A Sequence of Albin Type Continuous Martingales with Brownian Marginals and Scaling

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Dedicated to Lester Dubins (1921–2010) to whom the third author owes a lot.

Abstract Closely inspired by Albin's method which relies ultimately on the duplication formula for the Gamma function, we exploit Gauss' multiplication formula to construct a sequence of continuous martingales with Brownian marginals and scaling.

Keywords Martingales · Brownian marginals

1 Motivation and Main Results

(1.1) Knowing the law of a "real world" random phenomena, i.e. random process, $(X_t, t \geq 0)$ is often extremely difficult and in most instances, one avails only of the knowledge of the 1-dimensional marginals of $(X_t, t \geq 0)$. However, there may be many different processes with the same given 1-dimensional marginals.

In the present paper, we make explicit a sequence of continuous martingales $(M_m(t), t \geq 0)$ indexed by $m \in \mathbb{N}$ such that for each m:

(i) $(M_m(t), t \geq 0)$ enjoys the Brownian scaling property: for any $c > 0$,

$$(M_m(c^2 t), t \geq 0) \stackrel{(law)}{=} (c M_m(t), t \geq 0)$$

(ii) $M_m(1)$ is standard Gaussian.

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Note that, combining (i) and (ii), we get, for any $t > 0$

$$M_m(t) \overset{(law)}{=} B_t,$$

where $(B_t, t \geq 0)$ is a Brownian motion, i.e. M_m admits the same 1-dimensional marginals as Brownian motion.

(1.2) Our main result is the following extension of Albin's construction [1] from $m = 1$ to any integer m.

Theorem 1. *Let $m \in \mathbb{N}$. Then, there exists a continuous martingale $(M_m(t), t \geq 0)$ which enjoys (i) and (ii) and is defined as follows:*

$$M_m(t) = X_t^{(1)} \dots X_t^{(m+1)} Z_m \tag{1}$$

where $(X_t^{(i)}, t \geq 0)$, for $i = 1, \dots, m+1$, are independent copies of the solution of the SDE

$$dX_t = \frac{1}{m+1} \frac{dB_t}{X_t^m}; \quad X_0 = 0 \tag{2}$$

and, furthermore, Z_m is independent from $(X^{(1)}, \dots, X^{(m+1)})$ and

$$Z_m \overset{(law)}{=} (m+1)^{1/2} \left(\prod_{j=0}^{m-1} \beta\left(\frac{1+2j}{2(m+1)}, \frac{m-j}{m+1} \right) \right)^{\frac{1}{2(m+1)}} \tag{3}$$

where $\beta(a, b)$ denotes a beta variable with parameter (a, b) with density

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1} 1_{[0,1]}(x)$$

and the beta variables on the right-hand side of (3) are independent.

Remark. For $m = 1$, $Z_1 = \sqrt{2}\left(\beta(\frac{1}{4}, \frac{1}{2})\right)^{1/4}$ and we recover the distribution of $Y := Z_1$ given by (2) in [1].

(1.3) For the convenience of the reader, we also recall that, if one drops the continuity assumption when searching for martingales $(M(t); t \geq 0)$ satisfying (i) and (ii), then, the Madan-Yor construction [5] based on the “Azéma-Yor under scaling" method provides such a martingale.

Precisely, starting from a Brownian motion $(B_u, u \geq 0)$ and denoting $S_u = \sup_{s \leq u} B_s$, introduce the family of stopping times

$$\tau_t = \inf\{u, S_u \geq \psi_t(B_u)\}$$

where ψ_t denotes the Hardy-Littlewood function associated with the centered Gaussian distribution μ_t with variance t, i.e.

$$\psi_t(x) = \frac{1}{\mu_t([x,\infty[)} \int_x^\infty y \exp\left(-\frac{y^2}{2t}\right) \frac{dy}{\sqrt{2\pi t}}$$
$$= \sqrt{t} \exp\left(-\frac{x^2}{2t}\right) / \mathcal{N}(x/\sqrt{t})$$

where $\mathcal{N}(a) = \int_a^\infty \exp(-\frac{y^2}{2})dy$. Then, $M_t = B_{\tau_t}$ is a martingale with Brownian marginals.

Another solution has been given by Hamza and Klebaner [4].

(1.4) In Sect. 3, we prove that Theorem 1 is actually the best we can do in our generalisation of Albin's construction: we cannot generalize (1) by allowing the $X^{(i)}$'s to be solution of (2) associated to different m_i's.

Finally, we study the asymptotic behavior of $X_t^{(1)} \dots X_t^{(m+1)}$ as $m \longrightarrow \infty$.

2 Proof of Theorem 1

Step 1: For $m \in \mathbb{R}$ and $c \in \mathbb{R}$, we consider the stochastic equation:

$$dX_t = c\frac{dB_t}{X_t^m}, \quad X_0 = 0.$$

This equation has a unique weak solution which can be defined as a time-changed Brownian motion

$$(X_t) \overset{(law)}{=} W(\alpha^{(-1)}(t))$$

where W is a Brownian motion starting from 0 and $\alpha^{(-1)}$ is the (continuous) inverse of the increasing process

$$\alpha(t) = \frac{1}{c^2} \int_0^t W_u^{2m} du.$$

We look for $k \in \mathbb{N}$ and c such that $(X_t^{2k}, t \geq 0)$ is a squared Bessel process of some dimension d. It turns out, by application of Itô's formula, that we need to take $k = m + 1$ and $c = \frac{1}{m+1}$. Thus, we find that $(X_t^{2(m+1)}, t \geq 0)$ is a squared Bessel process with dimension $d = k(2k-1)c^2 = \frac{2m+1}{m+1}$.

Note that the law of a BESQ(d) process at time 1 is well known to be that of $2\gamma_{d/2}$, where γ_a denotes a gamma variable with parameter a. Thus, we have:

$$|X_1| \overset{(law)}{=} \left(2\gamma_{\frac{2m+1}{2(m+1)}}\right)^{\frac{1}{2(m+1)}} \tag{4}$$

Step 2: We now discuss the scaling property of the solution of (2). From the scaling property of Brownian motion, it is easily shown that, for any $\lambda > 0$, we get:

$$(X_{\lambda t}, t \geq 0) \overset{(law)}{=} (\lambda^{\alpha} X_t, t \geq 0)$$

with $\alpha = \frac{1}{2(m+1)}$, that is, the process $(X_t, t \geq 0)$ enjoys the scaling property of order $\frac{1}{2(m+1)}$.

Step 3: Consequently, if we multiply $m + 1$ independent copies of the process $(X_t, t \geq 0)$ solution of (2), we get a process

$$Y_t = X_t^{(1)} \dots X_t^{(m+1)}$$

which is a martingale and has the scaling property of order $\frac{1}{2}$.

Step 4: Finally, it suffices to find a random variable Z_m independent of the processes $X_t^{(1)}, \dots, X_t^{(m+1)}$ and which satisfies:

$$N \overset{(law)}{=} X_1^{(1)} \dots X_1^{(m+1)} Z_m \tag{5}$$

where N denotes a standard Gaussian variable. Note that the distribution of any of the $X_1^{(i)}$'s is symmetric. We shall take $Z_m \geq 0$; thus, the distribution of Z_m shall be determined by its Mellin transform $\mathcal{M}(s) = \mathbb{E}(Z_m^s)$. From (5), $\mathcal{M}(s)$ satisfies:

$$\mathbb{E}[(2\gamma_{1/2})^{s/2}] = \left(\mathbb{E}\left[(2\gamma_{d/2})^{s/2(m+1)}\right]\right)^{m+1} \mathcal{M}(s)$$

with $d = \frac{2m+1}{m+1}$, that is:

$$2^{s/2} \frac{\Gamma(\frac{1+s}{2})}{\Gamma\left(\frac{1}{2}\right)} = 2^{s/2} \left(\frac{\Gamma\left(\frac{d}{2} + \frac{s}{2(m+1)}\right)}{\Gamma(\frac{d}{2})} \right)^{m+1} \mathcal{M}(s)$$

that is precisely:

$$\frac{\Gamma(\frac{1+s}{2})}{\Gamma\left(\frac{1}{2}\right)} = \left(\frac{\Gamma\left(\frac{2m+1+s}{2(m+1)}\right)}{\Gamma(\frac{2m+1}{2(m+1)})} \right)^{m+1} \mathcal{M}(s). \tag{6}$$

Now, we recall Gauss multiplication formula ([2], see also [3])

$$\Gamma(kz) = \frac{k^{kz-1/2}}{(2\pi)^{\frac{k-1}{2}}} \prod_{j=0}^{k-1} \Gamma\left(z + \frac{j}{k}\right) \tag{7}$$

which we apply with $k = m + 1$ and $z = \frac{1+s}{2(m+1)}$. We then obtain, from (7)

$$\frac{\Gamma(\frac{1+s}{2})}{\sqrt{\pi}} = \frac{(m+1)^{s/2}}{(2\pi)^{m/2}} \frac{1}{\sqrt{\pi}} \prod_{j=0}^{m} \Gamma\left(\frac{1+s+2j}{2(m+1)}\right) \tag{8}$$

$$= (m+1)^{s/2} \prod_{j=0}^{m} \left(\frac{\Gamma\left(\frac{1+s+2j}{2(m+1)}\right)}{\Gamma\left(\frac{1+2j}{2(m+1)}\right)}\right) \tag{9}$$

since the two sides of (8) are equal to 1 for $s = 0$. We now plug (9) into (6) and obtain

$$(m+1)^{s/2} \prod_{j=0}^{m} \left(\frac{\Gamma\left(\frac{1+s+2j}{2(m+1)}\right)}{\Gamma\left(\frac{1+2j}{2(m+1)}\right)}\right) = \left(\frac{\Gamma\left(\frac{2m+1+s}{2(m+1)}\right)}{\Gamma\left(\frac{2m+1}{2(m+1)}\right)}\right)^{m+1} \mathcal{M}(s) \tag{10}$$

We note that for $j = m$, the same term appears on both sides of (10), thus (10) may be written as:

$$(m+1)^{s/2} \prod_{j=0}^{m-1} \left(\frac{\Gamma\left(\frac{1+s+2j}{2(m+1)}\right)}{\Gamma\left(\frac{1+2j}{2(m+1)}\right)}\right) = \left(\frac{\Gamma\left(\frac{2m+1+s}{2(m+1)}\right)}{\Gamma\left(\frac{2m+1}{2(m+1)}\right)}\right)^{m} \mathcal{M}(s) \tag{11}$$

In terms of independent gamma variables, the left-hand side of (11) equals:

$$(m+1)^{s/2} \mathbb{E}\left[\left(\prod_{j=0}^{m-1} \gamma^{(j)}_{\frac{1+2j}{2(m+1)}}\right)^{\frac{s}{2(m+1)}}\right] \tag{12}$$

whereas the right-hand side of (11) equals:

$$\mathbb{E}\left[\left(\prod_{j=0}^{m-1} \gamma^{(j)}_{\frac{1+2m}{2(m+1)}}\right)^{\frac{s}{2(m+1)}}\right] \mathcal{M}(s) \tag{13}$$

where the $\gamma^{(j)}_{a_j}$ denote independent gamma variables with respective parameters a_j.

Now, from the beta-gamma algebra, we deduce, for any $j \le m-1$:

$$\gamma^{(j)}_{\frac{1+2j}{2(m+1)}} \overset{(law)}{=} \gamma^{(j)}_{\frac{1+2m}{2(m+1)}} \beta\left(\frac{1+2j}{2(m+1)}, \frac{m-j}{m+1}\right).$$

Thus, we obtain, again by comparing (12) and (13):

$$\mathcal{M}(s) = (m+1)^{s/2}\,\mathbb{E}\left[\left(\prod_{j=0}^{m-1}\beta\left(\frac{1+2j}{2(m+1)},\frac{m-j}{m+1}\right)\right)^{\frac{s}{2(m+1)}}\right]$$

which entails:

$$\mathbb{E}[Z_m^s] = (m+1)^{s/2}\,\mathbb{E}\left[\left(\prod_{j=0}^{m-1}\beta\left(\frac{1+2j}{2(m+1)},\frac{m-j}{m+1}\right)\right)^{\frac{s}{2(m+1)}}\right]$$

that is, equivalently,

$$Z_m \overset{(law)}{=} (m+1)^{1/2}\left(\prod_{j=0}^{m-1}\beta\left(\frac{1+2j}{2(m+1)},\frac{m-j}{m+1}\right)\right)^{\frac{1}{2(m+1)}}$$

3 Some Remarks About Theorem 1

3.1 A Further Extension

We tried to extend Theorem 1 by taking a product of independent martingales $X^{(i)}$, solution of (2) with different m_i's. Here are the details of our attempt. We are looking for the existence of a variable Z such that the martingale

$$M(t) = \left(\prod_{j=0}^{p-1} X_t^{(m_j)}\right) Z$$

satisfies the properties i) and ii). Here $p, (m_j)_{0\le j\le p-1}$ are integers and $X^{(m_j)}$ is the solution of the EDS (2) associated to m_j, the martingales being independent for j varying. In order that M enjoys the Brownian scaling property, we need the following relation

$$\sum_{j=0}^{p-1}\frac{1}{m_j+1} = 1. \tag{14}$$

Following the previous computations, see (6), the Mellin transform $\mathcal{M}(s)$ of Z should satisfy

$$\frac{\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \left(\prod_{j=0}^{p-1}\frac{\Gamma\left(\frac{2m_j+1+s}{2(m_j+1)}\right)}{\Gamma\left(\frac{2m_j+1}{2(m_j+1)}\right)}\right)\mathcal{M}(s). \tag{15}$$

We recall (see (9)) the Gauss multiplication formula

$$\frac{\Gamma\left(\frac{1+s}{2}\right)}{\sqrt{\pi}} = p^{s/2} \prod_{j=0}^{p-1} \left(\frac{\Gamma\left(\frac{1+s+2j}{2p}\right)}{\Gamma\left(\frac{1+2j}{2p}\right)} \right) \tag{16}$$

To find $\mathcal{M}(s)$ from (15), (16), we give some probabilistic interpretation:

$$\frac{\Gamma\left(\frac{1+s+2j}{2p}\right)}{\Gamma\left(\frac{1+2j}{2p}\right)} = \mathbb{E}\left[\gamma_{(1+2j)/2p}^{s/2p}\right]$$

whereas

$$\frac{\Gamma\left(\frac{2m_j+1+s}{2(m_j+1)}\right)}{\Gamma\left(\frac{2m_j+1}{2(m_j+1)}\right)} = \mathbb{E}\left[\gamma_{(1+2m_j)/2(m_j+1)}^{s/2(m_j+1)}\right].$$

Thus, we would like to factorize

$$\gamma_{(1+2j)/2p}^{1/2p} \overset{(law)}{=} \gamma_{(1+2m_j)/2(m_j+1)}^{1/2(m_j+1)} z_{m_j,p}^{(j)} \tag{17}$$

for some variable $z_{m_j,p}^{(j)}$ to conclude that

$$Z = p^{1/2} \prod_{j=0}^{p-1} z_{m_j,p}^{(j)}.$$

It remains to find under which condition the identity (17) may be fulfilled. We write

$$\gamma_{(1+2j)/2p} \overset{(law)}{=} \gamma_{(1+2m_j)/2(m_j+1)}^{p/(m_j+1)} (z_{m_j,p}^{(j)})^{2p}. \tag{18}$$

Now, if $\frac{1+2j}{2p} < \frac{1+2m_j}{2(m_j+1)}$, we may apply the beta-gamma algebra to obtain

$$\gamma_{(1+2j)/2p} \overset{(law)}{=} \gamma_{(1+2m_j)/2(m_j+1)} \beta\left(\frac{1+2j}{2p}, \frac{1+2m_j}{2(m_j+1)} - \frac{1+2j}{2p}\right)$$

but in (18), we need to have on the right-hand side $\gamma_{(1+2m_j)/2(m_j+1)}^{p/(m_j+1)}$ instead of $\gamma_{(1+2m_j)/2(m_j+1)}$.

However, it is known that

$$\gamma_a \overset{(law)}{=} \gamma_a^c \gamma_{a,c}$$

for some variable $\gamma_{a,c}$ independent of γ_a for any $c \in (0, 1]$. This follows from the self-decomposable character of $\ln(\gamma_a)$. Thus, we seem to need $\frac{p}{m_j+1} \leq 1$. But, this condition is not compatible with (14) unless $m_j = m = p - 1$.

3.2 Asymptotic Study

We study the behavior of the product $X_1^{(1)} \dots X_1^{(m+1)}$, resp. Z_m, appearing in the right-hand side of the equality in law (5), when $m \longrightarrow \infty$. Recall from (4) that

$$|X_1| \overset{(law)}{=} \left(2\gamma_{\frac{2m+1}{2(m+1)}}\right)^{\frac{1}{2(m+1)}}.$$

We are thus led to consider the product

$$\Theta_{a,b,c}^{(p)} = \left(\prod_{i=1}^{p} \gamma_{a-b/p}^{(i)}\right)^{c/p}$$

where in our set up of Theorem 1, $p = m+1, a = 1, b = c = 1/2$.

$$\begin{aligned}
\mathbb{E}\left[\left(\Theta_{a,b,c}^{(p)}\right)^s\right] &= \prod_{i=1}^{p} \mathbb{E}\left[\left(\gamma_{a-b/p}^{(i)}\right)^{cs/p}\right] \\
&= \left(\frac{\Gamma\left(a - \frac{b}{p} + \frac{cs}{p}\right)}{\Gamma\left(a - \frac{b}{p}\right)}\right)^p \\
&= \exp\left[p\left(\ln\left(\Gamma\left(a + \frac{cs-b}{p}\right)\right) - \ln\left(\Gamma\left(a - \frac{b}{p}\right)\right)\right)\right] \\
&\underset{p\to\infty}{\longrightarrow} \exp\left(\frac{\Gamma'(a)}{\Gamma(a)} cs\right).
\end{aligned}$$

Thus, it follows that

$$\Theta_{a,b,c}^{(p)} \underset{p\to\infty}{\overset{\mathbb{P}}{\longrightarrow}} \exp\left(\frac{\Gamma'(a)}{\Gamma(a)} c\right),$$

implying that

$$|X_1^{(1)} \dots X_1^{(m+1)}| \underset{m\to\infty}{\overset{\mathbb{P}}{\longrightarrow}} \exp(-\gamma/2) \tag{19}$$

and

$$\exp(-\gamma/2) Z_m \underset{m\to\infty}{\overset{(law)}{\longrightarrow}} |N|. \tag{20}$$

where $\gamma = -\Gamma'(1)$ is the Euler constant.

We now look for a central limit theorem for $\Theta_{a,b,c}^{(p)}$. We consider the limiting distribution of

$$\sqrt{p}\left\{\frac{c}{p}\sum_{i=1}^{p} \ln\left(\gamma_{a-b/p}^{(i)}\right) - c\frac{\Gamma'(a)}{\Gamma(a)}\right\}.$$

$$
\begin{aligned}
&\mathbb{E}\left(\exp\left[cs\sqrt{p}\left\{\frac{1}{p}\sum_{i=1}^{p}\ln\left(\gamma_{a-b/p}^{(i)}\right)-\frac{\Gamma'(a)}{\Gamma(a)}\right\}\right]\right)\\
&=\mathbb{E}\left[\prod_{i=1}^{p}\left(\gamma_{a-b/p}^{(i)}\right)^{cs/\sqrt{p}}\right]\exp\left(-cs\sqrt{p}\frac{\Gamma'(a)}{\Gamma(a)}\right)\\
&=\mathbb{E}\left[\left(\gamma_{a-b/p}^{(i)}\right)^{cs/\sqrt{p}}\right]^{p}\exp\left(-cs\sqrt{p}\frac{\Gamma'(a)}{\Gamma(a)}\right)\\
&=\left(\frac{\Gamma\left(a-\frac{b}{p}+\frac{cs}{\sqrt{p}}\right)}{\Gamma\left(a-\frac{b}{p}\right)}\right)^{p}\exp\left(-cs\sqrt{p}\frac{\Gamma'(a)}{\Gamma(a)}\right)\\
&=\exp\left[p\left(\ln\left(\Gamma\left(a-\frac{b}{p}+\frac{cs}{\sqrt{p}}\right)\right)-\ln\left(\Gamma\left(a-\frac{b}{p}\right)\right)\right)-cs\sqrt{p}\frac{\Gamma'(a)}{\Gamma(a)}\right]\\
&=\exp\left(\frac{c^2s^2}{2}(\ln(\Gamma))''(a)+O(m^{-1/2})\right)
\end{aligned}
$$

We thus obtain that

$$
\sqrt{p}\left\{\frac{c}{m}\sum_{i=1}^{m}\ln(\gamma_{a-b/m}^{(i)})-c\frac{\Gamma'(a)}{\Gamma(a)}\right\}\overset{(law)}{\longrightarrow}N(0,\sigma^2) \tag{21}
$$

where $N(0,\sigma^2)$ denotes a centered Gaussian variable with variance:

$$
\sigma^2=c^2(\ln(\Gamma))''(a)=c^2\left[\frac{\Gamma''(a)}{\Gamma(a)}-\left(\frac{\Gamma'(a)}{\Gamma(a)}\right)^2\right].
$$

or, equivalently

$$
\left(\Theta_{a,b,c}^{(p)}\exp\left(\frac{\Gamma'(a)}{\Gamma(a)}c\right)\right)^{\sqrt{p}}\underset{p\to\infty}{\overset{(law)}{\longrightarrow}}\exp(N(0,c^2(\ln(\Gamma))''(a))). \tag{22}
$$

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