Approximation Algorithms for Intersection Graphs

Frank Kammer, Torsten Tholey, and Heiko Voepel

Institut für Informatik, Universität Augsburg, D-86135 Augsburg, Germany
{kammer,tholey,voepel}@informatik.uni-augsburg.de

Abstract. We study three complexity parameters that in some sense measure how chordal-like a graph is. The similarity to chordal graphs is used to construct simple polynomial-time approximation algorithms with constant approximation ratio for many $\mathcal{NP}$-hard problems, when restricted to graphs for which at least one of the three complexity parameters is bounded by a constant. As applications we present approximation algorithms with constant approximation ratio for maximum weighted independent set, minimum (independent) dominating set, minimum vertex coloring, maximum weighted clique, and minimum clique partition for large classes of intersection graphs.

1 Introduction

Complexity parameters can help to solve many NP-hard problems of theoretical or practical importance on a subclass of instances for which the chosen parameter is very small. Treewidth is one of the classical complexity parameters studied in graph theory. Graphs of bounded treewidth have a tree-like structure that allows a generalization of efficient algorithms for hard problems on trees to graphs of bounded treewidth. In particular, all decision problems that can be expressed in monadic second-order logic can be solved in polynomial time on graphs of bounded treewidth [3,8].We study three complexity parameters that all generalize in some kind another class of graphs, namely chordal graphs. One of them is new, whereas the others also appear in [34] and [25], but were not analyzed in detail in these papers. See Section 2 for a detailed definition of the complexity parameters. Like trees, chordal graphs have a simple structure that facilitates the solution of a large number of NP-hard problems. For example, there are linear time algorithms on chordal graphs for maximum clique (MC), for minimum clique partition (MCP) [15], for maximum weighted independent set (MWIS) [13], and for minimum vertex coloring (MVC).Thus, it seems natural to search for a generalization of chordal graphs. In doing so, we obtain new approximation algorithms for the problems above on big graph classes containing many intersection graph classes such as t-interval graphs, circular-arc graphs, (unit) disk graphs, and intersection graphs of regular polygons or of arbitrary polygons of so-called bounded fatness. In general, intersection graphs are useful subclasses of graphs with several practical applications. See [17] or [18] for an overview of applications on these graphs. It is not surprising that, for small

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graph classes such as unit disk graphs, one can achieve better results than by our new algorithms designed for bigger classes of graphs. Nevertheless, also on small graph classes such as disk graphs we obtain new results for some of the problems above as well as for minimum dominating set (MDS) and minimum independent dominating set (MIDS).

Table 1 summarizes the best previously known and new approximation results for the intersection graphs of disks, regular polygons, fat objects, t-intervals, and t-fat-objects. MIS denotes the unweighted version of MWIS and MWC the weighted version of MC. By an *r-regular polygon* we mean a polygon with r corners placed on a cycle such that all pairs of consecutive corners of the polygon have the same distance. We assume that $r \in O(1)$. We define a set $\mathcal{C}$ of geometric objects in $\mathbb{R}^d$—i.e., a set of points in $\mathbb{R}^d$—to be a set of *fat objects* if the following holds: First of all, let us call the radius of a smallest d-dimensional ball containing the closure of a geometric object S in $\mathbb{R}^d$ the *size* of S. Moreover, let R be the size of the largest object in $\mathcal{C}$. $\mathcal{C}$ is called *fat* if there is a constant c such that, for each d-dimensional ball B of radius r with $0 < r \leq R$, there exist c points (possibly also outside B) such that every B-intersecting object $S \in \mathcal{C}$ of size at least r contains at least one of the c points. We also say that $\mathcal{C}$ has *fatness* c. $\mathcal{C}$ is called a *(c-)restricted set of fat objects* if in the condition above every B-intersecting object in $\mathcal{C}$ (with arbitrary size) contains at least one of the c points. By a *unit* set of objects—in opposite to *arbitrary*—we mean that each object must be a copy of each other object, i.e., it has to be of the same size and shape. However, unit and arbitrary objects may be rotated and moved to any position. An intersection graph G of t-intervals is an intersection graph, where each vertex represents a *t-interval*, i.e., the union of t intervals taken from a set S of intervals. By the intersection graph G of t-fat-objects we mean an intersection graph, where each vertex represents a *t-fat-object*, i.e., the union of t objects taken from a fat set S of objects. In both cases S is the *universe* of G.

As usual, disks and regular polygons should be defined in the plane $\mathbb{R}^2$, intervals in $\mathbb{R}$ and fat objects in $\mathbb{R}^d$, where we assume that $d = O(1)$. Concerning the results in table 1 including the hardness results, we assume that—beside an intersection graph itself—a representation of the intersection graph is given. More precisely, for the intersection graph of a set S of (1) disks, (2) r-regular polygons, (3) t-intervals, (4) fat objects, or (5) t-fat objects, we are given for each element in S its radius and the coordinates of its center in case 1, the coordinates of the center and at least one corner in case 2, the start and end point of each interval in case 3. In case 4, we should be given a representation that, for each pair X, Y of objects, each point $p \in \mathbb{R}^d$, and each d-dimensional ball B represented by the coordinates of its center and its radius $r \leq R$, supports the following computations in polynomial time: Decide whether X and Y intersect, whether X and B intersect, and whether p is contained in X. Moreover, determine the size s of X as well as the center of the ball with a radius s containing the closure of X, and find c points that are contained in every object of size $\geq r$ intersecting B. In case 5, each t-fat-object has a representation of its objects as described in case 4. The representations described are given explicitly in many applications.

Table 1. Approximation results. We use PA. and NP-h. as abbreviation for polynomial-time approximation algorithm and NP-hard, respectively. By n we denote the number of vertices of the intersection graph. [$*$] denotes a new result shown in this paper.

	disk	r-reg. polygon	fat objects	t-interval	t-fat-objects
MIS	arbitrary: PTAS [6,10] unit: PTAS [22] NP-h. [12]	arbitrary: PTAS [6,10] unit: PTAS [22] NP-h. [12]	fatness c: PTAS [6,10] unit: NP-h. [12]	$2t$-PA. [2] $t \geq 3$: $\mathcal{APX}$-h. [21,31]	fatness c: $2tc$-PA. [$*$] NP-h. [12]
MWIS	arbitrary: PTAS [10]	arbitrary: PTAS [10]	fatness c: PTAS [10]	$2t$-PA. [2]	fatness c: $2tc$-PA. [$*$]
MDS	arbitrary: PTAS [16] unit: PTAS [22] NP-h. [7]	c-restricted: c-PA. [$*$] unit: PTAS [22] NP-h. [7]	c-restricted: c-PA. [$*$] unit: NP-h. [7]	t^2-PA. [4] $t \geq 2$: $\mathcal{APX}$-h. [21,31]	c-restricted: tc-PA. [$*$] $t \geq 2$: $\mathcal{APX}$-h. [21,31]
MIDS	c-restricted: c-PA. [$*$] unit: PTAS [23] NP-h. [7]	c-restricted: c-PA. [$*$] unit: PTAS [23] NP-h. [7]	c-restricted: c-PA. [$*$] unit: NP-h. [7]		c-restricted: tc-PA. [$*$] $t \geq 2$: NP-h. [7]
MVC	arbitrary: 5-PA. [19,28,29] unit: 3-PA. [29] 4/3-PA. is NP -h. [7,14,24]	arbitrary: $O(1)$-PA. [27,34] unit: $O(1)$-PA. [29] NP-h. [14,24]	fatness c: c-PA. [34] unit: NP-h. [14,24]	$2t$-PA. [2]	fatness c: $2tc$-PA. [$*$]
MC	arbitrary: 8-PA. [$*$] unit: $\in$ P [7]	arbitrary: $O(1)$-PA. [$*$]	fatness c: c-PA. [$*$]	$\frac{t^2-t+1}{2}$-PA. [4] $4t$-PA. [$*$] $t \geq 3$: NP-h. [4]	fatness c: $2tc$-PA. [$*$] $t \geq 3$: NP-h. [4]
MWC	arbitrary: 8-PA. [$*$]	arbitrary: $O(1)$-PA. [$*$]	fatness c: c-PA. [$*$]	$\frac{t^2-t+1}{2}$-PA. [4] $4t$-PA. [$*$]	fatness c: $2tc$-PA. [$*$]
MCP	arbitrary: 8-PA. [$*$] unit: 3-PA. [5] PTAS [9,32]	arbitrary: $O(1)$-PA. [$*$]	fatness c: c-PA. [$*$]	$O(\log^2 n/$ $\log(1+1/t))$- PA. [$*$]	c-restricted: tc-PA. [$*$]

Very related to the graph classes considered in this paper is the so-called class of *sequentially k-independent graphs* introduced by Akcoglu, Aspnes, Das-Gupta, and Kao [1] and studied more extensively in a recent paper by Ye and Borodin [34]. We omit an exact definition of this graph class, but want to remark that even though the results of Ye et al. and our results are achieved completely independently, there are similarities between the papers. This indicates that our generalizations of chordal graphs are quite natural, but surprisingly have not been studied more extensively before. Other generalized classes of graphs including the intersection graphs of unit disks or r-regular polygons of unit size are graph classes of so-called *polynomially bounded growth* studied by Nieberg, Hurink and Kern [23,30]. Nieberg et al. presented a PTAS for MWIS, MDS and MIDS for these classes of graphs. However, graphs of polynomially bounded growth do not include the intersection graphs of arbitrary disks, arbitrary r-regular polygons, t-interval graphs, etc.

Our results include the first polynomial-time approximation algorithms with constant approximation ratio for maximum clique and minimum clique partition on disk graphs and on intersection graphs of r-regular polygons. We also present a polynomial-time approximation algorithm with constant approximation ratio for minimum dominating set on the intersection graphs of a restricted set of r-regular polygons. Recently, Erlebach and van Leeuwen [11] presented an approximation algorithm with constant approximation ratio for the same problem on an arbitrary set of r-regular polygons, however, they do not allow to rotate the polygons in contrast to this paper. Our results also imply an approximation algorithm with constant approximation ratio for minimum dominating set on intersection graphs of an arbitrary set of *non-rotated* r-regular polygons. With the introduction of the completely new graph class of k-perfect orientable graphs, we also can solve an open question posted by Butman et al. [4], namely to improve their approximation bound of $(t^2 - t + 1)/2$ for maximum clique on t-interval graphs. Our results lead to a $4t$-approximation. In general, our results also extend to intersection graphs of a restricted set of t-fat objects and further classes of graphs not discussed in this paper.

2 New Complexity Parameters

In this section, the following definitions introduce three complexity parameters. For each complexity parameter, we present examples of classes of intersection graphs for which the complexity parameter is bounded by a constant. For a set S of vertices in a graph G, let $G[S]$ be the subgraph of G induced by S.

Definition 1 (k-perfectly groupable). *A graph is* k-perfectly groupable *if the neighbors of each vertex v can be partitioned into k sets $S_1, \ldots, S_k$ such that $G[S_i \cup \{v\}]$ is a clique for each $i \in \{1, \ldots, k\}$.*

For each object S of a k-restricted set $\mathcal{C}$ of fat objects and a smallest ball B containing S, there exists a set P_S of k points such that every object in $\mathcal{C}$ intersecting B covers a point in P_S. For each S-intersecting and hence also B-intersecting object $S' \in \mathcal{C}$, choose one of the points in $S' \cap P_S$ as a representative. Then all S-intersecting objects having the same representative in P_S induce a clique in the intersection graph. Hence, the intersection graph of a k-restricted set of fat objects is k-perfectly groupable. Note that graphs of maximum degree k are also k-perfectly groupable. The set of the well studied $(k+1)$-clawfree graphs contains all k-groupable graphs. As we see in this section, unit disk graphs and unit square graphs are k-perfectly groupable for a suitable constant k.

Definition 2 (k-simplicial, k-simplicial elimination order, successor). *A graph G is* k-simplicial *if there is an order $v_1, \dots, v_n$ of the vertices of G such that, for each vertex v_i $(1 \le i \le n)$, the subset of neighbors of v_i contained in $\{v_j \mid j > i\}$ can be partitioned into k sets $S_1, \dots, S_k$ such that $G[S_j \cup \{v_i\}]$ is a clique for each $j \in \{1, \dots, k\}$. The vertices in $\{v_j \mid j > i, \{v_i, v_j\} \in E(G)\}$ are called the* successors *of v_i and the order above of the vertices in G is called a* k-simplicial elimination order.

The k-simplicial graphs are already defined in [25] and [34], whereas in the latter paper they are called $\tilde{G}(VCC_k)$. Let $\mathcal{C}$ be a set of fat objects $S_1, \dots, S_n$ ordered by non-decreasing size. Let k be the fatness of $\mathcal{C}$. Then, for each object S_i with $i \in \{1, \dots, n\}$, we can find k points such that every S_i-intersecting object in $\{S_{i+1}, \dots, S_n\}$ contains one of the k points. Let v_i be the vertex representing S_i in the intersection graph G of $\mathcal{C}$. Then $v_1, \dots, v_n$ defines a k-simplicial elimination order. Therefore, G is k-simplicial. Also note that disk graphs and square graphs are k-simplicial for a suitable constant k. Chordal graphs are exactly the 1-simplicial graphs. Moreover, every planar graph is 5-simplicial and every k-simplicial graph is sequentially k-independent (see [34]).

Definition 3 (k-perfectly orientable). *A graph G is called* k-perfectly orientable *if each edge $\{u_1, u_2\}$ of G can be assigned to exactly one of its endpoints u_1 and u_2 such that, for each vertex v, the vertices connected to v by edges assigned to v can be partitioned into k sets $S_1, \dots, S_k$ such that $G[S_i \cup \{v\}]$ is a clique for each $i \in \{1, \dots, k\}$. We write $a(\{u_1, u_2\}) = u_1$ if $\{u_1, u_2\}$ is assigned to u_1.*

We now show that the intersection graph $G = (V, E)$ of a set of t-fat-objects $\mathcal{C}$ with a universe of fatness c is $(t \cdot c)$-perfectly orientable. Let $V = \{v_1, \dots, v_n\}$ and, for each $i \in \{1, \dots, n\}$, let $\mathcal{S}_i$ be the union of t objects $S_{i,1}, \dots, S_{i,t}$ represented by v_i. Choose, for each edge $\{v_i, v_j\}$ in G with $i < j$, a pair $\{p, q\}$ of indices such that $S_{i,p}$ and $S_{j,q}$ intersect. Assign $\{v_i, v_j\}$ to v_i if the size of $S_{i,p}$ is smaller than the size of $S_{j,q}$ and to v_j otherwise. Then, for each vertex v_i, one can find $t \cdot c$ points such that each $\mathcal{S}_i$-intersecting t-fat-object $\mathcal{S}_j$ with $\{v_i, v_j\}$ being assigned to v_i must intersect S_i in at least one of the $t \cdot c$ points. Therefore, the set of vertices being endpoints of edges assigned to v_i can be partitioned into $\le t \cdot c$ cliques. This proves that G is $(t \cdot c)$-perfectly orientable. Note also that the intersection graphs of t-intervals are $2t$-perfectly orientable. For these graphs, an edge $\{v_i, v_j\}$ with $i < j$ is assigned to v_i if the t-interval represented by v_j intersects one of $2t$ endpoints of the intervals whose union is represented by v_i. Otherwise, $\{v_i, v_j\}$ is assigned to v_j.

We next present explicit upper bounds for the three complexity parameters on some special intersection graphs. Before that let us define the *inball* and the *outball* of a geometric object S to be a ball with largest radius contained in the closure of S and the ball with smallest radius containing the closure of S, respectively. The *center* of S is the center of its outball.

Theorem 4. *An intersection graph of t-squares, i.e., of unions of t (not necessarily axis-parallel) squares, is*

1. *10-perfectly groupable if $t = 1$ and if the squares are of unit size,*
2. *10-simplicial if $t = 1$, and*
3. *$10t$-perfectly orientable.*

Proof. For proving the first two cases, let G be the intersection graph of a set S of squares. It remains to show that, for a square Q of minimal side length ℓ, there are 10 points—called the *barriers* of Q—such that every Q-intersecting square Q' of length $\geq \ell$ must cover at least one of them. This fact also proves case 3 since the universe of a set of t-squares then has fatness 10.

We first describe our choice of the 10 barriers of Q. See also the left side of Fig. 1 for the following construction. Let b_1 and b_2 be the two perpendicular bisectors of the sides of Q. Choose two barriers x and y of Q as points on b_1 such that the part of b_1 inside Q is divided into three parts of equal length. We call these two points the *inner barriers* of Q. Let C be the curve surrounding Q that consists of all points having a distance of exactly ℓ to one of the inner barriers and a distance of at least ℓ to the remaining inner barrier. The remaining 8 barriers, called *outer barriers*, are almost equidistant points on C. More exactly, 4 outer barriers of Q are placed on the $2+2$ intersection points of C with b_1 and b_2. Choosing the other 4 outer barriers of Q is more sophisticated. Let x' and y' be the two points on b_1 having the same distance to the center of Q as to x and y, respectively. In addition, let $r_1, \ldots, r_4$ be the 4 rays starting from x' and y', respectively, and intersecting a corner of Q but neither b_1 nor b_2. The four remaining outer barriers are placed on the intersection points of C with the rays $r_1, \ldots, r_4$.

By a simple mathematical analysis one can show that the distance between any two consecutive outer barriers on C is strictly smaller than ℓ. It remains to show that each square of side length at least ℓ intersecting Q also covers one of the barriers of Q. Assume for a contradiction that we can find a square Q' of side length at least ℓ intersecting Q but none of the barriers of Q. W.l.o.g. we can assume that Q' has side length exactly ℓ since otherwise Q' also contains a smaller square intersecting Q. Let $\mathcal{H}$ be the convex hull of the outer barriers and let B be the largest circle contained in $\mathcal{H}$ such that B has the same center as Q. B and thus also $\mathcal{H}$ contain at least one corner of Q' since Q' intersects Q and B, and since a simple mathematical analysis shows that each chord of B with length at most l does not intersect Q. We now distinguish two cases.

Case 1: No side of Q' is completely contained in the convex hull $\mathcal{H}$ of the outer barriers. For each pair of consecutive outer barriers p and q on C, let us define $C_{p,q}$ to be the semi-circle inside $\mathcal{H}$ with endpoints p, q and hence having a diameter equal to the distance between p and q. See again the left side of Fig. 1. Let z be the corner of Q' inside B with the smallest distance to a point in Q. Note that the two sides of Q' ending in z are not completely contained in $\mathcal{H}$. Consequently, by Thales' theorem and the fact that Q' does not contain any

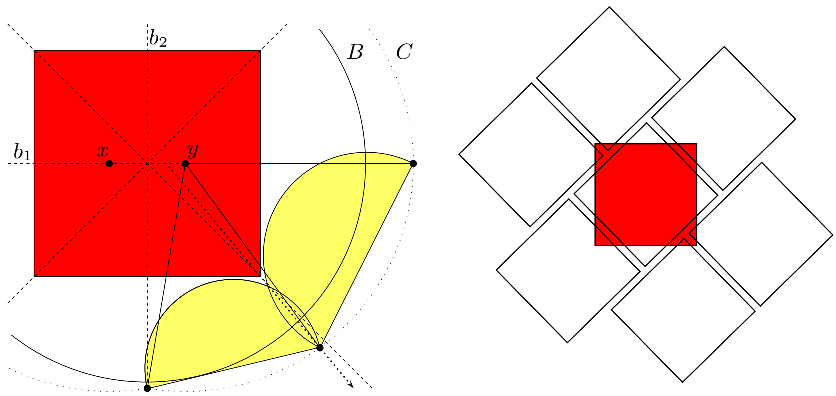

Fig. 1. The left side shows a square with some barriers, and on the right side, we see a square intersecting 7 disjoint squares

barriers there must be two consecutive outer barriers p and q on C such that z is contained in the face enclosed by $C_{p,q}$ and $\overline{pq}$. Again a simple mathematical analysis shows that none of our semi-circles intersects Q. Thus, neither z nor any other point of Q' is covered by Q. Contradiction.

Case 2: At least one side of Q' is completely contained in $\mathcal{H}$. Since each pair of consecutive outer barriers on C has a distance smaller than ℓ, the center q of Q' is inside $\mathcal{H}$.

By symmetry, w.l.o.g. we can assume that the distance between q and y is smaller or equal than the distance between q and x. Let $\mathcal{H}'$ be the convex hull of x and the outer barriers having a distance of at most ℓ to y. On the one hand, for each pair of consecutive barriers q_1 and q_2 on $\mathcal{H}'$, there is at most one corner in the face bounded by $\overline{q_1q_2}$ and the semi-circle outside $\mathcal{H}'$ with endpoints q_1 and q_2. On the other hand, at least one corner of Q' is outside $\mathcal{H}'$ since the inball of Q', which does not contain y, must intersect the border of $\mathcal{H}'$. Consequently, there are two sides s_1 and s_2 of Q' that have a common corner p outside $\mathcal{H}'$ and that intersect $\mathcal{H}'$ between to outer barriers, say q_1 and q_2.

Let T be the triangle with corners y, q_1 and q_2. Since Q' is a square of side length ℓ, since p is not covered by T, and since T is a triangle with two sides of length ℓ and with an s_1-intersecting side of length at most ℓ, y has to be inside Q'. Contradiction. □

Observation 5. *Some square graphs are not* 6*-perfectly groupable as shown on the right side of Fig. 1.*

Lemma 6. *The intersection graph of a set of rectangles, all having aspect ratio of* α*, is* $10\lceil\alpha\rceil$*-simplicial.*

Proof. Consider each rectangle as a set of $\lceil\alpha\rceil$ squares. For each rectangle r_1 replaced by squares of a size s_1, one can find $10\lceil\alpha\rceil$ points such that every r_1-intersecting square of size $s_2 \geq s_1$ replacing another rectangle r_2 must cover one of these points. Here we use the fact that each rectangle can be replaced by $\lceil \alpha \rceil$ unit squares.

Theorem 7. *Let c be a fixed constant and G be an intersection graph, where each vertex represents a union of t polygons taken from a universe of non-rotated c-regular polygons. Then G is $(t \cdot c)$-perfectly orientable.*

Proof. The intersection of two non-rotated c-regular polygons must contain at least one of the corners of the two polygons. Note that this does not hold for general rotated polygons. Let $\{v_1, \dots, v_n\}$ be the vertices of G. We assign an edge $\{v_i, v_j\}$ in G with $i < j$ to v_i if and only if one of the polygons in the union of polygons represented by v_i has a corner contained in the union of polygons represented by v_j. Otherwise, we assign it to v_j. The edges assigned to a vertex v can be partitioned into $t \cdot c$ sets such that the endpoints of the edges of each set induce a clique in G since we have one clique for each corner of the t polygons. □

Theorem 8. *Let G be the intersection graph of some geometric objects in $\mathbb{R}^d$. If the objects are convex and if, additionally, there is a constant k such that, for each object, the ratio between its size and the radius of its inball is bounded by k, then G is $(\frac{3}{2}\sqrt{d\pi}(k+1))^d/\Gamma(d/2+1)$-simplicial, where Γ should denote the Gamma function. If the ratio between the largest size of the objects and the radius of a smallest inball of the objects is bounded by a constant k', G is $(\frac{3}{2}\sqrt{d\pi}(k'+1))^d/\Gamma(d/2+1)$-perfectly groupable (even in the case of non-convex objects).*

Proof. For proving the lemma we first show how to find, for a given ball B with radius $\leq R'$ and a real number $r > 0$, a set of points such that every ball b with radius at least r intersecting B must cover at least one of these points. Therefore, let us consider the d-dimensional space, paved with d-dimensional cubes of edge length $s = 2r/\sqrt{d}$ and volume $s^d = 2^d r^d d^{-\frac{d}{2}}$. Then, every ball b of radius at least r must contain at least one of their midpoints, as the cubes' diagonals have length $2r$. Furthermore, the distance between the center of a ball b of radius $\geq r$ intersecting B and B's center is at most $R' + r$. Hence it suffices to pave a ball of radius $R' + 2r$. To do this, we do not need more cubes than completely fit in a ball of radius $R' + 3r$. A ball of radius $R' + 3r$ has volume $(\sqrt{\pi}(R'+3r))^d/\Gamma(\frac{d}{2}+1)$ and hence the following number of cubes are enough:

$$\left\lfloor \frac{(\sqrt{\pi}(R'+3r))^d}{\Gamma(\frac{d}{2}+1)} \cdot \frac{1}{2^d r^d d^{-\frac{d}{2}}} \right\rfloor = \left\lfloor \left(\frac{\sqrt{d\pi}}{2} \left(\frac{R'}{r} + 3 \right) \right)^d / \Gamma\left(\frac{d}{2}+1\right) \right\rfloor$$

Let $\mathcal{S}$ be a set of geometric objects such that G is the intersection graph of $\mathcal{S}$. We first consider the case, where all objects are convex and where there is a k such that, for each object, the ratio between its size and the radius of its inball is bounded by k. Let S_1 be an object of $\mathcal{S}$ with smallest size R and let S_2 be an S_1-intersecting object in $\mathcal{S}$ with size $s_2 \geq R$. Choose S_2' as the image of a dilation of S_2 with an arbitrary point $p \in S_1 \cap S_2$ as center and scaling factor $\lambda = R/s_2 > 0$. Then—as S_2 is convex—every point covered by S_2' is also covered by S_2. Furthermore, the inball of S_2' having radius $r \geq R/k$ must be completely contained in the ball of radius $R' := 3R$ around the center of S_1. Now the considerations above imply that S_2'—and hence S_2—must cover the midpoint of at least one cube of edge length $s = 2r/\sqrt{d}$ completely contained in a ball of radius $R' + 3r$. If we number the vertices of G in an order such that the sizes of objects represented by the vertices do not decrease, we obtain a $(\frac{3}{2}\sqrt{d\pi}(k+1))^d/\Gamma(d/2+1))$-simplicial elimination order proving the claim.

Finally, let us consider the case, where the objects of $\mathcal{S}$ are not necessarily convex, but the ratio between the largest size of the objects and the radius of a smallest inball of the objects is bounded by a constant k'. Consider intersecting geometric objects S_1 (with size R_1) and S_2 (with size R_2 and inball radius r_2) in $\mathcal{S}$. Then the considerations above imply, that the inball of S_2 must completely lie inside the ball of radius $R' := R_1 + 2R_2$ around the center of S_1. With $\frac{R'}{r_2} = \frac{R_1+2R_2}{r_2} \leq 3k'$ the second part of the lemma follows immediately. □

Theorem 9. *An intersection graph of t-disks, i.e., of unions of t disks, is*

1. *8-perfectly groupable if $t = 1$ and if the squares are of unit size,*
2. *8-simplicial if $t = 1$, and*
3. *8t-perfectly orientable.*

The theorem above can be shown with a proof similar to Theorem 4. Due to space limitations, we only want to remark that one can choose barriers—defined as in the proof of Theorem 4—of a disk with radius r as follows: One barrier is placed on the center of the disk and the remaining 7 barriers are placed equidistant on a circle of radius $3/2r$ with the same center than the disk.

3 Relations and Recognition

In the following we study the relations between the complexity parameters defined in the last section to each other and the NP-hardness of determining their minimal possible value.

Observation 10. *Each k-perfectly groupable graph is k-simplicial since any ordering of the vertices defines a k-simplicial elimination order. Conversely, an n-vertex star, i.e., an n-vertex tree with $n-1$ leaves, is not k-perfectly groupable for all $k < n-1$, but it is 1-simplicial.*

Lemma 11. *A k-simplicial graph is also k-perfectly orientable, but for every $n \in \mathbb{N}$ with $n \geq 12$, there exists a 2-perfectly orientable graph with n vertices that is not ℓ-simplicial for all $\ell < \lfloor\sqrt{n/3}\rfloor$.*

Proof. Let G be a k-simplicial graph having a k-simplicial elimination order $v_1, \ldots, v_n$. If all edges incident to a vertex v and one of its successors are assigned to v, the endpoints $u \neq v$ of the edges assigned to v can be partitioned into k

sets $S_1, \ldots, S_k$ such that $G[S_i \cup \{v\}]$ is a clique for every $i \in \{1, \ldots, k\}$. In other words, G is k-perfectly orientable.

Let us choose an arbitrary $n \in \mathbb{N}$ with $n \geq 12$ and let $k = \lfloor \sqrt{n/3} \rfloor$. We now construct a 2-perfectly orientable graph $G = (V, E)$ with n vertices that is not ℓ-simplicial for any $\ell < k$. The vertices of this graph are divided into three disjoint sets S_0, S_1 and S_2 of size k^2 and, if $n - 3k^2 > 0$, a further set $R = V \setminus (S_0 \cup S_1 \cup S_1)$ of isolated vertices. Each set S_i ($i \in \{0, 1, 2\}$) is divided into k subsets $S_{i,1}, \ldots, S_{i,k}$ of size k. For each $i \in \{0, 1, 2\}$ and each $j \in \{1, \ldots, k\}$, we introduce edges between each pair of vertices contained in the same subset $S_{i,j}$ and assign each of these edges arbitrarily to one of its endpoints. Let us define a numbering on the vertices of $S_{i,j}$ such that we can refer to the h-th vertex of $S_{i,j}$. For each $i \in \{0, 1, 2\}$ and each $h, j \in \{1, \ldots, k\}$, we additionally introduce edges between the h-th vertex u of $S_{i,j}$ and all vertices of $S_{(i+1) \bmod 3,h}$. We assign them to u. The constructed graph G is 2-perfectly orientable since the endpoints of an edge assigned to a vertex u being the h-th vertex of a subset $S_{i,j}$ belong to one of the two cliques induced by the vertices of $S_{i,j}$ and $S_{(i+1) \bmod 3,h}$. However, u is also adjacent to k vertices in $S_{(i-1) \bmod 3}$. Since there is no edge between a vertex in $S_{(i-1) \bmod 3,j_1}$ and a vertex in $S_{(i-1) \bmod 3,j_2}$ for $j_1 \neq j_2$, G cannot be ℓ-simplicial for any $\ell < k$. □

A graph has *inductive degree* k if it can be obtained from a single vertex by repeatedly adding a new vertex with k edges. Then we can easily conclude:

Lemma 12. *All graphs of inductive degree k are k-simplicial and therefore also k-perfectly orientable.*

Note that an important subclass of the graphs of inductive degree k is the exentsively studied class of graphs of treewidth k (not defined in this paper).

Observation 13. *The n-vertex clique is an example for a 1-perfectly groupable graph G that does not have treewidth $n-2$. Conversely, the n-vertex star is a graph with treewidth 1 that is not $(n-2)$-perfectly groupable.*

Lemma 14. *It is* NP*-hard to decide, for a tuple (G, k) of graph G and an integer k, whether G is k-perfectly groupable, k-simplical, or k-perfectly orientable.*

Proof. In this version of the paper we only proof the result for k-perfectly orientable graphs. The proofs for the other graph classes are based on similiar reductions. Given an n-vertex graph $G = (V, E)$ as an instance of the minimum clique partition problem, we add a set V' of $nk+1$ new vertices to G and connect each new vertex to each vertex in V. Let G' be the graph obtained. We next show that G' is k-perfectly orientable if G has clique partition of size at most k. For this purpose, assign all incident edges of a vertex $v' \in V'$ to v' and edges $e \in E$ to an arbitrary endpoint of e. Then a vertex v together with the endpoints of edges assigned to $v \in V \cup V'$ induce k cliques, i.e., G' is k-perfectly orientable.

Conversely, let us assume that G' is k-perfectly orientable and let $a : E \rightarrow V \cup V'$ be a suitable assignment of the edges to their endpoints. For each vertex $v \in V$ at most k of the $nk+1$ new edges incident to v can be assigned by a to v

since there are no edges between two vertices of V'. Thus, there is at least one $v' \in V'$ with all its edges assigned to itself. Thus, G must have a clique partition of size at most k.

For each constant k, one can use a fixed parameter algorithm for the MCP, e.g., see [20], to decide in polynomial time whether a graph G is k-groupable.

4 Algorithms

We present now polynomial time approximation algorithms for several NP-hard problems on graph classes with one of the three complexity parameters bounded by a constant. We implicitly assume that we are given an explicit *representation* of a graph as a k-perfectly groupable, k-simplicial, or k-perfectly orientable graph G. By that we mean that we are given, for each vertex v, a partition of its neighbors, of its successors, and of the vertices connected to v by edges assigned to v, respectively, into k sets $S_1, \ldots, S_k$ such that $G[S_i \cup \{v\}]$ is a clique for all $i \in \{1, \ldots, k\}$. In addition, we are given a k-simplicial elimination order in the case of a k-simplicial graph and, for each vertex of G, the edges assigned to it in the case of a k-perfectly orientable graph. These representations are sufficient even for intersection graphs. We do not need the explicit representations as intersection graphs described in Section 1, but we can use them to construct our new representations in polynomial time (see also the Theorems 4 and 9).

Theorem 15. *On k-perfectly groupable graphs, minimum dominating set and minimum independent dominating set can be k-approximated in polynomial time.*

Proof. As a k-approximative solution on a k-perfectly groupable graph G we output a maximal—not necessarily maximum—independent set S of G. To prove correctness, let us consider a minimum (independent) dominating set S_{opt} of G. For all $v \in S \setminus S_{\text{opt}}$, there must be a neighbor of v in S_{opt}. However, each such neighbor cannot cover more than k vertices of S, since G is k-perfectly groupable. Consequently, S is an independent dominating set of size at most $k|S_{\text{opt}}|$. □

Theorem 16. *Minimum clique partition, maximum weighted independent set, and maximum weighted clique, are k-approximable on k-simplicial and on k-perfectly groupable graphs in polynomial time.*

Proof (minimum clique partition). Given a graph G and a k-simplicial elimination order $v_1, \ldots, v_n$ for G, we first compute the graph G' obtained by removing v_1 and its neighbors from G. We then solve the problem recursively on G'. Let S' be the collection of vertex sets obtained as a solution for G'. Note that the graph induced by the removed vertices can be partitioned into a set Z of at most k cliques. We output $S = S' \cup Z$ as a solution for G. Note that v_1 is not incident to any vertex of G'. This guarantees that the difference between the size of a clique partition for G and for G' is at least 1. Thus, the clique partition obtained uses at most k times as many cliques as an optimal clique partition for G. □

Proof (maximum weighted independent set). See [1], [27], or [34]. □

Proof (maximum weighted clique). Given a k-simplicial graph, choose, for each vertex v, a clique C_v of maximal weight among the cliques obtained from one of the k cliques induced by v and the successors of v. Return the clique with maximal weight among the cliques in $\{C_v \mid v \in V\}$. This solution has approximation ratio k since a maximum weighted clique C_{opt} must also contain a vertex v with C_{opt} consisting only of v and a subset of its successors. □

Theorem 17. *On k-perfectly orientable n-vertex graphs, there are polynomial-time algorithms with approximation ratio*

1. *$2k$ for maximum weighted independent set, minimum vertex coloring and maximum weighted clique.*
2. *$O(\log^2 n/\log(1+1/k))$ for minimum clique partition.*

For the following proofs let $G = (V, E)$ be a k-perfectly orientable n-vertex graph, and for each $u \in V$, let $V_{u,1}, \ldots, V_{u,k}$ be k pairwise disjoint vertex sets such that their union are the neighbors of u and such that $C_{u,i} = G[V_{u,i} \cup \{u\}]$ is a clique for all $1 \leq i \leq k$. Moreover, define $\mathcal{C} = \{C_{u,i} \mid u \in V, 1 \leq i \leq k\}$.

The proof for maximum weighted independent set bases on the ideas including the local ratio technique of [2] and is omitted here.

Proof (minimum vertex coloring). Construct an order $v_1, \ldots, v_n$ of the vertices of G such that, for each vertex v_i ($i \in \{1, \ldots, n\}$), at least half of the edges in $G[\{v_i, \ldots, v_n\}]$ being adjacent to v_i are assigned to v_i. We now want to color the vertices $v_n, \ldots, v_1$ in this order with numbers in $\{1, \ldots, n\}$. We color each vertex $v \in V$ with the smallest number different from the colors of all already colored neighbors of v. Concerning the approximation ratio, let us define, for each vertex v, D_v to be a set of vertices of maximal weight such that D_v consists only of successors of v with respect to the order above and such that $G[D_v]$ is a clique. Then, each vertex v of G obtains a color smaller or equal $2k|D_v| + 1$, whereas an optimal coloring must color v and its neighbors with at least $|D_v| + 1$ colors. Therefore, the coloring obtained is a $2k$-approximation.

Proof (maximum weighted clique). As a $2k$-approximative solution, return the clique $C \in \mathcal{C}$ of maximal weight. Let us compare the weight of C with the weight of a maximal clique C_{OPT} of G. The subgraph of G induced by the vertices of C_{OPT} contains at least one vertex u for which the sum of the weights of the neighbors not being endpoints of edges assigned to u does not exceed the sum of the weights of the neighbors being endpoints of edges assigned to u. Thus, the weight of C is at most a factor $2k$ smaller than the weight of C_{OPT}. □

Proof (minimum clique partition). As part of our computation, we want to find a minimal number of cliques in $\mathcal{C}$ in polynomial time such that the union of their vertex sets is V. Unfortunately, this is an instance of the NP-hard set cover problem. However, using the Johnson's algorithm [26] we can find a subset of the cliques in $\mathcal{C}$ that covers V and that is at most a factor $O(\log |V|)$ larger than the minimal number of cliques in $\mathcal{C}$. We return this subset as an approximative solution. We achieve the approximation ratio $O(\log^2 |V|/\log \frac{2k}{2k-1}) =$

$O(\log^2 |V|/\log(1+\frac{1}{k}))$ since there is a clique partition of V using only cliques in $\mathcal{C}$ that uses $O(\log|V|/\log\frac{2k}{2k-1})$ as many cliques as a minimum clique partition C_{OPT} of $q \leq n$ arbitrary cliques $C_1, \ldots, C_q$ of G: Choose a vertex v of C_1 such that in the subgraph of G induced by the vertices of C_1 at least half of the edges adjacent to v are assigned to v. Remove the clique among $C_{v,1}, \ldots, C_{v,k}$ containing the largest number of not already deleted vertices in C_1. This decreases the number of vertices of C_1 by a factor of at least $1 - \frac{1}{2k} = \frac{2k-1}{2k}$. Repeat this step recursively until, after $O(\log|V|/\log\frac{2k}{2k-1})$ steps, C_1 contains no vertices any more. More precisely, when choosing a vertex v for which at least half of the adjacent edges are assigned to v, only count the edges not already being deleted. If we do the same for the remaining cliques, we obtain a clique partition with $O(q\log|V|/\log\frac{2k}{2k-1})$ cliques part of $\mathcal{C}$. □

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