2. The Perceptron Model

2.1 Introduction

The name of this chapter comes from the theory of neural networks. An accessible introduction to neural networks is provided in [83], but what these are is not relevant to our purpose, which is to study the underlying mathematics. Roughly speaking, the basic problem is as follows. What "proportion" of $\Sigma_N = \{-1,1\}^N$ is left when one intersects this set with many random half-spaces? A natural definition for a random half-space is a set $\{\mathbf{x} \in \mathbb{R}^N \; ; \; \mathbf{x} \cdot \mathbf{v} \geq 0\}$ where the random vector $\mathbf{v}$ is uniform over the unit sphere of $\mathbb{R}^N$. More conveniently one can consider the set $\{\mathbf{x} \in \mathbb{R}^N ; \mathbf{x} \cdot \mathbf{g} \geq 0\}$, where $\mathbf{g}$ is a standard Gaussian vector, i.e. $\mathbf{g} = (g_i)_{i \leq N}$, where g_i are independent standard Gaussian r.v.s. This is equivalent because the vector $\mathbf{g}/\|\mathbf{g}\|$ is uniformly distributed on the unit sphere of $\mathbb{R}^N$. Consider now M such Gaussian vectors $\mathbf{g}_k = (g_{i,k})_{i \leq N}$, $k \leq M$, all independent, the half-spaces

$$U_k = \{\mathbf{x} ; \; \mathbf{x} \cdot \mathbf{g}_k \geq 0\} = \Big\{\mathbf{x} \, , \, \sum_{i \leq N} g_{i,k} x_i \geq 0\Big\} \, ,$$

and the set

$$\Sigma_N \cap \bigcap_{k \leq M} U_k \, . \tag{2.1}$$

A given point of Σ_N has exactly a 50% chance to belong to U_k, so that

$$\mathsf{E}\,\mathrm{card}\Big(\Sigma_N \cap \bigcap_{k \leq M} U_k\Big) = 2^{N-M} \, . \tag{2.2}$$

The case of interest is when N becomes large and M is proportional to N, $M/N \to \alpha > 0$. A consequence of (2.2) is that if $\alpha > 1$ the set (2.1) is typically empty when N is large, because the expected value of its cardinality is $\ll 1$. When $\alpha < 1$, what is interesting is *not* however the expected value (2.2) of the cardinality of the set (2.1), but rather the typical value of this cardinality, which is likely to be smaller. Our ultimate goal is the computation of this typical value, which we will achieve only for α small enough.

A similar problem was considered in (0.2) where Σ_N is replaced by the sphere $\mathbb{S}_N$ of center 0 and radius $\sqrt{N}$. The situation with Σ_N is usually

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called the *binary perceptron*, while the situation with $\mathbb{S}_N$ is usually called the *spherical perceptron*. The spherical perceptron will motivate the next chapter. We will return to both the binary and the spherical perceptron in Volume II, in Chapter 8 and Chapter 9 respectively. Both the spherical and the binary perceptron admit another popular version, where the Gaussian r.v.s $g_{i,j}$ are replaced by independent Bernoulli r.v.s (i.e. independent random signs), and we will also study these. Thus we will eventually investigate a total of four related but different models. It is not very difficult to replace the Gaussian r.v.s by random signs; but it is very much harder to study the case of Σ_N than the case of the sphere.

Research Problem 2.1.1. (Level 3!) Prove that there exists a number α^* and a function $\varphi : [0, \alpha^*) \to \mathbb{R}$ with the following properties:

1 - If $\alpha > \alpha^*$, then as $N \to \infty$ and $M/N \to \alpha$ the probability that the set (2.1) is not empty is at most $\exp(-N/K(\alpha))$.

2 - If $\alpha < \alpha^*$, $N \to \infty$ and $M/N \to \alpha$, then

$$\frac{1}{N} \log \operatorname{card}\left(\Sigma_N \cap \bigcap_{k \le M} U_k\right) \to \varphi(\alpha) \tag{2.3}$$

in probability. Compute α^* and φ.

This problem is a typical example of a situation where one expects "regularity" as $N \to \infty$, but where it is unclear how to even start doing anything relevant. In Volume II, we will prove (2.3) when α is small enough, and we will compute $\varphi(\alpha)$ in that case. (We expect that the case of larger α is much more difficult.) As a corollary, we will prove that there exists a number $\alpha_0 < 1$ such that if $M = \lfloor \alpha N \rfloor$, $\alpha > \alpha_0$, then the set (2.1) is typically empty for N large, despite the fact that the expected value of its cardinality is $2^{N-M} \gg 1$.

One way to approach the (very difficult) problem mentioned above is to introduce a version "with a temperature". We observe that if $x \ge 0$ we have $\lim_{\beta \to \infty} \exp(-\beta x) = 0$ if $x > 0$ and $= 1$ if $x = 0$. Using this for $x = \sum_{k \le M} \mathbf{1}_{\{\boldsymbol{\sigma} \notin U_k\}}$ where $\boldsymbol{\sigma} \in \Sigma_N$ implies

$$\operatorname{card}\left(\Sigma_N \cap \bigcap_{k \le M} U_k\right) = \lim_{\beta \to \infty} \sum_{\boldsymbol{\sigma} \in \Sigma_N} \exp\left(-\beta \sum_{k \le M} \mathbf{1}_{\{\boldsymbol{\sigma} \notin U_k\}}\right), \tag{2.4}$$

so that to study (2.3) it should be relevant to use the Hamiltonian

$$-H_{N,M}(\boldsymbol{\sigma}) = -\beta \sum_{k \le M} \mathbf{1}_{\{\boldsymbol{\sigma} \notin U_k\}}. \tag{2.5}$$

If one can compute the corresponding partition function (and succeed in exchanging the limits $N \to \infty$ and $\beta \to \infty$), one will then prove (2.3).

More generally, we will consider Hamiltonians of the type

$$- H_{N,M}(\boldsymbol{\sigma}) = \sum_{k \le M} u\left(\frac{1}{\sqrt{N}} \sum_{i \le N} g_{i,k}\sigma_i\right) , \tag{2.6}$$

where u is a function, and where $(g_{i,k})$ are independent standard normal r.v.s. Of course the Hamiltonian depends on u, but the dependence is kept implicit. The role of the factor $N^{-1/2}$ is to make the quantity $N^{-1/2}\sum_{i\le N} g_{i,k}\sigma_i$ typically of order 1. There is no parameter β in the right-hand side of (2.6), since this parameter can be thought of as being included in the function u.

Since it is difficult to prove anything at all without using integration by parts we will always assume that u is differentiable. But if we want the Hamiltonian (2.6) to be a fair approximation of the Hamiltonian (2.5), we will have to accept that u' takes very large values. Then, in the formulas where u' occurs, we will have to show that somehow these large values cancel out. There is no magic way to do this, one has to work hard and prove delicate estimates (as we will do in Chapter 8). Another source of difficulty is that we want to approximate the Hamiltonian (2.5) for large values of β. That makes it difficult to bound from below a number of quantities that occur naturally as denominators in our computations.

On the other hand, there is a kind of beautiful "algebraic" structure connected to the Hamiltonian (2.6), which is uncorrelated to the analytical problems described above. We feel that it is appropriate, in a first stage, to bring this structure forward, and to set aside the analytical problems (to which we will return later). Thus, in this chapter we will assume a very strong condition on u, namely that for a certain constant D we have

$$\forall \ell\ ,\ 0 \le \ell \le 3\ ,\ \ |u^{(\ell)}| \le D\ . \tag{2.7}$$

Given values of N and M we will try to "describe the system generated by the Hamiltonian (2.6)" within error terms that become small for N large. We will be able to do this when the ratio $\alpha = M/N$ is small enough, $\alpha \le \alpha(D)$. The notation $\alpha = M/N$ will be used through this chapter and until Chapter 4.

Let us now try to give an overview of what will happen, without getting into details. We recall the notation $R_{\ell,\ell'} = N^{-1}\sum_{i\le N}\sigma_i^\ell\sigma_i^{\ell'}$. As is the case for the SK model, we expect that in the high-temperature regime we have

$$R_{1,2} \simeq q \tag{2.8}$$

for a certain number q depending on the system. Let us define

$$S_k = \frac{1}{\sqrt{N}}\sum_{i\le N} g_{i,k}\sigma_i \quad ; \quad S_k^\ell = \frac{1}{\sqrt{N}}\sum_{i\le N} g_{i,k}\sigma_i^\ell\ . \tag{2.9}$$

After one works some length of time with the system, one gets the irresistible feeling that (in the high-temperature regime) "the quantities S_k behave like individual spins", and (2.8) has to be complemented by the relation

$$\frac{1}{N} \sum_{k \le M} u'(S_k^1) u'(S_k^2) \simeq r \tag{2.10}$$

where r is another number attached to the system. Probably the reader would expect a normalization factor M rather than N in (2.10), but since we should think of M/N as $M/N \to \alpha > 0$, this is really the same. Also, the occurrence of u' will soon become clear.

We will use the cavity method twice. In Section 2.2 we "remove one spin" as in Chapter 1. This lets us guess what is the correct expression of q as a function of r. In Section 2.3, we then use the "cavity in M", comparing the system with the similar system where M has been replaced by $M-1$. This lets us guess what the expression of r should be as a function of q. The two relations between r and q that are obtained in this manner are called the "replica-symmetric equations" in physics. We prove in Section 2.4 that these equations do have a solution, and that (2.8) and (2.10) hold for these values of q and r. For N large and M/N small, we will then (approximately) compute the value of

$$p_{N,M}(u) = \frac{1}{N} \mathsf{E} \log \sum_{\boldsymbol{\sigma}} \exp(-H_{M,N}(\boldsymbol{\sigma})) , \tag{2.11}$$

(for the Hamiltonian defined by (2.6)) by an interpolation method motivated by the idea that the quantities S_k "behave like individual spins".

2.2 The Smart Path

It would certainly help to understand how the Hamiltonian (2.6) depends on the last spin. Let us write

$$S_k^0 = \frac{1}{\sqrt{N}} \sum_{i \le N-1} g_{i,k} \sigma_i ,$$

so that $S_k = S_k^0 + N^{-1/2} g_{N,k} \sigma_N$ and if u is differentiable,

$$\sum_{k \le M} u(S_k) = \sum_{k \le M} u(S_k^0) + \sigma_N \sum_{k \le M} \frac{g_{N,k}}{\sqrt{N}} u'(S_k^0) + \frac{\sigma_N^2}{2} \sum_{k \le M} \frac{g_{N,k}^2}{N} u''(S_k^0) + \cdots \tag{2.12}$$

The terms $\cdots$ are of lower order. We observe that $\sigma_N^2 = 1$. (This will no longer be the case in Chapter 3, when we will consider spins taking all possible values, so that σ_N^2 will no longer be constant.) We also observe that the r.v.s $g_{N,k}$ are independent. So it is reasonable according to the law of large numbers to expect that the third term on the right-hand side should behave like a constant and not influence the Hamiltonian. By the central limit theorem, one should expect the second term on the right-hand side of (2.12) to behave like

$\sigma_N Y$, where Y is a Gaussian r.v. independent of all the other r.v.s (Of course at some point we will have to guess what is the right choice for $r = \mathsf{E}Y^2$, but the time will come when this guess will be obvious.) Thus we expect that

$$\sum_{k \le M} u(S_k) \simeq \sum_{k \le M} u(S_k^0) + \sigma_N Y + \text{constant} . \tag{2.13}$$

Rather than using power expansions (which are impractical when we do not have a good control on higher derivatives) it is more fruitful to find a suitable interpolation between the left and the right-hand sides of (2.13). The first idea that comes to mind is to use the Hamiltonian

$$\sum_{k \le M} u\left(S_k^0 + \sqrt{\frac{t}{N}} g_{N,k} \sigma_N\right) + \sigma_N \sqrt{1-t}\, Y . \tag{2.14}$$

This is effective and was used in [157]. However, the variance of the Gaussian r.v. $S_k^0 + \sqrt{t/N} g_{N,k} \sigma_N$ depends on t; when differentiating, this creates terms that we will avoid by being more clever. Let us consider the quantity

$$\begin{aligned} S_{k,t} = S_{k,t}(\boldsymbol{\sigma}, \xi_k) &= S_k^0 + \sqrt{\frac{t}{N}} g_{N,k} \sigma_N + \sqrt{\frac{1-t}{N}} \xi_k \\ &= \frac{1}{\sqrt{N}} \sum_{i<N} g_{i,k} \sigma_i + \sqrt{\frac{t}{N}} g_{N,k} \sigma_N + \sqrt{\frac{1-t}{N}} \xi_k . \end{aligned} \tag{2.15}$$

In this expression, we should think of $(\xi_k)_{k \le M}$ not just as random constants ensuring that the variance of $S_{k,t}$ is constant but also as "new spins". That is, let $\boldsymbol{\xi} = (\xi_k)_{k \le M} \in \mathbb{R}^M$, and consider the Hamiltonian

$$- H_{N,M,t}(\boldsymbol{\sigma}, \boldsymbol{\xi}) = \sum_{k \le M} u(S_{k,t}) + \sigma_N \sqrt{1-t}\, Y . \tag{2.16}$$

The configurations are now points $(\boldsymbol{\sigma}, \boldsymbol{\xi})$ in $\Sigma_N \times \mathbb{R}^M$. Let us denote by γ the canonical Gaussian measure on $\mathbb{R}^M$. We define Gibbs' measure on $\Sigma_N \times \mathbb{R}^M$ by the formula

$$\langle f \rangle_t = \frac{1}{Z} \sum_{\boldsymbol{\sigma}} \int f(\boldsymbol{\sigma}, \boldsymbol{\xi}) \exp(-H_{N,M,t}(\boldsymbol{\sigma}, \boldsymbol{\xi})) \mathrm{d}\gamma(\boldsymbol{\xi}) ,$$

where f is a function on $\Sigma_N \times \mathbb{R}^M$ and where Z is the normalizing factor,

$$Z = \sum_{\boldsymbol{\sigma}} \int \exp(-H_{N,M,t}(\boldsymbol{\sigma}, \boldsymbol{\xi})) \mathrm{d}\gamma(\boldsymbol{\xi}) .$$

More generally for a function f on $(\Sigma_N \times \mathbb{R}^M)^n = \Sigma_N^n \times \mathbb{R}^{Mn}$, we define

$$\langle f\rangle_t = \frac{1}{Z^n}\sum_{\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n}\int\cdots\int f(\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n,\boldsymbol{\xi}^1,\ldots,\boldsymbol{\xi}^n)$$
$$\times\exp\Bigl(-\sum_{\ell\le n}H^\ell_{N,M,t}\Bigr)\mathrm{d}\gamma(\boldsymbol{\xi}^1)\cdots\mathrm{d}\gamma(\boldsymbol{\xi}^n)\;, \tag{2.17}$$

where Z is as above and

$$H^\ell_{N,M,t} = H_{N,M,t}(\boldsymbol{\sigma}^\ell,\boldsymbol{\xi}^\ell)\;. \tag{2.18}$$

Integration of $\boldsymbol{\xi}$ with respect to γ means simply that we think of $(\xi_k)_{k\le M}$ as independent Gaussian r.v.s and we take expectation. We recall **the convention that E_ξ denotes expectation with respect to all r.v.s labeled** ξ (be it with subscripts or superscripts). We thus rewrite (2.17) as

$$\langle f\rangle_t = \frac{1}{Z^n}\mathsf{E}_\xi\sum_{\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n} f(\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n,\boldsymbol{\xi}^1,\ldots,\boldsymbol{\xi}^n)\exp\Bigl(-\sum_{\ell\le n}H^\ell_{N,M,t}\Bigr)\;; \tag{2.19}$$

$$Z = \mathsf{E}_\xi\sum_{\boldsymbol{\sigma}}\exp(-H_{N,M,t}(\boldsymbol{\sigma},\boldsymbol{\xi}))\;.$$

In these formulas, $\boldsymbol{\xi}^\ell = (\xi^\ell_k)_{k\le M}$, ξ^ℓ_k are independent Gaussian r.v.s. One should think of $\boldsymbol{\xi}^\ell$ as being a "replica" of $\boldsymbol{\xi}$. In this setting, replicas are simply independent copies.

Exercise 2.2.1. Prove that when f depends on $\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n$, but not on $\boldsymbol{\xi}^1,\ldots,\boldsymbol{\xi}^n$, then $\langle f\rangle_t$ in (2.19) is exactly the average of f with respect to the Hamiltonian

$$-H = \sum_{k\le M}u_t\Bigl(\frac{1}{\sqrt{N}}\sum_{i\le N-1}g_{i,k}\sigma_i + \sqrt{\frac{t}{N}}g_{N,k}\sigma_N\Bigr) + \sigma_N\sqrt{1-t}Y\;,$$

where u_t is defined by

$$\exp u_t(x) = \mathsf{E}\exp u\Bigl(x+\sqrt{\frac{1-t}{N}}\xi\Bigr)\;, \tag{2.20}$$

for ξ a standard normal r.v.

The reader might wonder whether it is really worth the effort to introduce this present setting simply in order to avoid an extra term in Proposition 2.2.3 below, a term with which it is not so difficult to deal anyway. The point is that the mechanism of "introducing new spins" is fundamental and must be used in Section 2.3, so we might as well learn it now.

Consistently with our notation, if f is a function on $\Sigma_N^n\times\mathbb{R}^{Mn}$, we define

$$\nu_t(f) = \mathsf{E}\langle f\rangle_t\;;\;\nu'_t(f) = \frac{\mathrm{d}}{\mathrm{d}t}\,\nu_t(f)\;, \tag{2.21}$$

where $\langle f\rangle_t$ is given by (2.19).

We also write $\nu(f) = \nu_1(f)$. When f does not depend on the r.v.s $\boldsymbol{\xi}^\ell$, then $\nu(f) = \mathsf{E}\langle f\rangle$, where $\langle\cdot\rangle$ refers to Gibbs' measure with Hamiltonian (2.6). As in Chapter 1, we write $\varepsilon_\ell = \sigma_N^\ell$, and we recall the r.v. Y of (2.16).

Lemma 2.2.2. *Given a function f^- on Σ_{N-1}^n, and a subset I of $\{1,\ldots,n\}$, we have*

$$\nu_0\Big(f^-\prod_{\ell\in I}\varepsilon_\ell\Big) = \mathsf{E}\big((\mathrm{th}Y)^{\mathrm{card}I}\big)\nu_0(f^-) = \nu_0\Big(\prod_{\ell\in I}\varepsilon_\ell\Big)\nu_0(f^-)\ .$$

This lemma holds whatever the value of $r = \mathsf{E}Y^2$. The proof is identical to that of Lemma 1.6.2. The Hamiltonian $H_{N,M,0}$ decouples the last spin from the first $N-1$ spins, which is what it is designed to do.

We now turn to the computation of $\nu_t'(f)$. Throughout the chapter, we write $\alpha = M/N$. Implicitly, we think of N and M as being large but fixed. The model then depends on the parameters N and α (and of course of u). We recall the definition (2.15) of $S_{k,t}$, and consistently with the notation (2.18) we write

$$S_{k,t}^\ell = \frac{1}{\sqrt{N}}\sum_{i<N} g_{i,k}\sigma_i^\ell + \sqrt{\frac{t}{N}}g_{N,k}\varepsilon_\ell + \sqrt{\frac{1-t}{N}}\xi_k^\ell\ . \tag{2.22}$$

Proposition 2.2.3. *Assume that u is twice differentiable and let $r = \mathsf{E}Y^2$. Then for a function f on Σ_N^n, we have*

$$\nu_t'(f) = \mathrm{I} + \mathrm{II} \tag{2.23}$$

$$\begin{aligned}\mathrm{I} = \alpha &\sum_{1\le\ell<\ell'\le n} \nu_t\big(\varepsilon_\ell\varepsilon_{\ell'}u'(S_{M,t}^\ell)u'(S_{M,t}^{\ell'})f\big)\\ &- \alpha n\sum_{\ell\le n}\nu_t\big(\varepsilon_\ell\varepsilon_{n+1}u'(S_{M,t}^\ell)u'(S_{M,t}^{n+1})f\big)\\ &+ \alpha\frac{n(n+1)}{2}\nu_t\big(\varepsilon_{n+1}\varepsilon_{n+2}u'(S_{M,t}^{n+1})u'(S_{M,t}^{n+2})f\big)\ .\end{aligned} \tag{2.24}$$

$$\begin{aligned}\mathrm{II} = -r\bigg(&\sum_{1\le\ell<\ell'\le n}\nu_t(\varepsilon_\ell\varepsilon_{\ell'}f) - n\sum_{\ell\le n}\nu_t(\varepsilon_\ell\varepsilon_{n+1}f)\\ &+ \frac{n(n+1)}{2}\nu_t(\varepsilon_{n+1}\varepsilon_{n+2}f)\bigg)\ .\end{aligned} \tag{2.25}$$

The proposition resembles Lemma 1.6.3, so it should not be so scary anymore. As in Lemma 1.6.3, the complication is algebraic, and each of the terms I and II is made up of simple pieces. Moreover both terms have similar structures. This formula will turn out to be much easier to use than one might think at first. In particular one should observe that by symmetry, and since $\alpha = M/N$, in the expression for I we can replace the term $\alpha u'(S^\ell_{M,t})u'(S^{\ell'}_{M,t})$ by

$$\frac{1}{N}\sum_{k\le M} u'(S^\ell_{k,t})u'(S^{\ell'}_{k,t}) \ ,$$

so that if (2.10) is indeed correct, the terms I and II should have a good will to cancel each other out.

Proof. We could make this computation appear as a consequence of (1.40), but for the rest of the book we will change policy, and proceed directly, i.e. we write the value of the derivative and we integrate by parts. It is immediate from (2.19) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle f\rangle_t = \sum_{\ell\le n}\left\langle \frac{\mathrm{d}}{\mathrm{d}t}(-H^\ell_{N,M,t})f\right\rangle_t - n\left\langle \frac{\mathrm{d}}{\mathrm{d}t}(-H^{n+1}_{N,M,t})f\right\rangle_t \ , \tag{2.26}$$

and, writing g_k for $g_{N,k}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(-H^\ell_{N,M,t}) = \sum_{k\le M}\frac{1}{2\sqrt{N}}\left(\frac{g_k\varepsilon_\ell}{\sqrt{t}} - \frac{\xi^\ell_k}{\sqrt{1-t}}\right)u'(S^\ell_{k,t}) - \frac{\varepsilon_\ell Y}{2\sqrt{1-t}} \ . \tag{2.27}$$

We observe the symmetry for $k \le M$. All the values of k bring the same contribution. There are M of them, and $M/\sqrt{N} = \alpha\sqrt{N}$, so that

$$\nu'_t(f) = \mathrm{III} + \mathrm{IV} + \mathrm{V}$$

$$\mathrm{III} = \frac{\alpha}{2}\sqrt{\frac{N}{t}}\left(\sum_{\ell\le n}\nu_t\big(g_M\varepsilon_\ell u'(S^\ell_{M,t})f\big) - n\nu_t\big(g_M\varepsilon_{n+1}u'(S^{n+1}_{M,t})f\big)\right) \tag{2.28}$$

$$\mathrm{IV} = -\frac{\alpha}{2}\sqrt{\frac{N}{1-t}}\left(\sum_{\ell\le n}\nu_t\big(\xi^\ell_M u'(S^\ell_{M,t})f\big) - n\nu_t\big(\xi^{n+1}_M u'(S^{n+1}_{M,t})f\big)\right)$$

$$\mathrm{V} = -\frac{1}{2}\frac{1}{\sqrt{1-t}}\left(\sum_{\ell\le n}\nu_t(\varepsilon_\ell Yf) - n\nu_t(\varepsilon_{n+1}Yf)\right) .$$

It remains to integrate by parts in these formulas to get the result. The easiest case is that of the term IV, because "different replicas use independent copies of ξ". We write the explicit formula for $\langle \xi^\ell_M u'(S^\ell_{M,t})f\rangle_t$, that is

$$\langle \xi^\ell_M u'(S^\ell_{M,t})f\rangle_t = \frac{1}{Z^n}\mathsf{E}_\xi\left(\xi^\ell_M\sum_{\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^n} u'(S^\ell_{M,t})f(\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^n)\exp\Big(-\sum_{\ell\le n}H^\ell_{M,N,t}\Big)\right) ,$$

and we see that we only have to integrate by parts in the numerator. The dependence on ξ_M^ℓ is through $u'(S_{M,t}^\ell)$ and through the term $u(S_{M,t}^\ell)$ in the Hamiltonian and moreover

$$\frac{\partial S_{M,t}^\ell}{\partial \xi_M^\ell} = \sqrt{\frac{1-t}{N}} \ , \tag{2.29}$$

so that

$$\langle \xi_M^\ell u'(S_{M,t}^\ell) f \rangle_t = \sqrt{\frac{1-t}{N}} \big\langle (u''(S_{M,t}^\ell) + u'^2(S_{M,t}^\ell)) f \big\rangle_t \ ,$$

and therefore

$$\mathrm{IV} = -\frac{\alpha}{2}\bigg(\sum_{\ell \le n} \nu_t\big(((u''(S_{M,t}^\ell) + u'^2(S_{M,t}^\ell))f\big) - n\nu_t\big((u''(S_{M,t}^{n+1}) + u'^2(S_{M,t}^{n+1}))f\big)\bigg).$$

The second easiest case is that of V, because we have done the same computation (implicitly at least) in Chapter 1; since $\mathsf{E}Y^2 = r$, we have V = II. Of course, the reader who does not find this formula obvious should simply write

$$\nu_t(\varepsilon_\ell Y f) = \mathsf{E} Y \langle \varepsilon_\ell f \rangle_t \ ,$$

and carry out the integration by parts, writing the explicit formula for $\langle \varepsilon_\ell f \rangle_t$. To compute the term III, there is no miracle. We write

$$\nu_t(g_M \varepsilon_\ell u'(S_{M,t}^\ell) f) = \mathsf{E} g_M \langle \varepsilon_\ell u'(S_{M,t}^\ell) f \rangle_t$$

and we use the integration by parts formula $\mathsf{E}(g_M F(g_M)) = \mathsf{E}F'(g_M)$ when seeing $\langle \varepsilon_\ell u'(S_{M,t}^\ell) f \rangle_t$ as a function of g_M. The dependence on g_M is through the quantities $S_{M,v}^\ell$, and

$$\frac{\partial S_{M,v}^\ell}{\partial g_M} = \varepsilon_\ell \sqrt{\frac{t}{N}} \ .$$

Writing the (cumbersome) explicit formula for $\langle \varepsilon_\ell u'(S_{M,t}^\ell) f \rangle_t$, we get that

$$\frac{\partial}{\partial g_M} \langle \varepsilon_\ell u'(S_{M,t}^\ell) f \rangle_t = \sqrt{\frac{t}{N}} \bigg(\langle u''(S_{M,t}^\ell) f \rangle_t$$
$$+ \sum_{\ell' \le n} \langle \varepsilon_\ell \varepsilon_{\ell'} u'(S_{M,t}^\ell) u'(S_{M,t}^{\ell'}) f \rangle_t - n \langle \varepsilon_\ell \varepsilon_{n+1} u'(S_{M,t}^\ell) u'(S_{M,t}^{n+1}) f \rangle_t \bigg) \ .$$

The first term arises from the dependence of the factor $u'(S_{M,t}^\ell)$ on g_M and the other terms from the dependence of the Hamiltonian on g_M. Consequently we obtain

$$\nu_t(\varepsilon_\ell u'(S_{M,t}^\ell) f) = \sqrt{\frac{t}{N}} \bigg(\nu_t(u''(S_{M,t}^\ell) f)$$
$$+ \sum_{\ell' \le n} \nu_t(\varepsilon_\ell \varepsilon_{\ell'} u'(S_{M,t}^\ell) u'(S_{M,t}^{\ell'}) f) - n\nu_t(\varepsilon_\ell \varepsilon_{n+1} u'(S_{M,t}^\ell) u'(S_{M,t}^{n+1}) f) \bigg) \ .$$

Similarly we have

$$\frac{\partial}{\partial g_M}\langle \varepsilon_{n+1} u'(S^{n+1}_{M,t}) f\rangle_t = \sqrt{\frac{t}{N}}\Big(\langle u''(S^{n+1}_{M,t}) f\rangle_t + \sum_{\ell'\le n+1} \big\langle \varepsilon_{\ell'}\varepsilon_{n+1} u'(S^{\ell'}_{M,t}) u'(S^{n+1}_{M,t}) f\big\rangle_t - (n+1)\big\langle \varepsilon_{n+1}\varepsilon_{n+2} u'(S^{n+1}_{M,t}) u'(S^{n+2}_{M,t}) f\big\rangle_t\Big)\,,$$

and consequently

$$\nu_t(\varepsilon_{n+1} u'(S^{n+1}_{M,t}) f) = \sqrt{\frac{t}{N}}\Big(\nu_t(u''(S^{n+1}_{M,t}) f) + \sum_{\ell'\le n+1} \nu_t(\varepsilon_{\ell'}\varepsilon_{n+1} u'(S^{\ell'}_{M,t}) u'(S^{n+1}_{M,t}) f) - (n+1)\nu_t(\varepsilon_{n+1}\varepsilon_{n+2} u'(S^{n+1}_{M,t}) u'(S^{n+2}_{M,t}) f)\Big)\,.$$

Regrouping the terms, we see that III + IV = I. □

Exercise 2.2.4. Suppose that we had not been as sleek as we were, and that instead of (2.15) and (2.22) we had defined

$$S_{k,t} = S_{k,t}(\boldsymbol{\sigma}) = S^0_k + \sqrt{\frac{t}{N}} g_{N,k}\sigma_N = \frac{1}{\sqrt{N}}\sum_{i<N} g_{i,k}\sigma_i + \sqrt{\frac{t}{N}} g_{N,k}\sigma_N$$

and

$$S^\ell_{k,t} = \frac{1}{\sqrt{N}}\sum_{i<N} g_{i,k}\sigma^\ell_i + \sqrt{\frac{t}{N}} g_{N,k}\sigma^\ell_N\,.$$

Prove that then in the formula (2.23) we would get the extra term

$$\mathrm{VI} = \frac{\alpha}{2}\Big(\sum_{\ell\le n}\nu_t\Big(\big(u'(S^\ell_{M,t})^2 + u''(S^\ell_{M,t})\big) f\Big) - n\nu_t\Big(\big(u'(S^{n+1}_{M,t})^2 + u''(S^{n+1}_{M,t})\big) f\Big)\Big)\,.$$

2.3 Cavity in M

To pursue the idea that the terms I and II in (2.23) should nearly cancel out each other, the first thing to do is to try to make sense of the term I, and to understand the influence of the quantities $u'(S^\ell_{M,t})$. The quantities $S^\ell_{M,t}$ also occur in the Hamiltonian, and we should make this dependence explicit. For this we introduce a new Hamiltonian

$$-H_{N,M-1,t}(\boldsymbol{\sigma},\boldsymbol{\xi})=\sum_{k\le M-1}u(S_{k,t}(\boldsymbol{\sigma},\xi_k))+\sigma_N\sqrt{1-t}Y\ , \tag{2.30}$$

where the dependence on $\boldsymbol{\xi}$ is stressed to point out that it will be handled as in the case of the Hamiltonian (2.16), that is, an average $\langle\cdot\rangle_{t,\sim}$ with respect to this Hamiltonian will be computed with the formula (2.31) below. Let us first notice that, even though the right-hand side of (2.30) does not depend on ξ_M, we denote for simplicity of notation the Hamiltonian as a function of $\boldsymbol{\sigma}$ and $\boldsymbol{\xi}$. If f is a function on $\Sigma_N^n\times\mathbb{R}^{Mn}$, we then define

$$\langle f\rangle_{t,\sim}=\frac{1}{Z_\sim^n}\mathsf{E}_\xi\sum_{\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n}f(\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n,\boldsymbol{\xi}^1,\ldots,\boldsymbol{\xi}^n)\exp\Big(-\sum_{\ell\le n}H^\ell_{N,M-1,t}\Big)\ , \tag{2.31}$$

where

$$Z_\sim=\mathsf{E}_\xi\sum_{\boldsymbol{\sigma}}\exp(-H_{N,M-1,t}(\boldsymbol{\sigma},\boldsymbol{\xi}))\ ,$$

and where $H^\ell_{N,M-1,t}=H_{N,M-1,t}(\boldsymbol{\sigma}^\ell,\boldsymbol{\xi}^\ell)$. There of course E_ξ includes expectation in the r.v.s ξ^ℓ_M, even though the Hamiltonian does not depend on those. Since $-H^\ell_{N,M,t}=-H^\ell_{N,M-1,t}+u(S^\ell_{M,t})$, the identity

$$\begin{aligned}Z=\mathsf{E}_\xi\sum_{\boldsymbol{\sigma}}\exp(-H^1_{N,M,t})&=\mathsf{E}_\xi\sum_{\boldsymbol{\sigma}}\exp u(S^1_{M,t})\exp(-H^1_{N,M-1,t})\\&=Z_\sim\langle\exp u(S^1_{M,t})\rangle_{t,\sim}\end{aligned}$$

holds, and, similarly,

$$\begin{aligned}&\mathsf{E}_\xi\sum_{\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n}f(\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n,\boldsymbol{\xi}^1,\ldots,\boldsymbol{\xi}^n)\exp\Big(-\sum_{\ell\le n}H^\ell_{N,M,t}\Big)\\&=Z^n_\sim\Big\langle f\exp\sum_{\ell\le n}u(S^\ell_{M,t})\Big\rangle_{t,\sim}\ .\end{aligned}$$

Combining these two formulas with (2.31) yields that if f is a function on $\Sigma_N^n\times\mathbb{R}^{Mn}$, we have

$$\langle f\rangle_t=\frac{\big\langle f\exp\big(\sum_{\ell\le n}u(S^\ell_{M,t})\big)\big\rangle_{t,\sim}}{\langle\exp u(S^1_{M,t})\rangle^n_{t,\sim}}\ . \tag{2.32}$$

Our best guess now is that the quantities $S^\ell_{M,t}$, when seen as functions of the system with Hamiltonian (2.30), will have a jointly Gaussian behavior under Gibbs' measure, with pairwise correlation q, allowing us to approximately compute the right-hand side of (2.32) in Proposition 2.3.5 below. This again will be shown by interpolation. Let us consider a new parameter $0\le q\le 1$ and standard Gaussian r.v.s (ξ^ℓ) and z that are independent of all the other r.v.s already considered. (The reader will not confuse the r.v.s ξ^ℓ with the r.v.s ξ_M^ℓ.) Let us set

$$\theta^\ell = z\sqrt{q} + \xi^\ell\sqrt{1-q}\ . \tag{2.33}$$

Thus these r.v.s share the common randomness z and are independent given that randomness. For $0 \le v \le 1$ we define

$$S_v^\ell = \sqrt{v}S_{M,t}^\ell + \sqrt{1-v}\theta^\ell\ . \tag{2.34}$$

The dependence on t is kept implicit; when using S_v^ℓ we think of t (and M) as being fixed.

Let us pursue the idea that in (2.31), E_ξ denotes **expectation in all r.v.s labeled** ξ *including the variables* ξ^ℓ and let us further define with this convention

$$\nu_{t,v}(f) = \mathsf{E}\frac{\left\langle f\exp\left(\sum_{\ell\le n}u(S_v^\ell)\right)\right\rangle_{t,\sim}}{\langle\exp u(S_v^1)\rangle_{t,\sim}^n}\ . \tag{2.35}$$

Using (2.32) yields

$$\nu_{t,1}(f) = \nu_t(f)\ .$$

The idea of (2.35) is of course that in certain cases $\nu_{t,0}(f)$ should be much easier to evaluate than $\nu_t(f) = \nu_{t,1}(f)$ and that these quantities should be close to each other if q is appropriately chosen. Before we go into the details however, we would like to explain the pretty idea that is hidden behind this construction. The idea is simply that we consider ξ "as a new spin". To explain this, consider a spin system where the space of configurations is the collection of all triplets $(\boldsymbol{\sigma}, \boldsymbol{\xi}, \xi)$ for $\boldsymbol{\sigma} \in \Sigma_N$, $\boldsymbol{\xi} \in \mathbb{R}^M$ and $\xi \in \mathbb{R}$. Consider the Hamiltonian

$$-H(\boldsymbol{\sigma}, \boldsymbol{\xi}, \xi) = -H_{N,M-1,t}(\boldsymbol{\sigma}, \boldsymbol{\xi}) + u(S_v)\ ,$$

where $S_v = \sqrt{v}S_{M,t} + \sqrt{1-v}\theta$, for $\theta = z\sqrt{q} + \sqrt{1-q}\xi$. Then, for a function f of $\boldsymbol{\sigma}^1, \ldots, \boldsymbol{\sigma}^n, \boldsymbol{\xi}^1, \ldots, \boldsymbol{\xi}^n$ and $\xi^1, \ldots, \xi^n$ we can define a quantity $\langle f\rangle_{t,v}$ by a formula similar to (2.19) and (2.31). As in (2.32), we have

$$\langle f\rangle_{t,v} = \frac{\left\langle f\exp\left(\sum_{\ell\le n}u(S_v^\ell)\right)\right\rangle_{t,\sim}}{\langle\exp u(S_v^1)\rangle_{t,\sim}^n}\ ,$$

so that in fact $\nu_{t,v} = \mathsf{E}\langle\cdot\rangle_{t,v}$. Let us observe that the r.v. θ depends also on z, but this r.v. is not considered as a "new spin", but rather as "new randomness".

The present idea of considering ξ as a new spin is essential. As we mentioned on page 156, the idea of considering $\xi_1, \ldots, \xi_M$ as new spins was not essential, but since it is the same idea, we decided to make the minimal extra effort to use the setting of (2.19).

First, we reveal the magic of the computation of $\nu_{t,0}$.

Lemma 2.3.1. *Consider* $0 \le q \le 1$ *and define*

$$\widehat{r} = \mathsf{E}\left(\frac{\mathsf{E}_\xi u'(\theta)\exp u(\theta)}{\mathsf{E}_\xi \exp u(\theta)}\right)^2 , \tag{2.36}$$

where $\theta = z\sqrt{q} + \xi\sqrt{1-q}$ *for independent standard Gaussian r.v.s* z *and* ξ *and where* E_ξ *denotes expectation in* ξ *only. Consider a function* f *on* Σ_N^n. *This function might depend on the variables* ξ_k^ℓ *for* $k < M$ *and* $\ell \le n$*, but it does not depend on the randomness of the variables* ξ_M^ℓ *or* ξ^ℓ*. Then*

$$\nu_{t,0}(f) = \mathsf{E}\langle f\rangle_{t,\sim} , \tag{2.37}$$

and

$$\nu_{t,0}(u'(S_0^1)u'(S_0^2)f) = \widehat{r}\mathsf{E}\langle f\rangle_{t,\sim} . \tag{2.38}$$

In particular we have $\nu_{t,0}(u'(S_0^1)u'(S_0^2)f) = \widehat{r}\nu_{t,0}(f)$. If such an equality is nearly true for $v = 1$ rather than for $v = 0$, we are in good shape to use Proposition 2.2.3.

Proof. First we have

$$\Big\langle f \exp\sum_{\ell\le n} u(\theta^\ell)\Big\rangle_{t,\sim} = \langle f\rangle_{t,\sim}\mathsf{E}_\xi \exp\sum_{\ell\le n} u(\theta^\ell) . \tag{2.39}$$

This follows from the formula (2.31). The quantities θ^ℓ do not depend on the spins $\boldsymbol{\sigma}$, and their randomness "in the variables labeled ξ" is independent of the randomness of the other terms. Now, independence implies

$$\mathsf{E}_\xi \exp\sum_{\ell\le n} u(\theta^\ell) = (\mathsf{E}_\xi \exp u(\theta))^n .$$

Moreover $\langle \exp u(\theta)\rangle_{t,\sim} = \mathsf{E}_\xi \exp u(\theta)$, as (an obvious) special case of (2.39). This proves (2.37).

To prove (2.38), proceeding in a similar manner and using now that

$$\mathsf{E}_\xi\Big(u'(\theta^1)u'(\theta^2)\exp\sum_{\ell\le n} u(\theta^\ell)\Big) = \big(\mathsf{E}_\xi u'(\theta)\exp u(\theta)\big)^2\big(\mathsf{E}_\xi \exp u(\theta)\big)^{n-2} ,$$

we get

$$\begin{aligned}\nu_{t,0}(u'(S_0^1)u'(S_0^2)f) &= \mathsf{E}\,\frac{\big\langle f u'(\theta^1)u'(\theta^2)\exp\sum_{\ell\le n} u(\theta^\ell)\big\rangle_{t,\sim}}{\langle \exp u(\theta)\rangle_{t,\sim}^n}\\ &= \widehat{r}\mathsf{E}\langle f\rangle_{t,\sim} ,\end{aligned}$$

and this finishes the proof. □

We now turn to the proof that $\nu_{t,0}$ and $\nu_{t,1}$ are close. We recall that D is the constant of (2.7).

Lemma 2.3.2. *Consider a function f on Σ_N^n. This function depend on the variables ξ_k^ℓ for $k < M$ and $\ell \le n$, but it does not depend on the randomness of the variables $z, g_{i,M}, \xi_M^\ell$ or ξ^ℓ. Then if $B_v \equiv 1$ or $B_v = u'(S_v^1)u'(S_v^2)$, whenever $1/\tau_1 + 1/\tau_2 = 1$ we have*

$$\left|\frac{\mathrm{d}}{\mathrm{d}v}\nu_{t,v}(B_v f)\right| \le K(n,D)\left(\nu_{t,v}(|f|^{\tau_1})^{1/\tau_1}\nu_{t,v}(|R_{1,2}-q|^{\tau_2})^{1/\tau_2} + \frac{1}{N}\nu_{t,v}(|f|)\right). \tag{2.40}$$

Here $K(n,D)$ depends on n and D only.

Therefore the left-hand side is small if we can find q such that $R_{1,2} \simeq q$. The reason why we write a derivative in the left-hand side rather than a partial derivative is that when considering $\nu_{t,v}$ we always think of t as fixed.

Proof. The core of the proof is to compute $\mathrm{d}(\nu_{t,v}(B_v f))/\mathrm{d}v$ by differentiation and integration by parts, after which the bound (2.40) basically follows from Hölder's inequality. It turns out that if one looks at things the right way, there is a relatively simple expression for $\mathrm{d}(\nu_{t,v}(B_v f))/\mathrm{d}v$. We will not reveal this magic formula now. Our immediate concern is to explain in great detail the mechanism of integration by parts, that will occur again and again, and for this we decided to use a completely pedestrian approach, writing only absolutely explicit formulas.

First, we compute $\mathrm{d}(\nu_{t,v}(B_v f))/\mathrm{d}v$ by straightforward differentiation of the formula (2.35). In the case where $B_v = u'(S_v^1)u'(S_v^2)$, setting

$$S_v^{\ell\prime} = \frac{1}{2\sqrt{v}}S_{M,t}^\ell - \frac{1}{2\sqrt{1-v}}\theta^\ell ,$$

we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}v}(\nu_{t,v}(B_v f)) &= \nu_{t,v}\big(f S_v^{1\prime} u''(S_v^1)u'(S_v^2)\big) + \nu_{t,v}\big(f S_v^{2\prime} u'(S_v^1)u''(S_v^2)\big)\\ &\quad + \sum_{\ell\le n}\nu_{t,v}\big(f S_v^{\ell\prime}u'(S_v^\ell)u'(S_v^1)u'(S_v^2)\big)\\ &\quad - (n+1)\nu_{t,v}\big(f S_v^{n+1\prime}u'(S_v^{n+1})u'(S_v^1)u'(S_v^2)\big) .\end{aligned} \tag{2.41}$$

Of course the first term occurs because of the factor $u'(S_v^1)$ in B_v, the second term because of the factor $u'(S_v^2)$ and the other terms because of the dependence of the Hamiltonian on v. The rest of the proof consists in integrating by parts. In some sense it is a straight forward application of the Gaussian integration by parts formula (A.17). However, since we are dealing with complicated expressions, it will take several pages to fill in all the details. The notation is complicated, and this obscures the basic simplicity of the argument. Probably the ambitious reader should try to compute everything on her own in simple case, and look at our presentation only if she gets stuck.

Even though we have written the previous formula in a compact form using $\nu_{t,v}$, to integrate by parts we have to spell out the dependence of the

Hamiltonian on the variables S_v^ℓ by using the formula (2.35). For example, the first term in the right-hand side of (2.41) is

$$\mathsf{E}\,\frac{\big\langle f S_v^{1\prime} u''(S_v^1)u'(S_v^2)\exp\big(\sum_{\ell\le n} u(S_v^\ell)\big)\big\rangle_{t,\sim}}{\langle \exp u(S_v^1)\rangle_{t,\sim}^n}\ . \tag{2.42}$$

To keep the formulas manageable, let us write

$$w = w(\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n,\boldsymbol{\xi}^1,\ldots,\boldsymbol{\xi}^n) = \exp\Big(-\sum_{\ell\le n} H_{N,M-1,t}^\ell\Big)$$

and let us define

$$w_*^\ell = w_*(\boldsymbol{\sigma}^\ell,\boldsymbol{\xi}^\ell) = \exp(-H_{N,M-1,t}^\ell)\ .$$

These quantities are probabilistically independent of the randomness of the variables S_v^ℓ (which is why we introduced the Hamiltonian $H_{N,M-1,t}$ in the first place).

The quantity (2.42) is then equal to

$$\mathsf{E}\,\frac{\mathsf{E}_\xi \sum_{\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n} w S_v^{1\prime} C}{Z^n}\ , \tag{2.43}$$

where

$$Z = \mathsf{E}_\xi \sum_{\boldsymbol{\sigma}^1} w_*^1 \exp u(S_v^1)\ ,$$

and where

$$C = f u''(S_v^1)u'(S_v^2)\exp\Big(\sum_{\ell\le n} u(S_v^\ell)\Big)\ .$$

Let us now make an observation that will be used **many times**. The r.v. Z is independent of all the r.v.s labeled ξ, so that

$$\frac{\mathsf{E}_\xi \sum_{\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n} w\, S_v^{1\prime} C}{Z^n} = \mathsf{E}_\xi \frac{\sum_{\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n} w\, S_v^{1\prime} C}{Z^n}\ ,$$

and thus the quantity (2.43) is then equal to

$$\mathsf{E}\mathsf{E}_\xi \sum_{\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n} w\, S_v^{1\prime}\frac{C}{Z^n} = \mathsf{E} \sum_{\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n} w\, S_v^{1\prime}\frac{C}{Z^n}\ . \tag{2.44}$$

Let us now denote by E_0 integration in the randomness of $g_{i,M}$, ξ_M^ℓ, z and ξ^ℓ, given all the other sources of randomness. Therefore, since the quantities w do not depend on any of the variables $g_{i,M}$, ξ_k^ℓ, z or ξ^ℓ, the quantity (2.44) equals

$$\mathsf{E} \sum_{\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n} w\, \mathsf{E}_0 S_v^{1\prime}\frac{C}{Z^n}\ . \tag{2.45}$$

The main step in the computation is the calculation of the quantity $\mathsf{E}_0 S_v^{1\prime} C/Z^n$ by integration by parts. We advise the reader to study the elementary proof of Lemma 2.4.4 below as a preparation to this computation in a simpler setting. To apply the Gaussian integration by parts formula (A.17), we need to find a jointly Gaussian family $(g, z_1, \ldots, z_P)$ of r.v.s such that $g = S_v^{1\prime}$ and that C/Z^n is a function $F(z_1, \ldots, z_P)$ of $z_1, \ldots, z_P$. The first idea that comes to mind is to use for the r.v.s (z_p) the following family of variables, indexed by $\boldsymbol{\sigma}$ and ℓ,

$$\begin{aligned} z_{\boldsymbol{\sigma}}^\ell &= \sqrt{v} S_{M,t}(\boldsymbol{\sigma}, \xi_M^\ell) + \sqrt{1-v}\theta^\ell \\ &= \sqrt{v}\Bigg(\frac{1}{\sqrt{N}} \sum_{i<N} g_{i,M}\sigma_i + \sqrt{\frac{t}{N}} g_{N,M}\sigma_N + \sqrt{\frac{1-t}{N}}\xi_M^\ell\Bigg) \\ &\quad + \sqrt{1-v}(z\sqrt{q} + \xi^\ell\sqrt{1-q}) \ , \end{aligned}$$

where $\boldsymbol{\sigma} \in \Sigma_N$ takes all possible values and ℓ is an integer. Of course these variables depend on v but the dependence is kept implicit because we think now of v as fixed. We observe that

$$S_v^\ell = z_{\boldsymbol{\sigma}^\ell}^\ell \ , \tag{2.46}$$

so that we can think of C as a function of these quantities:

$$C = C_{\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n} = F_{\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n}((z_{\boldsymbol{\sigma}}^\ell)) \ , \tag{2.47}$$

where $F_{\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n}$ is the function of the variables $x_{\boldsymbol{\sigma}}^\ell$ given by

$$F_{\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n}((x_{\boldsymbol{\sigma}}^\ell)) = f(\boldsymbol{\sigma}^1, \ldots, \boldsymbol{\sigma}^n) u''(x_{\boldsymbol{\sigma}^1}^1) u'(x_{\boldsymbol{\sigma}^2}^2) \exp\Bigg(\sum_{\ell \le n} u(x_{\boldsymbol{\sigma}^\ell}^\ell)\Bigg) \ . \tag{2.48}$$

Condition (2.47) holds simply because to compute $F_{\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n}((z_{\boldsymbol{\sigma}^\ell}^\ell))$, we substitute $z_{\boldsymbol{\sigma}^\ell}^\ell = S_v^\ell$ to $x_{\boldsymbol{\sigma}^\ell}^\ell$ in the previous formula. This construction however does not suffice, because Z cannot be considered as a function of the quantities $z_{\boldsymbol{\sigma}}^\ell$: the effect of the expectation E_ξ is that "the part depending on the r.v.s labeled ξ has been averaged out". The part of $z_{\boldsymbol{\sigma}}^\ell$ that does not depend on the r.v.s labeled ξ is simply

$$y_{\boldsymbol{\sigma}} = \sqrt{v}\Bigg(\frac{1}{\sqrt{N}} \sum_{i<N} g_{i,M}\sigma_i + \sqrt{\frac{t}{N}} g_{N,M}\sigma_N\Bigg) + \sqrt{1-v}\sqrt{q} z \ .$$

Defining

$$\xi_*^\ell = \sqrt{v}\sqrt{\frac{1-t}{N}}\xi_M^\ell + \sqrt{1-v}\sqrt{1-q}\xi^\ell \ ,$$

we then have

$$z_{\boldsymbol{\sigma}}^\ell = y_{\boldsymbol{\sigma}} + \xi_*^\ell \ .$$

It is now possible to express Z as a function of the r.v.s $y_{\boldsymbol{\sigma}}$. This is shown by the formula

$$Z = F_1((y_{\boldsymbol{\sigma}}))\ ,$$

where F_1 is the function of the variables $x_{\boldsymbol{\sigma}}$ given by

$$F_1((x_{\boldsymbol{\sigma}})) = \mathsf{E}_\xi \sum_{\boldsymbol{\sigma}} w_*(\boldsymbol{\sigma}, \boldsymbol{\xi}^1) \exp u(x_{\boldsymbol{\sigma}} + \xi_*^1)\ . \tag{2.49}$$

Let us now define

$$\begin{aligned} z_{\boldsymbol{\sigma}}^{\ell\prime} &= \frac{1}{2\sqrt{v}} S_{M,t}(\boldsymbol{\sigma}, \xi_M^\ell) - \frac{1}{2\sqrt{1-v}} \theta^\ell \\ &= \frac{1}{2\sqrt{v}} \left(\frac{1}{\sqrt{N}} \sum_{i<N} g_{i,M}\sigma_i + \sqrt{\frac{t}{N}} g_{N,M}\sigma_N + \sqrt{\frac{1-t}{N}} \xi_M^\ell \right) \\ &\quad - \frac{1}{2\sqrt{1-v}} (\sqrt{q} z + \sqrt{1-q}\xi^\ell)\ , \end{aligned}$$

so that $S_v^{\ell\prime} = z_{\boldsymbol{\sigma}^\ell}^{\ell\prime}$. The family of all the r.v.s $z_{\boldsymbol{\sigma}}^\ell, y_{\boldsymbol{\sigma}}, \xi_*^\ell$, and $z_{\boldsymbol{\sigma}}^{\ell\prime}$ is a Gaussian family, and this is the family we will use to apply the integration by parts formula. In the upcoming formulas, the reader should take great care to distinguish between the quantities $z_{\boldsymbol{\sigma}}^{\ell\prime}$ and $z_{\boldsymbol{\sigma}}^{\ell'}$ (The position of the $\prime$ is not the same).

We note the relations

$$\mathsf{E}(\theta^\ell)^2 = 1 = \mathsf{E}(S_{M,t}(\boldsymbol{\sigma}, \xi_M^\ell))^2\ ;\ \ell \neq \ell' \Rightarrow \mathsf{E}\theta^\ell\theta^{\ell'} = q\ .$$

$$\ell \neq \ell' \Rightarrow \mathsf{E}S_{M,t}(\boldsymbol{\sigma}, \xi_M^\ell) S_{M,t}(\boldsymbol{\tau}, \xi_M^{\ell'}) = R^t(\boldsymbol{\sigma}, \boldsymbol{\tau}) := \frac{1}{N} \sum_{i<N} \sigma_i\tau_i + \frac{t}{N}\sigma_N\tau_N\ ,$$

so that

$$\mathsf{E}z_{\boldsymbol{\sigma}}^{\ell\prime} z_{\boldsymbol{\sigma}}^\ell = 0\ ;\ \ell \neq \ell' \Rightarrow \mathsf{E}z_{\boldsymbol{\sigma}}^{\ell\prime} z_{\boldsymbol{\tau}}^{\ell'} = \frac{1}{2}(R^t(\boldsymbol{\sigma}, \boldsymbol{\tau}) - q)\ , \tag{2.50}$$

and

$$\mathsf{E}z_{\boldsymbol{\sigma}}^{\ell\prime} y_{\boldsymbol{\tau}} = \frac{1}{2}(R^t(\boldsymbol{\sigma}, \boldsymbol{\tau}) - q)\ . \tag{2.51}$$

We will simply use the integration by parts formula (A.17) and these relations to understand the form of the quantity

$$\mathsf{E}_0 S_v^{1\prime} \frac{C}{Z^n} = \mathsf{E}_0 z_{\boldsymbol{\sigma}^1}^{1\prime} \frac{F_{\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n}((z_{\boldsymbol{\sigma}}^\ell))}{F_1((y_{\boldsymbol{\sigma}}))^n}\ . \tag{2.52}$$

Let us repeat that this integration by parts takes place given all the sources of randomness other than the r.v.s $g_{i,M}$, ξ_k^ℓ for $k < M$, z and ξ^ℓ (so that it is fine if f depends on some randomness independent of these). The exact result of the computation is not relevant now (it will be given in Chapter 9). For the present result we simply need the information that $\mathrm{d}\nu_{t,v}(B_v f)/\mathrm{d}v$ is a sum of terms of the type (using the notation $R^t_{\ell,\ell'} = R^t(\boldsymbol{\sigma}^\ell, \boldsymbol{\sigma}^{\ell'})$)

$$\nu_{t,v}(f(R^t_{\ell,\ell'} - q)A) , \tag{2.53}$$

where A is a monomial in the quantities $u'(S^m_v), u''(S^m_v), u^{(3)}(S^m_v)$ for $m \leq n+2$. So, let us perform the integration by parts in (2.52):

$$\begin{aligned}\mathsf{E}_0 z^{1\prime}_{\boldsymbol{\sigma}^1} \frac{F_{\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^n}((z^\ell_{\boldsymbol{\sigma}}))}{F_1((y_{\boldsymbol{\sigma}}))^n} &= \sum_{\boldsymbol{\tau},\ell} \mathsf{E}_0 z^{1\prime}_{\boldsymbol{\sigma}^1} z^\ell_{\boldsymbol{\tau}} \mathsf{E}_0 \frac{\partial F_{\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^n}}{\partial x^\ell_{\boldsymbol{\tau}}}((z^\ell_{\boldsymbol{\sigma}})) \frac{1}{F_1((y_{\boldsymbol{\sigma}}))^n} \\ &\quad - n \sum_{\boldsymbol{\tau}} \mathsf{E}_0 z^{1\prime}_{\boldsymbol{\sigma}^1} y_{\boldsymbol{\tau}} \mathsf{E}_0 \frac{\partial F_1}{\partial x_{\boldsymbol{\tau}}}((y_{\boldsymbol{\sigma}})) \frac{F_{\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^n}((z^\ell_{\boldsymbol{\sigma}}))}{F_1((y_{\boldsymbol{\sigma}}))^{n+1}} .\end{aligned}$$

It is convenient to refer to the last term in the above (or similar) formula "as the term created by the denominator" when performing the integration by parts in (2.52). (It would be nice to remember this, since we will often use this expression in our future attempts at describing at a high level computations similar to the present one.) We first compute this term. We observe that

$$\frac{\partial F_1}{\partial x_{\boldsymbol{\tau}}} = \mathsf{E}_\xi w_*(\boldsymbol{\tau}, \boldsymbol{\xi}^1) u'(x_{\boldsymbol{\tau}} + \xi^1_*) \exp u(x_{\boldsymbol{\tau}} + \xi^1_*) .$$

Therefore using (2.51) we see that the term created by the denominator in (2.52) is

$$-\frac{n}{2} \mathsf{E}_0 \sum_{\boldsymbol{\tau}} (R^t(\boldsymbol{\sigma}^1, \boldsymbol{\tau}) - q) \frac{F_{\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^n}((z^\ell_{\boldsymbol{\sigma}})) \mathsf{E}_\xi w_*(\boldsymbol{\tau}, \boldsymbol{\xi}^1) u'(y_{\boldsymbol{\tau}} + \xi^1_*) \exp u(y_{\boldsymbol{\tau}} + \xi^1_*)}{F_1((y_{\boldsymbol{\sigma}}))^{n+1}} .$$

Since $y_{\boldsymbol{\tau}} + \xi^1_* = z^1_{\boldsymbol{\tau}}$, the contribution of this term to (2.44) is then

$$-\frac{n}{2} \mathsf{E} \sum_{\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^n,\boldsymbol{\tau}} w(R^t(\boldsymbol{\sigma}^1, \boldsymbol{\tau}) - q) \frac{F_{\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^n}((z^\ell_{\boldsymbol{\sigma}})) \mathsf{E}_\xi w_*(\boldsymbol{\tau}, \boldsymbol{\xi}^1) u'(z^1_{\boldsymbol{\tau}}) \exp u(z^1_{\boldsymbol{\tau}})}{F_1((y_{\boldsymbol{\sigma}}))^{n+1}} . \tag{2.54}$$

Now,

$$\mathsf{E}_\xi w_*(\boldsymbol{\tau}, \boldsymbol{\xi}^1) u'(z^1_{\boldsymbol{\tau}}) \exp u(z^1_{\boldsymbol{\tau}}) = \mathsf{E}_\xi w_*(\boldsymbol{\tau}, \boldsymbol{\xi}^{n+1}) u'(z^{n+1}_{\boldsymbol{\tau}}) \exp u(z^{n+1}_{\boldsymbol{\tau}}) ,$$

so that, changing the name of $\boldsymbol{\tau}$ into $\boldsymbol{\sigma}^{n+1}$, and since $w^{n+1}_* = w_*(\boldsymbol{\sigma}^{n+1}, \boldsymbol{\xi}^{n+1})$, the quantity (2.54) is equal to (using (2.46) in the second line)

$$\begin{aligned}&= -\frac{n}{2} \mathsf{E} \sum_{\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^{n+1}} w(R^t_{1,n+1} - q) \frac{F_{\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^n}((z^\ell_{\boldsymbol{\sigma}})) \mathsf{E}_\xi w^{n+1}_* u'(z^{n+1}_{\boldsymbol{\sigma}^{n+1}}) \exp u(z^{n+1}_{\boldsymbol{\sigma}^{n+1}})}{F_1((y_{\boldsymbol{\sigma}}))^{n+1}} \\ &= -\frac{n}{2} \mathsf{E} \sum_{\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^{n+1}} w(R^t_{1,n+1} - q) \frac{C \mathsf{E}_\xi w^{n+1}_* u'(S^{n+1}_v) \exp u(S^{n+1}_v)}{Z^{n+1}} .\end{aligned}$$

In a last step we observe that in the above formula we can remove the expectation E_ξ. This is because the r.v.s labeled ξ that occur in this expectation (namely $\boldsymbol{\xi}^{n+1}$ and ξ^{n+1}) are independent of the other r.v.s labeled ξ that occur in C and w. In this manner we finally see that the contribution of this quantity to the computation of (2.42) is

$$
\begin{aligned}
&-\frac{n}{2}\mathsf{E}\sum_{\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^{n+1}} \frac{C(R^t_{1,n+1}-q)ww_*^{n+1}u'(S_v^{n+1})\exp u(S_v^{n+1})}{Z^{n+1}} \\
&= -\frac{n}{2}\nu_{t,v}\big(f(R^t_{1,n+1}-q)u''(S_v^1)u'(S_v^2)u'(S_v^{n+1})\big)\ .
\end{aligned}
$$

In a similar manner we compute the contribution in (2.52) of the dependence of $F_{\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n}$ on the variables $z^\ell_{\boldsymbol{\sigma}}$ at a given value of ℓ, i.e of the quantity

$$
\sum_{\boldsymbol{\tau}} \mathsf{E}_0 z^{1\prime}_{\boldsymbol{\sigma}^1} z^\ell_{\boldsymbol{\tau}} \mathsf{E}_0 \frac{\partial F_{\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n}}{\partial x^\ell_{\boldsymbol{\tau}}}((z^\ell_{\boldsymbol{\sigma}}))\frac{1}{F_1((y_{\boldsymbol{\sigma}}))^n}\ . \tag{2.55}
$$

We observe in particular from (2.48) that

$$
\frac{\partial F_{\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n}}{\partial x^\ell_{\boldsymbol{\tau}}}((z^\ell_{\boldsymbol{\sigma}})) = 0
$$

unless $\boldsymbol{\tau} = \boldsymbol{\sigma}^\ell$, so that the quantity (2.55) equals

$$
\mathsf{E}_0 z^{1\prime}_{\boldsymbol{\sigma}^1} z^\ell_{\boldsymbol{\sigma}^\ell} \mathsf{E}_0 \frac{\partial F_{\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n}}{\partial x^\ell_{\boldsymbol{\sigma}^\ell}}((z^\ell_{\boldsymbol{\sigma}}))\frac{1}{F_1((y_{\boldsymbol{\sigma}}))^n}\ . \tag{2.56}
$$

Since $\mathsf{E} z^{\ell\prime}_{\boldsymbol{\sigma}} z^\ell_{\boldsymbol{\sigma}} = 0$ by (2.50) we see that for $\ell = 1$ the contribution of this term is 0.

When $\ell \geq 3$, we have

$$
\frac{\partial F_{\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n}}{\partial x^\ell_{\boldsymbol{\tau}}}((x^\ell_{\boldsymbol{\sigma}})) = f(\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n)u''(x^1_{\boldsymbol{\sigma}^1})u'(x^2_{\boldsymbol{\sigma}^2})u'(x^\ell_{\boldsymbol{\sigma}^\ell})\exp\Big(\sum_{\ell\leq n} u(x^\ell_{\boldsymbol{\sigma}^\ell})\Big)\ ,
$$

so that the term (2.55) is simply

$$
\frac{1}{2}\nu_{t,v}\big(f(R^t_{1,\ell}-q)u''(S_v^1)u'(S_v^2)u'(S_v^\ell)\big)\ .
$$

If $\ell = 2$, there is another term because of the factor $u'(S_v^2)$, and this term is $\frac{1}{2}\nu_{t,v}\big(f(R^t_{1,2}-q)u''(S_v^1)u''(S_v^2)\big)$. So actually we have shown that

$$
\begin{aligned}
\nu_{t,v}(fS_v^{1\prime}u''(S_v^1)u'(S_v^2)) &= \frac{1}{2}\nu_{t,v}\big(f(R^t_{1,2}-q)u''(S_v^1)u''(S_v^2)\big) \\
&\quad + \frac{1}{2}\sum_{2\leq\ell\leq n}\nu_{t,v}\big(f(R^t_{1,\ell}-q)u''(S_v^1)u'(S_v^2)u'(S_v^\ell)\big) \\
&\quad - \frac{n}{2}\nu_{t,v}\big(f(R^t_{1,n+1}-q)u''(S_v^1)u'(S_v^2)u'(S_v^{n+1})\big)\ .
\end{aligned}
$$

We strongly suggest to the enterprising reader to compute now all the other terms of (2.41). This is the best way to really understand the mechanism at work. There is no difficulty whatsoever, this just requires patience.

Calculations similar to the previous one will be needed again and again. We will not anymore explain them formally as above. Rather, we will give the result of the computation with possibly a few words of explanation. It is worth making now a simple observation that helps finding the result of such a computation. It is the fact that from (2.51) we have

$$\mathsf{E} z_{\boldsymbol{\sigma}}^{\ell\prime} y_{\boldsymbol{\tau}} = \mathsf{E} z_{\boldsymbol{\sigma}}^{\ell\prime} z_{\boldsymbol{\tau}}^{n+1} .$$

In a sense this means that when performing the integration by parts, we obtain the same result as if Z were actually a function of the variables $z_{\boldsymbol{\sigma}}^{n+1}$. It is useful to formulate this principle as a heuristic rule:

$$\begin{gathered}\text{The result of the expectation } \mathsf{E}_\xi \text{ in the definition of } Z \text{ is somehow} \\ \text{“to shift the dependence of } Z \text{ in } S_v \text{ on a new replica” .}\end{gathered} \tag{2.57}$$

When describing in the future the computation of a quantity such as $\nu_{t,v}(f S_v^{1\prime} u''(S_v^1) u'(S_v^2))$ by integration by parts, we will simply say: we integrate by parts using the relations

$$\mathsf{E} S_v^{\ell\prime} S_v^{\ell} = 0 \; ; \; \mathsf{E} S_v^{\ell\prime} S_v^{\ell'} = \frac{1}{2}(R_{\ell,\ell'}^t - q) \; , \tag{2.58}$$

and we will expect that the reader has understood enough of the algebraic mechanism at work to be able to check that the result of the computation is indeed the one we give, and the heuristic rule (2.57) should be precious for this purpose. There are two more such calculations in the present chapter, and the algebra in each is much simpler than in the present case. As a good start to develop the understanding of this mechanism, the reader should at the very least check the following two formulas involved in the computation of (2.41):

$$\begin{aligned} &\nu_{t,v}\big(f S_v^{3\prime} u'(S_v^3) u'(S_v^1) u'(S_v^2)\big) \\ &= \frac{1}{2}\nu_{t,v}\big(f(R_{3,1}^t - q) u'(S_v^3) u''(S_v^1) u'(S_v^2)\big) \\ &+ \frac{1}{2}\nu_{t,v}\big(f(R_{3,2}^t - q) u'(S_v^3) u'(S_v^1) u''(S_v^2)\big) \\ &+ \frac{1}{2}\sum_{\ell\neq 3, \ell\leq n} \nu_{t,v}\big(f(R_{3,\ell}^t - q) u'(S_v^3) u'(S_v^1) u'(S_v^2) u'(S_v^\ell)\big) \\ &- \frac{n}{2}\nu_{t,v}\big(f(R_{3,n+1}^t - q) u'(S_v^3) u'(S_v^1) u'(S_v^2) u'(S_v^{n+1})\big) \; , \end{aligned}$$

and

$$
\begin{aligned}
&\nu_{t,v}\big(f S_v^{n+1\prime} u'(S_v^{n+1}) u'(S_v^1) u'(S_v^2)\big)\\
&= \frac{1}{2}\nu_{t,v}\big(f(R^t_{n+1,1}-q) u'(S_v^{n+1}) u''(S_v^1) u'(S_v^2)\big)\\
&+ \frac{1}{2}\nu_{t,v}\big(f(R^t_{n+1,2}-q) u'(S_v^{n+1}) u'(S_v^1) u''(S_v^2)\big)\\
&+ \frac{1}{2}\sum_{\ell\le n}\nu_{t,v}\big(f(R^t_{n+1,\ell}-q) u'(S_v^{n+1}) u'(S_v^1) u'(S_v^2) u'(S_v^\ell)\big)\\
&- \frac{n+1}{2}\nu_{t,v}\big(f(R^t_{n+1,n+2}-q) u'(S_v^{n+1}) u'(S_v^1) u'(S_v^2) u'(S_v^{n+2})\big)\ .
\end{aligned}
$$

We bound a term (2.53) by

$$
K(D)\nu_{t,v}(|f||R^t_{1,\ell'}-q|)\ ,
$$

and we write $|R^t_{\ell,\ell'}-q|\le |R_{\ell,\ell'}-q|+1/N$ to obtain the inequality

$$
\left|\frac{\mathrm{d}}{\mathrm{d}v}\nu_{t,v}(B_v f)\right| \le K(n,D)\bigg(\sum_{1\le\ell<\ell'\le n+2}\nu_{t,v}(|f||R_{\ell,\ell'}-q|)+\frac{1}{N}\nu_{t,v}(|f|)\bigg)\ . \tag{2.59}
$$

To conclude we use Hölder's inequality. □

Exercise 2.3.3. Let us recall the notation $S^\ell_{k,t}$ of Proposition 2.2.3 and define

$$
S^{\ell\prime}_{k,t} = \frac{1}{2\sqrt{N}}\left(\frac{g_k\varepsilon_\ell}{\sqrt{t}} - \frac{\xi^\ell_k}{\sqrt{1-t}}\right)\ ,
$$

so that (2.27) becomes

$$
\frac{\mathrm{d}}{\mathrm{d}t}(-H^\ell_{N,M,t}) = \sum_{k\le M} S^{\ell\prime}_{k,t} u'(S^\ell_{k,t}) - \frac{\varepsilon_\ell Y}{2\sqrt{1-t}}\ .
$$

Observe the relations

$$
\mathsf{E} S^{\ell\prime}_{k,t}S^\ell_{k,t}=0\ ;\ \mathsf{E} S^{\ell\prime}_{k,t}S^{\ell'}_{k,t}=\frac{1}{2N}\varepsilon_\ell\varepsilon_{\ell'} \text{ if } \ell\ne\ell'\ ;\ \mathsf{E} S^{\ell\prime}_{k,t}S^{\ell'}_{k',t}=0 \text{ if } k\ne k'\ . \tag{2.60}
$$

Get convinced that the previously described mechanism yields the formula (when $\ell\le n+1$)

$$
\begin{aligned}
\nu_t(S^{\ell\prime}_{k,t}u'(S^\ell_{k,t})f) = \frac{1}{2N}\bigg(&\sum_{\ell'\ne\ell,\ell'\le n+1}\nu_t(\varepsilon_\ell\varepsilon_{\ell'}u'(S^\ell_{k,t})u'(S^{\ell'}_{k,t})f)\\
&-(n+1)\nu_t(\varepsilon_\ell\varepsilon_{n+2}u'(S^\ell_{k,t})u'(S^{n+2}_{k,t})f)\bigg)\ .
\end{aligned}
$$

Then get convinced that the term I in (2.23) can be obtained "in one step" rather than by integrating by parts separately over the r.v.s $\xi_{k,\ell}$ and g_k as was done in the proof of Proposition 2.2.3.

To follow future computations it is really important to understand the difference between the situation (2.58) (where integration by parts "brings a factor $(R^t_{\ell,\ell'} - q)/2$ in each term") and the situation (2.60), where this integration by parts brings "a factor $\varepsilon_\ell \varepsilon_{\ell'}/2N$ in each term".

Let us point out that the constants $K(n,D)$ and $K(D)$ are simply avatars of our ubiquitous constant K, and they need not be the same at each occurrence. The only difference is that here we make explicit that these constants depend only on n and D (etc.) simply because this is easier to do when there are so few parameters. Of course, $K_1(D)$, etc. denote specific constants.

Lemma 2.3.4. *If $f \geq 0$ is a function on Σ_N^n we have*

$$\nu_{t,v}(f) \leq K(n,D)\nu_t(f) \,. \tag{2.61}$$

Proof. We use (2.40) with $B_v \equiv 1$, $\tau_1 = 1, \tau_2 = \infty$ to get

$$\left|\frac{\mathrm{d}}{\mathrm{d}v}\nu_{t,v}(f)\right| \leq K(n,D)\nu_{t,v}(f) \,.$$

We integrate and we use that $\nu_{t,1}(f) = \nu_t(f)$. □

Proposition 2.3.5. *Consider a function f on Σ_N^n. This function might be random, but it does not depend on the randomness of the variables $g_{i,M}, \xi_M^\ell, \xi^\ell$ or z. Then, whenever $1/\tau_1 + 1/\tau_2 = 1$, we have*

$$|\nu_t(fu'(S^1_{M,t})u'(S^2_{M,t})) - \widehat{r}\nu_t(f)| \leq K(n,D)\bigg(\nu_t(|f|^{\tau_1})^{1/\tau_1}\nu_t(|R_{1,2} - q|^{\tau_2})^{1/\tau_2} + \frac{1}{N}\nu_t(|f|)\bigg) \,. \tag{2.62}$$

This provides a good understanding of the term I of (2.23), provided we can find q such that the right-hand side is small.

Proof. We consider B_v as in Lemma 2.3.2, we write

$$|\nu_{t,1}(B_1 f) - \nu_{t,0}(B_0 f)| \leq \max_v \left|\frac{\mathrm{d}}{\mathrm{d}v}\nu_{t,v}(B_v f)\right| \,, \tag{2.63}$$

and we use (2.40) and (2.61) to get

$$|\nu_{t,1}(B_1 f) - \nu_{t,0}(B_0 f)| \leq \mathcal{B} \,, \tag{2.64}$$

where $\mathcal{B}$ is a term as in the right-hand side of (2.62). Thus in the case $B_v \equiv 1$, and since $\nu_{t,1} = \nu_t$, (2.37) and (2.64) imply that

$$|\nu_t(f) - \mathsf{E}\langle f\rangle_{t,\sim}| \leq \mathcal{B} \,. \tag{2.65}$$

In the case $B_v = u'(S^1_v)u'(S^2_v)$, (2.38) and (2.64) mean

$$\left|\nu_t\big(fu'(S^1_{M,t})u'(S^2_{M,t})\big) - \widehat{r}\,\mathsf{E}\langle f\rangle_{t,\sim}\right| \le \mathcal{B}$$

and combining with (2.65) finishes the proof. □

We now set $r = \alpha\widehat{r}$, and (2.62) implies

$$\begin{aligned}&\left|\alpha\nu_t\big(\varepsilon_\ell\varepsilon_{\ell'}fu'(S^\ell_{M,t})u'(S^{\ell'}_{M,t})\big) - r\nu_t(\varepsilon_\ell\varepsilon_{\ell'}f)\right|\\ &\le \alpha K(n,D)\Big(\nu_t(|f|^{\tau_1})^{1/\tau_1}\nu_t(|R_{1,2}-q|^{\tau_2})^{1/\tau_2} + \frac{1}{N}\nu_t(|f|)\Big)\,.\end{aligned}$$

Looking again at the terms I and II of Proposition 2.2.3, we have proved the following.

Proposition 2.3.6. *Consider a function f on Σ_N^n (that does not depend on any of the r.v.s $g_{i,M}, \xi^\ell, \xi^\ell_M$ or z). Then, whenever $1/\tau_1 + 1/\tau_2 = 1$, we have*

$$|\nu'_t(f)| \le \alpha K(D,n)\Big(\nu_t(|f|^{\tau_1})^{1/\tau_1}\nu_t(|R_{1,2}-q|^{\tau_2})^{1/\tau_2} + \frac{1}{N}\nu_t(|f|)\Big)\,. \tag{2.66}$$

The following is an obviously helpful way to relate ν and ν_t.

Lemma 2.3.7. *There exists a constant $K(D)$ with the following property. If $\alpha K(D) \le 1$, whenever $f \ge 0$ is a function on Σ_N^2 (that does not depend on any of the r.v.s $g_{i,M}, \xi^\ell, \xi^\ell_M$ or z), we have*

$$\nu_t(f) \le 2\nu(f)\,. \tag{2.67}$$

Proof. We use Proposition 2.3.6 with $\tau_1 = 1$ and $\tau_2 = \infty$ to see that

$$|\nu'_t(f)| \le \alpha K_1(D)\nu_t(f)\,,$$

from which (2.67) follows by integration if $\alpha K_1(D) \le \log 2$. □

2.4 The Replica Symmetric Solution

We recall the notation $\theta = z\sqrt{q} + \xi\sqrt{1-q}$ where z and ξ are independent standard Gaussian r.v.s, and that E_ξ denotes expectation in ξ only.

Theorem 2.4.1. *Given $D > 0$, there is a number $K(D)$ with the following property. Assume that the function u satisfies (2.7), i.e.*

$$\forall \ell \le 3\quad ,\qquad |u^{(\ell)}| \le D\,.$$

Then whenever $\alpha \le 1/K(D)$ the system of equations

$$q = \mathsf{E}\,\mathrm{th}^2(z\sqrt{r})\quad ;\quad r = \alpha\mathsf{E}\left(\frac{\mathsf{E}_\xi u'(\theta)\exp u(\theta)}{\mathsf{E}_\xi \exp u(\theta)}\right)^2 \tag{2.68}$$

in the unknown q and r has a unique solution, and

$$\nu\big((R_{1,2}-q)^2\big) \le \frac{L}{N}\,. \tag{2.69}$$

Proof. Let us write the second equation of (2.68) as $r = \alpha\hat{r} = \alpha\hat{r}(q)$. Differentiation and integration by parts show that $|\hat{r}'(q)| \le K(D)$ under (2.7). The function $r \mapsto \mathsf{E}\,\mathrm{th}^2(z\sqrt{r})$ has a bounded derivative; so the function $q \mapsto \psi(q) := \mathsf{E}\mathrm{th}^2(z\sqrt{\alpha\hat{r}(q)})$ has a derivative $\le \alpha K_2(D)$. Therefore if $2\alpha K_2(D) \le 1$ there is a unique solution to the equation $q = \psi(q)$ because then the function $\psi(q)$ is valued in $[0,1]$ with a derivative $\le 1/2$.

Symmetry among sites yields

$$\nu\big((R_{1,2}-q)^2\big) = \nu(f) \tag{2.70}$$

where $f = (\varepsilon_1\varepsilon_2 - q)(R_{1,2}-q)$, and we write

$$\nu(f) \le \nu_0(f) + \sup_{0<t<1} |\nu_t'(f)| \,. \tag{2.71}$$

Since $q = \mathsf{E}\,\mathrm{th}^2(z\sqrt{r}) = \mathsf{E}\,\mathrm{th}^2 Y$, Lemma 2.2.2 implies

$$\nu_0((\varepsilon_1\varepsilon_2 - q)(R^-_{1,2} - q)) = (\mathsf{E}\,\mathrm{th}^2 Y - q)\nu_0(R^-_{1,2} - q) = 0 \,,$$

and thus

$$\nu_0(f) = \frac{1}{N}\nu_0(1-\varepsilon_1\varepsilon_2 q) = \frac{1}{N}(1-q^2) \,. \tag{2.72}$$

To compute $\nu_t'(f)$, we use Proposition 2.3.6 with $n = 2$ and $\tau_1 = \tau_2 = 2$. Since $|f| \le 2|R_{1,2} - q|$, we obtain

$$|\nu_t'(f)| \le \alpha K(D)\left(\nu_t\big((R_{1,2}-q)^2\big) + \frac{1}{N}\nu(|f|)\right) \,. \tag{2.73}$$

We substitute in (2.71) and use (2.67) to get the relation

$$\nu(f) = \nu\big((R_{1,2}-q)^2\big) \le \alpha K(D)\Big(\nu\big((R_{1,2}-q)^2\big) + \frac{1}{N}\nu(|f|)\Big) + \frac{1}{N}(1-q^2) \,,$$

so that since $|f| \le 4$ we obtain

$$\nu\big((R_{1,2}-q)^2\big) \le \alpha K(D)\nu\big((R_{1,2}-q)^2\big) + \frac{K(D)(\alpha+1)}{N} \,. \qquad \square$$

One should observe that in the above argument we never used the uniqueness of the solutions of the equations (2.68) to obtain (2.69), only their existence. In turn, uniqueness of these solutions follows from (2.69).

One may like to think of the present model as a kind of "square". There are two "spin systems", one that consists of the σ_i and one that consists of the S_k. These are coupled: the σ_i determine the S_k and these in turn determine the behavior of the σ_i. This philosophy undermines the first proof of Theorem 2.4.2 below.

From now on in this section, q and r always denote the solutions of (2.68). We recall the definition (2.11)

$$p_{N,M}(u) = \frac{1}{N}\mathsf{E}\log\sum_{\boldsymbol{\sigma}}\exp(-H_{N,M}(\boldsymbol{\sigma}))\ ,$$

and we define

$$p(u) = -\frac{1}{2}r(1-q) + \mathsf{E}\log(2\mathrm{ch}(z\sqrt{r})) + \alpha\mathsf{E}\log\mathsf{E}_\xi\exp u(z\sqrt{q}+\xi\sqrt{1-q})\ . \tag{2.74}$$

Theorem 2.4.2. *Under the conditions of Theorem 2.4.1 we have*

$$|p_{N,M}(u) - p(u)| \le \frac{K(D)}{N}\ . \tag{2.75}$$

We will present two proofs of this fact.

First proof of Theorem 2.4.2. We start with the most beautiful proof, which is somewhat challenging. It implements through interpolation the idea that "the quantities S_k behave like individual spins". We consider independent standard Gaussian r.v.s $z, (z_k)_{k\le M}, (z'_i)_{i\le N}, (\xi_k)_{k\le M}$ and for $0<s<1$ the Hamiltonian

$$-H_{M,N,s} = \sum_{k\le M} u(\sqrt{s}S_k + \sqrt{1-s}\theta_k) + \sum_{i\le N}\sigma_i\sqrt{1-s}z'_i\sqrt{r} \tag{2.76}$$

where $\theta_k = z_k\sqrt{q}+\xi_k\sqrt{1-q}$. In this formula, we should think of z'_i and z_k as representing new randomness, and of ξ_k as representing "new spins", so that Gibbs averages are given by (2.19), and we define

$$p_{N,M,s} = \frac{1}{N}\mathsf{E}\log\mathsf{E}_\xi\sum_{\boldsymbol{\sigma}}\exp(-H_{M,N,s})\ .$$

The variables ξ_k are not the same as in Section 2.2; we could have denoted them by ξ'_k to insist on this fact, but we preferred simpler notation.

A key point of the present interpolation is that the equations giving the parameters q_s and r_s corresponding to the parameters q and r in the case $s=1$ are now

$$q_s = \mathsf{E}\mathrm{th}^2\left(\sqrt{s}z\sqrt{r_s}+\sqrt{1-s}z'\sqrt{r}\right) \tag{2.77}$$

$$r_s = \alpha\mathsf{E}\left(\frac{\mathsf{E}_\xi u'(\theta_s)\exp u(\theta_s)}{\mathsf{E}_\xi\exp u(\theta_s)}\right)^2 \tag{2.78}$$

where

$$\theta_s = \sqrt{s}(z\sqrt{q_s}+\xi\sqrt{1-q_s}) + \sqrt{1-s}(z'\sqrt{q}+\xi'\sqrt{1-q})\ .$$

To understand the formula (2.77) one should first understand what happens if we include the action of a random external field in the Hamiltonian, i.e. we add a term $h\sum_{i\le N}g_i\sigma_i$ (where g_i are i.i.d. standard Gaussian) to the right-hand side of (2.6). Then there is nothing to change to the proof of Theorem 2.4.1; only the first formula of (2.68) becomes

$$q = \mathsf{E}\,\mathrm{th}^2(z\sqrt{r} + hg)\ , \tag{2.79}$$

where g, z are independent standard Gaussian r.v.s. We then observe that the last term in (2.76) is an external field, that creates the term $\sqrt{1-s}z'\sqrt{r}$ in (2.77). The second term in the definition of θ_s is created by the terms $\sqrt{1-s}\theta_k$ in the Hamiltonian (2.76), a source of randomness "inside u".

The values $q_s = q$, $r_s = r$ are solutions of the equations (2.77) and (2.78), because for these values $\sqrt{s}z\sqrt{q_s} + \sqrt{1-s}z'\sqrt{q}$ is distributed like $z\sqrt{q}$ (etc.). One could easily check that the solution of the system of equations (2.77) and (2.78) is unique when $\alpha K(D) \leq 1$, but this is not needed.

We leave to the readers, as an excellent exercise for those who really want to master the present ideas, the task to prove (2.69) in the case of the Hamiltonian (2.76). Since we have already made the effort to understand the effect of the expectations E_ξ, there is really not much to change to the proof we gave.

So, with obvious notation, one has

$$\forall s \in [0,1]\ ,\ \nu_s\big((R_{1,2} - q)^2\big) \leq \frac{L}{N}\ . \tag{2.80}$$

Let us define

$$S_{k,s} = \sqrt{s}S_k + \sqrt{1-s}\theta_k\ ;\ S'_{k,s} = \frac{1}{2\sqrt{s}}S_k - \frac{1}{2\sqrt{1-s}}\theta_k\ ,$$

so that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}p_{N,M,s}(u) &= \frac{1}{N}\nu_s\left(\frac{\mathrm{d}}{\mathrm{d}s}(-H_{N,M,s})\right)\\ &= \frac{1}{N}\nu_s\Bigg(\sum_{k\leq M} S'_{k,s}u'(S_{k,s}) - \frac{1}{2\sqrt{1-s}}\sum_{i\leq N}\sigma_i z'_i\sqrt{r}\Bigg)\ . \end{aligned}\tag{2.81}$$

The next step is to integrate by parts. It should be obvious how to proceed for the integration by parts in z'_i; this gives

$$\frac{1}{N}\nu_s\Bigg(\frac{1}{2\sqrt{1-s}}\sum_{i\leq N}\sigma_i z'_i\sqrt{r}\Bigg) = \frac{r}{2}(1 - \nu_s(R_{1,2}))\ .$$

Let us now explain how to compute $\nu_s(S'_{k,s}u'(S_{k,s}))$. Without loss of generality we assume $k = M$. We make explicit the dependence of the Hamiltonian on $S_{M,s}$ by introducing the Hamiltonian

$$-H_{M-1,N,s} = \sum_{k\leq M-1} u(\sqrt{s}S_k + \sqrt{1-s}\theta_k) + \sum_{i\leq N}\sigma_i\sqrt{1-s}z'_i\sqrt{r}\ .$$

Denoting by $\langle\cdot\rangle_\sim$ an average for this Hamiltonian, we then have

$$\nu_s(S'_{M,s}u'(S_{M,s})) = \mathsf{E}\frac{\langle S'_{M,s}u'(S_{M,s})\exp u(S_{M,s})\rangle_\sim}{\langle \exp u(S_{M,s})\rangle_\sim} . \tag{2.82}$$

Let us denote as usual by an upper index ℓ the fact "that the spins are in the ℓ-th replica". For example, (since we think of ξ_k as a spin) $\theta_k^\ell = z_k\sqrt{q} + \xi_k^\ell\sqrt{1-q}$ where ξ_k^ℓ are independent standard Gaussian r.v.s, and $S_{k,s}^\ell = \sqrt{s}S_k^\ell + \sqrt{1-s}\theta_k^\ell$, and let us observe the key relations (where the reader will not confuse $S_{M,s}^{\ell\prime}$ with $S_{M,s}^{\ell'}$)

$$\mathsf{E}S_{M,s}^{\ell\prime}S_{M,s}^{\ell} = 0\ ;\ \ell \neq \ell' \Rightarrow \mathsf{E}S_{M,s}^{\ell\prime}S_{M,s}^{\ell'} = \frac{1}{2}(R_{\ell,\ell'} - q) .$$

Now we integrate by parts in (2.82). This integration by parts will take place given the randomness of $H_{M-1,N,s}$. We have explained in detail in the proof of Lemma 2.3.2 how to proceed. The present case is significantly simpler. There is only one term, "the term created by the denominator" (as defined page 168), and we obtain

$$\nu_s(S'_{M,s}u'(S_{M,s})) = -\frac{1}{2}\nu_s\big((R_{1,2}-q)u'(S_{M,s}^1)u'(S_{M,s}^2)\big) .$$

This illustrates again the principle (2.58) that the expectation E_ξ in the denominator "shifts the variables there to a new replica." Therefore we have found that

$$\frac{\mathrm{d}}{\mathrm{d}s}p_{N,M,s}(u) = -\frac{1}{2}\nu_s\bigg((R_{1,2}-q)\frac{1}{N}\sum_{k\le M}u'(S_{k,s}^1)u'(S_{k,s}^2)\bigg) - \frac{r}{2}(1-\nu_s(R_{1,2})) .$$

We will not use the fact that the contribution for each $k \le M$ is the same, but rather we regroup the terms as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}p_{N,M,s}(u) &= -\frac{r}{2}(1-q)\\ &\quad -\frac{1}{2}\nu_s\bigg((R_{1,2}-q)\bigg(\frac{1}{N}\sum_{k\le M}u'(S_{k,s}^1)u'(S_{k,s}^2) - r\bigg)\bigg).\end{aligned} \tag{2.83}$$

This formula should be compared to (1.65). There seems to be little hope to get any kind of positivity argument here. This is unfortunate because as of today, positivity arguments are almost our only tool to obtain low-temperature results.

We get, using the Cauchy-Schwarz inequality

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}s}p_{N,M,s}(u) + \frac{r}{2}(1-q)\right| &\le \nu_s\big((R_{1,2}-q)^2\big)^{1/2} \\ &\quad\times \nu_s\bigg(\bigg(\frac{1}{N}\sum_{k\le M}u'(S_{k,s}^1)u'(S_{k,s}^2) - r\bigg)^2\bigg)^{1/2} .\end{aligned} \tag{2.84}$$

From (2.80) we see that the right-hand side is $\leq K(D)/\sqrt{N}$; but to get the correct rate $K(D)/N$ (rather than $K(D)/\sqrt{N}$) in Theorem 2.4.2, we need to know the following, that is proved separately in Lemma 2.4.3 below:

$$\nu_s\Big(\Big(\frac{1}{N}\sum_{k\leq M} u'(S^1_{k,s})u'(S^2_{k,s}) - r\Big)^2\Big) \leq \frac{K(D)}{N} \,. \tag{2.85}$$

We combine with (2.80) to obtain from (2.84) that

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}p_{N,M,s}(u) + \frac{r}{2}(1-q)\right| \leq \frac{K(D)}{N}$$

so that, since $p_{N,M}(u) = p_{N,M,1}(u)$,

$$\Big|p_{N,M}(u) + \frac{r}{2}(1-q) - p_{N,M,0}(u)\Big| \leq \frac{K(D)}{N} \,.$$

As the spins decouple in $p_{N,M,0}(u)$, the computation of this quantity is straightforward and this yields (2.75). □

Lemma 2.4.3. *Inequality (2.85) holds under the conditions of Theorem 2.4.1.*

Proof. Let us write

$$f = \frac{1}{N}\sum_{k\leq M} u'(S^1_{k,s})u'(S^2_{k,s}) - r$$
$$f^- = \frac{1}{N}\sum_{k< M} u'(S^1_{k,s})u'(S^2_{k,s}) - r \,,$$

so that, using symmetry between the values of $k \leq M$,

$$\begin{aligned}\nu_s(f^2) &= \nu_s\big((\alpha u'(S^1_{M,s})u'(S^2_{M,s}) - r)f\big)\\ &\leq \nu_s\big((\alpha u'(S^1_{M,s})u'(S^2_{M,s}) - r)f^-\big) + \frac{K(D)}{N} \,.\end{aligned} \tag{2.86}$$

We extend Proposition 2.3.5 to the present setting of the Hamiltonian (2.76) to get

$$\begin{aligned}&\big|\nu_s\big((\alpha u'(S^1_{M,s})u'(S^2_{M,s}) - r)f^-\big)\big|\\ &\leq \alpha K(D)\Big(\nu_s\big((R_{1,2}-q)^2\big)^{1/2}\nu_s\big((f^-)^2\big)^{1/2} + \frac{1}{N}\Big) \,.\end{aligned}$$

Combining these, and since $2\sqrt{ab} \leq a+b$, for $\alpha K(D) \leq 1$ this yields

$$\nu_s(f^2) \leq \frac{1}{2}\nu_s\big((R_{1,2}-q)^2\big) + \frac{1}{2}\nu_s\big((f^-)^2\big) + \frac{K(D)}{N}$$

and since $|f^2 - (f^-)^2| \le K(D)/N$ we get

$$\nu_s(f^2) \le \frac{1}{2}\nu_s\big((R_{1,2} - q)^2\big) + \frac{1}{2}\nu_s(f^2) + \frac{K(D)}{N} ,$$

which completes the proof using (2.80). □

To prepare for the second proof of Theorem 2.4.2, let us denote by $F(\alpha, r, q)$ the right-hand side of (2.74), i.e.

$$F(\alpha, r, q) = -\frac{1}{2}r(1-q) + \mathsf{E}\log(2\mathrm{ch}(z\sqrt{r})) + \alpha\mathsf{E}\log\mathsf{E}_\xi\exp u(\theta) ,$$

where $\theta = z\sqrt{q} + \xi\sqrt{1-q}$ and let us think of this quantity as a function of three unrelated variables. For convenience, we reproduce the equations (2.68):

$$q = \mathsf{E}\,\mathrm{th}^2(z\sqrt{r}) \quad ; \quad r = \alpha\mathsf{E}\left(\frac{\mathsf{E}_\xi u'(\theta)\exp u(\theta)}{\mathsf{E}_\xi \exp u(\theta)}\right)^2 . \tag{2.87}$$

Lemma 2.4.4. *The conditions (2.87) mean respectively that $\partial F/\partial r = 0$, $\partial F/\partial q = 0$.*

Proof. This is of course calculus, differentiation and integration by parts, but it would be nice to *really* understand why this is true. We give the proof in complete detail, but we suggest as a simple exercise that the reader tries first to figure out these details by herself.

Integration by parts yields

$$\frac{\partial F}{\partial r} = \frac{1}{2}\left(q - 1 + \frac{1}{\sqrt{r}}\mathsf{E}\,z\mathrm{th}z\sqrt{r}\right) = \frac{1}{2}\left(q - 1 + \mathsf{E}\,\frac{1}{\mathrm{ch}^2(z\sqrt{r})}\right)$$

so that $\partial F/\partial r = 0$ if

$$q = 1 - \mathsf{E}\,\frac{1}{\mathrm{ch}^2(z\sqrt{r})} = \mathsf{E}\,\mathrm{th}^2(z\sqrt{r}) .$$

Next, if

$$\theta = z\sqrt{q} + \xi\sqrt{1-q}, \;\; \theta' = \frac{z}{2\sqrt{q}} - \frac{\xi}{2\sqrt{1-q}} ,$$

we have

$$\frac{\partial F}{\partial q} = \frac{r}{2} + \frac{\alpha}{2}\mathsf{E}\left(\theta'\frac{u'(\theta)\exp u(\theta)}{\mathsf{E}_\xi\exp u(\theta)}\right) . \tag{2.88}$$

To integrate by parts, we observe that $F_1(z) = \mathsf{E}_\xi\exp u(\theta)$ does not depend on ξ and

$$\frac{\mathrm{d}F_1}{\mathrm{d}z} = \frac{\mathrm{d}}{\mathrm{d}z}\,\mathsf{E}_\xi\exp u(z\sqrt{q} + \xi\sqrt{1-q}) = \sqrt{q}\mathsf{E}_\xi\, u'(\theta)\exp u(\theta) .$$

We appeal to the integration by parts formula (A.17) to find, since $\mathsf{E}(\theta'\theta) = 0$, $\mathsf{E}(\theta' z) = 1/\sqrt{q}$ that

$$\begin{aligned}\mathsf{E}\left(\theta' \frac{u'(\theta)\exp u(\theta)}{F_1(z)}\right) &= -\mathsf{E}\left(\frac{1}{F_1(z)^2} u'(\theta)\exp u(\theta) \mathsf{E}_\xi(u'(\theta)\exp u(\theta))\right) \\ &= -\mathsf{E}\,\frac{(\mathsf{E}_\xi u'(\theta)\exp u(\theta))^2}{(\mathsf{E}_\xi \exp u(\theta))^2}\ ,\end{aligned}$$

so that by (2.88), $\partial F/\partial q = 0$ if and only if the second part of (2.87) holds. □

If q and r are now related by the conditions (2.87), for small α they are functions $q(\alpha)$ and $r(\alpha)$ of α (since, as shown by Theorem 1.4.1 the equations (2.87) have a unique solution). The quantity $F(\alpha, r(\alpha), q(\alpha))$ is function $F(\alpha)$ of α alone, and

$$\frac{\mathrm{d}F}{\mathrm{d}\alpha} = \frac{\partial F}{\partial \alpha} + \frac{\partial F}{\partial q}\frac{\mathrm{d}q}{\mathrm{d}\alpha} + \frac{\partial F}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}\alpha} = \frac{\partial F}{\partial \alpha}\ ,$$

since $\partial F/\partial q = \partial F/\partial r = 0$ when $q = q(\alpha)$ and $r = r(\alpha)$. Therefore

$$F'(\alpha) = \mathsf{E}\log \mathsf{E}_\xi \exp u(\theta)\ . \tag{2.89}$$

Second proof of Theorem 2.4.2. We define $Z_{N,M} = \sum_{\boldsymbol{\sigma}} \exp(-H_{N,M}(\boldsymbol{\sigma}))$, and we note the identity

$$Z_{N,M+1} = Z_{N,M}\left\langle \exp u\left(\frac{1}{\sqrt{N}}\sum_{i\le N} g_{i,M+1}\sigma_i\right)\right\rangle$$

so that

$$p_{N,M+1}(u) - p_{N,M}(u) = \frac{1}{N}\mathsf{E}\log\left\langle \exp u\left(\frac{1}{\sqrt{N}}\sum_{i\le N} g_{i,M+1}\sigma_i\right)\right\rangle\ . \tag{2.90}$$

To compute the right-hand side of (2.90) we introduce

$$S_v = \sqrt{\frac{v}{N}}\sum_{i\le N} g_{i,M+1}\sigma_i + \sqrt{1-v}\,\theta\ ,$$

where $\theta = z\sqrt{q} + \xi\sqrt{1-q}$, where (I almost hesitate to say it again) z and ξ are independent standard Gaussian r.v.s, and where q is as in (2.68) for $\alpha = M/N$ (so that the value of q depends on M). We set

$$\varphi(v) = \mathsf{E}\log \mathsf{E}_\xi\langle \exp u(S_v)\rangle\ .$$

As usual E_ξ denotes expectation in all the r.v.s labeled ξ. Here this expectation is not built in the bracket $\langle\cdot\rangle$, in contrast with what we did e.g in (2.35), so that it must be written explicitly.

We note that

$$\varphi(1) = N(p_{N,M+1}(u) - p_{N,M}(u)) \; ; \; \varphi(0) = \mathsf{E} \log \mathsf{E}_\xi \exp u(\theta) \; .$$

With obvious notation we have

$$\varphi'(v) = \mathsf{E} \frac{\mathsf{E}_\xi \langle S'_v \exp u(S_v) \rangle}{\mathsf{E}_\xi \langle \exp u(S_v) \rangle} = \mathsf{E} \frac{\langle S'_v \exp u(S_v) \rangle}{\mathsf{E}_\xi \langle \exp u(S_v) \rangle} \; .$$

We then integrate by parts, exactly as in (2.82). This yields the formula

$$\varphi'(v) = -\frac{1}{2} \mathsf{E} \frac{\langle (R_{1,2} - q) u'(S_v^1) u'(S_v^2) \exp(u(S_v^1) + u(S_v^2)) \rangle}{\mathsf{E}_\xi \langle \exp(u(S_v^1) + u(S_v^2)) \rangle} \; , \tag{2.91}$$

where S_v^ℓ is defined as S_v, but replacing ξ by ξ^ℓ and $\boldsymbol{\sigma}$ by $\boldsymbol{\sigma}^\ell$. Now (2.69) implies

$$|\varphi'(v)| \le K(D) \nu(|R_{1,2} - q|) \le K(D) \nu\big((R_{1,2} - q)^2\big)^{1/2} \le \frac{K(D)}{\sqrt{N}} \; .$$

This bound unfortunately does not get the proper rate. To get the proper bound in $K(D)/N$ in (2.75) one must replace the bound

$$|\varphi(1) - \varphi(0)| \le \sup |\varphi'(v)|$$

by the bound

$$|\varphi(1) - \varphi(0) - \varphi'(0)| \le \sup |\varphi''(v)| \; . \tag{2.92}$$

A new differentiation and integration by parts in (2.91) bring out in each term a new factor $(R_{\ell,\ell'} - q)$, so that using (2.69) we now get

$$|\varphi''(v)| \le K(D) \nu\big((R_{1,2} - q)^2\big) \le \frac{K(D)}{N} \; .$$

As a special case of (2.91),

$$\varphi'(0) = -\frac{1}{2} \widehat{r} \nu(R_{1,2} - q) \; .$$

We shall prove later (when we learn how to prove central limit theorems in Chapter 9) the non-trivial fact that $|\nu(R_{1,2} - q)| \le K(D)/N$, and (2.92) then implies

$$\left| p_{N,M+1}(u) - p_{N,M}(u) - \frac{1}{N} \mathsf{E} \log \mathsf{E}_\xi \exp u(\theta) \right| \le \frac{K(D)}{N^2} \; . \tag{2.93}$$

One can then recover the value of $p_{N,M}(u)$ by summing these relations over M. This is a non-trivial task, since the value of q (and hence of θ) depends on M.

Let us recall the function $F(\alpha)$ of (2.89). It is tedious but straightforward to check that $F''(\alpha)$ remains bounded as $\alpha K(D) \le 1$, so that (2.89) yields

$$\left|F\left(\frac{M+1}{N}\right) - F\left(\frac{M}{N}\right) - \frac{1}{N}\mathsf{E}\log \mathsf{E}_\xi \exp u(\theta)\right| \le \frac{K(D)}{N^2} \; .$$

Comparing with (2.93) and summing over M then proves (2.75) (and even better, since the summation is over M, we get a bound $\alpha K(D)/N$). This completes the second proof of Theorem 2.4.2. □

It is worth noting that the first proof of Theorem 2.4.2 provides an easy way to discover the formula (2.74), but that this formula is much harder to guess if one uses the second proof. In some sense the first proof of Theorem 2.4.2 is more powerful and more elegant than the second proof. However we will meet situations (in Chapters 3 and 4) where it is not immediate to apply this method (and whether this is possible remains to be investigated). In these situations, we shall use instead the argument of the second proof of Theorem 2.4.2.

2.5 Exponential Inequalities

Our goal is to improve the control of $R_{1,2} - q$ from second to higher moments.

Theorem 2.5.1. *Given D, there is a number $K(D)$ such that if u satisfies (2.7), i.e. $|u^{(\ell)}| \le D$ for all $0 \le \ell \le 3$ then for $\alpha K(D) \le 1$, we have*

$$\forall\, k \ge 0\,, \quad \nu\big((R_{1,2} - q)^{2k}\big) \le \left(\frac{64k}{N}\right)^k \; . \tag{2.94}$$

Proof. It goes by induction over k, and is nearly identical to that of Proposition 1.6.7.

For $1 \le n \le N$, we define $A_n = N^{-1}\sum_{n\le i\le N}(\sigma_i^1\sigma_i^2 - q)$, and the induction hypothesis is that for each $n \le N$,

$$\nu(A_n^{2k}) \le \left(\frac{64k}{N}\right)^k \; . \tag{2.95}$$

To perform the induction from k to $k+1$, we can assume $n < N$, for (2.95) holds if $n = N$. Using symmetry between sites yields

$$\nu(A_n^{2k+2}) = \frac{N-n+1}{N}\nu(f)\;,$$

where

$$f = (\varepsilon_1\varepsilon_2 - q)A_n^{2k+1} \; .$$

Thus

$$\nu(A_n^{2k+2}) \le |\nu_0(f)| + \sup_t |\nu_t'(f)| \ . \tag{2.96}$$

We first study the term $\nu_0(f)$. Consider

$$A' = \frac{1}{N} \sum_{n \le i \le N-1} (\sigma_i^1 \sigma_i^2 - q) \ .$$

Since by Lemma 2.2.2 we have $\nu_0((\varepsilon_1\varepsilon_2 - q)A'^{2k+1}) = 0$, using the inequality

$$|x^{2k+1} - y^{2k+1}| \le (2k+1)|x-y|(x^{2k} + y^{2k})$$

for $x = A_n$ and $y = A'$ we get, since $|x - y| \le 2/N$ and $|\varepsilon_1\varepsilon_2 - q| \le 2$,

$$|\nu_0(f)| \le \frac{4(2k+1)}{N}\bigl(\nu_0(A_n^{2k}) + \nu_0(A'^{2k})\bigr) \ .$$

We use (2.67), the induction hypothesis, and the observation that since $n < N$, we have

$$\nu(A'^{2k}) = \nu(A_{n+1}^{2k})$$

to obtain

$$|\nu_0(f)| \le \frac{16(2k+1)}{N}\left(\frac{64k}{N}\right)^k \le \frac{2k+1}{4(k+1)}\left(\frac{64(k+1)}{N}\right)^{k+1} \ . \tag{2.97}$$

To compute $\nu_t'(f)$ we use Proposition 2.3.6 with $n = 4, \tau_1 = (2k+2)/(2k+1)$, $\tau_2 = 2k+2$ and (2.67) to get

$$|\nu_t'(f)| \le \alpha K(D)\left(\nu(A_n^{2k+2})^{1/\tau_1}\nu\bigl((R_{1,2}-q)^{2k+2}\bigr)^{1/\tau_2} + \frac{1}{N}\nu(|A_n|^{2k+1})\right) .$$

Using the inequality $x^{1/\tau_1}y^{1/\tau_2} \le x + y$ for $x = \nu(A_n^{2k+2})$ and $y = \nu((R_{1,2} - q)^{2k+2})$ this implies

$$|\nu_t'(f)| \le \alpha K(D)\left(\nu(A_n^{2k+2}) + \nu\bigl((R_{1,2}-q)^{2k+2}\bigr) + \frac{1}{N}\nu(|A_n|^{2k+1})\right) .$$

Combining with (2.96) and (2.97) we get if $\alpha K(D) \le 1/4$,

$$\begin{aligned}\nu(A_n^{2k+2}) \le{}& \frac{1}{4}\left(\nu(A_n^{2k+2}) + \nu\bigl((R_{1,2}-q)^{2k+2}\bigr)\right) \\ &+ \frac{2k+1}{4(k+1)}\left(\frac{64(k+1)}{N}\right)^{k+1} + \frac{1}{N}\nu(|A_n|^{2k+1}) \ .\end{aligned} \tag{2.98}$$

Since $|A_n| \le 2$ and hence $|A_n|^{2k+1} \le 2A_n^{2k}$, the induction hypothesis implies that the last term of (2.98) is at most

$$\frac{1}{32(k+1)}\left(\frac{64(k+1)}{N}\right)^{k+1} ,$$

so the sum of the last 2 terms is at most

$$\frac{1}{2}\left(\frac{64(k+1)}{N}\right)^{k+1} .$$

Since $A_1 = R_{1,2} - q$, considering first the case $n = 1$ provides the required inequality in that case. Using back this inequality in (2.98) provides the required inequality for all values of n. □

The following extends Lemma 2.4.3. Its proof is pretty similar to that of Theorem 2.5.1, and demonstrates the power of this approach. The reader who does not enjoy the argument should skip the forthcoming proof and make sure she does not miss the pretty Theorem 2.5.3. We denote by $K_0(D)$ the constant of Theorem 2.5.1.

Theorem 2.5.2. *Assume that* u *satisfies (2.7) for a certain number* D*. Then there is a number* $K(D)$*, depending on* D *only, with the following property. For* $\alpha K_0(D) \leq 1$ *we have*

$$\forall k \geq 0\ ,\quad \nu\left(\left(\frac{1}{N}\sum_{j\leq M} u'(S_j^1)u'(S_j^2) - r\right)^{2k}\right) \leq \left(\frac{\alpha k K(D)}{N}\right)^k . \tag{2.99}$$

Proof. We recall the definition of $\widehat{r}$ given by (2.36), i.e.

$$\widehat{r} = \mathsf{E}\left(\frac{\mathsf{E}_\xi u'(\theta)\exp u(\theta)}{\mathsf{E}_\xi \exp u(\theta)}\right)^2 ,$$

so that with the notation (2.87) we have $r = \alpha\widehat{r}$. For $1 \leq n \leq M$ we define

$$C_n = \frac{1}{M}\sum_{n\leq j\leq M}(u'(S_j^1)u'(S_j^2) - \widehat{r}) .$$

Since $r = \alpha\widehat{r}$ and $1/N = \alpha/M$ the left-hand side of (2.99) is $\alpha^{2k}\nu(C_1^{2k})$.

We prove by induction over k that if $\alpha K_0(D) \leq 1$ then for a suitable number $K_1(D)$ we have for $k \geq 1$ and any $n \leq M$ that

$$\nu(C_n^{2k}) \leq \left(\frac{kK_1(D)}{M}\right)^k . \tag{2.100}$$

Using this for $n = 1$ concludes the proof. For $k = 0$ (2.100) is true if one then understands the right-hand side of (2.99) as being 1. The reader disliking this can instead start the induction at $k = 1$. To prove the case $k = 1$ it suffices to repeat the proof of Lemma 2.4.3 (while keeping a tighter watch on the dependence on α). For the induction step from k to $k+1$ we can assume that $n < M$, and we use symmetry among the values of j to obtain

$$\nu(C_n^{2k+2}) = \nu(f^\sim) , \tag{2.101}$$

where $f^{\sim} = (u'(S_M^1)u'(S_M^2) - \widehat{r})C_n^{2k+1}$. Let us define

$$C' = \frac{1}{M} \sum_{n \le j \le M-1} (u'(S_j^1)u'(S_j^2) - \widehat{r}) \ .$$

Using the inequality

$$|x^{2k+1} - y^{2k+1}| \le (2k+1)|x-y|(x^{2k} + y^{2k}) \tag{2.102}$$

for $x = C_n$ and $y = C'$, and since $|u'(S_M^1)u'(S_M^2) - \widehat{r}| \le 2D^2$, we obtain that for $f^* = (u'(S_M^1)u'(S_M^2) - \widehat{r})C'^{2k+1}$:

$$\nu(f^{\sim}) \le \nu(f^*) + \frac{2(2k+1)D^2}{M}(\nu(C_n^{2k}) + \nu(C'^{2k})) \ . \tag{2.103}$$

Since $n < M$, symmetry among the values of j implies $\nu(C'^{2k}) = \nu(C_{n+1}^{2k})$ and the induction hypothesis yields

$$\nu(f^{\sim}) \le \nu(f^*) + \frac{8(k+1)D^2}{M}\left(\frac{K_1(D)k}{M}\right)^k \ . \tag{2.104}$$

Next, we use (2.62) for $t = 1$, $f = C'^{2k+1}$ and $n = 2$. This is permitted because f does not depend on the randomness of ξ_M^ℓ, ξ^ℓ or $g_{i,M}$. We choose $\tau_1 = (2k+2)/(2k+1)$ and $\tau_2 = 2k+2$ to get

$$|\nu(f^*)| \le K_2(D)\left(\nu(C'^{2k+2})^{1/\tau_1}\nu\big((R_{1,2}-q)^{2k+2}\big)^{1/\tau_2} + \frac{1}{N}\nu(|C'|^{2k+1})\right) \ .$$

Since we work under the condition $\alpha K_0(D) \le 1$, we can as well assume that $\alpha \le 1$, so that $M \le N$ and

$$|\nu(f^*)| \le K_2(D)\left(\nu(C'^{2k+2})^{1/\tau_1}\nu\big((R_{1,2}-q)^{2k+2}\big)^{1/\tau_2} + \frac{1}{M}\nu(|C'|^{2k+1})\right) \ . \tag{2.105}$$

We recall the inequality $x^{1/\tau_1}y^{1/\tau_2} \le x + y$. Changing x to x/A and y to $A^{\tau_2/\tau_1}y$ in this inequality gives

$$x^{1/\tau_1}y^{1/\tau_2} \le \frac{x}{A} + A^{\tau_2/\tau_1}y \ .$$

Using this for $A = 2K_2(D)$, $x = \nu(C'^{2k+2})$ and $y = \nu((R_{1,2}-q)^{2k+2})$, we deduce from (2.105) that

$$|\nu(f^*)| \le \frac{1}{2}\nu(C'^{2k+2}) + K(D)^{2k+1}\nu\big((R_{1,2}-q)^{2k+2}\big) + \frac{K(D)}{M}\nu(|C'|^{2k+1}) \ . \tag{2.106}$$

We now use the inequality

$$|x^{2k+2} - y^{2k+2}| \le (2k+2)|x-y|(|x|^{2k+1} + |y|^{2k+1})$$

for $x = C'$ and $y = C_n$ to obtain

$$\nu(C'^{2k+2}) \le \nu(C_n^{2k+2}) + \frac{2(2k+2)D^2}{M}\big(\nu(|C'|^{2k+1}) + \nu(|C_n|^{2k+1})\big) .$$

We combine this with (2.106), we use that $|C_n|^{2k+1} \le 2D^2 C_n^{2k}$ and $|C'|^{2k+1} \le 2D^2 C'^{2k}$ and the induction hypothesis to get

$$|\nu(f^*)| \le \frac{1}{2}\nu(C_n^{2k+2}) + K(D)^{2k+2}\nu\big((R_{1,2}-q)^{2k+2}\big) + \frac{(k+1)K(D)}{M}\left(\frac{K_1(D)k}{M}\right)^k ,$$

and combining with (2.101) and (2.104) that

$$\nu(C_n^{2k+2}) \le \frac{1}{2}\nu(C_n^{2k+2}) + K(D)^{2k+2}\nu\big((R_{1,2}-q)^{2k+2}\big) + \frac{(k+1)K(D)}{M}\left(\frac{K_1(D)k}{M}\right)^k .$$

Finally we use (2.94) to conclude the proof that $\nu(C_n^{2k+2}) \le (K_1(D)(k+1)/M)^{k+1}$ if $K_1(D)$ has been chosen large enough. This completes the induction. □

The following central limit theorem describes the fluctuations of $p_{N,M}(u)$ (given by (2.11)). We recall that $a(k) = \mathsf{E}z^k$ where z is a standard Gaussian r.v. and that $O(k)$ denotes a quantity $A = A_N$ with $|A| \le KN^{-k/2}$ where K does not depend on N. We recall the notation $p(u)$ of (2.74),

$$p(u) = -\frac{1}{2}r(1-q) + \mathsf{E}\log(2\mathrm{ch}(z\sqrt{r})) + \alpha\mathsf{E}\log\mathsf{E}_\xi \exp u(z\sqrt{q} + \xi\sqrt{1-q}) .$$

Theorem 2.5.3. *Let*

$$b = \mathsf{E}(\log\mathrm{ch}(z\sqrt{r}))^2 - (\mathsf{E}\log\mathrm{ch}(z\sqrt{r}))^2 - qr .$$

Then for each $k \ge 1$ we have

$$\mathsf{E}(p_{N,M}(u) - p(u))^k = \left(\frac{b}{N}\right)^{k/2} a(k) + O(k+1) .$$

Proof. This argument resembles that in the proof of Theorem 1.4.11, and it would probably help the reader to review the proof of that theorem now. The present proof is organized a bit differently, avoiding the a priori estimate of Lemma 1.4.12. The interpolation method of the first proof of Theorem 2.4.2 is at the center of the argument, so the reader should feel comfortable with this proof in order to proceed. We recall the Hamiltonian (2.76) and we denote by $\langle\cdot\rangle_s$ an average for the corresponding Gibbs measure. In the proof $O(k)$ will denote a quantity $A = A_N$ such that $|A| \le KN^{-k/2}$ where K does not depend on N **or** s, and we will take for granted that Theorems 2.5.1 and 2.5.2 hold uniformly over s. (This fact is left as a good exercise for the reader.)

Consider the following quantities

$$\begin{aligned} A(s) &= \frac{1}{N}\log\sum_{\boldsymbol\sigma} \mathsf{E}_\xi \exp(-H_{N,M,s}(\boldsymbol\sigma)) \\ \mathrm{RS}(s) &= \mathsf{E}\log 2\mathrm{ch}(z\sqrt{r}) + \alpha\mathsf{E}\log\mathsf{E}_\xi \exp u(z\sqrt{q}+\xi\sqrt{1-q}) - \frac{s}{2}r(1-q) \\ V(s) &= A(s) - \mathrm{RS}(s) \\ b(s) &= \mathsf{E}(\log\mathrm{ch}(z\sqrt{r}))^2 - (\mathsf{E}\log\mathrm{ch}(z\sqrt{r}))^2 - rqs\ . \end{aligned}$$

The quantities $\mathsf{E}A(s)$, $\mathrm{RS}(s)$ and $b(s)$ are simply the quantities corresponding for the interpolating system respectively to the quantities $p_{N,M}(u)$, p_u, and b. Fixing k, we set

$$\psi(s) = \mathsf{E}V(s)^k\ .$$

We aim at proving by induction over k that $\psi(s) = (b(s)/N)^{k/2}a(k)+O(k+1)$, which, for $s=1$, proves the theorem. Consider $\varphi(s,a) = \mathsf{E}(A(s)-a)^k$, so that $\psi(s) = \varphi(s,\mathrm{RS}(s))$ and by straightforward differentiation $\partial\varphi/\partial s$ is given by the quantity

$$\frac{k}{2N}\mathsf{E}\left(\left\langle \sum_{j\le M}\left(\frac{S_j}{\sqrt{s}} - \frac{\theta_j}{\sqrt{1-s}}\right)u'(S_{j,s}) - \sum_{i\le N}\frac{\sigma_i}{\sqrt{1-s}}z_i'\sqrt{r}\right\rangle_s (A(s)-a)^{k-1}\right),$$

where $S_{j,s} = \sqrt{s}S_j + \sqrt{1-s}\theta_j$. Next, defining $S_{j,s}^\ell$ as usual we claim that $\partial\varphi/\partial s = \mathrm{I}+\mathrm{II}$, where

$$\mathrm{I} = \frac{k}{2}\mathsf{E}\left(\left\langle -\frac{1}{N}\sum_{j\le M}(R_{1,2}-q)u'(S_{j,s}^1)u'(S_{j,s}^2) - r(1-R_{1,2})\right\rangle_s (A(s)-a)^{k-1}\right)$$

and II is the quantity

$$\frac{k(k-1)}{2N}\mathsf{E}\left(\left\langle \frac{1}{N}\sum_{j\le M}(R_{1,2}-q)u'(S_{j,s}^1)u'(S_{j,s}^2) - rR_{1,2}\right\rangle_s (A(s)-a)^{k-2}\right)\ .$$

This follows by integrating by parts as in the proof of (2.83). The term I is created by the dependence of the bracket $\langle\cdot\rangle_s$ on the r.v.s S_j, θ_j and z_i', and the term II by the dependence on these variables of $A(s)$. We note the obvious identity $\mathrm{I} = \mathrm{III}+\mathrm{IV}$ where

$$\mathrm{III} = -\frac{k}{2}\mathsf{E}\left(\left\langle (R_{1,2}-q)\left(\frac{1}{N}\sum_{j\le M}u'(S_{j,s}^1)u'(S_{j,s}^2) - r\right)\right\rangle_s (A(s)-a)^{k-1}\right)$$

and

$$\mathrm{IV} = -\frac{kr(1-q)}{2}\mathsf{E}((A(s)-a)^{k-1}) .$$

Similarly we have also II = V + VI where V is the quantity

$$\frac{k(k-1)}{2N}\mathsf{E}\left(\left\langle (R_{1,2}-q)\left(\frac{1}{N}\sum_{j\le M} u'(S^1_{j,s})u'(S^2_{j,s})-r\right)\right\rangle_s (A(s)-a)^{k-2}\right)$$

and

$$\mathrm{VI} = -\frac{rq}{2N}k(k-1)\mathsf{E}((A(s)-a)^{k-2}) .$$

Now,

$$\psi'(s) = \frac{\mathrm{d}}{\mathrm{d}s}\varphi(s,\mathrm{RS}(s)) = \frac{\partial\varphi}{\partial s}(s,\mathrm{RS}(s)) + \mathrm{RS}'(s)\frac{\partial\varphi}{\partial a}(s,\mathrm{RS}(s)) . \tag{2.107}$$

Since $\mathrm{RS}'(s) = -r(1-q)/2$ and $\partial\varphi/\partial a(s,\mathrm{RS}(s)) = -k\mathsf{E}v(s)^{k-1}$, the second term of (2.107) cancels out with the term IV and we get

$$\psi'(s) = \mathrm{VII} + \mathrm{VIII} + \mathrm{IX} \tag{2.108}$$

where

$$\mathrm{VII} = -\frac{k}{2}\mathsf{E}\left(\left\langle (R_{1,2}-q)\left(\frac{1}{N}\sum_{j\le M} u'(S^1_{j,s})u'(S^2_{j,s})-r\right)\right\rangle_s V(s)^{k-1}\right)$$

$$\mathrm{VIII} = \frac{k(k-1)}{2N}\mathsf{E}\left(\left\langle (R_{1,2}-q)\left(\frac{1}{N}\sum_{j\le M} u'(S^1_{j,s})u'(S^2_{j,s})-r\right)\right\rangle_s V(s)^{k-2}\right)$$

$$\mathrm{IX} = -\frac{rq}{2N}k(k-1)\mathsf{E}V(s)^{k-2} .$$

The idea is that each of the factors $R_{1,2}-q$, $(N^{-1}\sum_{j\le M} u'(S^1_{j,s})u'(S^2_{j,s})-r)$ and $V(s)$ "counts as $N^{-1/2}$". This follows from Theorems 2.5.1 and 2.5.2 for the first two terms, but we have not proved it yet in the case of $V(s)$. (In the case of Theorem 1.4.11, the a priori estimate of Lemma 1.4.12 showed that $V(s)$ "counts as $N^{-1/2}$".) Should this be indeed the case, the terms VII and VIII will be of lower order $O(k+1)$. We turn to the proof that this is actually the case.

A first step is to show that

$$\mathrm{VII} \le \frac{K(k)}{N}(\mathsf{E}|V(s)|^k)^{\frac{k-1}{k}} \ ; \ \ \mathrm{VIII} \le \frac{K(k)}{N^2}(\mathsf{E}|V(s)|^k)^{\frac{k-2}{k}} . \tag{2.109}$$

In the case of VII, setting $A = R_{1,2}-q$ and

$$B = \frac{1}{N}\sum_{j\le M} u'(S^1_{j,s})u'(S^2_{j,s}) - r$$

we write, using Hölder's inequality and Theorems 2.5.1 and 2.5.2:

$$\begin{aligned}\mathsf{E}(\langle AB\rangle_s V(s)^{k-1}) &\le \mathsf{E}\langle A^{2k}\rangle_s^{1/2k}\mathsf{E}\langle B^{2k}\rangle_s^{1/2k}(\mathsf{E}|V(s)|^k)^{\frac{k-1}{k}}\\ &\le \frac{K(k)}{N}\mathsf{E}|V(s)|^k)^{\frac{k-1}{k}} .\end{aligned}$$

We proceed in a similar manner for VIII, i.e. we write that

$$\begin{aligned}\mathsf{E}(\langle AB\rangle_s V(s)^{k-1}) &\le \mathsf{E}\langle |A|^{k}\rangle_s^{1/k}\mathsf{E}\langle |B|^{k}\rangle_s^{1/k}(\mathsf{E}|V(s)|^k)^{\frac{k-2}{k}}\\ &\le \frac{K(k)}{N}(\mathsf{E}|V(s)|^k)^{\frac{k-2}{k}} ,\end{aligned}$$

and this proves (2.109).

Since $xy \le x^{\tau_1} + y^{\tau_2}$ for $\tau_2 = k/(k-2)$ and $\tau_1 = k/2$ we get

$$\frac{1}{N}(\mathsf{E}|V(s)|^k)^{\frac{k-2}{k}} \le \frac{1}{N^{k/2}} + \mathsf{E}|V(s)|^k .$$

This implies in particular

$$\mathrm{IX} \le \frac{K(k)}{N}(\mathsf{E}|V(s)|^k)^{\frac{k-2}{k}} \le K(k)\left(\frac{1}{N^{k/2}} + \mathsf{E}|V(s)|^k\right)$$

and

$$\mathrm{VIII} \le \frac{K(k)}{N}\left(\frac{1}{N^{k/2}} + \mathsf{E}|V(s)|^k\right) \le K(k)\left(\frac{1}{N^{k/2}} + \mathsf{E}|V(s)|^k\right) .$$

Next, we use that $xy \le x^{\tau_1} + y^{\tau_2}$ for $\tau_2 = k/(k-1)$ and $\tau_1 = k$ to get

$$\frac{1}{N}(\mathsf{E}|V(s)|^k)^{\frac{k-1}{k}} \le \frac{1}{N^k} + \mathsf{E}|V(s)|^k \le \frac{1}{N^{k/2}} + \mathsf{E}|V(s)|^k .$$

When k is even (so that $|V(s)|^k = V(s)^k$ and $\mathsf{E}|V(s)|^k = \psi(s)$) we have proved that

$$\psi'(s) \le K(k)\left(\frac{1}{N^{k/2}} + \psi(s)\right) . \tag{2.110}$$

Thus (2.110) and Lemma A.13.1 imply that

$$\psi(s) \le K(k)\left(\psi(0) + \frac{1}{N^{k/2}}\right) .$$

Since it is easy (as the spins decouple) to see that $\psi(0) \le K(k)N^{k/2}$, we have proved that for k even we have $\mathsf{E}V(s)^k = O(k)$. Since $\mathsf{E}|V(s)|^k \le (\mathsf{E}V(s)^{2k})^{1/2}$ this implies that $\mathsf{E}|V(s)|^k = O(k)$ for each k so that by (2.109) we have VII $= O(k+1)$ and VIII $= O(k+1)$. Thus (2.108) yields

$$\begin{aligned}\psi'(s) &= -\frac{rq}{2N}k(k-1)\mathsf{E}V(s)^{k-2} + O(k+1)\\ &= \frac{b'(s)}{N}\frac{k}{2}(k-1)\mathsf{E}V(s)^{k-2} + O(k+1) .\end{aligned}$$

As in Theorem 1.4.11, one then shows by induction over k that

$$\mathsf{E}V(s)^k = a(k)\left(\frac{b(s)}{N}\right)^{k/2} + O(k+1) ,$$

using that this is true for $s = 0$, which is again proved as in Theorem 1.4.11. □

Exercise 2.5.4. Rewrite the proof of Theorem 1.4.11 without using the a priori estimate of Lemma 1.4.12. This allows to cover the case where the r.v. h is not necessarily Gaussian.

Research Problem 2.5.5. (Level 1^+) Prove the result corresponding to Theorem 1.7.1 for the present model.

This problem has really two parts. The first (easier) part is to prove results for the present model. For this, the approach of "separating the numerator from the denominator" as explained in Section 9.1 seems likely to succeed. The second part (harder) is to find arguments that will carry over when we will have much less control over u as in Chapter 9. For this second part, the work is partially done in [100], but reaching only the rate $1/\sqrt{N}$ rather than the correct rate $1/N$.

Research Problem 2.5.6. (Level 2) For the present model prove the TAP equations.

These equations have two parts. One part expresses $\langle\sigma_i\rangle$ as a function of $(\langle u'(S_k)\rangle)_{k\le M}$, and one part expresses $\langle u'(S_k)\rangle$ as a function of $(\langle\sigma_i\rangle)_{i\le N}$. It is (perhaps) not too difficult to prove these equations when one has a good control over all derivatives of u, but it might be another matter to prove something as precise as Theorem 1.7.7 in the setting of Chapter 9.

2.6 Notes and Comments

The problems considered in this chapter are studied in [63] and [52].

It is predicted in [90] that the replica-symmetric solution holds up to α^*, so Problem 2.1.1 amounts to controlling the entire replica-symmetric (="high-temperature") region, typically a very difficult task.

It took a long time to discover the proof of Theorem 2.4.1. The weaker methods developed previously [148] for this model or for the SK and the Hopfield models just would not work. During this struggle, it became clear that the smart path method as used here was a better way to go for these three models.