On Łukasiewicz' Infinie-Valued Logic and Fuzzy_L

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Abstract. An algebra of Łukasiewicz' logic $\text{Ł}_\aleph$ is considered without the extra operations $P \odot Q = \max\{0, P+Q-1\}$, the *restricted difference*, and $P \oplus Q = \min\{1, P+Q\}$, the *restricted sum*, later named by "Łukasiewicz t-norm" and "Łukasiewicz t-conorm", respectively. The logic $\text{Ł}_\aleph$ is compared with the fuzzy logic Fuzzy_L with the set of truth-values $[0,1]$. It appears that Fuzzy_L is a Łukasiewicz-like many-valued logic with a dual pair of modifiers (or fuzzifiers). Otherwise, the only reasonable version of Fuzzy_L is $\text{Ł}_\aleph$, without the additional connectives corresponding to the "Łukasiewicz norms" which are not original operations of Łukasiewicz' logics.

Keywords: Lukasiewicz Many-valued Logic, Fuzzy Logic.

1 Introduction

The author would like to respect Łukasiewicz's work in many-valued logic by giving a comment on the straight connection between Łukasiewicz implication and original disjunction. The comment says that there is a natural and easy formal connection between Łukasiewicz disjunction max and implication. In this paper, it is done by using an algebraic approach being a counter currency case, rather than the well known mainstream proceeding mainly developed for studying formal fuzzy logic on the base of residuated lattices, BL-algebras, MV-algebras, etc. (*cf.* a comprehensive presentation about these tools in Hájek's book [2]). In the mainstream proceeding, Łukasiewicz implication is developed by the t-conorm $\oplus$ to be an S-implication. Hence, this is a circuitous route from a *different disjunction* operation to Łukasiewicz implication. The circuitous route is completed by the case that the *different disjunction* is $\oplus$, and min operation can be expressed by means of $\oplus$ and $\odot$ (*cf.* Bergmann's deduction (4) below), and the dual of min is max. Hence, max operation can be expressed by $\oplus$ and $\odot$ by means of duality of min and max (*cf.* (6) below).

Some confusion or miss-understanding about the role of *restricted difference* $\odot$ and *restricted sum* $\oplus$ have appeared in some papers. The difference between the connectives disjunction and conjunction of Łukasiewicz many-valued logic and these additional connectives has not been very clear. This is possible if we do not draw a strict border between the connectives of original Łukasiewicz logic

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and the additional connectives. In late 80's the author has seen some few papers considering fuzzy control where introducing Łukasiewicz many-valued logic, authors have asserted that $\odot$ and $\oplus$ are the original Łukasiewicz conjunction and disjunction, respectively. Maybe, the reason for this is that the norms $\odot$ and $\oplus$ were renamed as Łukasiewicz's t-norm and Łukasiewicz's t-conorm, respectively. Actually, the right thing would be that min and max operations are the original Łukasiewicz' t-norm and t-conorm. In Łukasiewicz' logic, the operations min and max already have had a similar role before as in fuzzy logic nowadays.

Anyway, the system $([0,1], \odot, \oplus, ', 0, 1)$ is a many-valued logic with the negation $p' \equiv 1 - p$ for all $p \in [0,1]$, and where the implication operation can be created to be an S-implication of the system, but this system is not a lattice if we do not create supremum and infimum operations. We consider this more in Conclusion.

1.1 A Starting Point to Łukasiewicz' Many-Valued Logic

We begin with Łukasiewicz' many-valued logic $\text{Ł}_{\aleph_1}$ having the closed unit interval $[0,1]$ as the set of truth values.[1]

As we know, Łukasiewicz chose the connectives of *negation* and *implication* as primitives. Let v be any valuation of $\text{Ł}_{\aleph}$, then the truth value evaluation rules for negation and implication are

$$v(\neg A) = 1 - v(A) \tag{Neg.}$$
$$v(A \to B) = \min\{1, 1 - v(A) + v(B)\} \tag{Impl.}$$

By means of these connectives, Łukasiewicz defined the other connectives by the rules

$$A \vee B \overset{\text{def}}{\Longleftrightarrow} (A \to B) \to B \tag{Disj.}$$
$$A \wedge B \overset{\text{def}}{\Longleftrightarrow} \neg(\neg A \vee \neg B) \tag{Conj.}$$
$$A \leftrightarrow B \overset{\text{def}}{\Longleftrightarrow} (A \to B) \wedge (B \to A) \tag{Eq.}$$

The truth value evaluation rules for these derived connectives are

$$\max\{v(A), v(B)\} \qquad \text{for } A \vee B, \tag{1}$$
$$\min\{v(A), v(B)\} \qquad \text{for } A \wedge B, \tag{2}$$
$$1 - |v(A) - v(B)| \qquad \text{for } A \leftrightarrow B \tag{3}$$

for any valuation v of $\text{Ł}_{\aleph_1}$.

1.2 A Starting Point of Fuzzy Many-Valued Logic Fuzzy_{L}

The truth evaluation rules and the idea for Fuzzy_{L} is taken from M. Bergmann [1]. She says:

[1] *Cf.* Rescher [9], p.36, and 337.

> "To specify a full fuzzy propositional logic, we begin with an assignment v of fuzzy truth values, between 0 and 1 inclusive, to the atomic formulas of the language. We call the set of real numbers between 0 and 1 inclusive the unit interval lattice $(\mathbb{I}, \leq)$. So we may say that for each atomic formula P, $v(P)$ is a member of $\mathbb{I}$, or in more concise notation, $v(P) \in [0,1]$. We can then use the same numeric clauses that we presented for Łukasiewicz 3-valued logic to obtain the Łukasiewicz fuzzy system Fuzzy_L:
>
> 1. $v(\neg P) = 1 - v(P)$
> 2. $v(P \wedge Q) = \min\{v(P), v(Q)\}$
> 3. $v(P \vee Q) = \max\{v(P), v(Q)\}$
> 4. $v(P \to Q) = \min\{1, 1 - v(P) + v(Q)\}$
> 5. $v(P \leftrightarrow Q) = \min\{1, 1 - v(P) + v(Q), 1 - v(Q) + v(P)\} = 1 - |v(P) - v(Q)|$
> 6. $v(P \vartriangle Q) = \max\{0, v(P) + v(Q) - 1\}$
> 7. $v(P \triangledown Q) = \min\{1, v(P) + v(Q)\}$
>
> Some of these connectives can be defined using the other connectives, as we did in Ł3. For example, ... we noted that $P \wedge Q$ is definable as $P \vartriangle (\neg P \triangledown Q)$ in Ł3, and
>
> $$\begin{aligned} v(P \vartriangle (\neg P \triangledown Q)) &= \max\{0, v(P) + \min(1, 1 - v(P) + v(Q)) - 1\} \\ &= \max\{0, \min(v(P) + 1 - 1, v(P) + 1 - v(P) + \\ &\quad v(Q) - 1)\} \\ &= \max\{0, \min(v(P), v(Q)\} = \min\{v(P), v(Q)\}.\text{"} \end{aligned} \quad (4)$$

By means of duality applied to Bergmann's result,

$$\min\{v(P), v(Q)\} = v(P \vartriangle (\neg P \triangledown Q)), \quad (5)$$

max operation can be expressed by means of $\vartriangle$ and $\triangledown$, too, i.e.,

$$\max\{v(P), v(Q)\} = 1 - [(1 - v(P)) \vartriangle (v(P) \triangledown (1 - v(Q)))]. \quad (6)$$

This can be verified by the definitions of $\vartriangle$ and $\triangledown$ given above.

Usually, the symbolization of $\vartriangle$ and $\triangledown$ used in the literature in the truth evaluation rules 6 and 7 are $\odot$ and $\oplus$, respectively. We now return to use this notation again.

If we have a fuzzy many-valued propositional logic, say, Fuzzy_L, where the truth evaluation rules are those from 1 to 7, how we must interpret the connectives $\vartriangle$ and $\triangledown$, or correspondingly, $\odot$ and $\oplus$, *bounded difference* and *bounded sum*, especially when we consider inferences (or arguments) in natural language? A t-norm refers to conjunction and a t-conorm to disjunction. But we have already conjunction and disjunction in Fuzzy_L, namely min and max. And, in general, we can manage with the connectives whose evaluation rules are 1 - 5 in the same way as in $Ł_{\aleph_1}$. In practice, the formula (4) is not so big case that it would necessarily take sides with the use of $\odot$ and $\oplus$ in the set of connectives in such logics like Fuzzy_L.

It seems that the additional connective $\oplus$ cannot be interpreted as *exclusive or*, i.e., as $P \underline{\vee} Q$ which can be expressed by $\vee$, $\wedge$, and $\neg$ as $(P \wedge \neg Q) \vee (\neg P \wedge Q)$. 'Exclusive or' is more crisp than 'inclusive or'. On the other hand, $\oplus$ is more "weakening" in its interpretation because, as t-conorms, the value $v(P \oplus Q)$ is equal to or bigger than that of $v(P \vee Q)$ for any valuation v in Fuzzy_L. This means that $\oplus$ can be used as a basis for constructing a certain *weakening* modifier. Similarly, as t-conorms, $v(P \odot Q) \leq v(P \wedge Q)$. Hence, $\odot$ can be used as a basis for constructing a certain *substantiating* modifier (*cf.*, for example, Mattila [5]). Hence, the interpretations in natural language for these connectives may be something like *definitely P and Q* for $P \odot Q$ and *something like P or Q* for $P \oplus Q$, because *definitely* can be understood as substantiating and *something like* a weakening expression. Of course, there may be alternative ways to translate $P \odot Q$ and $P \oplus Q$ into natural language, but the idea is that the translations are substantiating and weakening, respectively.

These kind of modifying expressions can be used as *fuzzifiers* (*cf.* Mattila [3], [4], [7]), and hence Fuzzy_L can be considered as a many-valued fuzzy logic with one dual pair of fuzzifiers.

2 An Alternative Algebraic Approach to $\text{Ł}_{\aleph}$

The attribute *alternative* means here that the approach we consider is alternative to the mainstream approach mentioned above. And here we do not need any additional tools like different t-conorms. A set of essential algebras involved in many-valued logics are considered e.g. in Rasiowa's book [8].

If we want to consider logics based on Łukasiewicz logic without residuated lattices, BL- and MV-algebras, etc., we may proceed as follows.

Consider an algebra

$$L_{\mathbb{I}} = ([0,1], \wedge, \vee, ', 0, 1) \tag{7}$$

Suppose that the algebra $L_{\mathbb{I}}$ has at least the following properties:

- $L_{\mathbb{I}}$ is a DeMorgan algebra.
- The binary operations *meet* $\wedge$ and *join* $\vee$ are commutative and associative.
- The operations $\wedge$ and $\vee$ are distributive to each others.
- The unary operation $'$ is a complementarity operation with the property of involution.

We may need a short analysis of the universe of discourse $[0,1]$ of the algebra $L_{\mathbb{I}}$. Because $[0,1]$ is a subset of the set of real numbers $\mathbb{R}$, $[0,1]$ has all the arithmetical properties as $\mathbb{R}$ has. Hence, $[0,1]$ is a metric space with the natural metric *distance* between any two points of $[0,1]$, i.e., the distance is

$$d(x,y) = |x - y|, \quad x, y \in [0,1]. \tag{8}$$

This formula (8) satisfies the general definition of the concept *metric*. We need it in the following consideration where we manipulate expressions involving maxima and minima.

In manipulating maxima and minima, the consideration can sometimes be done easier by using the following expressions for max and min operations:

$$\max\{x, y\} = \frac{x + y + |x - y|}{2}, \qquad \min\{x, y\} = \frac{x + y - |x - y|}{2} \tag{9}$$

First, consider the case where the operations of the algebra are min, max, and $1 - \cdot$.

Proposition 1. *If the operations are chosen such that for any* $x, y \in [0, 1]$, $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ *and* $x' = 1 - x$, *then the resulting algebra*

$$([0, 1], \min, \max, 1 - \cdot, 0, 1) \tag{10}$$

satisfies the conditions of the algebra (7).

Proof. Consider the operation $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$, where x and y are variables taking their values from the interval $[0, 1]$. Using the arithmetical formula for min operation, we have

$$\begin{aligned}
\min\{x, y\} &= \frac{x + y - \mid x - y \mid}{2} \\
&= \frac{2 - 1 - 1 + x + y - \mid 1 - 1 + x - y \mid}{2} \\
&= 1 - \frac{1 + 1 - x - y + \mid 1 - 1 + x - y \mid}{2} \\
&= 1 - \frac{(1 - x) + (1 - y) + \mid (1 - y) - (1 - x) \mid}{2} \\
&= 1 - \max\{\, (1 - x), (1 - y) \,\}.
\end{aligned} \tag{11}$$

From the formula (11), by replacing x by $1 - x$ and y by $1 - y$, and then solving $\max(x, y)$, the following formula follows:

$$\max\{x, y\} = 1 - \min\{\, 1 - x, 1 - y\}. \tag{12}$$

The formulas (11) and (12) show that max and min are dual of each other which is already a well known fact. Hence, DeMorgan laws hold in the algebra (10). Hence, (10) is a DeMorgan algebra.

It is well-known that the operations max and min are commutative and associative, and they are distributive to each others, and $x' = 1 - x$ for all $x \in [0, 1]$ is a complementarity operation with the property of involution. This completes the proof.

Łukasiewicz knew that the operations max and min are dual of each other. Actually, this property is easily found in the classical special case, i.e. using characteristic functions in presenting crisp sets. But the general proof for this is easily done by using the expressions (9) for max and min in such cases where in the universe of discourse a distance mertic is defined. This always holds at least for real numbers.

Second, consider the connection between max and Łukasiewicz implication (similar considerations are done in Mattila [6], but the following proposition 2 is not proved).

Proposition 2. *For all $x, y \in [0, 1]$,*

$$\max(x, y) = (x \underset{L}{\rightarrow} y) \underset{L}{\rightarrow} y, \tag{13}$$

where $x \underset{L}{\rightarrow} y$ is Łukasiewicz implication.

Proof. Consider disjunction operation $x \vee y = \max(x, y)$. Because $0 \leq x, y \leq 1$, using the arithmetical formula (9) for max, we have

$$\begin{aligned}
\max(x, y) &= \min\{\,1, \max(x, y)\,\} \\
&= \min\{\,1, \frac{x + y + \mid x - y \mid}{2}\,\} \\
&= \min\left\{\,1, \frac{2 - 1 - 1 + x + 2y - y + \mid 1 - 1 + x - y \mid}{2}\,\right\} \\
&= \min\left\{\,1, 1 - \frac{1 + (1 - x + y) - \mid 1 - (1 - x + y) \mid}{2} + y\,\right\} \\
&= \min\{\,1, 1 - \min(1, 1 - x + y) + y\,\}
\end{aligned}$$

By (Disj.) we know that

$$\max(x, y) = (x \underset{L}{\rightarrow} y) \underset{L}{\rightarrow} y \tag{14}$$

We find two similar min-structures in the formula

$$\min\{\,1, 1 - \min(1, 1 - x + y) + y\,\}$$

where one of the min-stuctures is inside part of the whole formula. If we denote the inner min-structure $\min(1, 1 - x + y)$ by z then the outer min-structure is $\min(1, 1 - z + y)$, i.e., the min-structures are really the same. The implication operations in (14) are situated in the same way. Hence, $\min(1, 1 - x + y)$ must be $x \underset{L}{\rightarrow} y$, by (Disj.). And the expression $\min(1, 1 - x + y)$ is similar to the truth value evaluation rule of Łukasiewicz implication (Impl.).

Of course, Łukasiewicz must have known the connection between maximum operation and his truth evaluation formula (Impl.) of the implication because without any knowledge about this, he would have not been sure that everything fits well together in his logic. But how he has inferred this is not known.

The result of the proof of the formula (13) shows that from the join operation max of our algebra we deduce a formula that expresses the rule of Łukasiewicz' implication, and this formula is the truth value evaluation rule in $Ł_{\aleph_1}$. Hence, we have shown that from our algebra (10) it is possible to derive similar rules as the truth value evaluation rules in $Ł_{\aleph_1}$.

Hence, the author's comment can be given in a more formal way: If the cases

1. $x' = 1 - x$;
2. $x \vee y = \max(x, y)$;

hold, then the other cases

3. $x \wedge y = \min(x, y)$;
4. $x \to y = \min(1, 1 - x + y)$;
5. $x \leftrightarrow y = (x \to y) \wedge (y \to x) = 1 - \mid x - y \mid$,

can be derived in the DeMorgan algebra $L_{\mathbb{I}} = ([0,1], \wedge, \vee, ', 0, 1)$.

The case 3 follows from the case 2 by duality and the case 4 follows from the case 2 by Prop. 2. The case 5 is deduced as follows:

$$
\begin{aligned}
x \leftrightarrow y &= \min\{(x \to y), (y \to x)\} \\
&= \frac{(x \to y) + (y \to x) - |(x \to y) - (y \to x)|}{2} \\
&= \frac{\min(1, 1 - x + y) + \min(1, 1 - y + x)}{2} \\
&\quad - \frac{\min(1, 1 - x + y) - \min(1, 1 - y + x)}{2} \\
&= \frac{4 - 2\,|x - y| - |-2x + 2y - |x - y| + |x - y||}{4} \\
&= \frac{4 - 4\,|x - y|}{4} = 1 - |x - y|\,.
\end{aligned}
$$

These cases are similar to the truth value evaluation rules for connected formulas in $Ł_{\aleph_1}$. Hence, if we want to use algebraic approach for $Ł_{\aleph_1}$ we need not necessarily to follow the mainstream using additional operations for studying the connections between the connectives in $Ł_{\aleph_1}$.

3 Conclusion

We return back to the logic Fuzzy_L. The primitive connectives are min as conjunction, max as disjunction, and $1 - \cdot$ as negation. If we want to avoid interpretational confusions about the role of the additional connectives $\odot$ and $\oplus$ we may reject them. Hence, the logics Fuzzy_L and $Ł_{\aleph_1}$ would be identical. Hence, one may have a feeling that the dual operation pair $\odot$ and $\oplus$ are in the system only for getting Łukasiewicz implication very easily to the system as an S-implication, by means of $\oplus$.

On the other hand, consider the system $([0,1], \odot, \oplus, ', 0, 1)$ where $\odot$ and $\oplus$ are the primitive binary connectives, even though they are not lattice operations on $[0,1]$. Hence, we can apply the formulas (4) or (6) in order to construct the join operation $\wedge$ and meet operation $\vee$, which are min and max, respectively. Now we have the case where $\odot$ and $\oplus$ are conjunction and disjunction, respectively. Now we have an interpretation problem with min and max. They are only additional operations in order to make the system to be a lattice $([0,1], \odot, \oplus, \min, \max, ', 0, 1)$ where the infimum operation $\wedge$ is min and the supremum operation $\vee$ is max. In any case, the implication operation can be constructed by means of the *primitive* connective $\oplus$ as an S-implication.

In both cases, it immediately seems that we have the same logic

$$([0,1], \odot, \oplus, \min, \max, ', 0, 1) \quad \text{or} \quad ([0,1], \min, \max, \odot, \oplus, ', 0, 1), \tag{15}$$

but the binary primitive connectives are different, and further, they have the same interpretation in natural language. If we want to have a many-valued fuzzy logic with the set of truth values $[0,1]$ being as close to $\mathrm{Ł}_{\aleph_1}$ as possible, we may choose this Łukasiewicz logic $\mathrm{Ł}_{\aleph_1}$ instead of $\mathrm{Fuzzy}_{\mathrm{L}}$. Hence, we have no interpretation problems.

Another alternative case would be such that in the algebraic approach, the operations max and min belog to the primitives (because they have no modifying effects), and $\oplus$ and $\odot$ are some dual modifier operations, because the t-norm $\odot$ has some substantiating properties and t-conorm $\oplus$ some weakening properties. So, we would have a *Łukasiewicz-like many-valued modifier logic*. On t-norm-based modifiers, see Mattila [5] and [6].

In general, a totally different problem is whether $\mathrm{Ł}_{\aleph_1}$ can be considered to be a fuzzy logic or not. Or, is some fuzzy-valued logic better?

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