Rational Closure for Defeasible Description Logics

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Abstract. In the field of non-monotonic logics, the notion of *rational closure* is acknowledged as a landmark, and we are going to see that such a construction can be characterised by means of a simple method in the context of propositional logic. We then propose an application of our approach to rational closure in the field of Description Logics, an important knowledge representation formalism, and provide a simple decision procedure for this case.

1 Introduction

A lot of attention has been dedicated to *non-monotonic* reasoning (see, *e.g.* [20]). Relatively less investigated is the application of such reasoning models to Description Logics (DLs) [3]. In what follows we take under consideration one central non-monotonic reasoning model, that is, the *rational closure* [28], and we are going to apply such a construction to $\mathcal{ALC}$, a significant and expressive representative of the various DLs.

The contributions of this work can be summarised as follows: (i) we provide a characterisation of rational closure in the context of propositional logic, based on classical entailment tests only and, thus, amenable of a simple implementation; and (ii) we apply this characterisation to the context of DLs (we provide a construct $C \mid\!\sim D$ stating 'an instance of the concept C, typically is an instance of the concept D'), inheriting a simple reasoning procedure to decide entailment under rational closure.

While there have been several non-monotonic extensions of DLs, such as [1, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 21, 22, 23, 24, 25, 27, 31, 32, 34, 35, 36], which integrate several kind of non-monotonic reasoning mechanism into DLs, to the best of our knowledge, none of them address specifically the issue to model rational closure in DLs. Somewhat related to our proposal are [11, 23], but, beside other points, both model rational consequence relations, while we refer to a rational consequence relation that is recognised as particularly well-behaved, that is, rational closure.

We proceed as follows: first, we present a particular construction of the rational closure, based on the default-assumption approach (see, *e.g.* [30, 33]); then we implement such a construction in $\mathcal{ALC}$; in the end, we conclude with a summary of our contribution and future issues we plan to address.

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2 Propositional Rational Closure

Consider a finitely generated classical propositional language ℓ, defined in the usual way [1]. We shall use δ to indicate *default formulae*. The symbols $\models$, $\mid\sim$ will represent different kinds of consequence relations. In particular, $\models$ will be the classical consequence relation and $\mid\sim$ a defeasible inference relation. An element of a consequence relation, $\Gamma\mid\sim C$, will be called a *sequent* and has to be read as 'If Γ, then typically C'.

To start with, a *conditional knowledge base* will be characterised by a pair $\langle\mathcal{T},\mathcal{B}\rangle$, where $\mathcal{T}$ is a set of formulae, representing certain knowledge, and $\mathcal{B}$ is a set of *sequents* $C\mid\sim D$, representing default information (see [28]).

Example 1. The typical 'penguin' example can be encoded as [2]: $\mathcal{K} = \langle\mathcal{T},\mathcal{B}\rangle$ with $\mathcal{T} = \{P \rightarrow B\}$ and $\mathcal{B} = \{P\mid\sim\neg F, B\mid\sim F\}$. □

Another way to formalise defeasible information may be based on the *default-assumption approach*, where a *default knowledge base* is a pair $\langle\mathcal{T},\Delta\rangle$, where now Δ is a set of *formulae* representing what the agent considers as typically true.

Example 2. The 'penguin' example can, for instance, be encoded as: $\mathcal{K} = \langle\mathcal{T},\Delta\rangle$ with $\mathcal{T} = \{P \rightarrow B\}$ and $\Delta = \{(B \rightarrow F) \wedge (P \rightarrow \neg F), P \rightarrow \neg F\}$. □

Our proposal, using the results of Freund [19], will consist in mapping a conditional knowledge base into a default knowledge base (*e.g.*, we will transform the KB in Example 1 into the KB of Example 2), and then we show a simple procedure to reason within the latter, by relying on a decision procedure for $\models$ only. We then suggest to transpose such an approach, into the framework of DLs.

We proceed next in this way: (i) first, we define the notion of *rational consequence relation* (see *e.g.* [29]) and we present the notion of *rational closure*, (ii) then, we briefly present the *default-assumption approach* and show how to map, by preserving rational closure, a conditional knowledge base into a default knowledge base. Eventually, we describe a procedure to build a *rational closure* using the default-assumption approach.

Rational Consequence Relations. A particularly appreciated non-monotonic consequence relation is represented by the class of *rational consequence relations* (see [28]). A consequence relation $\mid\sim$ is *rational* iff it satisfies the following properties:

(REF) $C\mid\sim C$ Reflexivity

(CT) $\dfrac{C\mid\sim D \quad C\wedge D\mid\sim F}{C\mid\sim F}$ Cut (Cumulative Trans.)

(RW) $\dfrac{C\mid\sim D \quad D\models F}{C\mid\sim F}$ Right Weakening

(CM) $\dfrac{C\mid\sim D \quad C\mid\sim F}{C\wedge D\mid\sim F}$ Cautious Monotony

(OR) $\dfrac{C\mid\sim F \quad D\mid\sim F}{C\vee D\mid\sim F}$ Left Disjunction

(LLE) $\dfrac{C\mid\sim F \quad \models C\leftrightarrow D}{D\mid\sim F}$ Left Logical Equival.

(RM) $\dfrac{C\mid\sim F \quad C\not\mid\sim\neg D}{C\wedge D\mid\sim F}$ Rational Monotony

Rational consequence relations represent a particular subclass of the *preferential inference relations* (see [26]), which are defined by the above properties (REF – OR), without (RM); these are generally considered as the core properties defining a satisfying

[1] We use $\neg,\wedge,\vee,\rightarrow$ as connectives, $C, D,\ldots$ as sentences, $\Gamma,\Delta,\ldots$ as finite sets of sentences, $\top$ and $\bot$ as $A\vee\neg A$ and $A\wedge\neg A$ for some A.

[2] Read B as 'Bird', P as 'Penguin' and F as 'Flying'.

non-monotonic inference relation. Hence, rational consequence relations are preferential relations characterised by the property (RM), which can be read as 'If C typically implies F, and we are not aware that C typically implies $\neg D$, then we are authorised to consider F as a typical consequence of $C \wedge D$'. (RM) is generally considered as the strongest form of monotonicity we can use in the characterisation of a reasoning system in order to formalise a well-behaved form of defeasible reasoning.

Semantically, rational consequence relations can be characterised by means of a particular kind of possible-worlds model, that is, ranked preferential models, but we shall not deepen the connection with such a semantical characterisation here (see [28]).

Rational Closure. Consider $\mathcal{B} = \{C_1 \mid\!\sim E_1, \ldots, C_n \mid\!\sim E_n\}$. We want the agent to be able to reason about its defeasible information, that is, to be able to derive new sequents from his conditional base. A way to derive new default information is by defining a closure operation $\mathbb{P}$ that, given $\mathcal{B}$, gives back a preferential consequence relation $\mid\!\sim$ containing the sequents in $\mathcal{B}$ and is closed under the rules (REF)–(OR). Such a closure operation under the rules (REF)–(OR) is unique (see [26], Corollary 1 and p.31). Formally, given $\mathcal{B}$, a sequent $C \mid\!\sim D$ is in its preferential closure $\mathbb{P}(\mathcal{B})$ iff it is derivable from $\mathcal{B}$ using the preferential rules (REF)–(OR). However, the preferential closure is generally considered too weak to be satisfactory, and so it is natural to look for stronger forms of closure. The closure under the rule (RM) is considered, between the interesting rules, the strongest one. Lehmann and Magidor have defined in [28] a rational closure operation $\mathbb{R}$ that satisfies a set of desiderata: namely, (i) $\mathbb{P}(\mathcal{B}) \subseteq \mathbb{R}(\mathcal{B})$; (ii) $C \mid\!\sim \bot \in \mathbb{R}(\mathcal{B})$ iff $C \mid\!\sim \bot \in \mathbb{P}(\mathcal{B})$; (iii) $\top \mid\!\sim C \in \mathbb{R}(\mathcal{B})$ iff $\top \mid\!\sim C \in \mathbb{P}(\mathcal{B})$; (iv) If $C \mid\!\sim F$ in $\mathbb{P}(\mathcal{B})$, and $C \mid\!\sim \neg D, C \wedge D \mid\!\sim F \notin \mathbb{P}(\mathcal{B})$ then $C \wedge D \mid\!\sim F \in \mathbb{R}(\mathcal{B})$ whenever is possible(see [28], Section 5, for the justification of these desiderata). We shall not describe Lehmann and Magidor's rational closure operation referring to [28]. However, we shall directly refer to a correspondent, more simple construction, based on the default-assumption approach and defined by Freund in [19].

Default-Assumption Consequence Relations and Rational Closure. Consider a default knowledge base $\mathcal{K} = \langle \mathcal{T}, \Delta \rangle$. If the agent is confronted with a piece of information Γ, representing what actually holds, then he has to 'merge' the information in Γ with his background theory $\mathcal{T}$ and his default information Δ. Such an interaction is determined by a consistency check, formalised referring to the notion of *maxiconsistent subset*. Formally, let Δ, Φ be two sets of formulae, then Ψ is a *Φ-maxiconsistent subset* of Δ iff (i) $\Psi \subseteq \Delta$; (ii) $\Psi \not\models \neg(\bigwedge \Phi)$; and (iii) there is no set Ψ' such that $\Psi \subset \Psi' \subseteq \Delta$, and $\Psi' \not\models \neg(\bigwedge \Phi)$. Now, to determine what the agent presumes to be true in a situation in which Γ holds, he takes under consideration all the $(\mathcal{T} \cup \Gamma)$-maxiconsistent subsets of Δ, *i.e.* he considers all the default information that is compatible with what he knows to be true. That is, we say that D is a *default-assumption consequence* of the premise set Γ, given a background theory $\mathcal{T}$ and a set of default-assumptions Δ, written $\Gamma \mid\!\sim_{\langle \mathcal{T}, \Delta \rangle} D$, if and only if D is a classical consequence of the union of Γ with $\mathcal{T}$ and whichever $(\mathcal{T} \cup \Gamma)$-maxiconsistent subset of Δ, *i.e.*

$$\Gamma \mid\!\sim_{\langle \mathcal{T}, \Delta \rangle} D \textit{ iff } (\mathcal{T} \cup \Gamma \cup \Delta') \models D \textit{ for every } (\mathcal{T} \cup \Gamma)\textit{-maxiconsistent } \Delta' \subseteq \Delta \ .$$

As next, we want to characterize the rational closure by means of the default-assumption construction, *i.e.* we start from a defeasible KB $\langle \mathcal{T}, \mathcal{B} \rangle$ and from it we build a correspondent default KB $\langle \widetilde{\mathcal{T}}, \widetilde{\Delta} \rangle$. So, consider $\langle \mathcal{T}, \mathcal{B} \rangle$, with $\mathcal{B} = \{C_1 \mid\!\sim E_1, \ldots, C_n \mid\!\sim E_n\}$. The steps for the construction of $\langle \widetilde{\mathcal{T}}, \widetilde{\Delta} \rangle$ (obtained combining the results in [19] with some results from [6]) are the following.

Step 1. We translate $\mathcal{T}$ into a sequential form and add it to $\mathcal{B}$, that is, we move from a characterisation $\langle \mathcal{T}, \mathcal{B} \rangle$ to $\langle \emptyset, \mathcal{B}' \rangle$, where $\mathcal{B}' = \mathcal{B} \cup \{\neg C \mid\!\sim \bot \mid C \in \mathcal{T}\}$. Intuitively, C is valid is equivalent to saying that its negation is an absurdity ($\neg C \mid\!\sim \bot$) ([6], Section 6.5).

Step 2. We define $\Gamma_{\mathcal{B}'}$ as the set of the *materializations* of the sequents in $\mathcal{B}'$, *i.e.* the material implications corresponding to such sequents: $\Gamma_{\mathcal{B}'} = \{C \rightarrow D \mid C \mid\!\sim D \in \mathcal{B}'\}$. Also, we indicate by $\mathfrak{A}_{\mathcal{B}'}$ the set of the antecedents of the sequents in $\mathcal{B}'$: $\mathfrak{A}_{\mathcal{B}'} = \{C \mid C \mid\!\sim D \in \mathcal{B}'\}$.

Step 3. Now we define an *exceptionality ranking* of sequents with respect to (w.r.t.) $\mathcal{B}'$:

Step 3.1. Lehmann and Magidor [28] call a formula C *exceptional* for a set of sequents $\mathcal{D}$ iff $\mathcal{D}$ preferentially entails $\top \mid\!\sim \neg C$ (*i.e.* $\top \mid\!\sim \neg C \in \mathbb{P}(\mathcal{D})$). $C \mid\!\sim D$ is said to be exceptional for $\mathcal{D}$ iff its antecedent C is exceptional for $\mathcal{D}$. Exceptionality of sequents can be decided based on $\models$ only (see [28], Corollary 5.22), as C is exceptional for a set of sequents $\mathcal{D}$ (*i.e.* $\top \mid\!\sim \neg C \in \mathbb{P}(\mathcal{D})$) iff $\Gamma_{\mathcal{D}} \models \neg C$.

Step 3.2. Given a set of sequents $\mathcal{D}$, indicate by $E(\mathfrak{A}_{\mathcal{D}})$ the set of the antecedents that result exceptional w.r.t. $\mathcal{D}$, that is $E(\mathfrak{A}_{\mathcal{D}}) = \{C \in \mathfrak{A}_{\mathcal{D}} \mid \Gamma_{\mathcal{D}} \models \neg C\}$, and with $E(\mathcal{D})$ the exceptional sequents in $\mathcal{D}$, *i.e.* $E(\mathcal{D}) = \{C \mid\!\sim D \in \mathcal{D} \mid C \in E(\mathfrak{A}_{\mathcal{D}})\}$. Obviously, for every $\mathcal{D}$, $E(\mathcal{D}) \subseteq \mathcal{D}$.

Step 3.3. We can construct iteratively a sequence $\mathcal{E}_0, \mathcal{E}_1 \ldots$ of subsets of the conditional base $\mathcal{B}'$ in the following way: $\mathcal{E}_0 = \mathcal{B}'$, $\mathcal{E}_{i+1} = E(\mathcal{E}_i)$. Since $\mathcal{B}'$ is a finite set, the construction will terminate with an empty set ($\mathcal{E}_n = \emptyset$) or a fixed point of E.

Step 3.4. Using such a sequence, we can define a ranking function r that associates to every sequent in $\mathcal{B}'$ a number, representing its level of exceptionality:

$$r(C \mid\!\sim D) = \begin{cases} i & \text{if } C \mid\!\sim D \in \mathcal{E}_i \text{ and } C \mid\!\sim D \notin \mathcal{E}_{i+1} \\ \infty & \text{if } C \mid\!\sim D \in \mathcal{E}_i \text{ for every } i \,. \end{cases}$$

Step 4. In Step 3, we defined the materialisation of $\mathcal{B}'$ and the rank of every sequent in it. Now,

Step 4.1. we can determine if $\mathcal{B}'$ is inconsistent. A conditional base is inconsistent if in its preferential closure we obtain the sequent $\top \mid\!\sim \bot$ (from this sequent we can derive any other sequent using RW and CM). Given the result in Step 3.1, we can check the consistency of $\mathcal{B}'$ using $\Gamma_{\mathcal{B}'}$: $\top \mid\!\sim \bot \in \mathbb{P}(\mathcal{B}')$ iff $\Gamma_{\mathcal{B}'} \models \bot$.

Step 4.2. if $\mathcal{B}'$ is consistent and given the ranking, we define the *background theory* $\widetilde{\mathcal{T}}$ of the agent as $\widetilde{\mathcal{T}} = \{\neg C \mid C \mid\!\sim D \in \mathcal{B}' \text{ and } r(C \mid\!\sim D) = \infty\}$ [3] (one may verify that $\mathcal{T} \subseteq \widetilde{\mathcal{T}}$).

Step 4.3. once we have $\widetilde{\mathcal{T}}$, we can also identify the set of sequents $\widetilde{\mathcal{B}}$, *i.e.*, the defeasible part of the information contained in $\mathcal{B}'$: $\widetilde{\mathcal{B}} = \{C \mid\!\sim D \in \mathcal{B}' \mid r(C \mid\!\sim D) < \infty\}$ (one may verify that $\widetilde{\mathcal{B}} \subseteq \mathcal{B}$).

[3] One may easily verify the correctness of this definition referring to the following results in [6]: the definition of *clash* (p.175), Corollary 7.5.2, Definition 7.5.2, and Lemma 7.5.5. It suffices to show that the set of the sequents with ∞ as ranking value represents the greatest clash of $\mathcal{B}$.

Essentially, so far we have moved the non-defeasible knowledge ‘hidden’ in $\mathcal{B}$ to $\mathcal{T}$.

Step 5. Now we build the default-assumption characterisation of the rational closure of $\langle \widetilde{\mathcal{T}}, \widetilde{B} \rangle$. To do so, we translate $\widetilde{\mathcal{B}}$ into a set of default-assumptions, *i.e.* a set of formulae, $\widetilde{\Delta}$. Specifically, given the rank value of the sequents in $\widetilde{\mathcal{B}}$, we construct a set of default assumptions $\widetilde{\Delta} = \{\delta_0, \ldots, \delta_n\}$ (with n the highest rank-value in $\widetilde{\mathcal{B}}$), with

$$\delta_i = \bigwedge \{C \to D \mid C \mid\!\sim D \in \widetilde{\mathcal{B}} \text{ and } r(C \mid\!\sim D) \geq i\} \ . \quad (1)$$

Following this construction, presented by Freund in [19], we obtain a set of default formulae, each one associated with a rank value, s.t. every default formula is classically derivable from the preceding ones, that is, $\delta_i \models \delta_{i+1}$, for $0 \leq i < n$.

Step 6. Given the background theory $\widetilde{\mathcal{T}}$ and the default-assumption set $\widetilde{\Delta}$, we associate to the agent the pair $\langle \widetilde{\mathcal{T}}, \widetilde{\Delta} \rangle$ according to the steps defined so far.

Using [19], Theorem 24, we can prove that the default-assumption characterisation of the agent by means of the pair $\langle \widetilde{\mathcal{T}}, \widetilde{\Delta} \rangle$ is equivalent to the rational closure of the pair $\langle \mathcal{T}, \mathcal{B} \rangle$ defined by Lehmann and Magidor. That is,

Proposition 1. $\Gamma \mid\!\sim_{\langle \widetilde{\mathcal{T}}, \widetilde{\Delta} \rangle} D$ *iff* $\Gamma \mid\!\sim D \in \mathbb{R}(\mathcal{B}')$.

As a consequence, using the following knowledge base transformations

$$\mathcal{K} = \langle \mathcal{T}, \mathcal{B} \rangle \rightsquigarrow \langle \emptyset, \mathcal{B}' \rangle \rightsquigarrow \langle \widetilde{\mathcal{T}}, \widetilde{B} \rangle \rightsquigarrow \langle \widetilde{\mathcal{T}}, \widetilde{\Delta} \rangle \ , \quad (2)$$

we can characterise the rational closure of $\langle \mathcal{T}, \mathcal{B} \rangle$ via $\langle \widetilde{\mathcal{T}}, \widetilde{\Delta} \rangle$ by means of Proposition 1. Note that, given the Eq. (1), the default set $\widetilde{\Delta}$ is linearly ordered by $\delta_0 \models \delta_1 \models \ldots \models \delta_n$. Hence, given a set of premises Γ there will be *just one* $(\Gamma \cup \widetilde{\mathcal{T}})$-maxiconsistent subset of $\widetilde{\Delta}$, represented by a δ_i and every δ_j with $j \geq i$. However, since every such δ_j is classically implied by δ_i, we can associate to the set Γ just the default formula δ_i. Hence we can show that

Proposition 2. $\Gamma \mid\!\sim_{\langle \widetilde{\mathcal{T}}, \widetilde{\Delta} \rangle} D$ *iff* $\Gamma \cup \widetilde{\mathcal{T}} \cup \{\delta_i\} \models D$*, where* δ_i *is the first* $(\Gamma \cup \widetilde{\mathcal{T}})$*-consistent formula* [4] *of the sequence* $\langle \delta_0, \ldots, \delta_n \rangle$.

So, we have a simple method to decide defeasible consequence under rational closure. Given a defeasible knowledge base $\langle \mathcal{T}, \mathcal{B} \rangle$, certain facts Γ and a formula D,

1. Once for all, apply to $\langle \mathcal{T}, \mathcal{B} \rangle$ the transformations (2);
2. Given Γ, determine δ_i as the first $(\Gamma \cup \widetilde{\mathcal{T}})$-consistent formula of the sequence $\langle \delta_0, \ldots, \delta_n \rangle$.
3. Then decide if D follows under rational closure from Γ w.r.t. $\langle \mathcal{T}, \mathcal{B} \rangle$ by determining whether $\Gamma \cup \widetilde{\mathcal{T}} \cup \{\delta_i\} \models D$.

Furthermore, it is easily verified that all transformations (2) require at most $\mathcal{O}(|\mathcal{K}|)$ entailment tests and, thus, by Proposition 2,

Corollary 1. *Deciding defeasible consequence under rational closure is coNP-complete.*

Hence, the computational complexity does not increase w.r.t. classical entailment.

Let us illustrate the method with the following simple example.

[4] That is, $\widetilde{\mathcal{T}} \cup \Gamma \not\models \neg\delta_i$.

Example 3. Consider Example 1. By **Step 1** we transform $\mathcal{K}$ in $\mathcal{B}' = \{P \wedge \neg B \mid\!\sim \bot, P \mid\!\sim \neg F, B \mid\!\sim F\}$. By **Step 2**, the set of the materializations of $\mathcal{B}'$ is $\Gamma_{\mathcal{B}'} = \{P \wedge \neg B \rightarrow \bot, P \rightarrow \neg F, B \rightarrow F\}$, with $\mathfrak{A}_{\mathcal{B}'} = \{P \wedge \neg B, P, B\}$. By **Step 3**, we obtain the following exceptionality ranking over the sequents: $\mathcal{E}_0 = \{P \wedge \neg B \mid\!\sim \bot, P \mid\!\sim \neg F, B \mid\!\sim F\}$, $\mathcal{E}_1 = \{P \wedge \neg B \mid\!\sim \bot, P \mid\!\sim \neg F\}, \mathcal{E}_2 = \{P \wedge \neg B \mid\!\sim \bot\}$ and $\mathcal{E}_3 = \{P \wedge \neg B \mid\!\sim \bot\}$. So the ranking value of the sequents is: $r(B \mid\!\sim F) = 0, r(P \mid\!\sim \neg F) = 1$ and $r(P \wedge \neg B \mid\!\sim \bot) = \infty$. By **Step 4**, from such a ranking, we obtain a background theory $\widetilde{\mathcal{T}} = \{\neg(P \wedge \neg B)\}$ (hence, the background theory and the defeasible part of the knowledge base were already correctly separated in the original $\mathcal{K}$), and, by **Step 5**, a default-assumption set $\widetilde{\Delta} = \{\delta_0, \delta_1\}$, with $\delta_0 := (B \rightarrow F) \wedge (P \rightarrow \neg F)$ and $\delta_1 := P \rightarrow \neg F$, as in Example 2.

Now, to check if a flying creature presumably is not a penguin (*i.e.*, $F \mid\!\sim \neg P$), we take our premise F and our background theory $\widetilde{\mathcal{T}} = \{\neg(P \wedge \neg B)\}$, and we look for the first default δ_i that is consistent with F and $\widetilde{\mathcal{T}}$, *i.e.* $\widetilde{\mathcal{T}} \cup \{F\} \not\models \neg \delta_i$, that is δ_0. Now we have simply to check if $F \wedge \neg(P \wedge \neg B) \wedge (B \rightarrow F) \wedge (P \rightarrow \neg F) \models \neg P$. Since this holds, we have $F \mid\!\sim_{\langle \widetilde{\mathcal{T}}, \widetilde{\Delta} \rangle} \neg P$. Similarly, with such a procedure we can obtain a series of desirable results, as $\neg F \mid\!\sim \neg B$, $\neg F \mid\!\sim \neg P$, $B \mid\!\sim \neg P$, $\neg B \mid\!\sim \neg P$, $B \wedge P \mid\!\sim \neg F$, $B \wedge green \mid\!\sim F$, $P \wedge black \mid\!\sim \neg F$. Instead, other counterintuitive connections are not valid, such as $B \wedge \neg F \mid\!\sim P$, $B \wedge \neg F \mid\!\sim \neg P$, or $P \mid\!\sim F$. □

3 Rational Closure in DLs

We consider a significant DL representative, namely $\mathcal{ALC}$ (see e.g. [3], Chap. 2). $\mathcal{ALC}$ corresponds to a fragment of first order logic, using monadic predicates, called *concepts*, and diadic ones, called *roles*. In order to stress the parallel between the procedure presented in Section 2 and the proposal in $\mathcal{ALC}$, we are going to use the same notation for the components playing an analogous role in the two construction: we use $C, D, E, \ldots$ to indicate *concepts*, instead of propositions, and $\models$ and $\mid\!\sim$ to indicate, respectively, the 'classical' consequence relation of $\mathcal{ALC}$ and a non-monotonic consequence relation in $\mathcal{ALC}$. δ will indicate a *default concept*, that is, a concept that we assume as applying to every individual, if not informed of the contrary. We have a finite set of *concept names* $\mathcal{C}$, a finite set of *role names* $\mathcal{R}$ and the set $\mathcal{L}$ of $\mathcal{ALC}$ *-concepts* is defined inductively as follows: (i) $\mathcal{C} \subset \mathcal{L}$; (ii) $\top, \bot \in \mathcal{L}$; (iii) $C, D \in \mathcal{L} \Rightarrow C \sqcap D, C \sqcup D, \neg C \in \mathcal{L}$; and (iii) $C \in \mathcal{L}, R \in \mathcal{R} \Rightarrow \exists R.C, \forall R.C \in \mathcal{L}$. Concept $C \rightarrow D$ is used as a shortcut of $\neg C \sqcup D$. The symbols $\sqcap$ and $\sqcup$ correspond, respectively, to the conjunction $\wedge$ and the disjunction $\vee$ of classical logic. Given a set of *individuals* $\mathcal{O}$, an *assertion* is of the form $a{:}C$ $(C \in \mathcal{L})$ or of the form $(a, b){:}R$ $(R \in \mathcal{R})$, respectively indicating that the individual a is an instance of concept C, and that the individuals a and b are connected by the role R. A *general inclusion axiom* (GCI) is of the form $C \sqsubseteq D$ $(C, D \in \mathcal{L})$ and indicates that any instance of C is also an instance of D. We use $C = D$ as a shortcut of the pair of $C \sqsubseteq D$ and $D \sqsubseteq C$.

From a FOL point of view, concepts, roles, assertions and GCIs, may be seen as formulae obtained by the following transformation

$$
\begin{array}{ll}
\tau(a{:}C) = \tau(a, C) & \tau(x, C \sqcap D) = \tau(x, C) \wedge \tau(x, D) \\
\tau((a, b){:}R) = R(a, b) & \tau(x, C \sqcup D) = \tau(x, C) \vee \tau(x, D) \\
\tau(C \sqsubseteq D) = \forall x.\tau(x, C) \rightarrow \tau(x, D) & \tau(x, \exists R.C) = \exists y.R(x, y) \wedge \tau(y, C) \\
\tau(x, A) = A(x) & \tau(x, \forall R.C) = \forall y.R(x, y) \rightarrow \tau(y, C)\,. \\
\tau(x, \neg C) = \neg \tau(x, C) &
\end{array}
$$

Now, a classical knowledge base is defined by a pair $\mathcal{K} = \langle \mathcal{A}, \mathcal{T} \rangle$, where $\mathcal{T}$ is a finite set of GCIs (a *TBox*) and $\mathcal{A}$ is a finite set of assertions (the *ABox*), whereas a *defeasible knowledge base* is represented by a triple $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{B} \rangle$, where additionally $\mathcal{B}$ is a finite set of sequents of the form $C \mid\!\sim D$ ('an instance of a concept C is typically an instance of a concept D'), with $C, D \in \mathcal{L}$.

Example 4. Consider Example 3. Just add a role $Prey$ in the vocabulary, where a role instantiation $(a, b){:}Prey$ is read as 'a preys for b', and add also two more concepts, I (Insect) and Fi (Fish). A defeasible KB is $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{B} \rangle$ with $\mathcal{A} = \{a{:}P, b{:}B, (a, c){:}Prey, (b, c){:}Prey\}$; $\mathcal{T} = \{P \sqsubseteq B, I \sqsubseteq \neg Fi\}$ and $\mathcal{B} = \{P \mid\!\sim \neg F,\ B \mid\!\sim F, P \mid\!\sim \forall Prey.Fi, B \mid\!\sim \forall Prey.I\}$. □

The particular structure of a defeasible KB allows for the 'isolation' of the pair $\langle \mathcal{T}, \mathcal{B} \rangle$, that we could call the *conceptual system* of the agent, from the information about the individuals (formalised in $\mathcal{A}$) that will play the role of the facts known to be true. In the next section we are going to work with the information about concepts $\langle \mathcal{T}, \mathcal{B} \rangle$ first, exploiting the immediate analogy with the homonymous pair of Section 2, then we will address the case involving individuals as well.

Construction of the Default-Assumption System. We apply to $\langle \mathcal{T}, \mathcal{B} \rangle$ an analogous transformation (2), in order to obtain from $\langle \mathcal{T}, \mathcal{B} \rangle$ a pair $\langle \widetilde{\mathcal{T}}, \widetilde{\Delta} \rangle$, where $\widetilde{\mathcal{T}}$ is a set of GCIs, representing the background knowledge, and $\widetilde{\Delta}$ is a set of concepts, playing the role of default-assumptions, that is, concepts that, modulo consistency, apply to each individual. Hence, starting with $\langle \mathcal{T}, \mathcal{B} \rangle$, we apply the following steps.

Step 1. Define $\mathcal{B}' = \mathcal{B} \cup \{C \sqcap \neg D \mid\!\sim \bot \mid C \sqsubseteq D \in \mathcal{T}\}$. Now our agent is characterised by the pair $\langle \emptyset, \mathcal{B}' \rangle$.

Step 2. Define $\Gamma_{\mathcal{B}'} = \{\top \sqsubseteq C \rightarrow D \mid C \mid\!\sim D \in \mathcal{B}'\}$, and define a set $\mathfrak{A}_{\mathcal{B}'}$ as the set of the antecedents of the conditionals in $\mathcal{B}'$, *i.e.* $\mathfrak{A}_{\mathcal{B}'} = \{C \mid C \mid\!\sim D \in \mathcal{B}'\}$.

Step 3. We determine the exceptionality ranking of the sequents in $\mathcal{B}'$ using the set of the antecedents $\mathfrak{A}_{\mathcal{B}'}$ and the materializations in $\Gamma_{\mathcal{B}'}$, where a concept C is *exceptional* w.r.t. a set of sequents $\mathcal{D}$ iff $\Gamma_{\mathcal{D}} \models \top \sqsubseteq \neg C$. The steps are the same of the propositional case (**Steps 3.1 – 3.4**), we just replace the expression $\Gamma_{\mathcal{D}} \models \neg C$ with the expression $\Gamma_{\mathcal{D}} \models \top \sqsubseteq \neg C$. In this way we define a ranking function r.

Step 4. From $\Gamma_{\mathcal{B}'}$ and the ranking function r we obtain two kinds of information. First (**Step 4.1.**), we can verify if the conceptual system of the agent is consistent, by checking the consistency of $\Gamma_{\mathcal{B}'}$. Then (**Steps 4.2.-4.3.**), we can define the real background theory and the defeasible information of the agent, respectively the sets $\widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{B}}$ as:

$$\widetilde{\mathcal{T}} = \{\top \sqsubseteq \neg C \mid C \mid\!\sim D \in \mathcal{B}' \text{ and } r(C \mid\!\sim D) = \infty\}$$
$$\widetilde{\mathcal{B}} = \{C \mid\!\sim D \mid C \mid\!\sim D \in \mathcal{B}' \text{ and } r(C \mid\!\sim D) < \infty\}\ .$$

Step 5. Again, we define the set of our 'default assumptions' by using the materialisation of the sequents in $\widetilde{\mathcal{B}}$ and the ranking function r. That is, $\widetilde{\Delta} = \{\delta_0, \ldots, \delta_n\}$, where

$$\delta_i = \bigsqcap\{C \rightarrow D \mid C \mid\!\sim D \in \widetilde{\mathcal{B}} \text{ and } r(C \mid\!\sim D) \geq i\}\ .$$

Hence, we obtain an analogous of the default-assumption characterisation defined in the propositional case by substituting the conceptual system $\langle \mathcal{T}, \mathcal{B} \rangle$ with the pair $\langle \widetilde{\mathcal{T}}, \widetilde{\Delta} \rangle$, where $\widetilde{\Delta}$ is a set of concepts, instead of a set of sequents $C \mid\!\sim D$. It is not difficult to see that $\widetilde{\Delta}$ presents the same characteristics described at the end of Section 3, that is, for every δ_i, $0 \leq i < n$, $\models \delta_i \sqsubseteq \delta_{i+1}$.

Closure Operation over Concepts. Consider now $\widetilde{\mathcal{T}} = \{\top \sqsubseteq C_1, \ldots, \top \sqsubseteq C_m\}$ and $\widetilde{\Delta} = \{\delta_0, \ldots, \delta_n\}$. We call $\mathfrak{T}$ the set of the concepts in $\widetilde{\mathcal{T}}$, that is, $\mathfrak{T} = \{C_1, \ldots, C_m\}$. Next we define the notion of default-assumption consequence relation between the concepts, that is, a relation $\mid\!\sim_{\langle \widetilde{\mathcal{T}}, \widetilde{\Delta} \rangle}$ that tells us what presumably follows from a finite set of concepts. Formally, E is a *default-assumption consequence* of the set of concepts Γ, given a background theory $\mathcal{T}$ and a set of default-assumptions Δ, written $\Gamma \mid\!\sim_{\langle \mathcal{T}, \Delta \rangle} E$, if and only if E is implied by the union of Γ with $\mathfrak{T}$ and every $(\mathfrak{T} \cup \Gamma)$-maxiconsistent subset of Δ, *i.e.*

$$\Gamma \mid\!\sim_{\langle \mathcal{T}, \Delta \rangle} E \textit{ iff } \models (\mathfrak{T} \cup \Gamma \cup \Delta') \sqsubseteq E \textit{ for every } (\mathfrak{T} \cup \Gamma)\textit{-maxiconsistent } \Delta' \subseteq \Delta \ .$$

Given that, also in the DL case, every element δ_i of the default set Δ classically implies the subsequent elements (for every i, $0 \leq i < n$, $\models \delta_i \sqsubseteq \delta_{i+1}$), we obtain, in exactly the same way as in the propositional case, the analogous of Proposition 2 that every $\mid\!\sim_{\langle \mathcal{T}, \Delta \rangle}$-sequent is determined by a single element of Δ. That is:

Proposition 3. *$\Gamma \mid\!\sim_{\langle \widetilde{\mathcal{T}}, \widetilde{\Delta} \rangle} D$ iff $\models \bigsqcap \Gamma \sqcap \bigsqcap \mathfrak{T} \sqcap \delta_i \sqsubseteq D$, where δ_i is the first $(\Gamma \cup \mathfrak{T})$-consistent formula*[5] *of the sequence $\langle \delta_0, \ldots, \delta_n \rangle$.*

Hence, as in the propositional case, we have an unique default-assumption extension at the level of concepts. From now on, talking about a default set Δ, we assume that it is a linearly ordered set $\Delta = \{\delta_0, \ldots, \delta_n\}$ s.t. for every i, $0 \leq i < n$, $\models \delta_i \sqsubseteq \delta_{i+1}$.

Now, the main point is: if $\mid\!\sim_{\langle \widetilde{\mathcal{T}}, \widetilde{\Delta} \rangle}$ has been generated from $\mathcal{K} = \langle \mathcal{T}, \mathcal{B} \rangle$, then $\mid\!\sim_{\langle \widetilde{\mathcal{T}}, \widetilde{\Delta} \rangle}$ is a rational consequence relation validating $\mathcal{K}$ (*i.e.*, if $C \sqsubseteq E \in \mathcal{T}$, then $C \sqcap \neg E \mid\!\sim_{\langle \widetilde{\mathcal{T}}, \widetilde{\Delta} \rangle} \bot$, and if $C \mid\!\sim E \in \mathcal{B}$, then $C \mid\!\sim_{\langle \widetilde{\mathcal{T}}, \widetilde{\Delta} \rangle} E$).

Proposition 4. *$\mid\!\sim_{\langle \widetilde{\mathcal{T}}, \widetilde{\Delta} \rangle}$ is a rational consequence relation validating $\mathcal{K} = \langle \mathcal{T}, \mathcal{B} \rangle$.*

This can be shown by noting that the analogous properties of the propositional rational consequence relation are satisfied, namely:

$$\text{(REF)}\quad C \mid\!\sim_{\langle \mathcal{T}, \Delta \rangle} C$$

$$\text{(LLE)}\ \frac{C \mid\!\sim_{\langle \mathcal{T}, \Delta \rangle} E \qquad \models C = D}{D \mid\!\sim_{\langle \mathcal{T}, \Delta \rangle} E} \qquad \text{(RW)}\ \frac{C \mid\!\sim_{\langle \mathcal{T}, \Delta \rangle} D \qquad \models D \sqsubseteq E}{C \mid\!\sim_{\langle \mathcal{T}, \Delta \rangle} E}$$

$$\text{(CT)}\ \frac{C \sqcap D \mid\!\sim_{\langle \mathcal{T}, \Delta \rangle} E \qquad C \mid\!\sim_{\langle \mathcal{T}, \Delta \rangle} D}{C \mid\!\sim_{\langle \mathcal{T}, \Delta \rangle} E} \qquad \text{(CM)}\ \frac{C \mid\!\sim_{\langle \mathcal{T}, \Delta \rangle} E \qquad C \mid\!\sim_{\langle \mathcal{T}, \Delta \rangle} D}{C \sqcap D \mid\!\sim_{\langle \mathcal{T}, \Delta \rangle} E}$$

$$\text{(OR)}\ \frac{C \mid\!\sim_{\langle \mathcal{T}, \Delta \rangle} E \qquad D \mid\!\sim_{\langle \mathcal{T}, \Delta \rangle} E}{C \sqcup D \mid\!\sim_{\langle \mathcal{T}, \Delta \rangle} E} \qquad \text{(RM)}\ \frac{C \mid\!\sim_{\langle \mathcal{T}, \Delta \rangle} D \qquad C \not\mid\!\sim_{\langle \mathcal{T}, \Delta \rangle} \neg E}{C \sqcap E \mid\!\sim_{\langle \mathcal{T}, \Delta \rangle} D}$$

[5] That is, $\not\models \bigsqcap \mathfrak{T} \sqcap \bigsqcap \Gamma \sqsubseteq \neg \delta_i$.

Let us work out the analogue of Example 3 in the DL context.

Example 5. Consider the KB of Example 4 without the ABox. Hence, we start with $\mathcal{K} = \langle \mathcal{T}, \mathcal{B} \rangle$. Then $\mathcal{K}$ is changed into $\mathcal{B}' = \{P \sqcap \neg B \mid\!\sim \bot, I \sqcap Fi \mid\!\sim \bot, P \mid\!\sim \neg F, B \mid\!\sim F, P \mid\!\sim \forall Prey.Fi, B \mid\!\sim \forall Prey.I\}$. The set of the materializations of $\mathcal{B}'$ is $\Gamma_{\mathcal{B}'} = \{\top \sqsubseteq P \wedge \neg B \rightarrow \bot, \top \sqsubseteq I \sqcap Fi \rightarrow \bot, \top \sqsubseteq P \rightarrow \neg F, \top \sqsubseteq B \rightarrow F, \top \sqsubseteq P \rightarrow \forall Prey.Fi, \top \sqsubseteq B \rightarrow \forall Prey.I\}$, with $\mathfrak{A}_{\mathcal{B}'} = \{P \wedge \neg B, I \sqcap Fi, P, B\}$. Following the procedure at **Step 3**, we obtain the exceptionality ranking of the sequents: $\mathcal{E}_0 = \{P \sqcap \neg B \mid\!\sim \bot, I \sqcap Fi \mid\!\sim \bot, P \mid\!\sim \neg F, B \mid\!\sim F, P \mid\!\sim \forall Prey.Fi, B \mid\!\sim \forall Prey.I\}$; $\mathcal{E}_1 = \{P \sqcap \neg B \mid\!\sim \bot, I \sqcap Fi \mid\!\sim \bot, P \mid\!\sim \neg F, P \mid\!\sim \forall Prey.Fi\}$; $\mathcal{E}_2 = \{P \sqcap \neg B \mid\!\sim \bot, I \sqcap Fi \mid\!\sim \bot\}$ and $\mathcal{E}_3 = \{P \sqcap \neg B \mid\!\sim \bot, I \sqcap Fi \mid\!\sim \bot\}$. Automatically, we have the ranking values of every sequent in $\mathcal{B}'$: namely, $r(B \mid\!\sim F) = r(B \mid\!\sim \forall Prey.I) = 0$; $r(P \mid\!\sim \neg F) = r(P \mid\!\sim \forall Prey.Fi) = 1$ and $r(P \sqcap \neg B \mid\!\sim \bot) = r(I \sqcap Fi \mid\!\sim \bot) = \infty$. From such a ranking, we obtain a background theory $\widetilde{\mathcal{T}} = \{\top \sqsubseteq \neg(P \wedge \neg B), \top \sqsubseteq \neg(I \sqcap Fi)\}$, and a default-assumption set $\widetilde{\Delta} = \{\delta_0, \delta_1\}$, with

$$\delta_0 = (B \rightarrow F) \sqcap (B \rightarrow \forall Prey.I) \sqcap (P \rightarrow \neg F) \sqcap (P \rightarrow \forall Prey.Fi)$$
$$\delta_1 = (P \rightarrow \neg F) \sqcap (P \rightarrow \forall Prey.Fi)\ .$$

Now by using Proposition 3, we obtain the analogue sequents as in the propositional case, and avoid the same undesirable ones. Moreover we can derive also sequents connected to the roles, such as $B \mid\!\sim \forall Prey.\neg Fi$ and $P \mid\!\sim \forall Prey.\neg I$. □

We conclude by noting that from a computational complexity point of view, as deciding entailment in $\mathcal{ALC}$ is EXPTIME-complete [16] [6], we obtain immediately that

Corollary 2. *Deciding* $C \mid\!\sim_{\langle \widetilde{\mathcal{T}}, \widetilde{\Delta} \rangle} D$ *in* $\mathcal{ALC}$ *is an EXPTIME-complete problem.*

More generally, defeasible consequence under rational closure inherits the computational complexity of entailment of the underlying DL language and, thus, *e.g.* is polynomial for the DL $\mathcal{EL}$ [2].

Closure Operation over Individuals. So far, we left out the ABox that we will consider next. Given $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \Delta \rangle$, we would like to infer whether a certain individual a is presumably an instance of a concept C or not. The basic idea remains to associate to every individual every default-assumption information that is consistent with our knowledge base. As we will see, the major problem to be addressed here is to guarantee the uniqueness of the default-assumption extension.

Example 6. Consider $\mathcal{K} = \langle \mathcal{A}, \emptyset, \Delta \rangle$, with $\mathcal{A} = \{(a, b){:}R\}$ and $\Delta = \{A \sqcap \forall R.\neg A\}$. Informally, if we apply the default to a first, we get $b{:}\neg A$ and we cannot apply the default to b, while if we apply the default to b first, we get $b{:}A$ and we cannot apply the default to a. Hence, we may have *two* extensions. □

The possibility of multiple extensions is due to the presence of the roles, that allow the transmission of information from an individual to another; if every individual was 'isolated', without role-connections, then the addition of the default information to each

[6] Recall that for any deterministic complexity class $\mathcal{C}$, $\mathcal{C} = co\mathcal{C}$, so, *e.g.* $EXPTIME = coEXPTIME$.

individual would have been a ‘local’ problem, treatable without considering the concepts associated to the other individuals in the dominion, and the default-assumption extension would have been unique. On the other hand, while considering a specific individual, the presence of the roles forces to consider also the information associated to other individuals in order to maintain the consistency of the knowledge base, and, as show in example 6, the addition of default information to one individual could prevent the association of default information to another.

Now, first of all, we will assume that $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \Delta \rangle$ has already been transformed into $\langle \mathcal{A}, \widetilde{\mathcal{T}}, \widetilde{\Delta} \rangle$, where $\langle \widetilde{\mathcal{T}}, \widetilde{\Delta} \rangle$ have been computed as in the previous section and, thus, the defaults in $\widetilde{\Delta} = \{\delta_0, \dots, \delta_n\}$ are ordered ($0 \leq i < n$, $\models \delta_i \sqsubseteq \delta_{i+1}$). We also assume that $\langle \mathcal{A}, \mathcal{T} \rangle$ is consistent, *i.e.* $\langle \mathcal{A}, \mathcal{T} \rangle \not\models a{:}\bot$, for any a. For the sake of this paper, we will assume that $\mathcal{T}$ is *unfoldable*, that is defined as follows: (i) $\mathcal{T}$ contains axioms of the form $A \sqsubseteq C$ or $A = C$, where A is a concept name and C a concept; (ii) for any concept name A, there is at most one axiom having A on the left-hand side; (iii) $\mathcal{T}$ is *acyclic*, *i.e.* there is no concept name A that depends on A [7]. Besides having a high practical interest, unfoldable TBoxes have the characteristics that they can be removed in the following way: given $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \Delta \rangle$, (i) replace any inclusion axiom $A \sqsubseteq C \in \mathcal{T}$ with $A = C \sqcap A'$, where A' is a new concept name; (ii) in $\mathcal{A}$ and Δ, replace recursively any occurrence of concept names with their definition in $\mathcal{T}$; and (iii) remove $\mathcal{T}$ from $\mathcal{K}$. Hence, we may assume that $\mathcal{K}$ is of the form $\mathcal{K} = \langle \mathcal{A}, \Delta \rangle$. We may also assume that any concept in $\mathcal{A}$ is in *Negation Normal Form*, that is, a negation may occur in front of a concept name only (this is achieved in the usual way by removing double negations and pushing negation inwards [8]). Without loss of generality, we will further assume that $\mathcal{A}$ is closed under the following ‘completion’ rules: (i) if $a{:}C \sqcap D \in \mathcal{A}$ then both $a{:}C$ and $a{:}D$ are in $\mathcal{A}$; (ii) if $a{:}\exists R.C \in \mathcal{A}$ then there are $(a, b){:}R$ and $b{:}C$ in $\mathcal{A}$; and (iii) if $a{:}\forall R.C$ and $(a, b){:}R$ are in $\mathcal{A}$ then so is $b{:}C$. In this way, $\mathcal{A}$ contains all the information that is shared among all models of $\mathcal{A}$. Now, with $\mathcal{O}_\mathcal{A}$ we indicate the individuals occurring in $\mathcal{A}$. Given $\mathcal{K} = \langle \mathcal{A}, \Delta \rangle$ (recall that $\Delta = \{\delta_0, \dots, \delta_n\}$), we say that a knowledge base $\widetilde{\mathcal{K}} = \langle \mathcal{A}_\Delta \rangle$ is a *default-assumption extension* of $\mathcal{K}$ iff

- $\widetilde{\mathcal{K}}$ is classically consistent and $\mathcal{A} \subseteq \mathcal{A}_\Delta$.
- For any $a \in \mathcal{O}_\mathcal{A}$, $a{:}C \in \mathcal{A}_\Delta \setminus \mathcal{A}$ iff $C = \delta_i$ for some i and for every δ_h, $h < i$, $\mathcal{A}_\Delta \cup \{a{:}\delta_h\} \models \bot$.
- There is no $\mathcal{K}' \supset \widetilde{\mathcal{K}}$ satisfying these conditions.

Essentially, we assign to any individual $a \in \mathcal{O}_\mathcal{A}$, the strongest default to it.

Example 7. Referring to Example 6, consider $\mathcal{K} = \langle \mathcal{A}, \Delta \rangle$, with $\mathcal{A} = \{(a, b) : R\}$ and $\Delta = \{A \sqcap \forall R.\neg A, \top\}$. Then we have two default-assumption extensions, namely $\widetilde{\mathcal{K}}_1 = \mathcal{A} \cup \{a{:}A, a{:}\forall R.\neg A, b{:}\top\}$ and $\widetilde{\mathcal{K}}_2 = \mathcal{A} \cup \{b{:}A, b{:}\forall R.\neg A, a{:}\top\}$. □

A simple procedure to obtain extensions is as follows:

[7] *A depends directly on B* iff there is an axiom in $\mathcal{T}$ having A in the left-hand side and B in the right-hand side. The relation *depends on* is defined as the transitive closure of the relation *depends directly on*.

[8] Note that $\neg\forall R.C$ is the same as $\exists R.\neg C$.

1. fix a linear order $s = \langle a_1, \ldots, a_m \rangle$ of the individuals in $\mathcal{O}_{\mathcal{A}}$;
2. for any individual a_j processed in this order, consider the first default δ_i such that $\mathcal{A} \cup \{a_j{:}\delta_i\}$ is consistent;
3. update $\mathcal{A}$ by adding $a{:}\delta_i$ to it and process the next individual.

It can be shown that

Proposition 5. *Given a linear order of the individuals in $\mathcal{K}$, the above procedure determines a default-assumption extension of $\mathcal{K}$. Vice-versa, every default-assumption extension of $\mathcal{K}$ corresponds to the knowledge base generated by some linear order of the individuals in $\mathcal{K}$.*

For instance, related to Example 7, $\widetilde{\mathcal{K}}_1$ is obtained from the order $\langle a, b \rangle$, while $\widetilde{\mathcal{K}}_2$ is obtained from the order $\langle b, a \rangle$.

Example 8. Refer to Example 4 and 5, and let $\mathcal{K} = \{\mathcal{A}, \mathcal{T}, \Delta\}$, where $\mathcal{A} = \{a{:}P$, $b{:}B$, $(a,c){:}Prey$, $(b,c){:}Prey\}$, $\mathcal{T} = \{P = B \sqcap B', I = \neg Fi \sqcap I'\}$, $\Delta = \{\delta_0, \delta_1\}$, $\delta_0 = (B \rightarrow F) \sqcap (B \rightarrow \forall Prey.I) \sqcap (P \rightarrow \neg F) \sqcap (P \rightarrow \forall Prey.Fi)$ and $\delta_1 = (P \rightarrow \neg F) \sqcap (P \rightarrow \forall Prey.Fi)$. After expanding the TBox and 'applying' the completion rules to $\mathcal{A}$, we get $\mathcal{K} = \{\mathcal{A}, \Delta\}$, where $\mathcal{A} = \{a{:}B \sqcap B', a{:}B, a{:}B', b{:}B,$ $(a,c){:}Prey, (b,d){:}Prey\}$, $\Delta = \{\delta_0, \delta_1\}$, $\delta_0 = (B \rightarrow F) \sqcap (B \rightarrow \forall Prey.(\neg Fi \sqcap I')) \sqcap ((B \sqcap B') \rightarrow \neg F) \sqcap ((B \sqcap B') \rightarrow \forall Prey.Fi)$ and $\delta_1 = ((B \sqcap B') \rightarrow \neg F) \sqcap ((B \sqcap B') \rightarrow \forall Prey.Fi)$. If we consider an order where a is considered before b then we associate δ_1 to a, and consequently c is presumed to be a fish and we are prevented in the association of δ_0 to b. If we consider b before a, c is not a fish and we cannot apply δ_1 to a. □

Now, if we fix a priori a linear order s on the individuals, we may define a consequence relation depending on the default-assumption extension generated from it: we say that $a{:}C$ is a *defeasible consequence* of $\mathcal{K}$, written $\mathcal{K} \Vdash_s a{:}C$, iff $\widetilde{\mathcal{K}} \models a{:}C$, where $\widetilde{\mathcal{K}}$ is the default-assumption extension generated from $\mathcal{K}$ based on the order s.

For instance, related to Example 7 and order $s_1 = \langle a, b \rangle$, we may infer that $\mathcal{K} \Vdash_{s_1} a{:}A$, while with order $s_2 = \langle b, a \rangle$, we may infer that $\mathcal{K} \Vdash_{s_2} b{:}A$.

The interesting point of such a consequence relation is that it satisfies the properties of a *rational* consequence relation in the following way.

REF_{DL} $\langle \mathcal{A}, \Delta \rangle \Vdash_s a{:}C$ for every $a{:}C \in \mathcal{A}$

$$LLE_{DL} \quad \frac{\langle \mathcal{A} \cup \{b{:}D\}, \Delta \rangle \Vdash_s a{:}C \quad \models D = E}{\langle \mathcal{A} \cup \{b{:}E\}, \Delta \rangle \Vdash_s a{:}C}$$

$$RW_{DL} \quad \frac{\langle \mathcal{A}, \Delta \rangle \Vdash_s a{:}C \quad \models C \sqsubseteq D}{\langle \mathcal{A}, \Delta \rangle \Vdash_s a{:}D}$$

$$CT_{DL} \quad \frac{\langle \mathcal{A} \cup \{b{:}D\}, \Delta \rangle \Vdash_s a{:}C \quad \langle \mathcal{A}, \Delta \rangle \Vdash_s b{:}D}{\langle \mathcal{A}, \Delta \rangle \Vdash_s a{:}C}$$

$$CM_{DL} \quad \frac{\langle \mathcal{A}, \Delta \rangle \Vdash_s a{:}C \quad \langle \mathcal{A}, \Delta \rangle \Vdash_s b{:}D}{\langle \mathcal{A} \cup \{b{:}D\}, \Delta \rangle \Vdash_s a{:}C}$$

$$OR_{DL} \quad \frac{\langle \mathcal{A} \cup \{b{:}D\}, \Delta \rangle \Vdash_s a{:}C \quad \langle \mathcal{A} \cup \{b{:}E\}, \Delta \rangle \Vdash_s a{:}C}{\langle \mathcal{A} \cup \{b{:}D \sqcup E\}, \Delta \rangle \Vdash_s a{:}C}$$

$$RM_{DL} \quad \frac{\langle \mathcal{A}, \Delta \rangle \Vdash_s a{:}C \quad \langle \mathcal{A}, \Delta \rangle \nVdash_s b{:}\neg D}{\langle \mathcal{A} \cup \{b{:}D\}, \Delta \rangle \Vdash_s a{:}C}$$

We can show that

Proposition 6. *Given $\mathcal{K}$ and a linear order s of the individuals in $\mathcal{K}$, the consequence relation $\Vdash_s$ satisfies the properties* $\mathrm{REF_{DL}} - \mathrm{RM_{DL}}$.

Note that from a computational complexity point of view, as entailment w.r.t. a $\mathcal{ALC}$ ABox is PSPACE-complete, we get immediately

Proposition 7. *Deciding $\mathcal{K} \Vdash_s a{:}C$ in $\mathcal{ALC}$ with unfoldable TBox is a PSPACE-complete problem.*

We conclude by illustrating a case with a unique extension.

Example 9. Consider the KB in Example 8, where $(b,c){:}Prey$ is replaced with $(b,d){:}Prey$. Then, whatever is the order on the individuals, we obtain the following association between the default formulae and the individuals: $a{:}\delta_1$, $b{:}\delta_0$, $c{:}\delta_0$, and $d{:}\delta_0$. Using the information in these defaults, we obtain a unique default-assumption extension. □

The above example suggest yet another consequence relation, namely based on the intersection of every default-assumption extension: given $\mathcal{K}$, we say that $a{:}C$ is a *strict defeasible consequence* of $\mathcal{K}$, written $\mathcal{K} \Vdash a{:}C$, iff $\mathcal{K} \Vdash_s a{:}C$ for any linear order s of the individuals in $\mathcal{K}$. Note that $\Vdash$ may not necessarily satisfy the properties of a rational consequence relation. However, if the default-assumption extension is unique then $\Vdash$ satisfies the properties $\mathrm{REF}_{\mathrm{DL}} - \mathrm{RM}_{\mathrm{DL}}$.

4 Conclusions

We have presented a non-monotonic extension for the DL $\mathcal{ALC}$, focussing on the migration of the properties of a rational closure consequence relation at the propositional level towards the DL level. We provided first an algorithmic propositional definition that we then adapted to the DL case. In particular, we have defined a consequence relation $C \mid\!\sim_{\langle \tilde{\mathcal{T}}, \tilde{\Delta} \rangle} D$ among concepts and have shown that it is a rational consequence relation. We then defined a consequence relation $\mathcal{K} \Vdash_s a{:}C$ among an unfoldable KB and assertions that, under a given linear order s of the individuals in $\mathcal{K}$, is a rational consequence relation as well. Note that here s denotes a priority on the individuals on which to focus the attention, which is different from approaches such as [11, 23] in which the order indicates that an individual is more typical than another one. Interestingly, both consequence relations we have defined additionally inherit the same computational complexity of the underlying DL language.

Besides trying to extend our method to more expressive DL languages, we conjecture the validity, as in the propositional case (see [28,19]), of a representation result connecting rational consequence relations, default assumptions consequence relations (with Δ linearly ordered by $\models$ as above) and semantical models with a modular (*i.e.* reflexive, transitive and complete) typicality relation defined over the individuals.

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