Towards the Frontier between Decidability and Undecidability for Hyperbolic Cellular Automata

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Abstract. In this paper, we look at two ways to implement one dimensional cellular automata into hyperbolic cellular automata in three contexts: the pentagrid, the heptagrid and the dodecagrid, these tilings being classically denoted by $\{5,4\}$, $\{7,3\}$ and $\{5,3,4\}$ respectively. As an application, this may give a hint for the boundary between decidable and undecidable problems for hyperbolic cellular automata.

Keywords: Cellular automata, weak universality, decidability, hyperbolic spaces, tilings.

1 Introduction

In this paper, we look at the possibility to embed one-dimensional cellular automata, $1D$- for short, into hyperbolic cellular automata in the pentagrid, the heptagrid or the dodecagrid which are denoted by $\{5,4\}$, $\{7,3\}$ and $\{5,3,4\}$ respectively. We consider $1D$-cellular automata which are deterministic and whose number of cells is infinite. This will have consequences on the border between a decidable and an undecidable halting problem for a large class of hyperbolic cellular automata.

First, we shall prove a general theorem, and then we shall try to strengthen it at the price of a restriction on the set of cellular automata which we wish to embed in the case of the pentagrid.

The first theorem says:

Theorem 1. *There is a uniform algorithm to transform a deterministic* $1D$*-cellular automaton with* n *states into a deterministic cellular automaton in the pentagrid, the heptagrid or the dodecagrid with, in each case,* $n+1$ *states. Moreover, the cellular automaton obtained by the algorithm is rotation invariant.*

Later on, as we consider deterministic cellular automata only, we drop this precision. This theorem has a lot of corollaries, in particular we get this one, about weak universality:

A. Kučera and I. Potapov (Eds.): RP 2010, LNCS 6227, pp. 120–132, 2010.

Corollary 1. *There is a weakly universal cellular automaton in the pentagrid, in the heptagrid and in the dodecagrid which is weakly universal and which has three states exactly, one state being the quiescent state. Moreover, the cellular automaton is rotation invariant.*

We prove Theorem 1 and Corollary 1 in Section 2. In particular, we remind the notion of rotation invariance, especially for the $3D$ case. In Section 3, we strengthen the results, but this needs a restriction on the cellular automata under consideration in the case of the pentagrid.

2 Proof of Theorem 1 and Its Corollary

The idea of theorem 1 is very simple. Consider a one-dimensional cellular automaton A. The support of the cells of A is transported into a structure of the hyperbolic grid which we consider as a **line** of tiles. In each one of the three tilings which we shall consider, we define the line of tiles in a specific way. We examine these case, one after the other.

2.1 Pentagrid and Heptagrid

In the case of the pentagrid and of the heptagrid, it is the set of cells such that one side of the cells is supported by the same line of the hyperbolic plane. In the pentagrid, it is a line of the tiling, which is supported by a side of a cell, fixed once for all. In the heptagrid, it is what we call a mid-point line, a notion for which we refer the reader to [3]. Indeed, it was there proved that the mid-points of two contiguous sides of a heptagon define a line which cuts the other tiles of the heptagrid at the mid-points of two contiguous sides.

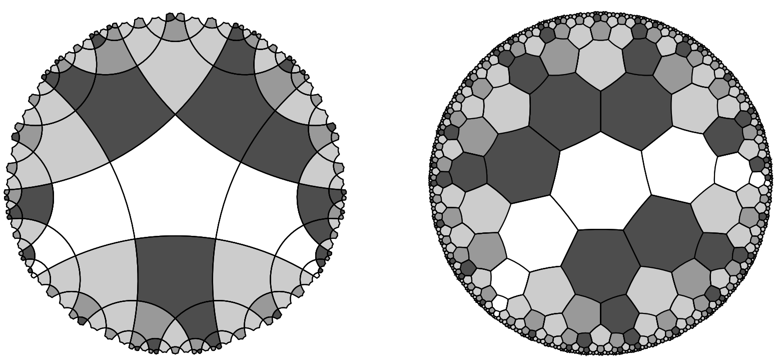

Fig. 1. Implementation of a cellular automaton in the pentagrid, left-hand side, and in the heptagrid, right-hand side. The white cells represent the line of tiles used for the $1D$-CA. The gray cells represent the cells which receive the new state.

In each case, the just defined line is called the **guideline** of the implementation. The guideline is illustrated in Figure 1. In this figure, the line on which we implement the $1D$ cellular automaton is represented by the white cells along the guideline. Note that a white cell has exactly two white neighbours. The cells are generated by the shift along the guideline which transforms one of the neighbours of the cell on the line into the cell itself.

In the figure, the white colour is assumed to represent the n states of the original automaton. The gray cells represent the additional state which is different from the n original ones. In the figure, there are three hues of gray which allow us to represent the tree structure of the tiling. These different hues represent the same state.

From the figure, it is plain that we have the following situation: white cells have exactly two white cells among their neighbours, the cell itself not being taken into account. In the pentagrid, a gray cell has at most one white cell in its neighbourhood. In the heptagrid, it has at most two white neighbours. Accordingly, this difference is enough to define the implementation of the rules in the pentagrid and in the heptagrid.

We can make this more precise as follows: denote the format of a rule by $\eta_0\eta_1...\eta_\alpha\eta_0^1$ with $\alpha \in \{5,7\}$ and where η_0 is the current state of the cell, η_i is the current state of neighbour i of the cell and η_0^1 is the new state of the cell, obtained after the rule was applied. We remind the reader that the neighbour i is the cell which shares the side i of the cell. We assume that the rules are **rotation invariant**. This means that if π is a circular permutation on $\{1..5\}$ and if $\eta_0\eta_1...\eta_\alpha\eta_0^1$ is a rule of the automaton, $\eta_0\eta_{\pi(1)}...\eta_{\pi(\alpha)}\eta_0^1$ is also a rule of the automaton. Now, as we assume the rules to be invariant, the numbering has only to be fixed according to the orientation: we consider that it increases from 1 to α as we clockwise turn around the tile. Which side is number 1 is not important. However, for the convenience of the reader, we shall fix it in a way which will be the most convenient for us.

In [4], we fixed the numbering in a rather uniform way. It consists in fixing number 1 to one side of the central cell. Then, for all other cells, number 1 is given to the side shared with the father: we remind the reader that the central cell is the father of the roots of the sectors which are displayed around itself.

Here, we keep this general setting for most cells except the white cells and those which share a side with a white cell. As the white cells are put along a linear structure defined by their guideline, we can order them. Accordingly, starting from now on, any white cell has one white left-hand side neighbour exactly and one white right-hand side neighbour exactly. Looking at Figure 1, the white left-hand side neighbour of a white cell is indeed on its left-hand side, both in the pentagrid and in the heptagrid. Now, we number the cells in such a way that the white left-hand side neighbour of a cell shares the side α of the cell. This fixes the cell with number 1 and, consequently, all the other neighbours. In the pentagrid, a gray cell c is in contact with at most one white cell. In this case, we consider that the white cell is number 5 for c. In the heptagrid, a gray cell c

is in contact with at most two white cells. We decide that they are numbered 6 and 7 for c or, in case of a unique white neighbour, that it is numbered 7 for c.

The rules for a white cell are: $\eta_0\mathbf{b}\mathbf{b}^a\eta_{2+a}\mathbf{b}^a\eta_\alpha\eta_0^1$, with $a = 1$ or 2 for the pentagrid, the heptagrid respectively. Moreover, $\eta_{2+a}\eta_0\eta_\alpha \rightarrow \eta_0^1$ is the unique rule of A which can be associated to the cell. For a gray cell which is in contact with a white cell, the rule is $\mathbf{b}\mathbf{b}^{2+2a}\eta_\alpha\mathbf{b}$ or, in the case of the heptagrid, it is the rule $\mathbf{b}\mathbf{b}^6\eta_\alpha\mathbf{b}$. Now, the rule for a gray cell which is not in contact with a white cell is: $\mathbf{b}\mathbf{b}^\alpha\mathbf{b}$.

Accordingly, we proved Theorem 1 for what are the grid of the hyperbolic plane which we considered. It can easily be proved that the same result holds for all the grids of the hyperbolic plane of the form $\{p, 4\}$ and $\{p+2, 3\}$, with $p \geq 5$.

2.2 In the Dodecagrid

In the dodecagrid, we use the representation introduced in [7]. We briefly remind it the reader for his/her convenience.

In fact, we consider the projection of the dodecahedra on a plane which is defined by a fixed face of one of them: this will be the plane of reference Π_0. The trace of the tiling on Π_0 is a copy of the pentagrid. So that, using a projection of each dodecahedron which is in contact with Π_0 and on the same half-space it defines which we call the half-space **above** Π_0, we obtain a representation of

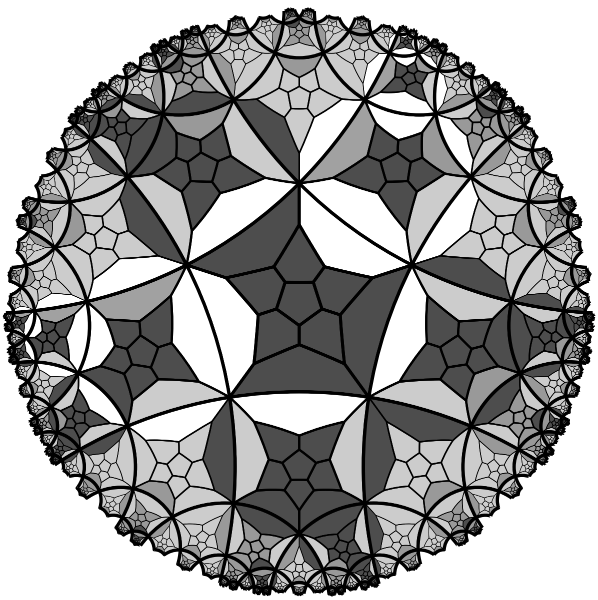

Fig. 2. Implementation of a cellular automaton in the dodecagrid. The white cells represent the line of tiles used for the one-dimensional CA. The gray cells represent the cells which receive the new state.

the line which is given by Figure 2. Indeed, the projection of each dodecahedron on this face looks like a Schlegel diagram, see [7,3] for more details on this tool dating from the 19^{th} century. Figure 3 also illustrates this representation for one dodecahedron.

Accordingly, the guideline is simply a line of the pentagrid which lies on Π_0. On the figure, we can see that the line which implements the one-dimensional cellular automaton is represented by the white cells, the other cells which receive the new state being gray. This line of white cells will be also called the **white line**. As in Figure 1, the different hues of gray are used in order to show the spanning trees of the pentagrid, dispatched around the central cell.

To define the rules of a cellular automaton, we also introduce a numbering of the faces of a dodecahedron which will allow us to number the neighbours. This numbering is given by Figure 3.

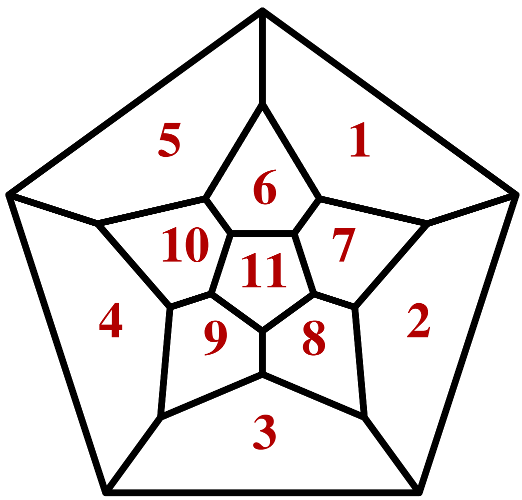

Fig. 3. The numbering of the faces of dodecahedron. Face 0 is delimited by the biggest pentagon of the figure.

Accordingly, the format of a rule is of the form $\eta_0^0\eta_0...\eta_{11}\eta_0^1$. Now, as the rules are assumed to be rotation invariant, which face receives number 1 is not important. However, for the convenience of the reader, we shall adopt the following convention. For all the white cells, we consider that the face which is on Π_0 is face 0. Accordingly, the numbers should appear in Figure 2 as they appear in Figure 3. Moreover, we consider that the other face of the cell which is in contact with the guideline is face 5.

Now, we come to the notion of **rotation invariant** cellular automata on the dodecagrid. We say that a motion in the hyperbolic $3D$ space is a **positive invariant displacement** of the dodecahedron if it leaves the dodecahedron globally invariant and if it preserves orientation. We shall say later positive displacement. Let μ be a positive invariant displacement of the dodecahedron. We denote by $\mu(i)$ the number of the face which is the image of face i under μ. Now, let $\rho = \eta_0^0\eta_0...\eta_{11}\eta_0^1$ be a rule. We say that $\eta_0^0\eta_{\mu(0)}...\eta_{\mu(11)}\eta_0^1$ is a **rotated form** of ρ. Say that $\eta_0^0\eta_0...\eta_{11}$ is the **context** or ρ. Then, we shall say that $\eta_0^0\eta_{\mu(0)}...\eta_{\mu(11)}$ is a rotated image of the context of ρ. Now, a cellular automaton is called **rotation invariant** if and only if two rules having contexts which are rotated forms of each other always produce the same new state.

Now, there are 60 positive displacements of the dodecahedron. They constitute a group which is called the icosahedral rotation group and which is isomorphic to A_5, the group of permutations on 5 elements whose signature is positive. It is well known that A_5 is a simple non-abelian group. This means that representations are difficult and that there is no canonical way to do that. In [10], we provided a simple algorithm to enumerate the positive displacements of the dodecahedron which we call the **rotation algorithm**.

Using this algorithm, we can define all the rotated forms of the context of a rule. These forms are words on the alphabet of the states of the cellular automaton and we can order these words lexicographically. As this order is total, there is a smallest element which we call the **minimal form** of the context. Similarly, we can define the **minimal form** of a rule. This allows us to obtain the following result:

Lemma 1 (see [7]) *A cellular automaton on the dodecagrid is rotation invariant if and only if for any pair of rules, if their minimal forms have the same context, they have the same new state too.*

Now, checking this property can easily be performed thanks to the rotation algorithm.

As we already indicated, we decided that face 0 of the cells belonging to the line of the implementation are on Π_0 and that the other face which has a side on the guideline is face 5. As a consequence, a white cell is in contact with two white neighbours by its faces 1 and 4. We decide that the face 1 of a cell is the same as the face 4 of the next white neighbour and, accordingly, its face 4 is the same as the face 1 of the other white neighbour. This allows to define two directions on the white line. The direction from left to right on the one-dimensional cellular automaton is, by convention,the direction from face 1 to face 4 of the same cell.

For the proof of Theorem 1 in the case of the dodecagrid, the rules for a gray cell have the form $\mathbf{b}\eta_0...\eta_{11}\mathbf{b}$ with all states in $\eta_0...\eta_{11}$ being $\mathbf{b}$ except, possibly, one of them. From the just defined convention on the numbering of the faces of the white cells, the rules for a white cell are of the form $\eta_0\mathbf{b}\eta_1\mathbf{bb}\eta_4\mathbf{bbbbbbb}\eta_0^1$, where $\eta_1\eta_0\eta_4 \rightarrow \eta_0^1$ is the rule of the one-dimensional cellular automaton.

Now, as the gray cells have at most one white neighbour and as the white cells have two white neighbours exactly, the difference between the rules is clearly recognizable.

This completes the proof of Theorem 1. ∎

Now, the proof of Corollary 1 is very easy: it is enough to apply the theorem to the elementary cellular automaton defined by rule 110 which is now known to be weakly universal, see [1,13].

3 Refinement of Theorem 1

Now, we shall prove that, under particular hypotheses in the case of the pentagrid and no restriction in the case of the heptagrid and of the dodecagrid, a $1D$ cellular automaton with n states can be simulated by a hyperbolic cellular automaton with n states too.

In order to formulate this hypothesis, consider a one-dimensional deterministic cellular automaton A. Say that a state s of A is **fixed** in the context x, y in this order, if the rule $xsy \to s$ belongs to the table of transitions of A. As an example, a **quiescent** state for A, usually denoted by 0, is fixed in the context $0, 0$. Now, we say that A is a **fixable** cellular automaton if it has a quiescent state 0 and another state, denoted by 1, such that 0 is also fixed in the context $1, 0$ and 1 is fixed in the context $0, 0$.

We can now formulate the following results:

Theorem 2. *There is an algorithm which transforms any fixable* $1D$ *cellular automaton* A *with* n *states into a rotation invariant cellular automaton* B *in the pentagrid with* n *states too, such that* B *simulates* A *on a line of the pentagrid.*

Theorem 3. *There is an algorithm which transforms any deterministic* $1D$ *cellular automaton* A *with* n *states into a rotation invariant deterministic cellular automaton* B *in the heptagrid, the dodecagrid respectively, with* n *states too, such that* B *simulates* A *on a line of the heptagrid, the dodecagrid respectively.*

We have not the room to produce a full proof of these theorems: we refer the reader to [8] for such a proof. Here, we simply sketch the outlines of the proof of Theorem 2.

Consider the left-hand side picture of Figure 4. The white colour is still used to represent any state of the automaton A. Now, the gray colour represents the quiescent state 0, and the black one represents the state 1 which is fixed in the context $0, 0$. We also assume that 0 is fixed in the context $1, 0$.

We shall consider all the neighbours of the central cell. Its black neighbour will be numbered by 1, and the others from 2 to 5, increasing as we clockwise turn around the cell. We also consider the cells which just has one vertex in common with the central cell. All the other cells are in quiescent state or they belong to the white line or are neighbouring a cell belonging to this line. In this latter case, such a cell is obtained from one of those we consider around the central cell by a shift along the guideline.

Define B with n states represented by different letters from those used from A. We fix a bijection between the states of A and those of B in which $\mathbf{B}$ is associated to the state 1 of A and $\mathbf{W}$ is associated to the state 0 of A.

Consider the configuration around the central cell. If we write the states of the cell and then those of its neighbours according to the order of their numbers, we get the following word: $Y\mathbf{BW}Z\mathbf{W}X$, where X, Y, Z are taken among the states of B. Now, if $Z = \mathbf{B}$, we can start from this neighbour in state $\mathbf{B}$ which has number 3, and we get the word $Y\mathbf{BW}X\mathbf{BW}$ in which we see $\mathbf{B}$ in position 4. If $X = \mathbf{B}$, then we get the word $Y\mathbf{BBW}Z\mathbf{W}$ in which we see $\mathbf{B}$ in position 2. In both case, the configuration around the cell is different from the one we obtain by starting from position 1. We can synthesise this information as follows:

		1	2	3	4	5
0	Y	**B**	**W**	Z	**W**	X
		B	**W**	X	**B**	**W**
		B	**B**	**W**	Z	**W**

The first line corresponds to the configuration which triggers the application of the rule of A corresponding to $XYZ \rightarrow X'$. Clearly, as already noticed with the positions of the fixed **B** and **W**, the other lines do not correspond to the application of a rule of A.

We shall do this for all the neighbours of the central cell, and in Table 1, we can see all the possible configurations for the neighbours of the central cell.

Table 1. Table of the configurations around the central cell in the pentagrid for the automaton B

		1	2	3	4	5
0	Y	**B**	**W**	Z	**W**	X
		B	**W**	X	**B**	**W**
		B	**B**	**W**	Z	**W**
1_1	**B**	Y	**W**	**W**	**W**	**W**
2_1	**W**	**B**	**W**	**W**	**W**	**W**
1_2	**W**	Y	**W**	**W**	**W**	**B**
		B	**W**	**W**	**W**	**B**
		B	**B**	**W**	**W**	**B**
		B	**W**	**W**	**W**	**W**
2_2	**B**	**W**	**W**	**W**	**W**	Z
		B	**W**	**W**	**W**	**W**
1_3	Z	Y	**B**	**W**	T	**W**
		B	**B**	**W**	T	**W**
		B	**W**	T	**W**	Y
		B	**W**	Y	**B**	**W**

		1	2	3	4	5
2_3	**W**	Z	**W**	**W**	**W**	**W**
1_4	**W**	Y	**W**	**W**	**W**	**W**
2_4	**W**	**W**	**W**	**W**	**W**	X
		B	**W**	**W**	**W**	**W**
1_5	X	Y	**W**	U	**B**	**W**
		B	**W**	U	**B**	**W**
		B	**B**	**W**	Y	**W**
		B	**W**	Y	**W**	U
2_5	**W**	X	**W**	**W**	**W**	**B**
		B	**W**	**W**	**W**	**B**
		B	**B**	**W**	**W**	**W**
		B	**W**	**W**	**W**	**W**

In Table 1, we indicate the coordinate of the cell which we represent together with its state. Then, if there are states as U, X, Y, Z, T, we also represent the case when one of this variable takes the value **B** and we represent the configuration around the cell when this **B** is put onto position 1.

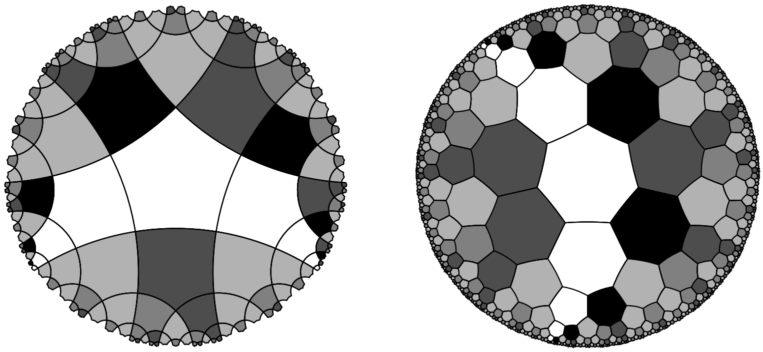

Fig. 4. Implementation of a $1D$ cellular automaton in the pentagrid, left-hand side, and in the heptagrid, right-hand side. The white cells represent the line of tiles used for the $1D$-CA. The gray cells represent the cells which receive a particular state among the states of the $1D$-CA.

The left-hand side picture of Figure 4 allows us to check the correctness of Table 1. We refer the reader to [8] for a further checking.

This completes the proof of Theorem 2.

Table 2. Table of the configurations around the central cell in the pentagrid for the automaton B

		1	2	3	4	5	6	7
0	*Y*	*X*	**B**	**W**	**B**	*Z*	**W**	**W**
		B	**B**	**W**	**B**	*Z*	**W**	**W**
		B	**W**	**B**	*Z*	**W**	**W**	*X*
		B	*Z*	**W**	**W**	*X*	**B**	**W**
		B	**W**	**W**	*X*	**B**	**W**	**B**
1_1	*X*	*Y*	**W**	**W**	*U*	**B**	**W**	**B**
		B	**W**	**W**	*U*	**B**	**W**	**B**
		B	**B**	**W**	**B**	*Y*	**W**	**W**
		B	**W**	**B**	*Y*	**W**	**W**	*U*
		B	*Y*	**W**	**W**	*U*	**B**	**W**
1_2	**B**	*Y*	*X*	**W**	**W**	**W**	**W**	**W**
		B	*X*	**W**	**W**	**W**	**W**	**W**
		B	**W**	**W**	**W**	**W**	**W**	*Y*
1_3	**W**	*Y*	**B**	**W**	**W**	**W**	**W**	**B**
		B	**B**	**W**	**W**	**W**	**W**	**B**
		B	**W**	**W**	**W**	**W**	**B**	*Y*
1_4	**B**	*Y*	**W**	**W**	**W**	**W**	**W**	*Z*
		B	**W**	**W**	**W**	**W**	**W**	*Z*
		B	*Y*	**W**	**W**	**W**	**W**	**W**
1_5	*Z*	*Y*	**B**	**W**	**B**	*T*	**W**	**W**
		B	**B**	**W**	**B**	*T*	**W**	**W**
		B	**W**	**B**	*T*	**W**	**W**	*Y*
		B	*T*	**W**	**W**	*Y*	**B**	**W**
		B	**W**	**W**	*Y*	**B**	**W**	**B**
1_6	**W**	*Y*	*Z*	**W**	**W**	**W**	**W**	**W**
		B	*Z*	**W**	**W**	**W**	**W**	**W**
		B	**W**	**W**	**W**	**W**	**W**	*Y*
1_7	**W**	*Y*	**W**	**W**	**W**	**W**	**W**	*X*
		B	**W**	**W**	**W**	**W**	**W**	*X*
		B	*Y*	**W**	**W**	**W**	**W**	**W**

Now, let us turn to the case of the heptagrid. In this case, the situation is in some sense easier as it requires no special hypothesis on the deterministic $1D$ cellular automaton. Indeed, the fact is that due to the number of neighbours, there is a way to differentiate the cells belonging to the white line from those which do not. As mentioned in Subsection 2.1, the white line is now implemented along a mid-point line of the heptagrid which is fixed, once for all. As in the case of the pentagrid, the white colour represents any state of automaton A. Now, we assume that A has at least two states, 0 and 1. In the right-hand side picture of Figure 4, these states are represented in gray and in black respectively. As in the case of the pentagrid, we use different hues of gray in order to make visible the tree structure which spans the tiling. Now, it is easy to see that the configurations allowing the application of a rule of A are reached only in the case of cells of the white line and that for these cells, among the rotated contexts, exactly one is compatible with the application of a rule of A. This can be checked on the figure and we report this examination in Table 2.

Looking at each entry of the table attached to a cell, we can see that there is at most a single configuration which is compatible with the application of a rule of A. Moreover, the admissible configuration occurs only for the cells which are on the white line and never for the others. Accordingly, the rule which consists in applying the rule of A when there is one for that and to leave the current state unchanged otherwise works more easily here. This completes the proof of Theorem 3 in the case of the heptagrid.

Let us now look at the same problem in the case of the dodecagrid. This, time, we can take advantage of a bigger number of neighbours and of their spatial display to strengthen the difference between a cell of the white line which is implemented as indicated in Subsection 2.2 and the cells which does not belong to this line. The way in which we establish this difference is illustrated by Figure 6. In this figure, the white colour represents the states of B which, by construction, are in bijection with those of A. As previously, the gray colour is associated with the state **W** which corresponds to the quiescent state 0 of A, and the black colour is associated with the state **B** which corresponds to the state 1 of A.

Now, each cell of the white line has four black neighbours. Numbering the cells as indicated in Subsection 2.2, the faces with a black neighbour are: 0, 3, 9 and 10. Figure 5 represents a cut in the plane of the face 4 of a white cell and it makes it easy to understand the configuration. Due to the fact that face 0 is on the plane Π_0, we can see only three black faces on the cells of the white line in Figure 6.

We refer the reader to [8] in order to check that the cells which do not belong to the white line have at most two black neighbours.

This completes the proof of Theorem 3.

Now, we can see that from Theorem 3 we have as an immediate corollary:

Corollary 2. *There is a weakly universal rotation invariant cellular automaton on the heptagrid, as well as in the dodecagrid with two states exactly.*

In both cases, we apply the construction defined in the proof of Theorem 3 to the elementary cellular automaton with rule 110. Now, if we look at the transitions of

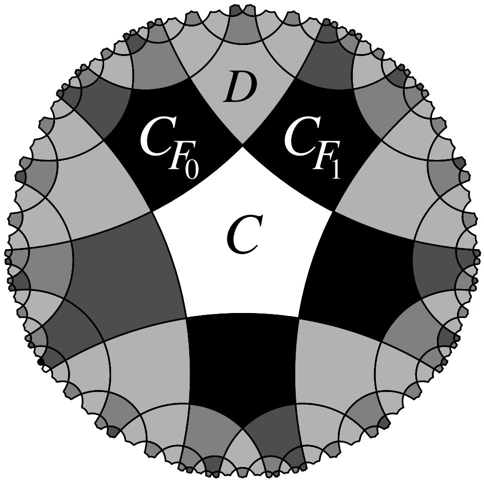

Fig. 5. Implementation of a cellular automaton in the heptagrid. The white cells represent the line of tiles used for the $1D$ CA. The gray cells represent the cells which receive a particular state among the states of the $1D$ CA.

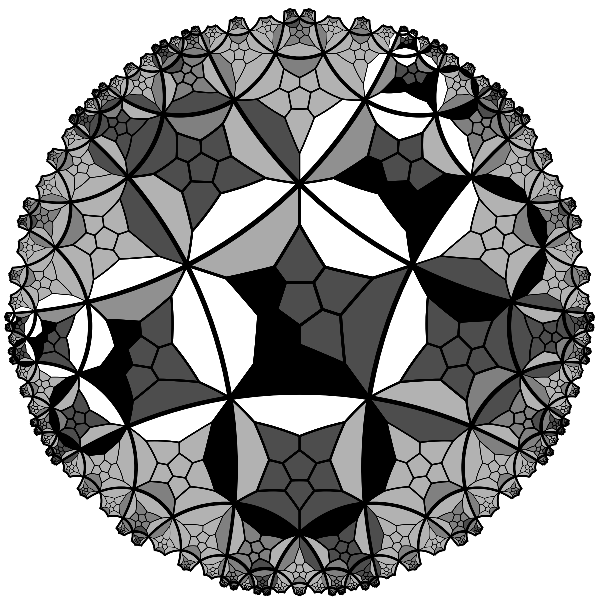

Fig. 6. Implementation of a cellular automaton in the dodecagrid. The white cells represent the line of tiles used for the $1D$ CA. The gray cells represent the cells which receive a particular state among the states of the $1D$ CA.

rule 110, we can see that 0 is a quiescent state, that it is fixed for the context $1, 0$ and that 1 is fixed for the context $0, 0$. This proves that the elementary cellular automaton with rule 110 is fixable. Consequently, applying Theorem 2 to this $1D$-cellular automaton, we get:

Corollary 3. *There is a weakly universal rotation invariant cellular automaton on the pentagrid with two states exactly.*

4 Conclusion

With this result, we reached the frontier between decidability and weak universality for cellular automata in hyperbolic spaces: starting from 2 states there are weakly universal such cellular automata, with 1 state, there are none, which is trivial.

We can remark that the argument of Theorem 3 can be applied to all tilings of the form $\{p, 4\}$ and $\{p+2, 3\}$ with $p \geq 5$.

We can also remark that the result proved in this paper suffers the same defect as the result indicated in [7] with 3 states. The results proved in this paper can be obtained in a not too complicate manner by an appropriate implementation of rule 110 which is weakly universal, as already mentioned. In the case of the dodecagrid, the author proved a similar result with 3 states but involving a much more elementary construction which is also an actual $3D$ construction. In the case of the heptagrid, he obtained 4 states with an actual planar construction, see [5,9] and the best result known for the pentagrid is 9 states, see [12], again with elementary tools and using an actual planar construction. What can be done in this direction is also an interesting question.

Accordingly, there is some work ahead, probably the hardest as we are now so close to the goal.

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