Fast Generalized Bruhat Decomposition

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Abstract. The deterministic recursive pivot-free algorithms for computing the generalized Bruhat decomposition of the matrix in the field and for the computation of the inverse matrix are presented. This method has the same complexity as algorithm of matrix multiplication, and it is suitable for the parallel computer systems.

1 Introduction

An LU matrix decomposition without pivoting is a decomposition of the form $A = LU$, a decomposition with partial pivoting has the form $PA = LU$, and decomposition with full pivoting (Trefethen and Bau) has the form $PAQ = LU$, where L and U are lower and upper triangular matrices, P and Q is a permutation matrix.

French mathematician Francois Georges René Bruhat was the first who worked with matrix decomposition in the form $A = VwU$, where V and U are nonsingular upper triangular matrices and w is a matrix of permutation. Bruhat decomposition plays an important role in algebraic group theory. The generalized Bruhat decomposition was introduced and developed by D.Grigoriev[1],[2]. He uses the Bruhat decomposition in the form $A = VwU$, where V and U are upper triangular matrices but they may be singular when the matrix A is singular. In the papers [3] and [4], there was analyzed the sparsity pattern of triangular factors of the Bruhat decomposition of a nonsingular matrix over a field.

Fast matrix multiplication and fast block matrix inversion were discovered by Strassen [5]. The complexity of Strassen's recursive algorithm for block matrix inversion is the same as the complexity of an algorithm for matrix multiplication. But in this algorithm it is assumed that principal minors are invertible and leading elements are nonzero as in the most of direct algorithms for matrix inversion. There are known other recursive methods for adjoint and inverse matrix computation, which have the complexity of matrix multiplications([6]-[8]).

In a general case, it is necessary to find suitable nonzero elements and to perform permutations of matrix columns or rows. Bunch and Hopkroft suggested such algorithm with full pivoting for matrix inversion [9].

The permutation operation is not a very difficult operation in the case of sequential computations by one processor, but it is a difficult operation in the case

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of parallel computations, when different blocks of a matrix are disposed in different processors. A matrix decomposition without permutations is needed for parallel computation for construction of efficient and fast computational schemes.

The problem of obtaining pivot-free algorithm was studied in [10],[11] by S. Watt. He presented the algorithm that is based on the following identity for a nonsingular matrix: $A^{-1} = (A^T A)^{-1} A^T$. Here A^T is the transposed matrix to A, and all principal minors of the matrix $A^T A$ are nonzero. This method is useful for making an efficient parallel program with the help of Strassen's fast decomposition of inverse matrix for dense nonsingular matrix over the field of zero characteristic when field elements are represented by the float numbers. Other parallel matrix algorithms are developed in [12] - [15].

This paper is devoted to the construction of the pivot-free matrix decomposition method in a common case of singular matrices over a field of arbitrary characteristic. The decomposition will be constructed in the form $LAU = E$, where L and U are lower and upper triangular matrices, and E is a truncated permutation matrix, which has the same rank as the matrix A. Then the generalized Bruhat decomposition may easily be obtained using the matrices L, E, and U. This algorithm has the same complexity as matrix multiplication and does not require pivoting. For singular matrices, it allows the obtaining of a nonsingular block of the biggest size, the echelon form, and kernel of matrix. The preliminary variants of this algorithm were developed in [16] and [17].

2 Preliminaries

We introduce some notations that will be used in the following sections.

Let F be a field, $F^{n\times n}$ be an $n \times n$ matrix ring over F, S_n be a permutation group of n elements. Let P_n be a multiplicative semigroup in $F^{n\times n}$ consisting of matrices A having exactly rank(A) nonzero entries, all of them equal to 1. We call P_n the permutation semigroup because it contains the permutation group of n elements S_n and all their truncated matrices.

The semigroup $D_n \subset P_n$ is formed by the diagonal matrices. So $|D_n|=2^n$ and the identity matrix $\mathbf{I}$ is the identity element in D_n, S_n and P_n.

Let $W_{i,j} \in P_n$ be a matrix, which has only one nonzero element in the position (i, j). For an arbitrary matrix E of P_n, which has the rank $n - s$ $(s = 0, ..n)$ we shall denote by $i_{\overline{E}} = \{i_1, .., i_s\}$ the ordered set of zero row numbers and $j_{\overline{E}} = \{j_1, .., j_s\}$ the ordered set of zero column numbers.

Definition 1. *Let $E \in P_n$ be the matrix of the rank $n - s$, let $i_{\overline{E}} = \{i_1, .., i_s\}$ and $j_{\overline{E}} = \{j_1, .., j_s\}$ are the ordered set of zero row numbers and zero columns number of the matrix E. Let us denote by $\overline{E}$ the matrix*

$$\overline{E} = \sum_{k=1,..s} W_{i_k,j_k}$$

and call it the complimentary matrix for E. For the case $s = 0$ we put $\overline{E} = 0$.

It is easy to see that $\forall E \in P_n: \; E + \overline{E} \in S_n$, and $\forall I \in D_n: \; I + \overline{I} = \mathbf{I}$. Therefore, the map $I \mapsto \overline{I} = \mathbf{I} - I$ is the involution, and we have $I\overline{I} = 0$. We can define the partial order on D_n: $I < J \Leftrightarrow J - I \in D_n$. For each matrix $E \in P_n$ we shall denote by

$$I_E = EE^T \text{ and } J_E = E^T E$$

the diagonal matrix: $I_E, J_E \in D_n$. The unit elements of the matrix I_E show nonzero rows of the matrix E and the unit elements of the matrix J_E show nonzero columns of the matrix E. Therefore, we have several zero identities:

$$E^T \overline{I}_E = \overline{I}_E E = E \overline{J}_E = \overline{J}_E E^T = 0. \tag{1}$$

For any pair $I, J \in D_n$ let us denote the subset of matrices $F^{n\times n}$

$$F^{n\times n}_{I,J} = \{B : B \in F^{n\times n}, IBJ = B\}.$$

We call them (I, J)-zero matrix. It is evident that $F^{n\times n} = F^{n\times n}_{\mathbf{I},\mathbf{I}}$, $0 \in \cup_{I,J} F^{n\times n}_{I,J}$, and if $I_2 < I_1$ and $J_2 < J_1$ then $F^{n\times n}_{I_2,J_2} \subset F^{n\times n}_{I_1,J_1}$.

Definition 2. *We shall call the factorization of the matrix* $A \in F^{n\times n}_{I,J}$

$$A = L^{-1} E U^{-1}, \tag{2}$$

LEU-decomposition if $E \in P_n$, L *is a nonsingular lower triangular matrix,* U *is an upper unitriangular matrix, and*

$$L - \overline{I}_E \in F^{n\times n}_{I,I_E}, \quad U - \overline{J}_E \in F^{n\times n}_{J_E,J}. \tag{3}$$

If (2) is the LEU-decomposition we shall write

$$(L, E, U) = \mathcal{LU}(A),$$

Sentence 1. *Let* $(L, E, U) = \mathcal{LU}(A)$ *be the LEU-decomposition of matrix* $A \in F^{n\times n}_{I,J}$ *then*

$$L = \overline{I}_E + ILI_E, \; U = \overline{J}_E + J_E U J, \; E \in F^{n\times n}_{I,J}, \tag{4}$$
$$L^{-1} = \overline{I}_E + L^{-1} I_E, \; U^{-1} = \overline{J}_E + J_E U^{-1}.$$

Proof. The first and second equalities follow from (3). To prove the property of matrix E we use the commutativity of diagonal semigroup D_n:

$$E = LAU = (\overline{I}_E + ILI_E) IAJ (\overline{J}_E + J_E U J) = I(\overline{I}_E + LI_E I) A (\overline{J}_E + J J_E U) J.$$

To prove the property of matrix L^{-1} let us consider the identity

$$\mathbf{I} = L^{-1} L = L^{-1}(\overline{I}_E + LI_E) = L^{-1}\overline{I}_E + \mathbf{I} I_E$$

Therefore, $L^{-1}\overline{I}_E = \overline{I}_E$ and $L^{-1} = L^{-1}(\overline{I}_E + I_E) = \overline{I}_E + L^{-1} I_E$. The proof of the matrix U^{-1} property may be obtained similarly.

Sentence 1 states the property of matrix E, which may be written in the form $I_E < I$, $J_E < J$. We shall call it *the property of immersion*. On the other hand, each zero row of the matrix E goes over to the unit column of matrix L and each zero column of the matrix E goes over to the unit row of matrix U.

Let us denote by $\mathcal{E}_n$ the permutation matrix $W_{1,n} + W_{2,n-1} + .. + W_{n,1} \in S_n$. It is easy to see that if the matrix $A \in F^{n\times n}$ is lower-(upper-) triangular, then the matrix $\mathcal{E}_n A \mathcal{E}_n$ is upper- (lower-) triangular.

Sentence 2. *Let $(L, E, U) = \mathcal{LU}(A)$ be the LEU-decomposition of matrix $A \in F^{n\times n}$, then the matrix $\mathcal{E}_n A$ has the generalized Bruhat decomposition $V_1 w V_2$ and*

$$V_1 = \mathcal{E}_n(L^{-1} - \overline{I}_E)\mathcal{E}_n,\ w = \mathcal{E}_n(E + \overline{E}),\ V_2 = (U^{-1} - \overline{J}_E).$$

Proof. As far as L^{-1} is a lower triangular matrix, and U^{-1} is an upper triangular matrix we see that V_1 and V_2 are upper triangular matrices. Matrix w is a product of permutation matrices so w is a permutation matrix. One easily checks that $V_1 w V_2 = \mathcal{E}_n L^{-1} E U^{-1} = \mathcal{E}_n A$.

Examples

For any matrix $I \in D_n$, $E \in P_n$, $0 \neq a \in F$ the product $(aI + \overline{I})I\ \mathbf{I}$ is a LEU decompositions of matrix aI and the product $(aI_E + \overline{I}_E)E\ \mathbf{I}$ is a LEU decompositions of matrix aE.

3 Algorithm of *LEU* Decomposition

Theorem 1. *For any matrix $A \in F^{n\times n}$ of size $n = 2^k$, $k \geq 0$ a LEU-decomposition exists. For computing such decomposition it is enough to compute 4 LEU-decompositions, 17 multiplications and several permutations for the matrices of size $n = 2^{k-1}$.*

Proof. For the matrix of size 1×1, when $k = 0$, we can write the following LEU decompositions

$$\mathcal{LU}(0) = (1, 0, 1) \text{ and } \ \mathcal{LU}(a) = (a^{-1}, 1, 1),\ \text{if}\ \ a \neq 0.$$

Let us assume that for any matrix of size n we can write a LEU decomposition, and let the given matrix $A \in F^{2n\times 2n}_{I,J}$ have the size $2n$. We shall construct a LEU decomposition of matrix A.

First of all we shall subdivide the matrices A, I, J and a desired matrix E into four equal blocks:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, I = \text{diag}(I_1, I_2), J = \text{diag}(J_1, J_2), E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix},$$

and denote

$$I_{ij} = E_{ij} E_{ij}^T,\quad J_{ij} = E_{ij}^T E_{ij}\quad \forall i, j \in \{1, 2\}. \tag{6}$$

Let

$$(L_{11}, E_{11}, U_{11}) = \mathcal{LU}(A_{11}), \tag{7}$$

denote the matrices

$$Q = L_{11}A_{12},\ B = A_{21}U_{11}, \tag{8}$$

$$A^1_{21} = B\overline{J}_{11},\ A^1_{12} = \overline{I}_{11}Q,\ A^1_{22} = A_{22} - BE^T_{11}Q. \tag{9}$$

Let

$$(L_{12}, E_{12}, U_{12}) = \mathcal{LU}(A^1_{12}) \text{ and } (L_{21}, E_{21}, U_{21}) = \mathcal{LU}(A^1_{21}), \tag{10}$$

denote the matrices

$$G = L_{21}A^1_{22}U_{12},\ A^2_{22} = \overline{I}_{21}G\overline{J}_{12}. \tag{11}$$

Let us put

$$(L_{22}, E_{22}, U_{22}) = \mathcal{LU}(A^2_{22}), \tag{12}$$

and denote

$$W = (GE^T_{12}L_{12} + L_{21}BE^T_{11}),\ V = (U_{21}E^T_{21}G\overline{J}_{12} + E^T_{11}QU_{12}), \tag{13}$$

$$L = \begin{pmatrix} L_{12}L_{11} & 0 \\ -L_{22}WL_{11} & L_{22}L_{21} \end{pmatrix}, U = \begin{pmatrix} U_{11}U_{21} & -U_{11}VU_{22} \\ 0 & U_{12}U_{22} \end{pmatrix}. \tag{14}$$

We have to prove that

$$(L, E, U) = \mathcal{LU}(A). \tag{15}$$

As far as $L_{11}, L_{12}, L_{21}, L_{22}$ are lower triangular nonsingular matrices, and U_{11}, U_{12}, U_{21}, U_{22} are upper unitriangular matrices we can see in (10) that the matrix L is a lower triangular nonsingular matrix and the matrix U is upper unitriangular.

Let us show that $E \in P_{2n}$. As far as $E_{11}, E_{12}, E_{21}, E_{22} \in P_n$ and $A_{11} = I_1A_{11}J_1$, $A^1_{21} = B\overline{J}_{11}$, $A^1_{12} = \overline{I}_{11}Q$, $A^2_{22} = \overline{I}_{21}G\overline{J}_{12}$ and due to the Sentence 1 we obtain $E_{11} = I_{11}E_{11}J_{11}$, $E_{21} = E_{21}\overline{J}_{11}$, $E_{12} = \overline{I}_{11}E_{12}$, $E_{22} = \overline{I}_{21}E_{22}\overline{J}_{12}$.

Therefore, the unit elements in each of the four blocks of the matrix E are disposed in different rows and columns of the matrix E. So $E \in P_{2n}$, and next identities hold

$$E_{11}E^T_{21} = E_{11}J_{21} = J_{11}E^T_{21} = J_{11}J_{21} = 0, \tag{16}$$

$$E^T_{12}E_{11} = E^T_{12}I_{11} = I_{12}E_{11} = I_{12}I_{11} = 0, \tag{17}$$

$$E_{12}E^T_{22} = E_{12}J_{22} = J_{12}E^T_{22} = J_{12}J_{22} = 0, \tag{18}$$

$$E^T_{22}E_{21} = E^T_{22}I_{21} = I_{22}E_{21} = I_{22}I_{21} = 0. \tag{19}$$

We have to prove that $E = LAU$. This equation in block form consists of four block equalities:

$$\begin{aligned} &E_{11} = L_{12}L_{11}A_{11}U_{11}U_{21}; \\ &E_{12} = L_{12}L_{11}(A_{12}U_{12} - A_{11}U_{11}V)U_{22}; \\ &E_{21} = L_{22}(L_{21}A_{21} - WL_{11}A_{11})U_{11}U_{21}; \\ &E_{22} = L_{22}((L_{21}A_{22} - WL_{11}A_{12})U_{12} - (L_{21}A_{21} - WL_{11}A_{11})U_{11}V)U_{22}. \end{aligned} \tag{20}$$

Therefore, we have to prove these block equalities.

Let us note that from the identity $A_{11} = I_1 A_{11} J_1$ and Sentence 1 we get

$$L_{11} = \overline{I}_{11} + I_1 L_{11} I_{11},\ U_{11} = \overline{J}_{11} + J_{11} U_{12} J_1. \tag{21}$$

Sentence 1 together with equations $A^1_{12} = \overline{I}_{11} L_{11} A_{12}$, $A^1_{21} = A_{21} U_{11} \overline{J}_{11}$, $A^2_{22} = \overline{I}_{21} L_{21} (A_{22} - A_{21} U_{11} E^T_{11} L_{11} A_{12}) U_{12} \overline{J}_{12}$ give the next properties of L- and U-blocks:

$$\begin{array}{ll} L_{12} = \overline{I}_{12} + \overline{I}_{11} I_1 L_{12} I_{12}, & U_{12} = \overline{J}_{12} + J_{12} U_{12} J_2, \\ L_{21} = \overline{I}_{21} + I_2 L_{21} I_{21}, & U_{21} = \overline{J}_{21} + J_{21} U_{12} J_1 \overline{J}_{11}, \\ L_{22} = \overline{I}_{22} + \overline{I}_{21} I_2 L_{22} I_{22}, & U_{22} = \overline{J}_{22} + J_{22} U_{22} J_2 \overline{J}_{12}. \end{array} \tag{22}$$

The following identities can easily be checked now:

$$L_{12} E_{11} = E_{11},\ L_{12} I_{11} = I_{11}, \tag{23}$$

$$E_{11} U_{21} = E_{11},\ J_{11} U_{21} = J_{11}, \tag{24}$$

$$E_{12} U_{22} = E_{12},\ J_{12} U_{22} = J_{12}, \tag{25}$$

$$L_{22} E_{21} = E_{21},\ L_{22} I_{21} = I_{21}. \tag{26}$$

We shall use the following equalities,

$$L_{11} A_{11} U_{11} = E_{11}, L_{12} A^1_{12} U_{12} = E_{12}, L_{21} A^1_{21} U_{21} = E_{21}, L_{22} A^2_{22} U_{22} = E_{22}, \tag{27}$$

which follow from (7),(10), and (12), the equality

$$E_{11} V = I_{11} Q U_{12}, \tag{28}$$

which follows from the definition of the block V in (13), (24), (16) and (6), the equality

$$W E_{11} = L_{21} B J_{11}, \tag{29}$$

which follows from the definition of the block W in (13), (23), (17) and (6).

1. The first equality of (20) follows from (27), (23) and (24).

2. The right-hand side of the second equality of (20) takes the form $L_{12}(\mathbf{I} - I_{11}) Q U_{12} U_{22}$ due to (8), (27) and (28). To prove the second equality we use the definition of the blocks B and A^1_{12} in (8) and (9), then the second equality in (27) and identity (25): $L_{12}(\mathbf{I} - I_{11}) Q U_{12} U_{22} = L_{12} A^1_{12} U_{12} U_{22} = E_{12} U_{22} = E_{12}$.

3. The right-hand side of the third equality of (20) takes the form $L_{22} L_{21} B (\mathbf{I} - J_{11}) U_{21}$ due to definition of the block B (8), the first equality in (27) and (29). To prove the third equality we use the definition of the blocks A^1_{21} in (9), then the third equality in (27) and identity (26): $L_{22} L_{21} B \overline{J}_{11} U_{21} = L_{22} L_{21} A^1_{21} U_{21} = L_{22} E_{21} = E_{21}$.

4. The identity

$$E^T_{12} L_{12} = E^T_{12} L_{12} (I_{11} + \overline{I}_{11}) = E^T_{12} L_{12} \overline{I}_{11} \tag{30}$$

follows from (23) and (17).

We have to check that $(L_{21}A_{22} - WL_{11}A_{12})U_{12} = (L_{21}A_{22} - (GE_{12}^T L_{12} + L_{21}BE_{11}^T)Q)U_{12} = L_{21}(A_{22} - BE_{11}^T Q)U_{12} - GE_{12}^T L_{12}QU_{12} = L_{21}A_{22}^1 U_{12} - GE_{12}^T L_{12}\overline{I}_{11}QU_{12} = G - GE_{12}^T L_{12}A_{12}^1 U_{12} = G - GE_{12}^T E_{12} = G\overline{J}_{12}$, using the definitions of the blocks W in (13), A_{22}^1 and A_{12}^1 in (9), identity (28), the second equality in (27), and definition (6).

We have to check that

$$
\begin{aligned}
&-(L_{21}A_{21} - WL_{11}A_{11})U_{11}V = -(L_{21}A_{21}U_{11} - WE_{11})V \\
&= (-L_{21}B + L_{21}BJ_{11})V = -L_{21}B\overline{J}_{11}V \\
&= -L_{21}B\overline{J}_{11}(U_{21}E_{21}^T G\overline{J}_{12} + E_{11}^T QU_{12}) = -L_{21}A_{21}^1 U_{21}E_{21}^T G\overline{J}_{12} = -I_{21}G\overline{J}_{12}
\end{aligned}
$$

using the first equality in (27), the identity (29), the definitions of the blocks V in (13), (1), then the third equality in (27) and definition (6).

To prove the fourth equality we have to substitute obtained expressions into the right-hand side of the fourth equality:

$$L_{22}(G\overline{J}_{12} - I_{21}G\overline{J}_{12})U_{22} = L_{22}\overline{I}_{21}G\overline{J}_{12}U_{22} = L_{22}A_{22}^2 U_{22} = E_{22}.$$

For the completion of the proof of this theorem we have to demonstrate the special form of the matrices U and L: $L - \overline{I}_E \in F_{I,I_E}$ and $U - \overline{J}_E \in F_{J_E,J}$.

The matrix L is invertible and $I_E < I$, therefore, we have to prove that $L = \overline{I}_E + ILI_E$, where $I_E = \text{diag}(I_{11} + I_{12}, I_{21} + I_{22})$, $\overline{I}_E = \text{diag}(\overline{I}_{11}\overline{I}_{12}, \overline{I}_{21}\overline{I}_{22})$, $I = \text{diag}(I_1, I_2)$.

This matrix equality for matrix L (14) is equivalent to the four block equalities:

$$L_{12}L_{11} = I_1 L_{12}L_{11}(I_{11} + I_{12}) + \overline{I}_{11}\overline{I}_{12},\ 0 = I_1 0(I_{21} + I_{22}),$$

$$-L_{22}WL_{11} = -I_2 L_{22}WL_{11}(I_{11} + I_{12}),\ L_{22}L_{21} = I_2 L_{22}L_{21}(I_{21} + I_{22}) + \overline{I}_{21}\overline{I}_{22}.$$

To prove the first block equalities we have to multiply its left-hand side by the unit matrix in the form $\mathbf{I} = (I_1 + \overline{I}_1)$ from the left side and by the unit matrix in the form $\mathbf{I} = (I_{11} + I_{12}) + \overline{I}_{11}\overline{I}_{12}$ from the left side. Then we use the following identities to obtain in the left-hand side the same expression as in the right-hand side: $L_{11}\overline{I}_{11} = \overline{I}_{11}$, $L_{12}\overline{I}_{12} = \overline{I}_{12}$, $\overline{I}_1 L_{12}L_{11} = \overline{I}_1$, $\overline{I}_1(I_{11} + I_{12}) = 0$. The same idea may be used for proving the last block equality, but we must use other forms of unit matrix: $\mathbf{I} = (I_2 + \overline{I}_2)$, $\mathbf{I} = (I_{21} + I_{22}) + \overline{I}_{21}\overline{I}_{22}$.

The second block equality is evident.

Let us prove the third block equality. We have to multiply the left-hand side of the third block equality by the unit matrix in the form $\mathbf{I} = (I_2 + \overline{I}_2)$ from the left side and by the unit matrix in the form $\mathbf{I} = (I_{11} + I_{12}) + \overline{I}_{11}\overline{I}_{12}$ from the right side.

The block W is equal to the following expression by definitions (13), (11), and (8):

$$W = (L_{21}(A_{22} - A_{21}U_{11}E_{11}^T Q)U_{12}E_{12}^T L_{12} + L_{21}A_{21}U_{11}E_{11}^T).$$

We have to use in the left-hand side the equations $\overline{I}_2 L_{22} = \overline{I}_2$, $\overline{I}_2 L_{21} = \overline{I}_2$, $\overline{I}_2 A_{22} = 0$, $\overline{I}_2 A_{21} = 0$, and $L_{11}\overline{I}_{11} = \overline{I}_{11}$, $L_{12}\overline{I}_{12} = \overline{I}_{12}$, $E_{12}^T\overline{I}_{12} = 0$, $E_{11}^T\overline{I}_{11} = 0$.

The property of the matrix U: $U - \overline{J}_E \in F_{J_E,J}$ may be proved in the same way as the property of the matrix L.

Theorem 2. *For any matrix A of size $s(s \geq 1)$, an algorithm of LEU-decomposition exists, which has the same complexity as matrix multiplication.*

Proof. We have proved an existence of LEU-decomposition for matrices of size $2^k, k > 0$. Let $A \in F_{I,J}^{s \times s}$ be a matrix of size $2^{k-1} < s < 2^k$, A' be a matrix of size 2^k, which has in the left upper corner the submatrix equal to A, and all other elements equal zero. We can construct LEU-decomposition of matrix A': $(L', E', U') = \mathcal{LU}(A')$. According to the Sentence 1 the product $L'A'U' = E'$ has the form

$$\begin{pmatrix} L & 0 \\ 0 & \mathbf{I} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & \mathbf{I} \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}$$

Therefore, $LAU = E$ is a LEU decomposition of matrix A.

The total amount of matrix multiplications in (7)-(15) is equal to 17, and total amount of recursive calls is equal to 4. We do not consider multiplications of the permutation matrices, we can do these multiplications due to permutation of pointers for the blocks which are disposed at the local processors.

We can compute the decomposition of the 2×2 matrix by means of 5 multiplicative operations. Therefore, we obtain the following recurrence equality for complexity

$$t(n) = 4t(n/2) + 17M(n/2), t(2) = 5.$$

Let γ and β be constants, $3 \geq \beta > 2$, and let $M(n) = \gamma n^\beta + o(n^\beta)$ be the number of multiplication operations in one $n \times n$ matrix multiplication.

After summation from $n = 2^k$ to 2^1 we obtain

$$17\gamma(4^0 2^{\beta(k-1)} + \ldots + 4^{k-2} 2^{\beta 1}) + 4^{k-2} 5 = 17\gamma \frac{n^\beta - 2^{\beta-2} n^2}{2^\beta - 4} + \frac{5}{16} n^2.$$

Therefore, the complexity of the decomposition is

$$\sim \frac{17\gamma n^\beta}{2^\beta - 4}.$$

If A is an invertible matrix, then $A^{-1} = UE^T L$, and a recursive block algorithm of matrix inversion is written in expressions (7)–(15). This algorithm has the complexity of matrix multiplications.

4 Conclusion

The algorithms for finding the generalized Bruhat decomposition and matrix inversion are described. These algorithms have the same complexity as matrix multiplication and do not require pivoting. For singular matrices, they allow to obtain a nonsingular block of the biggest size. These algorithms may be used in any field, including real and complex numbers, finite fields and their extensions.

The proposed algorithms are pivot-free and do not change the matrix block structure. So they are suitable for parallel hardware implementation.

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