Chapter 11. Function Fields of One Variable over PAC Fields

We prove that if K is an ample field of cardinality m and E is a function field of one variable over K, then $\mathrm{Gal}(E)$ is semi-free of rank m (Theorem 11.7.1). It follows from Theorem 10.5.4 that if F is a finite extension of E, or an Abelian extension of E, or a proper finite extension of a Galois extension of E, or F is "contained in a diamond" over E, then $\mathrm{Gal}(F)$ is semi-free.

We apply the latter results to the case where K is PAC and $E = K(x)$, where x is an indeterminate. We construct a K-radical extension F of E in a diamond over E and conclude that F is Hilbertian and $\mathrm{Gal}(F)$ is semi-free and projective (Theorem 11.7.6), so $\mathrm{Gal}(F)$ is free. In particular, if K contains all roots of unity of order not divisible by $\mathrm{char}(K)$, then $\mathrm{Gal}(E)_{\mathrm{ab}}$ is free of rank equal to $\mathrm{card}(K)$ (Theorem 11.7.6).

11.1 Henselian Fields

We give a sufficient condition for the absolute Galois group of a Henselian field (M, v) to be projective. Our proof is valuation theoretic and starts almost from the basic definitions. In particular, we do not use the connection between projectivity and the vanishing of the Brauer groups.

Let p be a prime number and A an Abelian group. We say that A is p'**-divisible**, if for each $a \in A$ and every positive integer n with $p \nmid n$ there exists $b \in A$ such that $a = nb$. Note that if $p = 0$, then "p'-divisible" is the same as "divisible".

LEMMA 11.1.1: *Let p be 0 or a prime number, B a torsion free Abelian group, and A a p'-divisible subgroup of finite index. Then B is also p'-divisible.*

Proof: First suppose that $p = 0$ and let $m = (B : A)$. Then, for each $b \in B$ and a positive integer n there exists $a \in A$ such that $mb = mna$. Since B is torsion free, $b = na$. Thus, B is divisible.

Now suppose p is a prime number, let $mp^k = (B : A)$, with $p \nmid m$ and $k \geq 0$, and consider $b \in B$. Then $mp^k b \in A$. Hence, for each positive integer n with $p \nmid n$ there exists $a \in A$ with $mp^k b = mna$. Thus, $p^k b = na$. Since $p \nmid n$, there exist $x, y \in \mathbb{Z}$ such that $xp^k + yn = 1$. It follows from $xp^k b = xna$ that $b = n(xa + yb)$, as claimed. ☐

COROLLARY 11.1.2: *Let L/K be an algebraic extension, v a valuation of L, and $p = 0$ or p is a prime number. Suppose that $v(K^\times)$ is p'-divisible. Then $v(L^\times)$ is p'-divisible.*

Proof: Let $x \in L^\times$ and n a positive integer with $p \nmid n$. Then $v(K(x)^\times)$ is a torsion free Abelian group and $v(K^\times)$ is a subgroup of index at most

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$[L:K]$. Since $v(K^\times)$ is p'-divisible, Lemma 11.1.1 gives $y \in K(x)^\times$ such that $v(x) = nv(y)$. It follows that $v(L^\times)$ is p'-divisible. $\square$

Given a Henselian valued field (M, v), we use v also for its unique extension to M_s. We use a bar to denote the residue with respect to v of objects associated with M, let O_M be the valuation ring of M, and let $\Gamma_M = v(M^\times)$ be the value group of M.

PROPOSITION 11.1.3: *Let (M, v) be a Henselian valued field. Suppose $p = \mathrm{char}(M) = \mathrm{char}(\bar{M})$, $\mathrm{Gal}(\bar{M})$ is projective, and Γ_M is p'-divisible. Then $\mathrm{Gal}(M)$ is projective.*

Proof: We denote the **inertia field** of M by M_u. It is determined by its absolute Galois group: $\mathrm{Gal}(M_u) = \{\sigma \in \mathrm{Gal}(M) \mid v(\sigma x - x) > 0 \text{ for all } x \in M_s \text{ with } v(x) \geq 0\}$. The map $\sigma \mapsto \bar{\sigma}$ of $\mathrm{Gal}(M)$ into $\mathrm{Gal}(\bar{M})$ such that $\bar{\sigma}\bar{x} = \overline{\sigma x}$ for each $x \in O_{M_s}$ is a well defined epimorphism [Efr06, Thm. 16.1.1] whose kernel is $\mathrm{Gal}(M_u)$. It therefore defines an isomorphism

$$\mathrm{Gal}(M_u/M) \cong \mathrm{Gal}(\bar{M}). \tag{1}$$

CLAIM A: *$\bar{M}_u$ is separably closed.* Let $g \in \overline{M_u}[X]$ be a monic irreducible separable polynomial of degree $n \geq 1$. Then there exists a monic polynomial $f \in O_{M_u}[X]$ of degree n such that $\bar{f} = g$. We observe that f is also irreducible and separable. Moreover, if $f(X) = \prod_{i=1}^n (X - x_i)$ with $x_1, \ldots, x_n \in M_s$, then $g(X) = \prod_{i=1}^n (X - \bar{x}_i)$. Given $1 \leq i, j \leq n$ there exists $\sigma \in \mathrm{Gal}(M_u)$ such that $\sigma x_i = x_j$. By definition, $\bar{x}_j = \overline{\sigma x_i} = \bar{\sigma}\bar{x}_i = \bar{x}_i$. Since g is separable, $i = j$, so $n = 1$. We conclude that $\bar{M}_u$ is separably closed.

CLAIM B: *Each l-Sylow group of* $\mathrm{Gal}(M_u)$ *with $l \neq p$ is trivial.* Indeed, let L be the fixed field of an l-Sylow group of $\mathrm{Gal}(M_u)$ in M_s. If $l = 2$, then $\zeta_l = -1 \in L$. If $l \neq 2$, then $[L(\zeta_l) : L] | l - 1$ and $[L(\zeta_l) : L]$ is a power of l, so $\zeta_l \in L$.

Assume that $\mathrm{Gal}(L) \neq 1$. By the theory of finite l-groups, L has a cyclic extension L' of degree l. By the preceding paragraph and Kummer theory, there exists $a \in L$ such that $L' = L(\sqrt[l]{a})$. By Corollary 11.1.2, there exists $b \in L^\times$ such that $lv(b) = v(a)$. Then $c = \frac{a}{b^l}$ satisfies $v(c) = 0$. By Claim A, $\bar{L}$ is separably closed. Therefore, $\bar{c}$ has an lth root in $\bar{L}$. By Hensel's lemma, c has an lth root in L. It follows that a has an lth-root in L. This contradiction implies that $L = M_s$, as claimed.

Having proved Claim B, we consider again a prime number $l \neq p$ and let G_l be an l-Sylow subgroup of $\mathrm{Gal}(M)$. By the claim, $G_l \cap \mathrm{Gal}(M_u) = 1$, hence the map res: $\mathrm{Gal}(M) \to \mathrm{Gal}(M_u/M)$ maps G_l isomorphically onto an l-Sylow subgroup of $\mathrm{Gal}(M_u/M)$. By (1), G_l is isomorphic to an l-Sylow subgroup of $\mathrm{Gal}(\bar{M})$. Since the latter group is projective, so is G_l, i.e. $\mathrm{cd}_l(\mathrm{Gal}(M)) \leq 1$ [Ser79, p. 58, Cor. 2].

Finally, if $p \neq 0$, then $\mathrm{cd}_p(\mathrm{Gal}(M)) \leq 1$ [Ser79, p. 75, Prop. 3], because then $\mathrm{char}(M) = p$. It follows that $\mathrm{cd}(\mathrm{Gal}(M)) \leq 1$ [Ser79, p. 58, Cor. 2]. $\square$

11.2 Brauer Groups of Henselian Fields

We establish a short exact sequence for the Brauer group of a finite unramified extension of a Henselian field. That sequence will be used in the proof of Lemma 11.5.1.

Again, when (M, v) is a Henselian field, we denote its valuation ring by O_M, the maximal ideal of O_M by $\mathfrak{m}_M$, the group of units of M by U_M, the value group of (M, v) by Γ_M, and use a bar to denote reduction modulo $\mathfrak{m}_M$

PROPOSITION 11.2.1: *Let (M, v) be a Henselian valued field and (N, v) a finite Galois extension with a trivial inertia group. Set $G = \mathrm{Gal}(N/M)$. Then the G-module $1+\mathfrak{m}_N$ is G-cohomologically trivial, that is $H^i(G, 1+\mathfrak{m}_N) = 0$ for all positive integers i.*

Proof: By Subsection 9.3.13, it suffices to prove the following equalities:

$$\begin{aligned} (1 + \mathfrak{m}_N)^G &= \mathrm{norm}_{N/M}(1 + \mathfrak{m}_N) \\ Z^1(G, 1 + \mathfrak{m}_N) &= B^1(G, 1 + \mathfrak{m}_N). \end{aligned} \tag{1}$$

Since the right hand sides of (1) are contained in the left hand sides, it suffices to prove only the other inclusions. This is done in two parts.

PART A: *Proof that* $(1 + \mathfrak{m}_N)^G \le \mathrm{norm}_{N/M}(1 + \mathfrak{m}_N)$. Note that $(1 + \mathfrak{m}_N)^G = 1+\mathfrak{m}_M$. Thus, we have to prove that $1+\mathfrak{m}_M \le \mathrm{norm}_{N/M}(1+\mathfrak{m}_N)$.

Since $G_0(N/M) = 1$, (1) of Section 11.1 implies that the map $\sigma \mapsto \bar\sigma$ is an isomorphism $\mathrm{Gal}(N/M) \cong \mathrm{Gal}(\bar N/\bar M)$. By the normal basis theorem there exists $x \in O_N$ such that $\{\bar\sigma\bar x \mid \sigma \in G\}$ is a basis of $\bar N/\bar M$ [Lan93, p. 312 for the case where M is infinite and Jac64, p. 61 for M finite]. Then the elements σx, $\sigma \in G$, are linearly independent over M, so they form a basis of N/M. If $\mathrm{trace}_{N/M}(x) = 0$, then $\mathrm{trace}_{N/M}(\sigma x) = 0$ for each $\sigma \in G$, so $\mathrm{trace}_{N/M}(y) = 0$ for all $y \in N$. This contradiction to the fact that $\mathrm{trace}_{N/M}\colon N \to M$ is a nonzero M-linear function [Lan93, p. 286, Thm. 5.2] proves that $a = \mathrm{trace}_{N/M}(x) \ne 0$. Dividing x by a, we may assume that $\mathrm{trace}_{N/M}(x) = 1$.

Now let $n = [N : M] = |G|$ and consider $y \in \mathfrak{m}_M$ and the polynomial

$$f(Z) = -y + Z + a_2Z^2 + \cdots + a_{n-1}Z^{n-1} + \mathrm{norm}_{N/M}(x)Z^n$$

with $a_k = \sum_\sigma x^{\sigma_1} \cdots x^{\sigma_k}$, where σ ranges over all injections from $\{1, \ldots, k\}$ into G. In particular, $f \in O_M[Z]$. For each $z \in O_M$ we have

$$\begin{aligned} \mathrm{norm}_{N/M}(1 + xz) &= \prod_{\sigma\in G} (1 + x^\sigma z) \\ &= 1 + \mathrm{trace}_{N/M}(x)z + a_2z^2 + \cdots + a_{n-1}z^{n-1} + \mathrm{norm}_{N/M}(x)z^n \\ &= 1 + z + a_2z^2 + \cdots + a_{n-1}z^{n-1} + \mathrm{norm}_{N/M}(x)z^n, \end{aligned}$$

so $f(z) = \mathrm{norm}_{N/M}(1 + xz) - 1 - y$.

Since $y \in \mathfrak{m}_M$, we have

$$f(y) = a_2y^2 + \cdots + a_{n-1}y^{n-1} + \mathrm{norm}_{N/M}(x)y^n \equiv 0 \mod \mathfrak{m}_M^2$$

and

$$f'(y) = 1+2a_2y^2+\cdots+(n-1)a_{n-1}y^{n-2}+n\cdot\mathrm{norm}_{N/M}(x)y^{n-1} \equiv 1 \mod \mathfrak{m}_M^2.$$

The Henselianity of (M, v) gives a $z \in \mathfrak{m}_M$ with $f(z) = 0$, that is

$$\mathrm{norm}_{N/M}(1 + xz) = 1 + y,$$

as desired.

Part B: $Z^1(G, 1 + \mathfrak{m}_N) \leq B^1(G, 1 + \mathfrak{m}_N)$. Consider a 1-cocycle

$$a \in Z^1(G, 1 + \mathfrak{m}_N).$$

Then $a \in Z^1(G, N^\times)$. Since $H^1(G, N^\times) = 1$ (Hilbert's theorem 90, Subsection 9.3.17), there exists $b \in N^\times$ such that $a_\sigma = (\sigma - 1)b$ for each $\sigma \in G$. Since $v(N^\times) = v(M^\times)$, there exists $b' \in M^\times$ with $v(b') = v(b)$. Then $c = \frac{b}{b'}$ satisfies $v(c) = 0$ and $a_\sigma = (\sigma - 1)c$ for each $\sigma \in G$. Since $a_\sigma \in 1 + \mathfrak{m}_N$, we have $1 = (\bar{\sigma} - 1)\bar{c}$, hence $\bar{\sigma}\bar{c} = \bar{c}$ for all $\sigma \in G$. Therefore, $\bar{c} \in \bar{M}$, so there exists $c' \in O_M$ with $\overline{c'} = \bar{c}$. The element $d = \frac{c}{c'}$ is in $1 + \mathfrak{m}_N$ and satisfies $a_\sigma = (\sigma - 1)d$ for all $\sigma \in G$. This means that $a \in B^1(G, 1 + \mathfrak{m}_N)$, as contended. □

Proposition 11.2.1 has a series of consequences expressed in the following lemmas.

Lemma 11.2.2: *Let M, N, and G be as in Proposition 11.2.1. Then, for each positive integer i there is a natural isomorphism, $H^i(G, U_N) \cong H^i(G, \bar{N}^\times)$.*

Proof: The short exact sequence $1 \to 1 + \mathfrak{m}_N \to U_N \to \bar{N}^\times \to 1$ of G-modules, in which $U_N \to \bar{N}^\times$ is the reduction map, induces a natural long exact sequence

$$H^i(G, 1 + \mathfrak{m}_N) \to H^i(G, U_N) \to H^i(G, \bar{N}^\times) \to H^{i+1}(G, 1 + \mathfrak{m}_N)$$

(Subsection 9.3.4). The first and the fourth terms of that sequence are trivial by Proposition 11.2.1. Hence the second and the third terms of that sequence are naturally isomorphic. □

Lemma 11.2.3: *Let M, N, v, and G be as in Proposition 11.2.1. Then for each positive integer i there is a natural short exact sequence*

$$1 \to H^i(G, \bar{N}_v^\times) \to H^i(G, N^\times) \xrightarrow{v} H^i(G, \Gamma_M) \to 0.$$

In particular, for $i = 2$ the following short sequence is exact:

$$0 \to \mathrm{Br}(\bar{N}_v/\bar{M}_v) \to \mathrm{Br}(N/M) \to H^2(G, \Gamma_M) \to 0$$

Proof: The short exact sequence $1 \to U_N \to N^\times \xrightarrow{v} \Gamma_N \to 0$ gives rise to a long exact sequence

$$\cdots \xrightarrow{\delta} H^i(G, U_N) \to H^i(G, N^\times) \xrightarrow{v} H^i(G, \Gamma_N) \xrightarrow{\delta} \cdots. \tag{2}$$

By Lemma 11.2.2, we may replace $H^i(G, U_N)$ by $H^i(G, \bar{N}^\times)$. Since N/M is unramified, $\Gamma_N = \Gamma_M$. Hence, (2) simplifies to a long exact sequence

$$\cdots \xrightarrow{\delta} H^i(G, \bar{N}^\times) \to H^i(G, N^\times) \xrightarrow{v} H^i(G, \Gamma_M) \xrightarrow{\delta} \cdots \tag{3}$$

of cohomology groups. We have to prove that each of the homomorphisms δ is the zero map. This is equivalent to proving that the map v in (3) is surjective for each $i \geq 0$.

To this end we consider a finitely generated subgroup A of Γ_M. Since Γ_M is torsion free, A is free. Lifting free generators of A to elements of $M^\times$ gives generators of a subgroup B of $M^\times$ that v maps isomorphically onto A. Since G acts trivially both on M and on Γ_M, $v|_B$ is a G-isomorphism.

$$\begin{array}{ccc} N^\times & \xrightarrow{v} & \Gamma_N \\ \uparrow & & \| \\ M^\times & \xrightarrow{v} & \Gamma_M \\ \uparrow & & \uparrow \\ B & \xrightarrow{v} & A \end{array}$$

Ignoring the second row and taking cohomology gives a commutative diagram

$$\begin{array}{ccc} H^i(G, N^\times) & \xrightarrow{v} & H^i(G, \Gamma_M) \\ \uparrow & & \uparrow \\ H^i(G, B) & \xrightarrow{v} & H^i(G, A) \end{array}$$

in which the lower arrow v is an isomorphism. In particular, each element of $H^i(G, A)$ lies in the image of v. Since $H^i(G, \Gamma_M)$ is the inductive limit of all of the groups $H^i(G, A)$ (Subsection 9.3.10), the upper arrow of the preceding diagram is surjective. ☐

11.3 Local-Global Theorems for Brauer Groups

We establish a commutative diagram for the Brauer group of a generalized function field of one variable over a field K relating it to the product of the Brauer groups of the Henselizations.

Remark 11.3.1: Let K be a perfect field and F a generalized function field of one variable over K, that is a regular extension of K of transcendence degree 1. We denote the set of all equivalence classes of valuations of F that are trivial on K by $\mathbb{P}(F/K)$. We choose a representative $v_\mathfrak{p}$ in each $\mathfrak{p} \in \mathbb{P}(F/K)$ and a Henselian closure $F_\mathfrak{p}$ of F at $v_\mathfrak{p}$. Then the residue fields $\bar{F}_\mathfrak{p}$ of both F and $F_\mathfrak{p}$ are the same and so are the value groups $\Gamma_\mathfrak{p}$. We extend the residue map of $F_\mathfrak{p}$ to a place $x \mapsto \bar{x}$ of F_s onto $\tilde{K} \cup \{\infty\}$ that fixes the elements of $\tilde{K}$. Then the map $\sigma \mapsto \bar{\sigma}$ defined by $\bar{\sigma}\bar{x} = \overline{\sigma x}$ is an epimorphism of $\mathrm{Gal}(F_\mathfrak{p})$ onto $\mathrm{Gal}(\bar{F}_\mathfrak{p})$. In particular, $\bar{\sigma}x = \sigma x$ for each $\sigma \in \mathrm{Gal}(F_\mathfrak{p})$ and every $x \in \tilde{K}$, that is the map $\sigma \to \bar{\sigma}$ is the restriction map. It follows that $F_\mathfrak{p} \cap \tilde{K} = \bar{F}_\mathfrak{p}$. Moreover, if $\sigma \in \mathrm{Gal}(F_\mathfrak{p})$, then $\bar{\sigma}\bar{x} = \bar{x}$ for all $x \in F_s$ with $\bar{x} \in \tilde{K}$ if and only if $\sigma \in \mathrm{Gal}(F_\mathfrak{p}\tilde{K})$. Thus, $\mathrm{Gal}(F_\mathfrak{p}\tilde{K})$ is the inertia group of the extension of $\mathfrak{p}$ to F_s and the restriction map $\mathrm{Gal}(F_\mathfrak{p}\tilde{K}/F_\mathfrak{p}) \to \mathrm{Gal}(\bar{F}_\mathfrak{p})$ is an isomorphism. □

LEMMA 11.3.2: *Let F be a generalized function field of one variable over a field K and let p be a prime number. Suppose for each function field E of one variable over K in F the map*

$$\text{(1)} \qquad \mathrm{res}\colon \mathrm{Br}(E)_{p^\infty} \to \prod_{\mathfrak{p} \in \mathbb{P}(E/K)} \mathrm{Br}(E_\mathfrak{p})_{p^\infty}$$

is injective and its image lies in $\bigoplus_{\mathfrak{p} \in \mathbb{P}(E/K)} \mathrm{Br}(E_\mathfrak{p})_{p^\infty}$. Then the map

$$\text{(2)} \qquad \mathrm{res}\colon \mathrm{Br}(F)_{p^\infty} \to \prod_{\mathfrak{p} \in \mathbb{P}(F/K)} \mathrm{Br}(F_\mathfrak{p})_{p^\infty}$$

is injective.

Proof: Given an algebraic extension of fields $E \subseteq E'$, we denote the restriction map $\mathrm{Br}(E)_{p^\infty} \to \mathrm{Br}(E')_{p^\infty}$ by $\mathrm{res}^E_{E'}$. Now we consider a function field E of one variable over K in F, let $\mathfrak{p} \in \mathbb{P}(E/K)$, and let $x \in \mathrm{Br}(E_\mathfrak{p})_{p^\infty}$. Suppose $\mathrm{res}^{E_\mathfrak{p}}_{F_\mathfrak{q}}(x) = 0$ for each $\mathfrak{q} \in \mathbb{P}(F/K)$ over $\mathfrak{p}$. Let $\mathcal{E}$ be the set of all finite extensions of E in F. We prove there exists $E' \in \mathcal{E}$ such that $\mathrm{res}^{E_\mathfrak{p}}_{E'_\mathfrak{q}}(x) = 0$ for each $\mathfrak{q} \in \mathbb{P}(E'/K)$ lying over $\mathfrak{p}$.

To this end we recall that for each $E' \in \mathcal{E}$ the set of prime divisors of E'/K that lie over $\mathfrak{p}$ bijectively corresponds to the set of all $E_\mathfrak{p}$-isomorphisms of $E'E_\mathfrak{p}$ into E_s. If σ' is such an isomorphism and $\mathfrak{q}'$ is the corresponding prime divisor of E'/K, we choose $\sigma'(E'E_\mathfrak{p})$ as the Henselian closure $E'_{\mathfrak{q}'}$ of

E' at $\mathfrak{q}'$. This choice ensures that if E'' is a finite extension of E' in F and $\mathfrak{q}''$ is a prime divisor of E''/K that lies over $\mathfrak{q}'$, then $E'_{\mathfrak{q}'} \subseteq E''_{\mathfrak{q}''}$.

Now assume E has no extension E' as in the first paragraph of the proof. Then for each $E' \in \mathcal{E}$ the finite set $Q(E')$ of all prime divisors $\mathfrak{q} \in \mathbb{P}(E'/K)$ lying over $\mathfrak{p}$ such that $\mathrm{res}^{E_\mathfrak{p}}_{E'_\mathfrak{q}}(x) \neq 0$ is nonempty. If E'' is a finite extension of E' in F, then restriction of divisors maps $Q(E'')$ into $Q(E')$. Since the inverse limit of nonempty finite sets is nonempty [FrJ08, Cor. 1.1.4], there exists a set $\mathfrak{Q} = \{\mathfrak{q}_{E'} \in Q(E') \mid E' \in \mathcal{E}\}$ such that $\mathfrak{q}_{E'}$ is the restriction of $\mathfrak{q}_{E''}$ for all $E', E'' \in \mathcal{E}$ with $E' \subseteq E''$. The set $\mathfrak{Q}$ determines an element $\mathfrak{q}$ of $\mathbb{P}(F/K)$ such that $\mathrm{res}^{E_\mathfrak{p}}_{F_\mathfrak{q}}(x) \neq 0$, in contrast to the assumption made in the first paragraph of the proof.

Claim: *The map (2) is injective.* Otherwise, there exists $z \in \mathrm{Br}(F)_{p^\infty}$ such that $z \neq 0$ and $\mathrm{res}^F_{F_\mathfrak{q}}(z) = 0$ for every $\mathfrak{q} \in \mathbb{P}(F/K)$. Since F is the union of function fields E of one variable over K and $\mathrm{Br}(F)_{p^\infty}$ is the direct limit of the groups $\mathrm{Br}(E)_{p^\infty}$ (Subsections 9.3.10 and 9.3.18), there exist such a field E and an element $x \in \mathrm{Br}(E)_{p^\infty}$ with $x \neq 0$ and $\mathrm{res}^E_F(x) = z$. By our assumption on the image of the map (1), $\mathrm{res}^E_{E_\mathfrak{p}}(x) = 0$ for all but finitely many $\mathfrak{p} \in \mathbb{P}(E/K)$. We denote the exceptional set by P. For each $\mathfrak{p} \in P$ let $x_\mathfrak{p} = \mathrm{res}^E_{E_\mathfrak{p}}(x)$. Then $\mathrm{res}^{E_\mathfrak{p}}_{F_\mathfrak{q}}(x_\mathfrak{p}) = \mathrm{res}^F_{F_\mathfrak{q}}(z) = 0$ for each $\mathfrak{q} \in \mathbb{P}(F/K)$ lying over $\mathfrak{p}$. By what we have proved above, E has a finite extension $E(\mathfrak{p})$ in F such that $\mathrm{res}^{E_\mathfrak{p}}_{E(\mathfrak{p})_\mathfrak{q}}(x_\mathfrak{p}) = 0$ for each $\mathfrak{q} \in \mathbb{P}(E(\mathfrak{p})/K)$ lying over $\mathfrak{p}$. Let $E' = \prod_{\mathfrak{p} \in P} E(\mathfrak{p})$. Then E' is a finite extension of E in F and $\mathrm{res}^E_{E'_\mathfrak{q}}(x) = 0$ for each $\mathfrak{p} \in P$ and every $\mathfrak{q} \in \mathbb{P}(E'/K)$ lying over $\mathfrak{p}$. It follows from the definition of P that $\mathrm{res}^E_{E'_\mathfrak{q}}(x) = 0$ for each $\mathfrak{p} \in \mathbb{P}(E/K)$ and every $\mathfrak{q} \in \mathbb{P}(E'/K)$ lying over $\mathfrak{p}$. Finally, let $y = \mathrm{res}^E_{E'}(x)$. Then $\mathrm{res}^{E'}_F(y) = z \neq 0$, so $y \neq 0$. On the other hand, $\mathrm{res}^{E'}_{E'_\mathfrak{q}}(y) = 0$ for all $\mathfrak{q} \in \mathbb{P}(E'/K)$. This contradicts the injectivity of the map (1). □

Lemma 11.3.3: *In the notation of Remark 11.3.1 and with $\mathbb{P} = \mathbb{P}(F/K)$ there is a natural commutative diagram*

(3)

$$\begin{array}{ccccc} \mathrm{Br}(F) & \xrightarrow{\beta} & H^2(\mathrm{Gal}(K), (F\tilde{K})^\times) & \xrightarrow{\gamma} & H^2(\mathrm{Gal}(K), (F\tilde{K})^\times/\tilde{K}^\times) \\ \downarrow \mathrm{res} & & \downarrow \mathrm{res} & & \downarrow \\ \prod_{\mathfrak{p}\in\mathbb{P}} \mathrm{Br}(F_\mathfrak{p}) & \xrightarrow{\beta'} & \prod_{\mathfrak{p}\in\mathbb{P}} H^2(\mathrm{Gal}(\bar{F}_\mathfrak{p}), (F_\mathfrak{p}\tilde{K})^\times) & \xrightarrow{\gamma'} & \prod_{\mathfrak{p}\in\mathbb{P}} H^2(\mathrm{Gal}(\bar{F}_\mathfrak{p}), \Gamma_\mathfrak{p}) \end{array}$$

where β and β' are isomorphisms.

Proof: The inflation-restriction sequence for Brauer groups (Subsection 9.3.18) applied to $\mathrm{Gal}(F)$ and $\mathrm{Gal}(F\tilde{K})$ is

(4)

$$1 \to H^2(\mathrm{Gal}(F\tilde{K}/F), (F\tilde{K})^\times) \xrightarrow{\mathrm{inf}} H^2(\mathrm{Gal}(F), F_s^\times) \xrightarrow{\mathrm{res}} H^2(\mathrm{Gal}(F\tilde{K}), F_s^\times).$$

Since F/K is regular, the map res: $\mathrm{Gal}(F\tilde{K}/F) \to \mathrm{Gal}(K)$ is an isomorphism. By Proposition 9.4.6(b), $\mathrm{cd}(\mathrm{Gal}(F\tilde{K})) \le 1$, so $H^2(\mathrm{Gal}(F\tilde{K}), F_s^\times) = 1$ (Subsection 9.3.18). Thus, inf in (4) is an isomorphism. We denote its inverse by β to get the left upper map in Diagram (3). The homomorphism γ in (3) is induced by the quotient map $(F\tilde{K})^\times \to (F\tilde{K})^\times/\tilde{K}^\times$.

For each $\mathfrak{p} \in \mathbb{P}$ we replace F and K in the preceding argument by $F_\mathfrak{p}$ and $\bar{F}_\mathfrak{p}$, respectively, and use that $F_\mathfrak{p}/\bar{F}_\mathfrak{p}$ is a regular extension (Remark 11.3.1) to produce an isomorphism $\beta_\mathfrak{p}\colon \mathrm{Br}(F_\mathfrak{p}) \to H^2(\mathrm{Gal}(\bar{F}_\mathfrak{p}), (F_\mathfrak{p}\tilde{K})^\times)$ that commutes with the restriction map. Then we define β' as the product of all the $\beta_\mathfrak{p}$'s.

Similarly, for each $\mathfrak{p} \in \mathbb{P}$, the quotient map $(F_\mathfrak{p}\tilde{K})^\times \to (F_\mathfrak{p}\tilde{K})^\times/\tilde{K}^\times$ yields a homomorphism

$$\gamma_\mathfrak{p}\colon H^2(\mathrm{Gal}(\bar{F}_\mathfrak{p}), (F_\mathfrak{p}\tilde{K})^\times) \to H^2(\mathrm{Gal}(\bar{F}_\mathfrak{p}), (F_\mathfrak{p}\tilde{K})^\times/\tilde{K}^\times).$$

The valuation $v_\mathfrak{p}$ extended to $F_\mathfrak{p}\tilde{K}$ maps $(F_\mathfrak{p}\tilde{K})^\times$ onto the valuation group $\Gamma_\mathfrak{p}$ and vanish on $\tilde{K}^\times$. So it defines a homomorphism

$$\gamma'_\mathfrak{p}\colon H^2(\mathrm{Gal}(\bar{F}_\mathfrak{p}), (F_\mathfrak{p}\tilde{K})^\times/\tilde{K}^\times) \to H^2(\mathrm{Gal}(\bar{F}_\mathfrak{p}), \Gamma_\mathfrak{p}).$$

We let $\gamma' = \prod_{\mathfrak{p}\in\mathbb{P}} \gamma'_\mathfrak{p} \circ \gamma_\mathfrak{p}$. Finally, noting that $F\tilde{K} = F_\mathfrak{p}\tilde{K}$, the third vertical arrow in (3) is just $\prod_{\mathfrak{p}\in\mathbb{P}} \gamma'_\mathfrak{p} \circ \mathrm{res}$. □

11.4 Picard Groups

Let F be a function field of one variable over a perfect field K. Thus, F/K is a finitely generated regular extension of transcendence degree 1. Let $\mathbb{P} = \mathbb{P}(F/K)$ be the set of prime divisors of F/K (Remark 11.3.1). Using the notation of Remark 5.8.1, we recall that each $\mathfrak{a} \in \mathrm{Div}(F/K)$ has a unique representations as $\mathfrak{a} = \sum_{\mathfrak{p}\in\mathbb{P}} v_\mathfrak{p}(\mathfrak{a})\mathfrak{p}$, with integers $v_\mathfrak{p}(\mathfrak{a})$, all but finitely many are 0. In particular, $\mathrm{div}(f) = \sum_{\mathfrak{p}\in\mathbb{P}} v_\mathfrak{p}(f)\mathfrak{p}$ for each $f \in F^\times$. Thus, the map $\mathfrak{a} \mapsto (v_\mathfrak{p}(\mathfrak{a}))_{\mathfrak{p}\in\mathbb{P}}$ is a natural isomorphism,

$$\mathrm{Div}(F/K) \cong \bigoplus_{\mathfrak{p}\in\mathbb{P}} \Gamma_\mathfrak{p} \tag{1}$$

The map div: $F^\times \to \mathrm{Div}(F/K)$ is a homomorphism with $\mathrm{Ker}(\mathrm{div}) = K^\times$ [Deu73, p. 25]. The **Picard group** of F/K is the cokernel of div, also called the **group of divisor classes** of F/K, that is, $\mathrm{Pic}(F/K) = \mathrm{Div}(F/K)/\mathrm{div}(F^\times)$. Since $\mathrm{div}(F^\times) \cong F^\times/K^\times$, we get the following natural short exact sequence:

$$1 \to F^\times/K^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(F/K) \to \mathrm{Pic}(F/K) \to 0. \tag{2}$$

Recall that the group of divisors of degree 0, $\mathrm{Div}_0(F/K)$, contains $\mathrm{div}(F^\times)$ (Remark 5.8.1(a)). Hence, $\mathrm{Pic}_0(F/K) = \mathrm{Div}_0(F/K)/\mathrm{div}(F^\times)$ is a subgroup of $\mathrm{Pic}(F/K)$ and (2) yields the following natural short exact sequence:

$$(3) \qquad 1 \to F^\times/K^\times \xrightarrow{\mathrm{div}} \mathrm{Div}_0(F/K) \to \mathrm{Pic}_0(F/K) \to 0.$$

Analogous convention and rules hold for the function field $F\tilde{K}/\tilde{K}$. Here we write $\tilde{\mathbb{P}} = \mathbb{P}(F\tilde{K}/\tilde{K})$.

Lemma 11.4.1: *There is a natural isomorphism*

$$\mathrm{Div}(F\tilde{K}/\tilde{K}) \cong \bigoplus_{\mathfrak{p}\in\mathbb{P}} \mathrm{Ind}_{\mathrm{Gal}(\bar{F}_\mathfrak{p})}^{\mathrm{Gal}(K)}(\Gamma_\mathfrak{p})$$

of $\mathrm{Gal}(K)$*-modules.*

Proof: Consider a prime divisor $\mathfrak{p} \in \mathbb{P}$ and a prime divisor $\mathfrak{P} \in \tilde{\mathbb{P}}$ lying over $\mathfrak{p}$. We identify $\mathrm{Gal}(F\tilde{K}/\tilde{K})$ with $\mathrm{Gal}(K)$ via restriction. For each $\sigma \in \mathrm{Gal}(K)$ the prime divisor $\sigma\mathfrak{P}$ is the equivalence class of the valuation $v_{\sigma\mathfrak{P}}$ of $F\tilde{K}$ defined by $v_{\sigma\mathfrak{P}}(x) = v_\mathfrak{P}(\sigma^{-1}x)$. When σ ranges over $\mathrm{Gal}(K)$, the divisor $\sigma\mathfrak{P}$ ranges over all extensions of $\mathfrak{p}$ to $F\tilde{K}$. By Remark 11.3.1, the stabilizer of $\mathfrak{P}$ under this action is $\mathrm{Gal}(\bar{F}_\mathfrak{p})$. Hence, $\bigoplus_{\mathfrak{Q}|\mathfrak{p}} \Gamma_\mathfrak{Q} = \bigoplus_{\sigma\in S} \Gamma_{\sigma\mathfrak{P}}$, where S is a subset of $\mathrm{Gal}(K)$ satisfying $\mathrm{Gal}(K) = \bigcup_{\sigma\in S} \mathrm{Gal}(\bar{F}_\mathfrak{p})\sigma$. Note that for each $\mathfrak{Q} \in \tilde{\mathbb{P}}$ lying over $\mathfrak{p}$ the value group $\Gamma_\mathfrak{Q}$ is $\mathbb{Z}$, so we may identify it with $\Gamma_\mathfrak{p}$. It follows from Subsection 9.3.12 that $\bigoplus_{\mathfrak{Q}|\mathfrak{p}} \Gamma_\mathfrak{Q} = \bigoplus_{\sigma\in S} \Gamma_{\sigma\mathfrak{P}} = \bigoplus_{\sigma\in S} \Gamma_\mathfrak{p} = \mathrm{Ind}_{\mathrm{Gal}(\bar{F}_\mathfrak{p})}^{\mathrm{Gal}(K)}(\Gamma_\mathfrak{p})$. Consequently, $\mathrm{Div}(F\tilde{K}/\tilde{K}) \cong \bigoplus_{\mathfrak{p}\in\mathbb{P}} \bigoplus_{\mathfrak{Q}|\mathfrak{p}} \Gamma_\mathfrak{Q} = \bigoplus_{\mathfrak{p}\in\mathbb{P}} \mathrm{Ind}_{\mathrm{Gal}(\bar{F}_\mathfrak{p})}^{\mathrm{Gal}(K)}(\Gamma_\mathfrak{p})$, as claimed. □

Lemma 11.4.2: *Let G be a profinite group acting trivially on a discrete torsion free Abelian group A. Then $H^1(G, A) = \mathrm{Hom}(G, A) = 0$.*

Proof: The left equality follows from the definition of H^1 (Subsection 9.3.2). Each element of $\mathrm{Hom}(G, A)$ is a continuous homomorphism $f\colon G \to A$. Its image is a compact subgroup, so must be finite. Since A is torsion free, $f(G) = 0$. Therefore, $\mathrm{Hom}(G, A) = 0$. □

Lemma 11.4.3: *Let F be a function field of one variable over a perfect field K. Then there is a natural exact sequence*

$$(4) \qquad \begin{aligned} 0 &\to H^1(\mathrm{Gal}(K), \mathrm{Pic}(F\tilde{K}/\tilde{K})) \to H^2(\mathrm{Gal}(K), (F\tilde{K})^\times/\tilde{K}^\times) \\ &\to \bigoplus_{\mathfrak{p}\in\mathbb{P}(F/K)} H^2(\mathrm{Gal}(\bar{F}_\mathfrak{p}), \Gamma_\mathfrak{p}) \to H^2(\mathrm{Gal}(K), \mathrm{Pic}(F\tilde{K}/\tilde{K})) \\ &\qquad \to H^3(\mathrm{Gal}(K), (F\tilde{K})^\times/\tilde{K}^\times). \end{aligned}$$

Proof: As above we set $\mathbb{P} = \mathbb{P}(F/K)$ and start from the short exact sequence for $F\tilde{K}/\tilde{K}$ analogous to (2):

$$1 \to (F\tilde{K})^\times/K^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(F\tilde{K}/\tilde{K}) \to \mathrm{Pic}(F\tilde{K}/\tilde{K}) \to 0.$$

It induces a long exact sequence:

$$\begin{aligned} &H^1(\mathrm{Gal}(K), \mathrm{Div}(F\tilde{K}/\tilde{K})) \to H^1(\mathrm{Gal}(K), \mathrm{Pic}(F\tilde{K}/\tilde{K})) \\ &\to H^2(\mathrm{Gal}(K), (F\tilde{K})^\times/\tilde{K}^\times) \to H^2(\mathrm{Gal}(K), \mathrm{Div}(F\tilde{K}/\tilde{K})) \\ &\to H^2(\mathrm{Gal}(K), \mathrm{Pic}(F\tilde{K}/\tilde{K})) \to H^3(\mathrm{Gal}(K), (F\tilde{K})^\times/\tilde{K}^\times). \end{aligned} \tag{5}$$

By Lemma 11.4.1 and by Shapiro's Lemma (Subsection 9.3.12), we have for $i = 1, 2$ natural isomorphism

$$\begin{aligned} H^i(\mathrm{Gal}(K), \mathrm{Div}(F\tilde{K}/\tilde{K})) &\cong \bigoplus_{\mathfrak{p}\in\mathbb{P}} H^i(\mathrm{Gal}(K), \mathrm{Ind}_{\mathrm{Gal}(\bar{F}_\mathfrak{p})}^{\mathrm{Gal}(K)}(\Gamma_\mathfrak{p})) \\ &\cong \bigoplus_{\mathfrak{p}\in\mathbb{P}} H^i(\mathrm{Gal}(\bar{F}_\mathfrak{p}), \Gamma_\mathfrak{p}), \end{aligned}$$

where the action of $\mathrm{Gal}(\bar{F}_\mathfrak{p})$ on $\Gamma_\mathfrak{p}$ is trivial. Since $\Gamma_\mathfrak{p}$ is a torsion free discrete Abelian group, $H^1(\mathrm{Gal}(K), \mathrm{div}(F\tilde{K}/\tilde{K})) = 0$ (Lemma 11.4.2). Collecting this information into (5) gives the exact sequence (4). □

LEMMA 11.4.4: *Let F be a function field of one variable over a perfect field K and let p be a prime number. Then:*

(a) *The natural map*

$$H^1(\mathrm{Gal}(K), \mathrm{Pic}_0(F\tilde{K}/\tilde{K})) \to H^1(\mathrm{Gal}(K), \mathrm{Pic}(F\tilde{K}/\tilde{K}))$$

is surjective.

(b) *If F/K has a prime divisor of degree 1, then*

$$H^i(\mathrm{Gal}(K), \mathrm{Pic}_0(F\tilde{K}/\tilde{K}))_{p^\infty} = 0$$

for each $i > \mathrm{cd}_p(\mathrm{Gal}(K))$ and

(c) *there is a natural isomorphism*

$$H^i(\mathrm{Gal}(K), \mathrm{Pic}(F\tilde{K}/\tilde{K}))_{p^\infty} \cong H^{i-1}(\mathrm{Gal}(K), \mathbb{Q}/\mathbb{Z})_{p^\infty}$$

for each $i > \max(1, \mathrm{cd}_p(\mathrm{Gal}(K)))$.

Proof of (a): The definition of the Picard groups gives rise to a short exact sequence

$$0 \to \mathrm{Pic}_0(F\tilde{K}/\tilde{K}) \to \mathrm{Pic}(F\tilde{K}/\tilde{K}) \xrightarrow{\deg} \mathbb{Z} \to 0 \tag{6}$$

of $\mathrm{Gal}(K)$-modules. We consider a segment of the corresponding long exact sequence of cohomology groups:

$$H^1(\mathrm{Gal}(K), \mathrm{Pic}_0(F\tilde{K}/\tilde{K})) \to H^1(\mathrm{Gal}(K), \mathrm{Pic}(F\tilde{K}/\tilde{K})) \to H^1(\mathrm{Gal}(K), \mathbb{Z}).$$

Since $\mathrm{Gal}(K)$ acts trivially on $\mathbb{Z}$, Lemma 11.4.2 implies that $H^1(\mathrm{Gal}(K), \mathbb{Z}) = 0$, hence (a) is true.

Proof of (b): Let J be the Jacobian variety of F/K. By Subsection 6.3.1, $J(\tilde{K})$ is a divisible Abelian group. Hence, multiplication by p^n gives a short exact sequence:

$$0 \to J(\tilde{K})_{p^n} \to J(\tilde{K}) \xrightarrow{p^n} J(\tilde{K}) \to 0,$$

which in turn gives for each positive integer i a long exact sequence

$$\begin{aligned} H^i(\mathrm{Gal}(K), J(\tilde{K})_{p^n}) &\to H^i(\mathrm{Gal}(K), J(\tilde{K})) \\ &\xrightarrow{p^n} H^i(\mathrm{Gal}(K), J(\tilde{K})) \\ &\to H^{i+1}(\mathrm{Gal}(K), J(\tilde{K})_{p^n}). \end{aligned} \tag{7}$$

If $i > \mathrm{cd}_p(\mathrm{Gal}(K))$, then both the first and the last groups in (7) are zero. Therefore multiplication with p^n is an automorphism of $H^i(\mathrm{Gal}(K), J(\tilde{K}))$. In particular, $H^i(\mathrm{Gal}(K), J(\tilde{K}))_{p^\infty} = 0$. Finally, by Subsection 6.3.2,

$$\mathrm{Pic}_0(F\tilde{K}/\tilde{K}) \cong J(\tilde{K})$$

as $\mathrm{Gal}(K)$-modules. Consequently, $H^i(\mathrm{Gal}(K), \mathrm{Pic}_0(F\tilde{K}/\tilde{K}))_{p^\infty} = 0$, as claimed.

Proof of (c): For each $i \geq 0$ the short exact sequence (6) induces an exact sequence

$$\begin{aligned} H^i(\mathrm{Gal}(K), \mathrm{Pic}_0(F\tilde{K}/\tilde{K})) &\to H^i(\mathrm{Gal}(K), \mathrm{Pic}(F\tilde{K}/\tilde{K})) \\ &\to H^i(\mathrm{Gal}(K), \mathbb{Z}) \\ &\to H^{i+1}(\mathrm{Gal}(K), \mathrm{Pic}_0(F\tilde{K}/\tilde{K})). \end{aligned} \tag{8}$$

By Subsection 9.3.10, the p-primary part

$$\begin{aligned} H^i(\mathrm{Gal}(K), \mathrm{Pic}_0(F\tilde{K}/\tilde{K}))_{p^\infty} &\to H^i(\mathrm{Gal}(K), \mathrm{Pic}(F\tilde{K}/\tilde{K}))_{p^\infty} \\ &\to H^i(\mathrm{Gal}(K), \mathbb{Z})_{p^\infty} \\ &\to H^{i+1}(\mathrm{Gal}(K), \mathrm{Pic}_0(F\tilde{K}/\tilde{K}))_{p^\infty} \end{aligned}$$

of (8) is also exact. By (b), the first and the last groups in the latter sequence are zero if $i > \mathrm{cd}_p(\mathrm{Gal}(K))$. In addition, by Lemma 9.3.11, there is a natural isomorphism $H^i(\mathrm{Gal}(K), \mathbb{Z}) \cong H^{i-1}(\mathrm{Gal}(K), \mathbb{Q}/\mathbb{Z})$ if $i \geq 2$. Hence, there is a natural isomorphism as in (c) if $i > \max(1, \mathrm{cd}_p(\mathrm{Gal}(K)))$. $\square$

11.5 Fields of Cohomological Dimension at most 1

We analyze the exact sequence of Lemma 11.4.3 in the case where $\mathrm{cd}(\mathrm{Gal}(K)) \le 1$ and prove a local-global principle for Brauer groups of generalized function fields of one variable over perfect PAC fields.

LEMMA 11.5.1: *Let F be a generalized function field over a perfect field K with $\mathrm{cd}(\mathrm{Gal}(K)) \le 1$. Then:*
(a) *The natural homomorphism*

$$\gamma\colon H^2(\mathrm{Gal}(K),(F\tilde{K})^\times) \to H^2(\mathrm{Gal}(K),(F\tilde{K})^\times/\tilde{K}^\times)$$

induced by the quotient map $(F\tilde{K})^\times \to (F\tilde{K})^\times/\tilde{K}^\times$ is an isomorphism.
(b) $H^i(\mathrm{Gal}(K),(F\tilde{K})^\times/\tilde{K}^\times) = 0$ *for $i \ge 3$.*
(c) *For each $\mathfrak{p} \in \mathbb{P}(F/K)$, the valuation map*

$$H^2(\mathrm{Gal}(\bar{F}_\mathfrak{p}),(F_\mathfrak{p}\tilde{K})^\times) \to H^2(\mathrm{Gal}(\bar{F}_\mathfrak{p}),\Gamma_\mathfrak{p})$$

is an isomorphism.

Proof: The short exact sequence

$$1 \to \tilde{K}^\times \to (F\tilde{K})^\times \to (F\tilde{K})^\times/\tilde{K}^\times \to 1$$

of $\mathrm{Gal}(K)$-modules gives rise to an exact sequence

$$\text{(1)}\qquad \begin{aligned} &H^2(\mathrm{Gal}(K),\tilde{K}^\times) \to H^2(\mathrm{Gal}(K),(F\tilde{K})^\times) \\ &\qquad \to H^2(\mathrm{Gal}(K),(F\tilde{K})^\times/\tilde{K}^\times) \to H^3(\mathrm{Gal}(K),\tilde{K}^\times) \end{aligned}$$

of cohomology groups. Since $\mathrm{cd}(\mathrm{Gal}(K)) \le 1$, we have $H^2(\mathrm{Gal}(K),\tilde{K}^\times) \cong \mathrm{Br}(K) = 0$ (Subsection 9.3.18) and $H^3(\mathrm{Gal}(K),\tilde{K}^\times) = 0$ for each $i \ge 3$ (Subsection 9.3.15). Thus, (a) follows from (1). Moreover, (b) holds.

Finally, let $\mathfrak{p} \in \mathbb{P}(F/K)$ and apply Lemma 11.2.3 for $F_\mathfrak{p}$, $F_\mathfrak{p}\tilde{K}$, and $v_\mathfrak{p}$ rather than to M, N, and v. Recall that we have identified $\mathrm{Gal}(F_\mathfrak{p}\tilde{K}/F_\mathfrak{p})$ with $\mathrm{Gal}(\bar{F}_\mathfrak{p})$ (Remark 11.3.1). Hence, that lemma gives a short exact sequence

$$0 \to \mathrm{Br}(\bar{F}_\mathfrak{p}) \to H^2(\mathrm{Gal}(\bar{F}_\mathfrak{p}),(F_\mathfrak{p}\tilde{K})^\times) \to H^2(\mathrm{Gal}(\bar{F}_\mathfrak{p}),\Gamma_\mathfrak{p}) \to 0.$$

Now we use that $\mathrm{Gal}(\bar{F}_\mathfrak{p})$ as a closed subgroup of $\mathrm{Gal}(K)$ has cohomological dimension at most 1 to deduce that $\mathrm{Br}(\bar{F}_\mathfrak{p}) = 0$ and conclude the proof of (c). ☐

LEMMA 11.5.2: *Let F be a generalized function field over a perfect field K with $\mathrm{cd}(K) \le 1$. Then there is a natural commutative square*

$$\begin{array}{ccc} \mathrm{Br}(F) & \longrightarrow & H^2(\mathrm{Gal}(K),(F\tilde{K})^\times/\tilde{K}^\times) \\ \big\downarrow{\scriptstyle \mathrm{res}} & & \big\downarrow \\ \prod_{\mathfrak{p}\in\mathbb{P}(F/K)} \mathrm{Br}(F_\mathfrak{p}) & \longrightarrow & \prod_{\mathfrak{p}\in\mathbb{P}(F/K)} H^2(\mathrm{Gal}(\bar{F}_\mathfrak{p}),\Gamma_\mathfrak{p}) \end{array}$$

where the horizontal arrows are isomorphisms.

Proof: By Lemma 11.5.1, the maps γ and γ' of Lemma 11.3.3 are isomorphisms. Hence, so are the maps $\gamma\circ\beta$ and $\gamma'\circ\beta'$ of the latter lemma. Therefore, the diagram of Lemma 11.3.3 shrinks to the diagram of our lemma. □

LEMMA 11.5.3: *Let F be a function field of one variable over a perfect field K. Suppose F/K has a prime divisor of degree* 1 *and* $\mathrm{cd}(\mathrm{Gal}(K)) \le 1$. *Then there exists a natural exact sequence*

$$\begin{aligned} 0 \to H^1(\mathrm{Gal}(K), \mathrm{Pic}(F\tilde{K}/\tilde{K})) &\to \mathrm{Br}(F) \\ &\xrightarrow{\mathrm{res}} \bigoplus_{\mathfrak{p}\in\mathbb{P}(F/K)} \mathrm{Br}(F_{\mathfrak{p}}) \\ &\to \mathrm{Hom}(\mathrm{Gal}(K), \mathbb{Q}/\mathbb{Z}) \to 0 \end{aligned}$$

Proof: We apply Lemma 11.5.2 to replace

$$H^2(\mathrm{Gal}(K), (F\tilde{K})^\times/\tilde{K}^\times) \text{ and } \bigoplus_{\mathfrak{p}\in\mathbb{P}(F/K)} H^2(\mathrm{Gal}(\bar{F}_{\mathfrak{p}}), \Gamma_{\mathfrak{p}})$$

in the exact sequence of Lemma 11.4.3 by $\mathrm{Br}(F)$ and $\bigoplus_{\mathfrak{p}\in\mathbb{P}(F/K)} \mathrm{Br}(F_{\mathfrak{p}})$, respectively. Since $\mathrm{cd}(\mathrm{Gal}(K)) \le 1$ and each cohomology group of positive degree is the sum of its primary parts, Lemma 11.4.4(c) implies that

$$H^2(\mathrm{Gal}(K), \mathrm{Pic}(F\tilde{K}/\tilde{K})) \cong H^1(\mathrm{Gal}(K), \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(\mathrm{Gal}(K), \mathbb{Q}/\mathbb{Z}).$$

By Lemma 11.5.1(b), $H^3(\mathrm{Gal}(K), (F\tilde{K})^\times/\tilde{K}^\times) = 0$. Consequently, the exact sequence of Lemma 11.4.3 becomes the sequence of our lemma. □

LEMMA 11.5.4: *Let F be a function field of one variable over a perfect PAC field K. Then there is a natural exact sequence*

$$0 \to \mathrm{Br}(F) \xrightarrow{\mathrm{res}} \bigoplus_{\mathfrak{p}\in\mathbb{P}(F/K)} \mathrm{Br}(F_{\mathfrak{p}}) \to \mathrm{Hom}(\mathrm{Gal}(K), \mathbb{Q}/\mathbb{Z}) \to 0$$

Proof: Let J be the Jacobian variety of F/K. Since K is PAC,

$$H^1(\mathrm{Gal}(K), J(\tilde{K})) = 0.$$

(Subsection 6.3.3). By Subsection 6.3.2, $\mathrm{Pic}_0(F\tilde{K}/\tilde{K}) \cong J(\tilde{K})$ as $\mathrm{Gal}(K)$-modules. Hence,

$$H^1(\mathrm{Gal}(K), \mathrm{Pic}_0(F\tilde{K}/\tilde{K})) = 0.$$

Therefore, by Lemma 11.4.4(a), $H^1(\mathrm{Gal}(K), \mathrm{Pic}(F\tilde{K}/\tilde{K})) = 0$. Consequently, the exact sequence of Lemma 11.5.3 shortens to the exact sequence of the present lemma. □

Using lemma 11.3.2, we extract the following result for generalized function fields from Lemma 11.5.4:

PROPOSITION 11.5.5 (Efrat): *Let F be a generalized function field of one variable over a perfect PAC field K. Then the restriction map*

$$\mathrm{res}\colon \mathrm{Br}(F) \to \prod_{\mathfrak{p}\in\mathbb{P}(F/K)} \mathrm{Br}(F_{\mathfrak{p}})$$

is injective

COROLLARY 11.5.6: *Let F be a generalized function field of one variable over a perfect PAC field K. Suppose* $\mathrm{Gal}(F_{\mathfrak{p}})$ *is projective for each* $\mathfrak{p} \in \mathbb{P}(F/K)$. *Then* $\mathrm{Gal}(F)$ *is projective.*

Proof: For each $\mathfrak{p} \in \mathbb{P}(F/K)$ the group $\mathrm{Gal}(F_{\mathfrak{p}})$ is projective, hence $\mathrm{cd}(\mathrm{Gal}(F_{\mathfrak{p}})) \leq 1$ (Subsection 9.3.16). Since K is perfect, $\mathrm{Br}(F_{\mathfrak{p}}) = 0$ (Subsection 9.3.18). Therefore, by Proposition 11.5.5, $\mathrm{Br}(F) = 0$.

The same conclusion holds for every finite separable extension F' of F, because the algebraic closure of K in F' is perfect and PAC [FrJ08, Corollary 11.2.5] and closed subgroups of projective groups are projective [FrJ08, Prop. 22.4.7]. By Subsection 9.3.18, $\mathrm{Gal}(F)$ is projective. ☐

11.6 Radical Extensions

We call an algebraic extension F/E of fields of characteristic p **radical** if for each $a \in E$ and every positive integer n with $p \nmid n$ there exists $x_{a,n} \in F$ such that $x_{a,n}^n = a$ and $F = E(x_{a,n})_{a\in E,\, p\nmid n}$. The following conjecture is a variant of a conjecture of Bogomolov-Positselski [BoP05, Conjecture 1.1]:

CONJECTURE 11.6.1: *Let E be an extension of a field K with* $\mathrm{trans.deg}(E/K) = 1$ *and F an algebraic extension of E. Suppose F contains a radical algebraic extension of E. Then* $\mathrm{Gal}(F)$ *is projective.*

We prove Conjecture 11.6.1 in the special case where K is PAC. It turns out that in this case it suffices to adjoin much less radicals to E than demanded by the definition of the radical extension.

Definition 11.6.2: K-radical extensions. Let E/K be a function field of one variable and F an algebraic extension of E. In the notation of Remark 11.3.1 we say that F/E is a **K-radical extension** if for each $\mathfrak{p} \in \mathbb{P}(E/K)$ and for each positive integer n with $\mathrm{char}(K) \nmid n$ there exists an element $x_{\mathfrak{p},n} \in F$ such that $x_{\mathfrak{p},n}^n \in E$, $v_{\mathfrak{p}}(x_{\mathfrak{p},n}^n) = 1$, and $F = K(x_{\mathfrak{p},n})_{\mathfrak{p}\in\mathbb{P}(E/K),\mathrm{char}(K)\nmid n}$.

In particular, if F/E is a radical extension, then F/E is also a K-radical extension. ☐

Definition 11.6.3: Let K be a field of characteristic p and F an extension of K of transcendence degree 1. We say that F has **p'-divisible K-functional valuation groups** if the value group of F at each valuation trivial on K is p'-divisible.

Note that in that case each algebraic extension F' of F also has p'-divisible K-functional valuation groups (Remark 11.1.2). ☐

LEMMA 11.6.4: *Let p be either 0 or a prime number and let Γ be an additive subgroup of $\mathbb{Q}$. Suppose $\frac{1}{n} \in \Gamma$ for each positive integer n with $p \nmid n$. Then Γ is p'-divisible.*

Proof: We consider $\gamma \in \Gamma$. If $p = 0$, we write $\gamma = \frac{a}{b}$, with $a \in \mathbb{Z}$ and $b \in \mathbb{N}$. Given $n \in \mathbb{N}$, we have $\frac{\gamma}{n} = a \cdot \frac{1}{nb} \in \Gamma$.

If $p > 0$, we write $\gamma = \frac{a}{bp^k}$, where $a \in \mathbb{Z}$, $b \in \mathbb{N}$, $k \in \mathbb{Z}$, and $p \nmid a, b$. Let $n \in \mathbb{N}$ with $p \nmid n$. If $k < 0$, then $\frac{\gamma}{n} = ap^{-k} \cdot \frac{1}{nb} \in \Gamma$. If $k > 0$, we may choose $x, y \in \mathbb{Z}$ such that $xp^k + ynb = 1$. Then $\frac{\gamma}{n} = \frac{a}{nbp^k} = \frac{axp^k + aynb}{nbp^k} = ax \cdot \frac{1}{nb} + by \cdot \frac{a}{bp^k} \in \Gamma$, as claimed. □

LEMMA 11.6.5: *Let E/K be a function field of one variable of characteristic p and F a K-radical extension of E. Then F has a p'-divisible K-functional valuation groups.*

Proof: Let $\mathfrak{p} \in \mathbb{P}(E/K)$ and consider a valuation w of F extending $v_\mathfrak{p}$. Thus, $w(y) = v_\mathfrak{p}(y)$ for each $y \in E$. Since F is an algebraic extension of E the value group Γ of w is contained in $\mathbb{Q}$. On the other hand, for each $\mathfrak{p} \in \mathbb{P}(E/K)$ and each n not divisible by p there is $x_{\mathfrak{p},n} \in F$ such that $\frac{1}{n} = \frac{1}{n} v_\mathfrak{p}(x_{\mathfrak{p},n}^n) = w(x_{\mathfrak{p},n}) \in \Gamma$. By Lemma 11.6.4, Γ is p'-divisible. □

PROPOSITION 11.6.6: *Let K be a PAC field of characteristic p, F an extension of K of transcendence degree 1 with p'-divisible K-functional valuation groups. Then $\mathrm{Gal}(F)$ is projective.*

Proof: By assumption, the value group of each valuation of F/K is p'-divisible. Hence, so is the value group of each valuation of every algebraic extension F' of F, therefore also of each Henselian closure of F'.

By Ax-Roquette, each algebraic extension of a PAC field is again PAC [FrJ08, Cor. 11.2.5]. Hence, we may first replace K by K_{ins} and F by FK_{ins} to assume that K is perfect. Then, we may replace K by $F \cap \tilde{K}$ to assume that F is a generalized function field of one variable over K.

Now we consider a prime divisor $\mathfrak{p}$ of F/K and its Henselization $F_\mathfrak{p}$. The residue field $\bar{F}_\mathfrak{p}$ is an algebraic extension of K, so $\bar{F}_\mathfrak{p}$ is PAC. Hence, by [FrJ08, Thm. 11.6.2], $\mathrm{Gal}(\bar{F}_\mathfrak{p})$ is projective. It follows from Proposition 11.1.3 that $\mathrm{Gal}(F_\mathfrak{p})$ is projective. By Corollary 11.5.6, $\mathrm{Gal}(F)$ is projective. □

COROLLARY 11.6.7: *Let K be a PAC field, E a function field of one variable over K, and F an algebraic extension of a K-radical extension of E. Then $\mathrm{Gal}(F)$ is projective.*

Proof: Let $p = \mathrm{char}(K)$. By Lemma 11.6.5 and Definition 11.6.3, F has a p'-divisible K-functional valuation groups. Hence, by Proposition 11.6.6, $\mathrm{Gal}(F)$ is projective. □

11.7 Semi-Free Absolute Galois Groups

The chapter culminates with its main new result. We construct for each PAC field K of cardinality m an algebraic extension F of $K(x)$ in $K_{\mathrm{cycl}}(x)_{\mathrm{ab}}$ such that F is Hilbertian and $\mathrm{Gal}(F) \cong \hat{F}_m$. If K contains all roots of unity, then $\mathrm{Gal}(K(x)_{\mathrm{ab}}) \cong \hat{F}_m$. The latter result, can be considered as an analog of a well known conjecture of Shafarevich saying that $\mathrm{Gal}(\mathbb{Q}_{\mathrm{ab}}) \cong \hat{F}_\omega$.

First of all we apply the main result of Chapter 8 to the absolute Galois group of a function field of one variable over an ample field.

THEOREM 11.7.1: *Let E be a function field of one variable over an ample field K of cardinality m. Then* $\mathrm{Gal}(E)$ *is semi-free of rank m.*

Proof: We choose a separating transcendence element x for E/K. Since K is ample, m is infinite and $m = \mathrm{card}(K(x)) = \mathrm{card}(E)$. Hence, $\mathrm{rank}(\mathrm{Gal}(K(x))) \leq m$. By Proposition 8.6.3, each finite split embedding problem for $\mathrm{Gal}(K(x))$ with a nontrivial kernel has m linearly disjoint solutions. Thus, $\mathrm{Gal}(K(x))$ is semi-free of rank m (Remark 10.1.6). Since $\mathrm{Gal}(E)$ is an open subgroup of $\mathrm{Gal}(K(x))$, $\mathrm{Gal}(E)$ is semi-free of rank m (Lemma 10.4.1). □

The combination of Theorems 10.5.8 and 11.7.1 gives the following result:

THEOREM 11.7.2: *Let K be an ample field with* $\mathrm{rank}(\mathrm{Gal}(K)) = m$, *$E$ a function field of one variable over K, and F a separable algebraic extension of E. Then* $\mathrm{Gal}(F)$ *is a semi-free profinite group of rank m in each of the following cases:*
(a) $[F : E] < \infty$.
(b) $\mathrm{weight}(F/E) < m$.
(c) *F/E is small.*
(d) *F is contained in an E-diamond.*
(e) *F is a proper finite extension of an extension E_0 of E and E_0 is contained in a Galois extension N of E that does not contain F.*
(f) *F is a proper finite extension of a Galois extension of E.*
(g) *F/E is Abelian.*

The next construction will allow us to move from a function field F of one variable to infinite extensions of F that are not too large.

LEMMA 11.7.3: *Let E be a function field of one variable over a field K, F a finite extension of E, and $\mathfrak{p}$ a prime divisor of E/K tamely and totally ramified in F. Then F is a regular extension of K.*

Proof: The extension F/E is separable, because $\mathfrak{p}$ is tamely and totally ramified in F. Since E/K is separable, also F/K is separable.

It remains to prove that K is algebraically closed in F. Thus, it suffices to prove that $F \cap EL = E$ for each finite extension L of K. Indeed, let L_0 be the maximal separable extension of K in L. Then $\mathfrak{p}$ is unramified in EL_0. Hence, $F \cap EL_0 = E$ and each extension $\mathfrak{p}'$ of $\mathfrak{p}$ to EL_0 is tamely and totally ramified in FL_0. Since EL/EL_0 is purely inseparable, $\mathfrak{p}'$ is either unramified or wildly ramified in EL. Therefore, $FL_0 \cap EL = EL_0$. Consequently, $F \cap EL = E$. $\square$

Given a field E and a prime number p, we write $E_{\mathrm{ab}}^{(p')}$ for the maximal Abelian extension of E of degree prime to p.

Construction 11.7.4: Special K-radical extensions. Let K be a field of characteristic p and infinite cardinality m. Let x be a variable and set $E = K(x)$. We denote the set of all monic irreducible polynomials of $K[x]$ by $\mathcal{F}$. Let $\mathcal{F} = \dot{\bigcup}_{i=1}^{r} \mathcal{F}_i$ be a partition of $\mathcal{F}$ such that $\mathrm{card}(\mathcal{F}_i) = \mathcal{F} = m$ for $i = 1, \ldots, r$. For each i we choose a wellordering $\mathcal{F}_i = (f_{i,\alpha})_{\alpha<m}$. Then, for each $\alpha < m$ and every positive integer n with $p \nmid n$ we choose a root $(f_{1,\alpha} \cdots f_{r,\alpha})^{1/n}$ in E_s such that if $n = dd'$, then $\big((f_{1,\alpha} \cdots f_{r,\alpha})^{1/n}\big)^d = (f_{1,\alpha} \cdots f_{r,\alpha})^{1/d'}$. Then we consider the separable algebraic field extension

$$F_0 = E\big((f_{1,\alpha} \cdots f_{r,\alpha})^{1/n}\big)_{\alpha<m,\, p\nmid n}$$

of E and call F_0 a **special K-radical extension** of E. Note that $F_0 K_{\mathrm{cycl}}$ is an Abelian extension of $K_{\mathrm{cycl}}(x)$ of degree not divisible by p. Hence, $F_0 \subseteq K_{\mathrm{cycl}}(x)_{\mathrm{ab}}^{(p')}$.

In the special case where $r = 1$, the presentation of F_0 is simplified to $F_0 = E(f^{1/n})_{f\in\mathcal{F},\, p\nmid n}$. $\square$

LEMMA 11.7.5: *Let K, x, E, and F_0 be as in Construction 11.7.4. Then:*

(a) *F_0/E is a K-radical extension (Definition 11.6.2).*

(b) *F_0/K is regular, thus F_0/K is a generalized function field of one variable.*

(c) *Every extension F of F_0 in $K_{\mathrm{cycl}}(x)_{\mathrm{ab}}^{(p')}$ is contained in an E-diamond, hence F is Hilbertian.*

(d) *If K contains no primitive root of order l for some prime number $l \neq \mathrm{char}(K)$, then F_0/E is not Galois.*

Proof of (a): For each prime divisor $\mathfrak{p} \neq \mathfrak{p}_{x,\infty}$ of $K(x)/K$ there exist (unique) $1 \le j \le r$ and $\alpha < m$ such that $v_{\mathfrak{p}}(f_{j,\alpha}) = 1$. Since the $\mathcal{F}_i$'s are disjoint, $v_{\mathfrak{p}}(f_{i,\alpha}) = 0$ if $i \neq j$. For $p \nmid n$, let $x_{\mathfrak{p},n} = (f_{1,\alpha} \cdots f_{r,\alpha})^{1/n}$. Then $x_{\mathfrak{p},n} \in F_0$, $x_{\mathfrak{p},n}^n \in E$, and $v_{\mathfrak{p}}(x_{\mathfrak{p},n}^n) = 1$. Next, for $\mathfrak{p} = \mathfrak{p}_{x,\infty}$ we set $\mathfrak{p}' = \mathfrak{p}_{x,0}$ and $x_{\mathfrak{p},n} = x_{\mathfrak{p}',n}^{-1}$. Then $x_{\mathfrak{p},n} \in F_0$, $x_{\mathfrak{p},n}^n \in E$, and $v_{\mathfrak{p}}(x_{\mathfrak{p},n}^n) = 1$. Finally, by construction, F_0 is the field obtained from E by adjoining all $x_{\mathfrak{p},n}$ where $\mathfrak{p} \in \mathbb{P}(E/K)$ and $p \nmid n$. Thus, F_0 is a K-radical extension of E.

Proof of (b): Every finite extension E' of E in F_0 is contained in a field

$$E_r = E(f_1^{1/n_1}, \ldots, f_r^{1/n_r}),$$

where $f_1, \ldots, f_r$ are distinct elements of $\mathcal{F}$ and $n_1, \ldots, n_r$ are positive integers not divisible by p. Inductively assume $E_{r-1} = E(f_1^{1/n_1}, \ldots, f_{r-1}^{1/n_{r-1}})$ is a regular extension of K. For $i = 1, \dots, r$, let v_i be the valuation of E/K satisfying $v_i(f_i) = 1$. Then $v_i(f_j) = 0$ for $i \neq j$. By [FrJ08, Example 2.3.8], v_r is unramified in E_{r-1}. Let w be an extension of v_r to E_{r-1}. Then $w(f_r) = 1$, so again by [FrJ08, Example 2.3.8], w tamely and totally ramifies in E_r. By Lemma 11.7.3, E_r/K is regular. Consequently, F_0 is a regular extension of K.

Proof of (c): Let N_1 be the field obtained from E by adjoining all roots of unity ζ_n and all roots $x^{1/n}$ with $p \nmid n$. Let N_2 be the field obtained from E by adjoining all ζ_n and all roots $f^{1/n}$ with $f \in \mathcal{F} \smallsetminus \{x\}$ and $p \nmid n$. Then both N_1 and N_2 are Galois extensions of E and $N_1 N_2 = K_{\mathrm{cycl}}(x)_{\mathrm{ab}}^{(p')}$, so $F \subseteq N_1 N_2$. Moreover, $\mathfrak{p}_{x,1}$ is ramified in F_0 but unramified in N_1, so $F \not\subseteq N_1$. Similarly, $\mathfrak{p}_{x,0}$ is ramified in F_0 but unramified in N_2, so $F_0 \not\subseteq N_2$, hence $F \not\subseteq N_2$. Thus, F is contained in a diamond over E. By [FrJ08, Thm. 13.4.2], E is Hilbertian. Hence, by Haran's diamond theorem [FrJ08, Thm. 13.8.3], F is Hilbertian.

Proof of (d): Now we assume that $\zeta_l \notin K$ for some prime number $l \neq p$. Then, by (b), $\zeta_l \notin F_0$. Let $f = f_{1,0} \cdots f_{r,0}$. Then $f^{1/l} \in F_0$. If F_0/E is Galois, then also $\zeta_l f^{1/l} \in F_0$, hence $\zeta_l \in F_0$. It follows from this contradiction that F_0 is not a Galois extension of E. □

THEOREM 11.7.6: *Let K be a PAC field of characteristic p and cardinality m and let F_0 be a special K-radical extension of $E = K(x)$ (Construction 11.7.4). Then:*

(a) *Every extension F of F_0 in $K_{\mathrm{cycl}}(x)_{\mathrm{ab}}^{p'}$ is Hilbertian and* $\mathrm{Gal}(F) \cong \hat{F}_m$.
(b) *If K contains no primitive root of order l for some prime number $l \neq p$, then F_0/E is not Galois.*
(c) *If K contains all roots of unity, then E_{ab} is a Hilbertian field with* $\mathrm{Gal}(E_{\mathrm{ab}}) \cong \hat{F}_m$.

Proof: By Lemma 11.7.5(a), F_0 is indeed a K-radical extension of E. Let F be as in (a). By Lemma 11.7.5(c), F is contained in an E-diamond, in particular F is Hilbertian. By Theorem 11.7.1, $\mathrm{Gal}(E)$ is semi-free of rank m. Hence, by Theorem 11.7.2(d), $\mathrm{Gal}(F)$ is semi-free of rank m. By Corollary 11.6.7, $\mathrm{Gal}(F)$ is projective. Hence, by Proposition 10.1.14, $\mathrm{Gal}(F)$ is free of rank m as claimed in (a).

Statement (b) is a special case of Lemma 11.7.5(d). To prove (c) note that since F_0 is generated by radicals of elements of E and all roots of unity of order prime to p are contained in E, we have $F_0 \subseteq E_{\mathrm{ab}}$. In particular, E_{ab} is an Abelian extension of F. Since F is Hilbertian, so is E_{ab} [FrJ08, Thm. 16.11.3]. Since $\mathrm{Gal}(F)$ is isomorphic to $\hat{F}_m$, so is $\mathrm{Gal}(E_{\mathrm{ab}})$ [FrJ08, Cor. 25.4.8]. □

Remark 11.7.7: Note that Theorem 11.7.6(c) follows already from the results of David Harbater quoted in the second paragraph of Section 10.6. Indeed according to those results, if K is a PAC field that contains all roots of unity, then $\mathrm{Gal}(K(x))$ is quasi-free of rank m, hense so is $\mathrm{Gal}(K(x)_{\mathrm{ab}})$. In addition, by Corollary 11.6.7, $\mathrm{Gal}(K(x)_{\mathrm{ab}})$ is projective. Hence, by Proposition 9.4.7, $\mathrm{Gal}(K(x)_{\mathrm{ab}}) \cong \hat{F}_m$.

The condition that K contains all roots of unity of order not divisible by $\mathrm{char}(K)$ is necessary for Theorem 11.7.6(c) to hold. In fact given an odd prime number l, we have examples of Hilbertian PAC fields K that contain all roots of unity of order not divisible by n with $\zeta_l \notin K$ such that $\mathrm{Gal}(K(x)_{\mathrm{ab}})$ is not projective. In particular $\mathrm{Gal}(K(x)_{\mathrm{ab}})$ is not free. We will publish those examples elsewhere. □

Example 11.7.8: Starting from a PAC field K of cardinality m, Theorem 11.7.6 gives an extension F of $K(x)$ in $K(x)_{\mathrm{ab}}$ such that $\mathrm{Gal}(F) \cong \hat{F}_m$ and F is Hilbertian. It is however not clear to us whether F is ample. We suspect it is not.

However, [GeJ01, Thm. 2.6] gives an example of a Hilbertian field F with $\mathrm{Gal}(F) \cong \hat{F}_\omega$ (in particular, $\mathrm{Gal}(F)$ is projective), but F is nonample. □

Notes

Proposition 11.1.3 about the projectivity of $\mathrm{Gal}(M)$ for a Henselian field M under appropriate assumptions on the residue field and the value group reproduces [JaP09, Lemma 1.3].

The results about the cohomology of local Galois groups appearing in Section 11.2 are taken from [Pop88, §2].

Sections 11.3, 11.4, and 11.5 are a work out of part of Efrat's work [Efr01]. The main result of [Efr01] we use is Proposition 11.5.5.

Lemma 11.3.2 is a special case of a more general lemma on a local-global principle for the Brauer group of a field that is a directed union of fields satisfying a local-global principle for their Brauer groups (see [Pop88, Lemma 4.4], or [Efr01, Lemma 3.3]).

Proposition 11.6.6 is [JaP09, Lemma 1.4].

One of the main results of the chapter is Theorem 11.7.1. It also appears as [BHH10, Thm. 7.2]. The proof of the latter theorem is an adjustment of the proof of [HaS05, Thm. 4.3] about quasi-freeness.

We note that [BHH10, Section 8] gives an account of Construction 11.7.4 and of Theorem 11.7.6 with a reference to our book. That work also refers to Theorem 11.7.1 (see [BHH08, comment following the proof of Thm. 7.2]).