In the weighted Voronoi diagram of the union of red and blue disks, the cell of every disk u is nonempty. Moreover, c_u (the center of u) belongs only to* $\mathbf{cell}(u)$*.*

Proof. We will show that c_u is only in $\mathbf{cell}(u)$. Suppose for the sake of contradiction that $c_u \in \mathbf{cell}(v)$ such that $u \neq v$. This means that $\mathbf{wvd}(c_u, v) \leq \mathbf{wvd}(c_u, u) = d(c_u, c_u) - r_u = -r_u$. So, $-r_u \geq \mathbf{wvd}(c_u, v) = d(c_u, c_v) - r_v \Rightarrow r_v \geq d(c_u, c_v) + r_u$. This implies that u is contained in v, and since the two disks are not the same, the containment is proper. But this is a contradiction, since no disk in $R \cup B$ contains another such disk. □

The Graph. Any cell in the WVD of $R \cup B$ is star-shaped with respect to the center of the corresponding disk. That is, for every point $y \in \mathbf{cell}(u)$, the segment $\overline{c_u y}$ is contained within $\mathbf{cell}(u)$.

The graph for the locality condition is simply the dual of the WVD of $R \cup B$. That is, for each cell in the WVD there is a vertex, and there is an edge between two vertices if and only if their corresponding cells share a boundary in the diagram (that is, if and only if there is a point in the plane equidistant from the two disks). The graph is planar – exploiting the fact that the cells are star-shaped, the edges can easily be drawn so that no two edges intersect [2].

Corollary 1. *The dual of the power diagram of $R \cup B$ is a planar graph.*

Because every red and blue disk has a nonempty cell in the WVD, every such disk will also have a corresponding vertex in our planar graph. We are now ready to show that for each $d \in \mathcal{D}$, there is a disk u from amongst the red dominators of d and a disk v amongst the blue dominators of d such that $\mathbf{cell}(u)$ and $\mathbf{cell}(v)$ share a boundary in the WVD. This would then imply that their corresponding vertices in the graph share an edge, completing the proof of Lemma 1. For simplicity, if there is an edge connecting the vertex corresponding to $\mathbf{cell}(u)$ and the vertex corresponding to $\mathbf{cell}(v)$, then we will simply say there is an edge connecting u and v.

Lemma 3. *In the dual graph of the weighted Voronoi diagram for $R \cup B$, for an arbitrary input disk $u \in \mathcal{D}$, there is an edge between some red dominator of u and some blue dominator of u.*

Proof. Consider the WVD of $R \cup B$. Without loss of generality, assume $c_u \in \mathbf{cell}(r)$ for some $r \in R$. Now, r must be a dominator of u, because r is the closest disk in $R \cup B$ to c_u. If r does not dominate u, u is not dominated by any disk in $R \cup B$ which contradicts the fact that both R and B are dominating sets.

Let b denote a closest blue disk to c_u, that is $\mathbf{wvd}(c_u, b) \leq \mathbf{wvd}(c_u, b')$ for all other blue disks b'. Note that b must dominate u, because if it did not, then no blue disks would dominate u. This would contradict the fact that B is a dominating set. Also, note that for any disk $d \in \mathcal{D}$ such that $\mathbf{wvd}(c_u, d) \leq \mathbf{wvd}(c_u, b)$, d must intersect with u.

If $\mathbf{wvd}(c_u, b) = \mathbf{wvd}(c_u, r)$, we are done, since then there is an edge in the dual graph incident on r and b. So, let us assume that $\mathbf{wvd}(c_u, b) > \mathbf{wvd}(c_u, r)$.

We will walk from c_u to c_b along the straight line segment $\overline{c_u c_b}$. The proof strategy is that during this walk, we will be crossing red cells and at some point before reaching c_b we will enter a blue cell, in particular, $\mathbf{cell}(b)$. We must have entered this cell from a red cell $\mathbf{cell}(r')$ which shares a boundary with $\mathbf{cell}(b)$, and thus $\{r', b\}$ is an edge in our planar graph. Moreover, we will argue that r' necessarily dominates u, completing the proof.

As seen in the proof of Lemma 2, $c_b \in \mathbf{cell}(b)$, and thus we will enter $\mathbf{cell}(b)$ at some point in time along our walk from c_u to c_b. Let x be the point at which we first enter $\mathbf{cell}(b)$. Then x is on the boundary of $\mathbf{cell}(b)$ and $\mathbf{cell}(r')$ for some $r' \in R \cup B$. If $r' = r$, we are done. Otherwise, we have

$$\mathbf{wvd}(c_u, r') < d(c_u, x) + \mathbf{wvd}(x, r') = d(c_u, x) + \mathbf{wvd}(x, b) = \mathbf{wvd}(c_u, b).$$

(Here the strictness of the first inequality comes from our non-degeneracy assumption which implies that $c_{r'}$ cannot lie on the line through c_u and c_b.) Now, it must be the case that $r' \in R$ because $\mathbf{wvd}(c_u, r') < \mathbf{wvd}(c_u, b)$ and b is the closest blue disk to c_u. This also implies that r' must dominate u. See Figure 2 for an illustration.

Therefore $\mathbf{cell}(b)$ and $\mathbf{cell}(r')$ share a boundary implying that the edge $\{b, r'\}$ is in our graph. Moreover, b is blue, r' is red, and both dominate u, which completes the proof. □

Together, Corollary 1 and Lemma 3 prove Lemma 1.

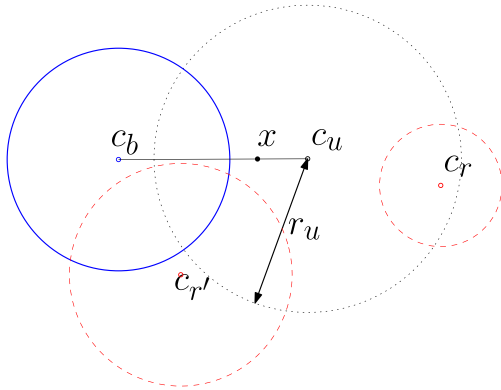

Fig. 2. Proof of Lemma 3. The dotted disk is u with center c_u and radius r_u. The two red disks r and r' are shown as dashed disks with centers c_r and $c_{r'}$, respectively. The only blue disk b is shown as a solid disk with center c_b.

3 The Weighted Dominating Set Case

In this section, we study a classical generalization of the dominating set problem. Each disk u now has an associated rational weight, w_u. The goal is to find a dominating set D having the lowest cost, that is, $\mathbf{wt}\,(D) = \sum_{u \in D} w_u$ be as small as possible. We will prove the following theorem:

Theorem 2. *Given a disk graph, $G = (V, E)$ of n weighted disks D in the plane, there is a randomized algorithm that produces a dominating set $V' \subseteq V$, and $\boldsymbol{wt}\,(V') \leq 2^{O(\log^* n)} \cdot$ OPT, w.h.p., where OPT denotes the cost of an optimal solution.*

The high-level structure of the algorithm is as follows: we first solve a natural linear programming relaxation, followed by a randomized rounding step; this step allows us to ignore the weights of the disks in the sampling (pruning) stage. In the rounding step, we make several copies of the disks to ensure that two properties hold. First, every disk in D is covered by at least n of the copies. Second, the weight of the copies is $O(n \cdot \lambda^*)$, where λ^* is the objective function value of an optimal LP solution. Following this step, we recursively apply a randomized pruning step where we remove some of the copies according to the algorithm given in the proof of Theorem 3 while ensuring that the remaining copies are a dominating set of D. The main goal of the pruning step is to remove some of the copies while approximately preserving the ratio of the cost of the remaining copies to the "depth" of the disks in D with respect to the remaining copies. We recursively apply the pruning step until the disks in D are covered by only a constant number of the remaining copies; the depth of our recursion is $\Theta(\log^* n)$. We can then show that the expected weight of our final dominating set is at worst $2^{O(\log^* n)} \cdot \lambda^*$.

First, we define some terms that are used in the remaining part of the section. Given a disk v and a set of disks S, we say that v is L-covered by S if there are exactly L disks in S each of which intersects v. In other words, neighborhood of v in S has size L. We will make use of the following lemma, which is our main contribution to the weighted case:

Lemma 4. *Let S be a set of m disks, and $1 \leq L \leq m$ an integer. Let Q be another (possibly infinite) set of disks. There are $O(m \cdot L^2)$ disks of Q that intersect distinct subsets of S each of size at most L.*

Proof. We first define a few concepts that we use in the proof. We focus on subsets $S' \subseteq S$ of size at most L and disks of Q whose neighborhood is precisely one of these subsets; let us denote this subset of Q by Q'. For a set $S' \subseteq S$ of size at most L, and a pair of disks $u, v \in Q'$, we say that u and v are related if they both intersect every disk in S' and no other disk of $S \setminus S'$, i.e. $u \cap S = v \cap S = S'$. So we have an equivalence relation on Q' where each equivalence class corresponds to a set $S' \subseteq S$. We wish to bound the number of these equivalence classes. Let these subsets of S be $\{S_1, S_2, \ldots, S_t\}$, and correspondingly, t equivalence classes $\{Q_1, Q_2, \ldots, Q_t\}$, where each disk in Q_i intersects every disk in S_i, and no other disk of $S \setminus S_i$. Consider any set Q_i and an arbitrary disk $v \in Q_i$. By scaling and/or translating v we can obtain a disk v' with the following property: v' has the same neighborhood as all the disks in Q_i and is sharing a single point with three, two, or one disk in S_i and is intersecting all the other disks in more than one point; for the cases when v' is touching a single disk in S_i, or two disks in S_i, we continue to translate and scale v' so that it touches two disks outside of S_i, or one disk outside of S_i, respectively. Without loss of generality, we assume that S has four special disks whose borders form the North, South, East, and West boundary, respectively, of the region that contains the input disks. We call these special disks N, S, E, W, respectively. Such a transformed disk, v', that touches exactly three disks is referred to as v_i. We say that a disk d is *canonical* with respect to a set of disks D' if there are three distinct disks in D' such that d intersects the three disks at only one point each. Note that each v_i is a canonical disk with respect to the set S. We say that a canonical disk v is *κ-canonical* with respect to a set of disks D' if at most κ disks from D' intersect the interior of v. Therefore, each of the canonical disks v_i that we defined are L-canonical disks. It is easy to see that t is within a constant factor of the number of L-canonical disks with respect to S. For each v_i, the set of disks that shares exactly one point with it is called the *defining set* of v_i and every disk of S_i that shares more than one point with v_i is said to be in the *conflict set* of v_i. Note that the defining set of v_i has at least one disk from S_i, but at most two remaining disks can be from outside S_i. We will upper bound the number of L-canonical disks with respect to S (and hence upper bound t) by choosing a random sample $S' \subseteq S$ and calculating the expected number of 0-canonical disks with respect to S'. This technique dates back to that of Clarkson [7].

Let us choose a random subset $S' \subseteq S$ using k independent trials in which we pick each disk from S with uniform probability, while we add N, S, E, W in S' with probability 1. Now, for a fixed v_i to be a 0-canonical disk in S', its defining set must have been picked in S', and its conflict set must not be in S'. The probability of this event is at least

$$\left(\frac{k}{m}\right)^3 \cdot \left(1 - \frac{L}{m}\right)^k$$

Thus, the expected number of disks among $v_1, \ldots, v_t$ (L-canonical disks for the sets) that are 0-canonical disks for S' is at least

$$t \cdot \left(\frac{k}{m}\right)^3 \cdot \left(1 - \frac{L}{m}\right)^k$$

We will show that the maximum number of 0-canonical disks for S' is $O(k)$.

Claim. For a set S' of disks of size k, the maximum number of 0-canonical disks induced is $O(k)$.

Proof. We will bound the number of 0-canonical disks by the number of Voronoi vertices of a weighted Voronoi diagram with k sites in which the sites are represented by the k centers of disks in S', and the weight of each site is the radius of the corresponding disk. Every Voronoi vertex is equidistant from the disks of the regions sharing that vertex. So each Voronoi vertex in the Voronoi diagram corresponds to the center of a disk that touches the boundary of exactly three disks of S' (disks corresponding to the three regions defining that vertex) and does not intersect any other disk of S'. Since the number of Voronoi vertices of a Voronoi diagram having k sites is bounded linearly in k, the number of of canonical disks that touch three disks of S' are thus bounded linearly in k as well. This leads to the final bound of $O(k)$ on the maximum number of canonical disks that S' admits. □

According to the claim, the maximum number of 0-canonical disks for S' is $O(k)$. So,

$$t \cdot \left(\frac{k}{m}\right)^3 \cdot \left(1 - \frac{L}{m}\right)^k \leq c_1 k,$$

for some constant $c_1 > 0$. Choosing $k = \frac{2m}{L}$ yields $t \leq c' m L^2$. □

We prove the following variant of a theorem of Varadarajan in [19].

Theorem 3. *Given a disk graph $G = (V, E)$ and set of n weighted disks $D \subseteq V$ in the plane s.t. D dominates V, there is a randomized algorithm that produces a subset $D' \subseteq D$, such that for any disk $v \in V$, if v is L-covered in D, then v is at least* $\log L$*-covered in D' and* $\boldsymbol{Pr}[d \in D'] \leq \frac{c \cdot \log L}{L}$.

Proof. We only describe a randomized process that selects a subset, D' of disks such that any disk $v \in V$ that is covered by D in the range $[L, 2L]$, v is at least $\log L$-covered in D'. Let $N_m = D$, and let $\mathcal{C}_m$ denote the set of equivalence classes of disks in V such that each class intersects at most $2L$ disks of D. Note that since the disks in one equivalence class of V have the same neighborhood in D, if we obtain a set D' that at least $\log L$-covers one disk in that class, then all the disks in that class are also at least $\log L$-covered. Therefore, we can assume we have one representative disk from each class and our goal is to at least $\log L$-cover these disks. We use this fact crucially in our analysis. By Lemma 4, $|\mathcal{C}_m| \leq c' \cdot n_m L^2$, $n_m = |N_m|$. So, there is a disk d_m that covers at most $2c'L^2$ classes of $\mathcal{C}_m$. Find such a disk $d_m \in N_m$, and recursively compute a sequence for $N_{m-1} = N_m \setminus \{d_m\}$, and append the sequence to d_m. That is, in the arrangement of N_{m-1} we consider the classes $\mathcal{C}_{m-1}$ whose coverage in N_{m-1} is at most $2L$. The recursion stops when there are fewer than L disks remaining, at which point, we compute an arbitrary sequence of the remaining set of disks.

Let σ be the *reverse* of this sequence, that is, $\sigma = (d_1, d_2, \ldots, d_m)$. When considering disk d_j, we make an instant decision about including it in our cover or not. Call a disk $d_j \in N_j$ *forced* if for some disk $v \in \mathcal{C}_j$, not including d_j *will not* $\log L$-cover v, whose coverage in N_m is in $[L, 2L]$. Otherwise, if d_j is not forced, we add it to D' with probability $\frac{c \cdot \log L}{L}$. We will upper bound the probability of d_j being forced – we will show that it is at most $O(1/L)$.

Observe that if a disk d_j is forced because of v, then all the disks $d_{j'}$ (with $j' \geq j$) that cover v are also forced, and the number of such disks is at most $\log L - 1$ (otherwise d_j won't be forced). So it is sufficient to upper bound the probability of a disk d_i being the first disk forced because of v. Let us denote this event by $\mathcal{E}_i(v)$. Since from among the disks that cover v at most the last $\log L$ disks can be forced, the probability of one of these $\log L$ disks being forced is at most $\log L$ times the probability that one of the disks before it is the "first" forced disk because of v. We use $\mathcal{E}_i$ to denote the event that d_i is the first disk forced because of some disk that it covers. We omit the proof of the following claim from this extended abstract.

Claim.

$$\mathbf{Pr}\left[\mathcal{E}_i(v)\right] \leq \frac{1}{L^4}$$

Note that any disk $d_{i'}$ that occurs before d_i in σ if $d_{i'}$ is forced for a disk v' that is not covered by d_i, which forces a disk d_k which occurs after d_i in σ and that d_k also covers v, then that event has no bearing on the event of d_i being a first forced disk for v. So, to upper bound the probability that some d_j is a forced disk for a fixed disk v, we sum over all valid indices $i < j$ with d_i being the first forced disk because of v, and obviously there are at most $\log L$ of them,

$$\mathbf{Pr}\left[\begin{smallmatrix}\text{some } d_j \text{ is}\\ \text{forced by}\\ \text{disk } v\end{smallmatrix}\right] \leq \sum_i \frac{1}{L^4} \leq \frac{1}{L^3}.$$

Since there are at most $2c'L^2$ classes of $\mathcal{C}_j$ having coverage in the range $[L, 2L]$ that are covered by d_j, d_j can be a forced addition for any one of the at most $2c'L^2$ representative disks. So,

$$\mathbf{Pr}\left[\begin{smallmatrix}d_j \text{ is forced}\\ \text{for some}\\ \text{disk } v \in \mathcal{C}_j\end{smallmatrix}\right] \leq \frac{2c'}{L}.$$

The probabilistic algorithm finds a dominating set $D' \subseteq D$ where the probability of a given disk being in D' is at most $\frac{c \cdot \log L}{L}$ and each disk $v \in V$ that is covered in the range $[L, 2L]$ by D, is at least $\log L$-covered in D'. We repeat the process for points that are between $2L$ and $4L$ deep, and so on. Note that the probability of a disk being in D' is still the same. □

3.1 Proof of Theorem 2

Let the input instance be a disk graph based on a set of disks D. For any disk $d \in D$, let $N[d]$ denote the set of neighbors of d in the graph, inclusive. Consider the following natural LP relaxation for the weighted dominating set problem:

$$\text{(LP)} \min \sum_{d \in D} w_d x_d$$

subject to,

$$\sum_{d' : d' \in N[d]} x_{d'} \geq 1, \forall d \in D$$
$$x_d \geq 0, \forall d \in D$$

After solving the LP relaxation, we create a set D_0 of disks as follows. For each disk d such that $x_d \geq \frac{1}{2n}$, we add $\lfloor \frac{x_d}{1/(2n)} \rfloor$ copies of d to D_0. Each copy of d inherits its original cost. For each disk d with $x_d < \frac{1}{2n}$, we don't add any copy to D_0. It is easily verified that $\mathbf{wt}\,(D_0) \leq 2n \cdot \lambda^*$, where λ^* is the objective function value of the optimal LP solution. Furthermore, we have that each disk $d \in D$ is n-covered by D_0.

In the next phase, our algorithm will recursively apply Theorem 3 to obtain a successively sparse dominating set. For the ith application of the theorem, we set $L_i = \log L_{i-1}$, for $i = 2, 3, \ldots, t$ to obtain a set $D_i \subseteq D_{i-1}$. For the first application, we set $L_1 = n$. Details of the approximation ratio are omitted from this extended abstract.

4 Concluding Remarks and Open Questions

Given the negative result of Marx [15] which shows that even for the simple case of unweighted unit disk graph, an EPTAS for the problem would contradict the *exponential time hypothesis* [13][2], it is unlikely that the dependence of $1/\varepsilon$ as an exponent of n on the running time for the PTAS can be improved to, say, $f(1/\varepsilon) \cdot n^{O(1)}$. However, the running time of the local search PTAS is $n^{O(1/\varepsilon^2)}$. Can this be improved to $n^{O(1/\varepsilon)}$? In our work, we have made no attempt to improve the running time.

For the weighted case, we are only able to show a constant integrality gap for the lower bound despite numerous attempts. Thus, we believe that the right upper bound for the approximation factor is $O(1)$.

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[2] Marx [15] actually shows something stronger.

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