Chapter 10.
Semi-Free Profinite Groups

We have already pointed out that a profinite group G of an infinite rank m is free of rank m if (and only if) G is projective and every finite split embedding problem for G with a nontrivial kernel has m solutions (Proposition 9.4.7). Dropping the condition on G to be projective leads to the notion of a "quasi-free profinite group" (Section 10.6).

A somewhat stronger condition is that of a "semi-free profinite group". We say that G is **semi-free** if every finite split embedding problem for G with a nontrivial kernel has m independent solutions (Definition 10.1.5). The advantage of the latter notion on the former one is that the known conditions on a closed subgroup of a free profinite group of rank m to be free of rank m go over to semi-free groups. Indeed, even the method of proof that applies twisted wreath products goes over from free profinite groups to semi-free profinite groups (Section 10.3). Thus, every open subgroup of a semi-free group G is semi-free (Lemma 10.4.1), every normal closed subgroup N of G with G/N Abelian is semi-free, every proper open subgroup of a closed normal subgroup of G is semi-free, and in general every closed subgroup M of G that is "contained in a diamond" is semi-free (Theorem 10.5.3).

An application of the diamond theorem to function fields of one variable over PAC fields appears in the next chapter.

10.1 Independent Subgroups

We introduce a group theoretic counterpart of "linear disjointness of fields".

Definition 10.1.1: Independent subgroups. Let F be a profinite group.

(a) Open subgroups $M_1,\ldots,M_n$ of F are F**-independent**, if

$$(F:\bigcap_{i=1}^{n} M_i)=\prod_{i=1}^{n}(F:M_i).$$

(b) A family $\mathcal{M}$ of open subgroups of F is F**-independent** if $M_1,\ldots,M_n$ are F-independent, for all distinct $M_1,\ldots,M_n\in\mathcal{M}$. □

Remark 10.1.2:

(a) Let F be a profinite group, μ the normalized Haar measure of F, and $M_1,\ldots,M_n$ open subgroups of F. Then $M_1,\ldots,M_n$ are F-independent if and only if $M_1,\ldots,M_n$ are μ-independent, that is $\mu(\bigcap_{i=1}^n M_i)=\prod_{i=1}^n \mu(M_i)$ [FrJ08, Lemma 18.3.7].

(b) Suppose in the notation of (a) that $M_1,\ldots,M_n$ are normal. Then $M_1,\ldots,M_n$ are F-independent if and only if $F/\bigcap_{i=1}^n M_i\cong\prod_{i=1}^n F/M_i$.

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(c) Let $L_1, \dots, L_n$ be finite separable extensions of a field K. Then $\mathrm{Gal}(L_1), \dots, \mathrm{Gal}(L_n)$ are $\mathrm{Gal}(K)$-independent if and only if $L_1, \dots, L_n$ are linearly disjoint over K [FrJ08, Lemma 18.5.1]. □

The following basic rules for F-independency of open subgroups of a profinite group F can be deduced from the corresponding properties of linear disjointness [FrJ08, Section 2.5], using a realization of F as a Galois group of a Galois extension [FrJ08, Cor. 1.3.4]. We give here direct proofs.

LEMMA 10.1.3: *Let $M_1, \dots, M_n$ be open subgroups of a profinite group F. Then:*

(a) $(F : \bigcap_{i=1}^n M_i) \le \prod_{i=1}^n (F : M_i)$.

(b) *M_1, M_2 are F-independent if and only if $(M_1 : M_1 \cap M_2) = (F : M_2)$.*

(c) *Suppose $M_1 \triangleleft F$. Then M_1, M_2 are F-independent if and only if $F = M_1 M_2$.*

(d) *Let $M_1 \le N_1 \le F$. Then M_1, M_2 are F-independent if and only if N_1, M_2 are F-independent and $M_1, N_1 \cap M_2$ are N_1-independent (the* **tower property***).*

(e) *$M_1, \dots, M_n$ are F-independent if and only if $M_1, \dots, M_{n-1}$ are F-independent and $\bigcap_{i=1}^{n-1} M_i$, M_n are F-independent.*

(f) *Suppose $M_i \le N_i \le F$ for $i = 1, \dots, n$. If $M_1, \dots, M_n$ are F-independent, then so are $N_1, \dots, N_n$.*

Proof of (a): The map $F/\bigcap_{i=1}^n M_i \to \prod_{i=1}^n F/M_i$ of quotient spaces defined by $f\bigcap_{i=1}^n M_i \mapsto (fM_1, \dots, fM_n)$ is injective, hence (a) is true.

Proof of (b): The statement follows from the identity $(F : M_1)(M_1 : M_1 \cap M_2) = (F : M_1 \cap M_2)$.

Proof of (c): The assumption $M_1 \triangleleft F$ implies that $(M_2 : M_1 \cap M_2) = (M_1 M_2 : M_1) \le (F : M_1)$. Now we apply (b) with the indices 1 and 2 exchanged to conclude (c).

Proof of (d): First assume that N_1, M_2 are F-independent and $M_1, N_1 \cap M_2$ are N_1-independent. Then, by (b),

(1)
$$(N_1 : N_1 \cap M_2) = (F : M_2) \text{ and } (N_1 : M_1 \cap M_2) = (N_1 : M_1)(N_1 : N_1 \cap M_2).$$

Hence,

$$\begin{aligned}(F : M_1 \cap M_2) &= (F : N_1)(N_1 : M_1 \cap M_2) \\ &= (F : N_1)(N_1 : M_1)(N_1 : N_1 \cap M_2) = (F : M_1)(F : M_2),\end{aligned}$$

so M_1, M_2 are F-independent.

Conversely, suppose $(F : M_1 \cap M_2) = (F : M_1)(F : M_2)$. Then, by (a),

$$\begin{aligned}(F : M_1)(F : M_2) &= (F : M_1 \cap M_2) = (F : N_1)(N_1 : M_1 \cap M_2) \\ &\le (F : N_1)(N_1 : M_1)(N_1 : N_1 \cap M_2) \le (F : M_1)(F : M_2).\end{aligned}$$

Hence, $(N_1 : M_1 \cap M_2) = (N_1 : M_1)(N_1 : N_1 \cap M_2)$, so by definition, $M_1, N_1 \cap M_2$ are N_1-independent. Also, $(N_1 : N_1 \cap M_2) = (F : M_2)$, so by (b), N_1, M_2 are F-independent.

Proof of (e): First we suppose $M_1, \ldots, M_{n-1}$ are F-independent and $\bigcap_{i=1}^{n-1} M_i$, M_n are F-independent. Then, by (b),

$$\begin{aligned}&(F : M_1 \cap \cdots \cap M_n)\\ &= (F : M_1 \cap \cdots \cap M_{n-1})(M_1 \cap \cdots \cap M_{n-1} : M_1 \cap \cdots \cap M_n)\\ &= (F : M_1) \cdots (F : M_{n-1})(F : M_n),\end{aligned}$$

so $M_1, \ldots, M_n$ are F-independent.

Conversely, suppose $M_1, \ldots, M_n$ are F-independent. Then, by (a),

$$\prod_{i=1}^{n}(F : M_i) = (F : \bigcap_{i=1}^{n} M_i) \le (F : \bigcap_{i=1}^{n-1} M_i)(F : M_n) \le \prod_{i=1}^{n-1}(F : M_i) \cdot (F : M_n),$$

hence $(F : \bigcap_{i=1}^{n} M_i) = (F : \bigcap_{i=1}^{n-1} M_i)(F : M_n)$, as desired.

Proof of (f): By definition, $M_1, \ldots, M_{n-1}$ are F-independent. Hence, by an induction hypothesis, $N_1, \ldots, N_{n-1}$ are F-independent. By the tower property (d), $N_n, \bigcap_{i=1}^{n-1} M_i$ are F-independent. Hence, by (d) again, $N_n, \bigcap_{i=1}^{n-1} N_i$ are F-independent. It follows from (e) that $N_1, \ldots, N_n$ are F-independent. □

Lemma 10.1.4: *Let $\mathcal{M} = (M_\alpha \mid \alpha < \lambda)$ be a transfinite sequence of open normal subgroups of a profinite group F. Suppose $M_\kappa \bigcap_{\alpha<\kappa} M_\alpha = F$ for each $\kappa < \lambda$. Then $\mathcal{M}$ is F-independent.*

Proof: Let $\alpha_1 < \cdots < \alpha_n$ be ordinal numbers smaller than λ. By assumption, $\bigcap_{\alpha<\alpha_n} M_\alpha$, M_{α_n} are F-independent. Hence, by Lemma 10.1.3(d), $\bigcap_{i=1}^{n-1} M_{\alpha_i}, M_{\alpha_n}$ are F-independent. By an induction hypothesis on n,

$$M_{\alpha_1}, \ldots, M_{\alpha_{n-1}}$$

are F-independent. Hence, by Lemma 10.1.3(e), $M_{\alpha_1}, \ldots, M_{\alpha_n}$ are F-independent. Consequently, $\mathcal{M}$ is F-independent. □

Definition 10.1.5: Semi-free profinite group. Solutions of an embedding problem

$$(\varphi\colon F \to A,\ \alpha\colon B \to A) \tag{2}$$

of a profinite group F are **independent** if their kernels are $\mathrm{Ker}(\varphi)$-independent.

Note that if the kernel of (2) is trivial, then α is an isomorphism and $\psi = \alpha^{-1} \circ \varphi$ is a solution. In this case, if for each i in a set I we set

$\psi_i = \psi$, then $\mathrm{Ker}(\psi_i) = \mathrm{Ker}(\varphi)$, so $\{\mathrm{Ker}(\psi_i) \mid i \in I\}$ are $\mathrm{Ker}(\psi)$-independent. Therefore, $\{\psi_i \mid i \in I\}$ is a set of independent solutions of (2).

A profinite group F of infinite rank m is **semi-free** if every finite split embedding problem for F with a nontrivial kernel has m independent solutions. □

Remark 10.1.6: Let M be a profinite group and m an infinite cardinal number. Suppose each finite split embedding problem with a nontrivial kernel of M has m independent solutions. In particular, if G is a nontrivial finite group, then the embedding problem $(M \to 1, G \to 1)$ has m independent solutions. Thus, M has m independent open normal subgroups M_i with $M/M_i \cong G$. In particular $\mathrm{rank}(M) \geq m$ [FrJ08, Lemma 17.1.2]. It follows that if in addition, $\mathrm{rank}(M) \leq m$, then $\mathrm{rank}(M) = m$ and M is semi-free. □

Lemma 10.1.7: *Let F be a profinite group and $\mathcal{M}$ an infinite family of pairwise F-independent normal open subgroups of F. Then $\mathcal{M}$ contains an F-independent subfamily $\mathcal{M}_0$ of cardinality* $\mathrm{card}(\mathcal{M})$.

Proof: By Zorn's lemma, $\mathcal{M}$ has a maximal F-independent subfamily $\mathcal{M}_0$. We prove that $\mathrm{card}(\mathcal{M}_0) = \mathrm{card}(\mathcal{M})$.

Otherwise, $\mathrm{card}(\mathcal{M}_0) < \mathrm{card}(\mathcal{M})$. Let $\mathcal{M}_1$ be the family of all finite intersections of elements of $\mathcal{M}_0$. If $\mathcal{M}_0$ is finite, then so is $\mathcal{M}_1$. If $\mathcal{M}_0$ is infinite, then $\mathrm{card}(\mathcal{M}_1) = \mathrm{card}(\mathcal{M}_0)$. In both cases, $\mathrm{card}(\mathcal{M}_1) < \mathrm{card}(\mathcal{M})$.

Next we denote the family of all subgroups of F that contain a group belonging to $\mathcal{M}_1$ by $\mathcal{M}_2$. Again, if $\mathcal{M}_1$ is finite, then so is $\mathcal{M}_2$. If $\mathcal{M}_1$ is infinite, then $\mathrm{card}(\mathcal{M}_2) = \mathrm{card}(\mathcal{M}_1)$. In both cases, $\mathrm{card}(\mathcal{M}_2) < \mathrm{card}(\mathcal{M})$.

By Lemma 10.1.3(d), each open proper subgroup of F contains at most one group $M \in \mathcal{M}$. Hence, there exists $M \in \mathcal{M}$ not contained in any proper subgroup of F that belongs to $\mathcal{M}_2$. We claim that the family $\mathcal{M}_0 \cup \{M\}$ is F-independent.

To prove the claim we consider $M_1, \ldots, M_n \in \mathcal{M}_0$. Then, $M_1, \ldots, M_n$ are F-independent. Moreover, $M \bigcap_{i=1}^n M_i \in \mathcal{M}_2$. Hence, by the choice of M we have $M \bigcap_{i=1}^n M_i = F$. By Lemma 10.1.3(c), $M, \bigcap_{i=1}^n M_i$ are F-independent. Therefore, by Lemma 10.1.3(e), $M_1, \ldots, M_n, M$ are F-independent. This completes the proof of the claim of the preceding paragraph and gives the desired contradiction to the maximality of $\mathcal{M}_0$. □

Taking into account Remark 10.1.6, Lemma 10.1.7 yields the following result:

Corollary 10.1.8: *Let m be an infinite cardinal number and F a profinite group of rank at most m. Then F is semi-free of rank m if and only if every finite split embedding problem with a nontrivial kernel has m pairwise independent solutions.*

Definition 10.1.9: Weight. Let M be a closed subgroup of a profinite group F. We define the **weight** of the quotient space F/M as 1 if M is open and as the cardinality of the set of all open subgroups of F that contain M if $(F:M)=\infty$.

Let m be an infinite cardinal number. If $L=\bigcap_{i\in I} L_i$, L_i is a closed subgroup of F, $\mathrm{weight}(F/L_i)<m$ for each $i\in I$, and $\mathrm{card}(I)<m$, then $\mathrm{weight}(F/L)<m$ [FrJ08, Lemma 25.2.1(b)].

If N is a closed subgroup of a closed subgroup M of F, and $\mathrm{weight}(F/M)$, $\mathrm{weight}(M/N)<m$, then $\mathrm{weight}(F/N)<m$ [FrJ08, Lemma 25.2.1(d)]. □

Definition 10.1.10: Small quotient spaces. Let M be a closed subgroup of a profinite group F. We denote the set of all open subgroups of F that contain M by $\mathrm{Open}(F/M)$. We say that the quotient space F/M is **small** if for each positive integer n, the set $\mathrm{Open}(F/M)$ has only finitely many groups of index at most n. In particular if $M\triangleleft F$, then the quotient group F/M is small in the sense of [FrJ08, p. 329].

Note that if M' is a closed subgroup of M and F/M' is small, then so is F/M. In particular, if $M'\triangleleft F$ and F/M' is finitely generated, then F/M' is a small group [FrJ08, Lemma 16.10.2], so F/M is also a small quotient space.

Let $\hat{M}=\bigcap_{\sigma\in F} M^\sigma$ be the **normal core** of M in F. If $F/\hat{M}$ is finitely generated, then by the preceding paragraph, F/M is small. If in addition N is an open normal subgroup of M, then there is an open normal subgroup L of F such that $L\cap\hat{M}\le N\cap\hat{M}$. Since $F/(L\cap\hat{M})$ embeds into $F/L\times F/\hat{M}$ and the latter group is finitely generated, $F/(L\cap\hat{M})$ is small, so F/N is small. □

LEMMA 10.1.11: *Let F be a profinite group, M a closed subgroup of F, and $\mathcal{E}$ an F-independent infinite family of open normal subgroups of F. If*

(a) $\mathrm{weight}(F/M)<\mathrm{card}(\mathcal{E})$, *or*

(b) *F/M is small and there exists a positive integer n such that $(F:E)\le n$ for each $E\in\mathcal{E}$,*

then there exists $E\in\mathcal{E}$ such that $EM=F$.

Proof: Every proper open subgroup F_0 of F contains at most one group of $\mathcal{E}$. In Case (a), $\mathrm{card}(\mathrm{Open}(F/M))<\mathrm{card}(\mathcal{E})$. Hence, there exists $E\in\mathcal{E}$ which is contained in no proper open subgroup of F that contains M. Therefore, $EM=F$.

In Case (b), $(F:EM)\le(F:E)\le n$, so only finitely many $E\in\mathcal{E}$ satisfy $EM<F$. Since $\mathcal{E}$ is infinite, there exists $E\in\mathcal{E}$ with $EM=F$. □

LEMMA 10.1.12: *Let F be a profinite group, M an open normal subgroup, and $\mathcal{E}$ an infinite F-independent family of open normal subgroups of F. Then, for each n there exist $E_1,\ldots,E_n\in\mathcal{E}$ such that $E_1,\ldots,E_n,M$ are F-independent.*

Proof: We assume inductively that $E_1,\ldots,E_{n-1}\in\mathcal{E}$ and $E_1,\ldots,E_{n-1},M$ are F-independent. Then $M'=E_1\cap\cdots E_{n-1}\cap M$ is an open subgroup of F. By Lemma 10.1.11, there exists $E_n\in\mathcal{E}$ with $E_nM'=F$. Hence, by Lemma 10.1.3, $E_1,\ldots,E_n,M$ are F-independent. This concludes the induction. □

We conclude this section with two results that line the notion of a "semi-free profinite group" to that of a "free profinite group".

PROPOSITION 10.1.13: *Let F be a free profinite group of infinite rank m. Then every finite embedding problem (2) for F with a nontrivial kernel has m independent solutions. In particular, F is semi-free.*

Proof: Let $\lambda < m$ be an ordinal number and assume $\{\gamma_\kappa \mid \kappa < \lambda\}$ are independent solutions of (2). We prove the existence of a solution γ_λ such that the set of solutions $\{\gamma_\kappa \mid \kappa \le \lambda\}$ of (2) is independent. Applying transfinite induction and Lemma 10.1.4, this will give m independent solutions of (2).

Let $E = \mathrm{Ker}(\varphi)$. Since $\alpha \circ \gamma_\kappa = \varphi$, we have $\mathrm{Ker}(\gamma_\kappa) \le E$, so $M = \bigcap_{\kappa<\lambda} \mathrm{Ker}(\gamma_\kappa) \le E$. Since $\mathrm{card}(\lambda) < m$ and the $\mathrm{Ker}(\gamma_\kappa)$'s are open in F, $\mathrm{weight}(F/M) < m$ (Definition 10.1.9). Set $\hat{A} = F/M$ and let $\hat\varphi\colon F \to \hat{A}$ be the quotient map. Then there exists an epimorphism $\bar\varphi\colon \hat{A} \to A$ such that $\bar\varphi \circ \hat\varphi = \varphi$. This gives rise to a commutative diagram

$$\begin{array}{ccc} & & F \\ & & \downarrow{\scriptstyle\hat\varphi} \\ \hat B & \xrightarrow{\hat\alpha} & \hat A \\ {\scriptstyle\hat\beta}\downarrow & & \downarrow{\scriptstyle\bar\varphi} \\ B & \xrightarrow{\alpha} & A \end{array}$$

in which the square is cartesian. By [FrJ08, Lemma 25.1.3], every finite embedding problem for F has m solutions. Hence, by [FrJ08, Lemma 25.1.5], there exists an epimorphism $\hat\gamma\colon F \to \hat{B}$ such that $\hat\alpha \circ \hat\gamma = \hat\varphi$. Thus, $\gamma = \hat\beta \circ \hat\gamma$ satisfies $\alpha \circ \gamma = \varphi$, that is γ is a solution of (2).

Let $D = \mathrm{Ker}(\gamma)$ and $N = \mathrm{Ker}(\hat\gamma)$. Then $D \cap M = N$, because if $x \in D \cap M$, then $\hat\beta(\hat\gamma(x)) = \gamma(x) = 1$ and $\hat\alpha(\hat\gamma(x)) = \hat\varphi(x) = 1$, hence $\hat\gamma(x) = 1$, so $x \in N$.

$$\begin{array}{ccc} & & F \\ & {\scriptstyle\hat\gamma}\swarrow & \downarrow{\scriptstyle\hat\varphi} \\ \hat B & \xrightarrow{\hat\alpha} & \hat A \\ {\scriptstyle\hat\beta}\downarrow & & \downarrow{\scriptstyle\bar\varphi} \\ B & \xrightarrow{\alpha} & A \end{array} \qquad\qquad \begin{array}{ccc} & & F \\ & & | \\ D & - & E \\ | & & | \\ N & - & M \end{array}$$

Then $E/D \cong \mathrm{Ker}(\alpha)$ and $M/N \cong \mathrm{Ker}(\hat\alpha)$. By [FrJ08, Lemma 22.2.5], $\mathrm{Ker}(\alpha) \cong \mathrm{Ker}(\hat\alpha)$. Hence, $(E : D) = (M : N)$. Therefore, by Lemma 10.1.3(b), D, M are E-independent. Consequently, γ is independent of the set of solutions $\{\gamma_\kappa \mid \kappa < \lambda\}$. ☐

The converse of Proposition 10.1.13 is a reformulation of Proposition 9.4.7.

PROPOSITION 10.1.14: *Let F be a projective semi-free profinite group of infinite rank m. Then F is a free profinite group.*

10.2 Fiber Products

Fiber products of pairs of profinite groups are introduced in [FrJ08, Sec. 13.7]. Here we consider fiber products of finitely many profinite groups and use them in the next section to construct wreath products.

Definition 10.2.1: Fiber products. For each $1 \le i \le n$ let $\alpha_i\colon H_i \to G$ be an epimorphism of profinite groups. The **fiber product** $\prod_G H_i = H_1 \times_G \cdots \times_G H_n$ with respect to the α_i's is the group

$$\prod\nolimits_G H_i = \{(h_1,\ldots,h_n) \in \prod_{i=1}^n H_i \mid \alpha_1(h_1) = \cdots = \alpha_n(h_n)\}.$$

For each $g \in G$ we can choose $h_i \in H_i$ with $\alpha_i(h_i) = g$. Hence, the projection $\mathrm{pr}_j\colon \prod_G H_i \to H_j$ on the jth coordinate is surjective. It follows that the homomorphism $\alpha^{(n)} = \alpha_j \circ \mathrm{pr}_j\colon \prod_G H_i \to G$ is independent of j and is surjective.

If all of the H_i's are the same group H and all of the α_i's are the same map α, we also write H_G^n for $\prod_G H_i$.

Note that the fiber product is associative in the following sense: There is a natural isomorphism $\prod_G H_i \cong (H_1 \times_G \cdots \times_G H_m) \times_G (H_{m+1} \times_G \cdots \times_G H_n)$ for each $1 \le m \le n$. □

Example 10.2.2: Let $G, A_1, \ldots, A_n$ be profinite groups. Suppose G acts on each A_i. Let $\alpha_i\colon G \ltimes A_i \to G$ be the projection on the first coordinate, $i = 1, \ldots, n$. Then $\prod_G (G \ltimes A_i)$ consists of all n-tuples $(\sigma a_1, \ldots, \sigma a_n)$ with $\sigma \in G$ and $a_i \in A_i$, $i = 1, \ldots, n$. For each j, the image of such an n-tuple under $\alpha_j \circ \mathrm{pr}_j$ is σ. On the other hand, G acts on $\prod A_i$ componentwise and the projection $G \ltimes \prod A_i \to G$ on the first coordinate maps an element $\sigma(a_1, \ldots, a_n)$ of $G \ltimes \prod A_i$ onto σ. We may therefore identify $G \ltimes \prod A_i$ with $\prod_G (G \ltimes A_i)$ and the projection on the first coordinate with $\alpha^{(n)}$.

In particular, if all of the A_i's are the same group A, we have $(G \ltimes A)_G^n = G \ltimes A^n$ and $\alpha^{(n)}\colon (G \ltimes A)_G^n \to G$ is the projection on the first coordinate. □

A key property of fiber products, in our setting, is that weak solutions ψ_i of embedding problems $(\varphi\colon F \to G, \alpha_i\colon H_i \to G)$, $i = 1, \ldots, n$, give rise to a canonical weak solution, $\psi = \prod_{i=1}^n \psi_i$, of the embedding problem $(\varphi\colon F \to G, \alpha^{(n)}\colon \prod_G H_i \to G)$. The homomorphism $\psi\colon F \to \prod_G H_i$ is defined by $\psi(x) = (\psi_1(x), \ldots, \psi_n(x))$. In particular, taking $F = G$ and $\varphi = \mathrm{id}$, we find that if all of the α_i's split, so does $\alpha^{(n)}$.

In the case where all of the H_i's are the same group H and all of the α_i's are the same map α, we prove that independency of solutions $\psi_1,\ldots,\psi_n$ of embedding problems is equivalent to a solution of the corresponding embedding problem onto H_G^n.

LEMMA 10.2.3: *Let $\mathcal{E} = (\varphi\colon F \to G, \alpha\colon H \to G)$ be a finite embedding problem of a profinite group F and $\psi_1,\ldots,\psi_n$ weak solutions of $\mathcal{E}$. Then $\psi_1,\ldots,\psi_n$ are independent solutions if and only if the weak solution $\psi\colon F \to H_G^n$ of the embedding problem $\mathcal{E}_n = (\varphi\colon F \to G, \alpha^{(n)}\colon H_G^n \to G)$ defined by $\psi(x) = (\psi_1(x),\ldots,\psi_n(x))$ is a solution of that problem, that is $\psi(F) = H_G^n$.*

Proof: Let $E = \mathrm{Ker}(\varphi)$, $M_i = \mathrm{Ker}(\psi_i)$, and $M = \bigcap_{i=1}^n M_i$. Then $(F : M_i) \le |H|$, $M = \mathrm{Ker}(\psi)$, and $(F : M) \le |H_G^n|$. By definition,

$$H_G^n = \{(h_1,\ldots,h_n) \in H^n \mid \alpha(h_1) = \cdots = \alpha(h_n)\} = \bigcup_{g\in G}^{\cdot} (\alpha^{-1}(g))^n,$$

so $|H_G^n| = |G|\big(\frac{|H|}{|G|}\big)^n$.

Now we assume that $\psi_1,\ldots,\psi_n$ are independent solutions of $\mathcal{E}$ and prove that ψ is a solution of $\mathcal{E}_n$. Since $\alpha\circ\mathrm{pr}_j\circ\psi = \alpha\circ\psi_j = \varphi$, it suffices to prove that $\psi(F) = H_G^n$. Indeed, $\psi_i(F) = H$ and $(E : M) = \prod_{i=1}^n (E : M_i)$, hence

$$\begin{aligned}|\psi(F)| = (F : M) = (F : E)(E : M) &= (F : E)\prod_{i=1}^n (E : M_i)\\ &= (F : E)\prod_{i=1}^n \frac{(F : M_i)}{(F : E)} = |G|\frac{|H|^n}{|G|^n} = |H_G^n|.\end{aligned}$$

Consequently, $\psi(F) = H_G^n$.

Conversely, suppose ψ is a solution of $\mathcal{E}_n$. Then by Lemma 10.1.3(a),

$$\begin{aligned}|H_G^n| = |\psi(F)| = (F : E)(E : M) &\le (F : E)\prod_{i=1}^n (E : M_i)\\ &= (F : E)\prod_{i=1}^n \frac{(F : M_i)}{(F : E)} \le |G|\frac{|H|^n}{|G|^n} = |H_G^n|,\end{aligned}$$

hence $(E : M) = \prod_{i=1}^n (E : M_i)$ and $(F : M_i) = |H|$ for each i. This means that $\psi_1,\ldots,\psi_n$ are independent solutions of $\mathcal{E}$. □

PROPOSITION 10.2.4: *If every finite split embedding problem of a profinite group F is solvable, then every finite split embedding problem with a nontrivial kernel has $\aleph_0$ independent solutions. In particular, if $\mathrm{rank}(F) = \aleph_0$, then F is semi-free.*

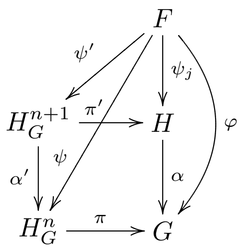
Proof: We consider a finite split embedding problem

$$\mathcal{E} = (\varphi\colon F \to G, \alpha\colon H \to G).$$

By induction we suppose $\psi_1, \ldots, \psi_n$ are independent solutions of $\mathcal{E}$. By Lemma 10.2.3, the map $\psi\colon F \to H_G^n$ defined by $\psi(x) = (\psi_1(x), \ldots, \psi_n(x))$ is a solution of the embedding problem $\mathcal{E}_n = (\varphi\colon F \to G,\, \pi\colon H_G^n \to G)$, where $\pi = \alpha \circ \mathrm{pr}_j$ is independent of j.

Let $\pi'\colon H_G^{n+1} \to H$ be the projection on the $(n+1)$th coordinate and $\alpha'\colon H_G^{n+1} \to H_G^n$ the projection on the first n-coordinates. Then we observe that the rectangle in diagram (1) is cartesian (Definition 10.2.1). Since α split, so does α' (comments preceding Lemma 10.2.3). By assumption, there exists an epimorphism $\psi'\colon F \to H_G^{n+1}$ such that $\alpha' \circ \psi' = \psi$.

(1)



Thus, $\alpha \circ \pi' \circ \psi' = \pi \circ \alpha' \circ \psi' = \pi \circ \psi = \varphi$. It follows that ψ' is a solution of the embedding problem $\mathcal{E}_{n+1} = (\varphi\colon F \to G,\, \alpha \circ \mathrm{pr}_i\colon H_G^{n+1} \to G)$ for each $1 \le i \le n+1$. By Lemma 10.2.3, $\mathcal{E}_{n+1}$ has independent solutions $\psi_1', \ldots, \psi_{n+1}'$ such that $\psi'(x) = (\psi_1'(x), \ldots, \psi_{n+1}'(x))$ for each $x \in F$. Since $\alpha' \circ \psi' = \psi$, we have $\psi_i'(x) = \psi_i(x)$ for $i = 1, \ldots, n$ and every $x \in F$. We set $\psi_{n+1} = \psi_{n+1}'$ to conclude that $\psi_1, \ldots, \psi_n, \psi_{n+1}$ are independent solutions of $\mathcal{E}$, as desired. □

10.3 Twisted Wreath Products

Following the works [Har99a] and [Har99b], twisted wreath products have been introduced in [FrJ08, Section 13.7] in order to prove the diamond theorem for Hilbertian fields [FrJ08, Thm. 13.8.3] and the diamond theorem for free profinite groups [FrJ08, Thm. 25.4.3]. Here we consider those products once more in order to generalize the latter theorem to a diamond theorem for semi-free profinite groups.

Definition 10.3.1: Twisted wreath product. Let A and G be finite groups and G_0 a subgroup of G. Suppose G_0 acts on A from the right and let $\mathrm{Ind}_{G_0}^G(A)$ be the set of all functions $f\colon G \to A$ such that $f(\sigma\tau) = f(\sigma)^\tau$ for all $\sigma \in G$ and $\tau \in G_0$. We make $\mathrm{Ind}_{G_0}^G(A)$ a group by the rule $(fg)(\sigma) = f(\sigma)g(\sigma)$ and let G acts on $\mathrm{Ind}_{G_0}^G(A)$ by $f^\sigma(\rho) = f(\sigma\rho)$ for all $\sigma, \rho \in G$. This gives rise to the semidirect product $G \ltimes \mathrm{Ind}_{G_0}^G(A)$ that we call the **twisted wreath product** of G and A over G_0 and denote by $G \mathrm{wr}_{G_0} A$. In particular, the map $\sigma f \mapsto \sigma$ for $\sigma \in G$ and $f \in \mathrm{Ind}_{G_0}^G(A)$ is a split epimorphism onto G with kernel $\mathrm{Ind}_{G_0}^G(A)$.

The map $\pi_0\colon \mathrm{Ind}_{G_0}^G(A) \to A$ defined by $\pi_0(f) = f(1)$ is an epimorphism. It commutes with the action of G_0. Indeed, for $f \in \mathrm{Ind}_{G_0}^G(A)$ and $\tau \in G_0$ we have $\pi_0(f^\tau) = f^\tau(1) = f(\tau) = f(1)^\tau = \pi_0(f)^\tau$. Thus, π_0 extend to an epimorphism $\pi\colon G_0 \ltimes \mathrm{Ind}_{G_0}^G(A) \to G_0 \ltimes A$ defined by $\pi(\tau f) = \tau f(1)$ giving rise to the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Ind}_{G_0}^G(A) & \longrightarrow & G\mathrm{wr}_{G_0}A & \longrightarrow & G & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \mathrm{Ind}_{G_0}^G(A) & \longrightarrow & G_0 \ltimes \mathrm{Ind}_{G_0}^G(A) & \longrightarrow & G_0 & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi} & & \| & & \\
1 & \longrightarrow & A & \longrightarrow & G_0 \ltimes A & \longrightarrow & G_0 & \longrightarrow & 1
\end{array}$$

We call π_0 and π the **Shapiro maps** of $\mathrm{Ind}_{G_0}^G(A)$ and $G_0 \ltimes \mathrm{Ind}_{G_0}^G(A)$, respectively.

If B is a normal subgroup of A invariant under G_0, then the action of G_0 on A induces an action on A/B and the quotient map $A \to A/B$ gives rise to epimorphisms $G_0 \ltimes A \to G_0 \ltimes A/B$, $\mathrm{Ind}_{G_0}^G(A) \to \mathrm{Ind}_{G_0}^G(A/B)$, $G_0 \ltimes \mathrm{Ind}_{G_0}^G(A) \to G_0 \ltimes \mathrm{Ind}_{G_0}^G(A/B)$, and $G\mathrm{wr}_{G_0}A \to G\mathrm{wr}_{G_0}A/B$. The second and the third maps commute with π. $\square$

Remark 10.3.2: Distributive law for twisted wreath products. Let G be a finite group, G_0 a subgroup, and $A_1, \ldots, A_n$ finite groups. Suppose G_0 acts on each A_i. Then $\mathrm{Ind}_{G_0}^G(\prod A_i) = \prod \mathrm{Ind}_{G_0}^G A_i$ and $\prod_G(G\mathrm{wr}_{G_0}A_i) \cong G\mathrm{wr}_{G_0} \prod A_i$, where G_0 acts on $\prod A_i$ componentwise.

Indeed, each element of $\mathrm{Ind}_{G_0}^G(\prod A_i)$ can be identified as an n-tuple $(f_1, \ldots, f_n)$ with $f_i \in \mathrm{Ind}_{G_0}^G A_i$. A combination of this observation with Example 10.2.2 gives $\prod_G(G\mathrm{wr}_{G_0}A_i) = G\mathrm{wr}_{G_0} \prod A_i$. Explicitly, each element of $\prod_G(G\mathrm{wr}_{G_0}A_i)$ has the form $(\sigma f_1, \ldots, \sigma f_n)$, where $\sigma \in G$ and $f_i \in \mathrm{Ind}_{G_0}^G(A_i)$, $i = 1, \ldots, n$. We identify that element with the element $\sigma(f_1, \ldots, f_n)$ of $G\mathrm{wr}_{G_0} \prod A_i$. $\square$

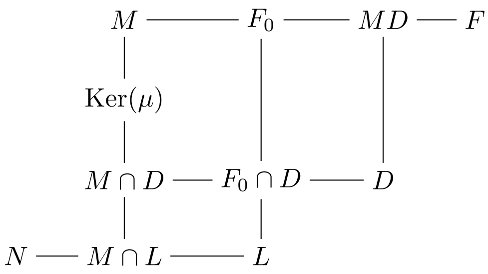
The next technical lemma induces finite split embedding problems for a closed subgroup M of a profinite group F to finite split embedding problems for F. Under an additional assumption, independent solutions of the problems of F yield independent solutions of the problems of M.

LEMMA 10.3.3: *Let F be a profinite group, M a closed subgroup and*

$$\mathcal{E}_1(A) = (\mu\colon M \to G_1,\ \alpha_1\colon G_1 \ltimes A \to G_1)$$

a finite split embedding problem for M. Let F_0 be an open subgroup of F, D and L open normal subgroups of F, and N a closed normal subgroup of

F as in the following diagram:



Set $G = F/L$, $G_0 = F_0/L$, and let $\varphi\colon F \to G$ and $\varphi_0\colon F_0 \to G_0$ be the quotient maps.

(a) *There exists an epimorphism $\bar\varphi_1\colon G_0 = F_0/L \to G_1 = M/\mathrm{Ker}(\mu)$ such that $\mu = \bar\varphi_1 \circ \varphi|_M$ (after identifying $M/M \cap L$ with ML/L).*

(b) *Let $\rho\colon G_0 \ltimes A \to G_1 \ltimes A$ be the extension of $\bar\varphi_1$ by id_A. Consider the finite split embedding problem*

$$\mathcal{E}(A) = (\varphi\colon F \to G,\ \beta\colon G\mathrm{wr}_{G_0}A \to G),$$

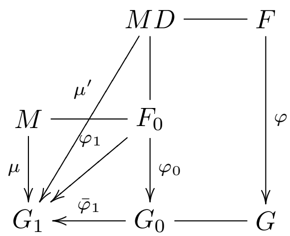
where G_0 acts on A via $\bar\varphi_1$ and β is the projection on the first coordinate. Let $\pi\colon G_0 \ltimes \mathrm{Ind}_{G_0}^{G}(A) \to G_0 \ltimes A$ be the Shapiro map.

For a positive integer n we assume that

(*) *no finite split embedding problem $\mathcal{E}(\bar A) = (\bar\varphi\colon F/N \to G,\ \bar\beta\colon G\mathrm{wr}_{G_0}\bar A \to G)$, where $\bar A$ is a nontrivial quotient of A^n and $\bar\varphi\colon F/N \to F/L = G$ is the quotient map, has a solution.*

Finally, let $\psi_1, \ldots, \psi_n$ be independent solutions of $\mathcal{E}(A)$. Then $\nu_i = \rho \circ \pi \circ \psi_i|_M$, $i = 1, \ldots, n$, are independent solutions of $\mathcal{E}_1(A)$.

Proof: Since $M \cap D \le \mathrm{Ker}(\mu)$, the map $\mu\colon M \to G_1$ extends to an epimorphism $\mu'\colon MD \to G_1$ by $\mu'(md) = \mu(m)$. In particular, μ' is trivial on L, so $\varphi_1 = \mu'|_{F_0}$ decomposes as $\varphi_1 = \bar\varphi_1 \circ \varphi_0$, where $\bar\varphi_1$ is an epimorphism from G_0 onto G_1.

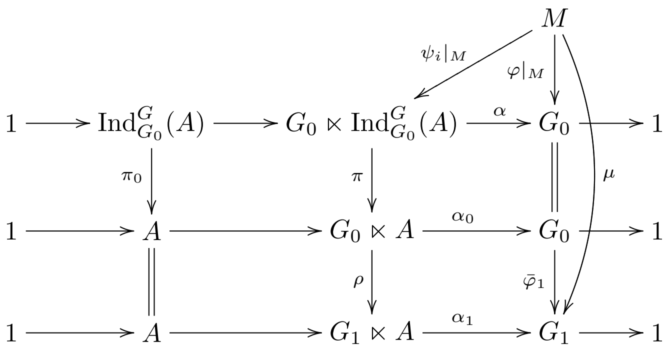


Since $\varphi_0 = \varphi|_{F_0}$, we have that $\mu = \bar\varphi_1 \circ \varphi|_M$. This proves (a).

The proof of (b) breaks up into three parts.

Part A: *The maps ν_i are weak solutions of $\mathcal{E}_1(A)$.* For each $1 \leq i \leq n$ we note that $\beta(\psi_i(M)) = \varphi(M) \leq \varphi(F_0) = G_0$, so $\psi_i(M) \leq \beta^{-1}(G_0) = G_0 \ltimes \mathrm{Ind}_{G_0}^G(A)$. Therefore, ν_i is well defined and the following diagram where α is the restriction of β to $G_0 \ltimes \mathrm{Ind}_{G_0}^G(A)$ and π_0 is the Shapiro map of $\mathrm{Ind}_{G_0}^G(A)$ is commutative:

(1)



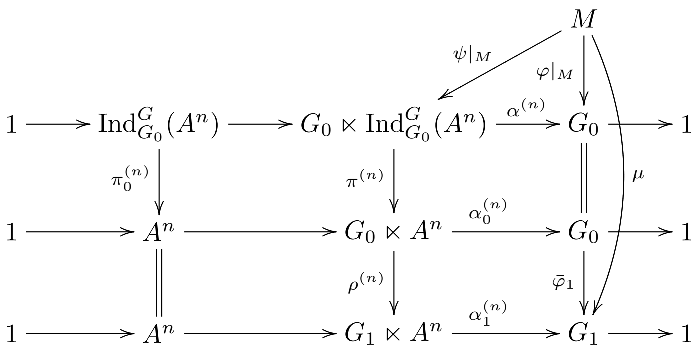
Thus, $\alpha_1 \circ \nu_i = \mu$, so ν_i is a weak solution of $\mathcal{E}_1(A)$. It suffices to prove that ν_i is surjective and $\nu_1, \ldots, \nu_n$ are independent.

Part B: *The map ν.* We use Remark 10.3.2 to identify $\mathrm{Ind}_{G_0}^G(A^n)$ with $\mathrm{Ind}_{G_0}^G(A)^n$, and $G\mathrm{wr}_{G_0}A^n$ with $(G\mathrm{wr}_{G_0}A)_G^n$, and denote the corresponding Shapiro maps by $\pi_0^{(n)}$ and $\pi^{(n)}$. By Lemma 10.2.3, $\psi_1, \ldots, \psi_n$ define a solution ψ of the embedding problem

$$\mathcal{E}(A^n) = (\varphi\colon F \to G,\ \beta^{(n)}\colon G\mathrm{wr}_{G_0}A^n \to G),$$

where $\beta^{(n)} = \mathrm{pr}_j \circ \beta$ for each $1 \leq j \leq n$. Replacing A in (1) by A^n, we get a commutative diagram

(2)



where $\alpha^{(n)}$, $\alpha_0^{(n)}$, and $\alpha_1^{(n)}$ are the projections on the first factor, and $\rho^{(n)}$ is the extension of $\bar{\varphi}_1$ by id_{A^n}. Identifying $G_1 \ltimes A^n$ with $(G_1 \ltimes A)_G^n$ (Example

10.2.2), we first observe that the combined map $\nu = \rho^{(n)} \circ \pi^{(n)} \circ \psi|_M$ satisfies $\nu = \nu_1 \times \cdots \times \nu_n$.

Indeed, by Remark 10.3.2, for each $x \in M$ there are $\sigma_0 \in G_0$ and $f_1, \ldots, f_n \in \mathrm{Ind}_{G_0}^G(A)$ such that

$$(\psi_1(x), \ldots, \psi_n(x)) = \psi(x) = \sigma_0(f_1, \ldots, f_n) = (\sigma_0 f_1, \ldots, \sigma_0 f_n).$$

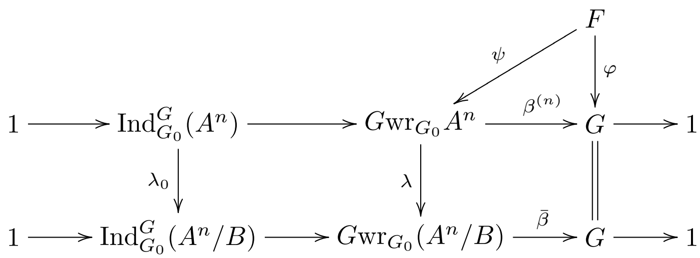
Hence,

$$\begin{aligned}
\nu(x) &= \rho^{(n)}(\pi^{(n)}(\psi(x))) = \rho^{(n)}(\pi^{(n)}(\sigma_0(f_1, \ldots, f_n))) \\
&= \rho^{(n)}(\sigma_0(f_1(1), \ldots, f_n(1))) \\
&= \bar{\varphi}_1(\sigma_0)(f_1(1), \ldots, f_n(1)) = (\rho(\sigma_0 f_1(1)), \ldots, \rho(\sigma_0 f_n(1))) \\
&= (\rho(\pi(\sigma_0 f_1)), \ldots, \rho(\pi(\sigma_0 f_n))) = (\rho(\pi(\psi_1(x))), \ldots, \rho(\pi(\psi_n(x)))) \\
&= (\nu_1(x), \ldots, \nu_n(x)),
\end{aligned}$$

as claimed.

In order to prove that $\nu_1, \ldots, \nu_n$ are surjective and independent, it suffices to prove that ν is surjective (Lemma 10.2.3). Since $\alpha_1^{(n)}(\nu(M)) = \mu(M) = G_1$, it suffices to prove that $A^n \le \nu(M)$. This will follow once we prove that $\pi^{(n)}(\psi(N)) = A^n$.

PART C: *A proof that* $\pi^{(n)}(\psi(N)) = A^n$. Since $N \le L = \mathrm{Ker}(\varphi)$, we have $\varphi(N) = 1$. In addition, $N \lhd F$, so $\psi(N) \le \mathrm{Ind}_{G_0}^G(A^n)$ and $\psi(N) \lhd G \mathrm{wr}_{G_0} A^n$. Hence, $\psi(N)$ is a normal G-invariant subgroup of $\mathrm{Ind}_{G_0}^G(A^n)$. Thus, $B = \pi^{(n)}(\psi(N))$ is a normal G_0-invariant subgroup of A^n. Therefore, G_0 acts on A^n/B. This gives a commutative diagram of two short exact sequences



in which λ_0 and λ are defined by the quotient map $A^n \to A^n/B$.

Now $\psi(N) \le (\pi^{(n)})^{-1}(B) = \{f \in \mathrm{Ind}_{G_0}^G(A^n) \mid f(1) \in B\}$ and, as mentioned above, $\psi(N)$ is a G-invariant subgroup of $\mathrm{Ind}_{G_0}^G(A^n)$. Hence,

$$\begin{aligned}
\psi(N) &\le \bigcap_{\sigma \in G} \{f \in \mathrm{Ind}_{G_0}^G(A^n) \mid f(1) \in B\}^\sigma \\
&= \bigcap_{\sigma \in G} \{f \in \mathrm{Ind}_{G_0}^G(A^n) \mid f(\sigma) \in B\} = \mathrm{Ker}(\lambda).
\end{aligned}$$

It follows that $\lambda \circ \psi$ defines an epimorphism $\bar{\psi}$: $F/N \to G\mathrm{wr}_{G_0}(A^n/B)$ that solves embedding problem $\mathcal{E}(\bar{A})$ with $\bar{A} = A^n/B$. By assumption, this cannot happen, unless $B = A^n$, as claimed. □

As a corollary to Lemma 10.3.3 we prove a sufficient condition for a closed subgroup of a semi-free profinite group to be semi-free.

PROPOSITION 10.3.4: *Let F be a semi-free profinite group of infinite rank m and let M be a closed subgroup of F. Suppose for every open normal subgroup D of F and for every finite group A there exists*

(a) *an open subgroup F_0, an open normal subgroup L, and a closed normal subgroup N of F such that*
(b) *$M \le F_0$, $L \le F_0 \cap D$, and $N \le M \cap L$; and*
(c) *no finite split embedding problem*

$$(\varphi\colon F/N \to F/L,\ \alpha\colon F/L\mathrm{wr}_{F_0/L}\bar{A} \to F/L),$$

where $\bar{A}$ is a nontrivial quotient of A^2, φ is the quotient map, and α is the projection on the first factor, is solvable.

Then M is semi-free of rank m.

Proof: By [FrJ08, Cor. 17.1.4], $\mathrm{rank}(M) \le \mathrm{rank}(F) = m$. Thus, by Corollary 10.1.8, it suffices to prove that every finite split embedding problem $\mathcal{E}_1(A) = (\mu\colon M \to G_1, \alpha_1\colon G_1 \ltimes A \to G_1)$ has m pairwise independent solutions.

To this end we choose a proper open normal subgroup D of F with $M \cap D \le \mathrm{Ker}(\mu)$. By assumption there exist subgroups F_0, L, and N as in (a) such that (b) and (c) hold. As in Lemma 10.3.3 we consider the finite split embedding problem $\mathcal{E}(A) = (\varphi\colon F \to G, \alpha\colon G\mathrm{wr}_{G_0}A \to G)$, where $G = F/L$, $G_0 = F_0/L$, φ is the quotient map, and α is the projection on the first factor.

By assumption, $\mathcal{E}(A)$ has a family Ψ of independent solutions of cardinality m. In particular, every pair of solutions in Ψ is independent. For each $\psi \in \Psi$ we consider the map $\nu = \rho \circ \pi \circ \psi|_M\colon M \to G_1 \ltimes A$, where π and ρ are as in (1). Note that Assumption (c) of our lemma implies Assumption $(*)$ of Lemma 10.3.3 with $n = 2$. Hence, by Lemma 10.3.3 in the case $n = 2$, $\{\rho \circ \pi \circ \psi|_M \mid \psi \in \Psi\}$ is a pairwise independent set of m well defined solutions of $\mathcal{E}_1(A)$, as desired. □

10.4 Closed Subgroups of Semi-free Profinite Groups

We prove in this section that the property of a profinite group F to be semi-free is inherited to each closed subgroup M which does not lie too deep in F. By that we mean that either the cardinality of all open subgroups of F that contain M is less than $\mathrm{rank}(F)$ or for each positive integer n there are only finitely many open subgroups that contain M. In particular, each open subgroup of F is semi-free. The proof of the latter statement uses the machinery of twisted wreath products developed in the preceding section. The proof of the two major statements is then reduced to the statement about open subgroups.

LEMMA 10.4.1: *Let M be an open subgroup of a semi-free profinite group F of rank m. Then M is semi-free of rank m.*

Proof: Let D be an open normal subgroup of F. We choose an open normal subgroup L of F with $L \leq M \cap D$ and set $F_0 = M$ and $N = L$. Then for each nontrivial finite group $\bar{A}$ on which F_0/L acts, the finite split embedding problem $(F/N \to F/L,\, F/L\mathrm{wr}_{F_0/L}\bar{A} \to F/L)$ has no solution because $|F/L\mathrm{wr}_{F_0/L}\bar{A}| > |F/N|$. It follows from Proposition 10.3.4 that M is semi-free of rank m. □

LEMMA 10.4.2: *Let M be a closed subgroup of a semi-free profinite group F of infinite rank m. Suppose* $\mathrm{weight}(F/M) < m$. *Then M is semi-free of rank m.*

Proof: Let $\mathcal{E}_M = (\mu\colon M \to G, \alpha\colon H \to G)$ be a finite split embedding problem with a nontrivial kernel. We use transfinite induction to construct for each $\lambda < m$ a solution ψ_λ of $\mathcal{E}_M$ such that the set $\{\psi_\lambda \mid \lambda < m\}$ of solutions is independent.

Let $M_1 = \mathrm{Ker}(\mu)$ and consider an ordinal number $\lambda < m$. Inductively suppose we have constructed an independent family $\{\psi_\kappa \mid \kappa < \lambda\}$ of solutions of $\mathcal{E}_M$. Then $\{\mathrm{Ker}(\psi_\kappa) \mid \kappa < \lambda\}$ is an M_1-independent family of open subgroups of M_1. By Definition 10.1.9, $N = \bigcap_{\kappa<\lambda} \mathrm{Ker}(\psi_\kappa)$ is a closed normal subgroup of M and $\mathrm{weight}(M/N) < m$, hence $\mathrm{weight}(F/N) < m$.

By [FrJ08, Lemma 1.2.5(c)], μ extends to an epimorphism $\varphi\colon E \to G$ for some open subgroup E of F containing M. In particular, $E_1 = \mathrm{Ker}(\varphi)$ satisfies $E_1 \cap M = M_1$. By Lemma 10.4.1, E is semi-free of rank m. Hence, the finite split embedding problem $\mathcal{E}_E = (\varphi\colon E \to G, \alpha\colon H \to G)$ has an independent family Ψ of solutions of cardinality m. Thus, $\{\mathrm{Ker}(\psi) \mid \psi \in \Psi\}$ is an E_1-independent family of open normal subgroups of E_1. Since E_1 is open in F, we have $\mathrm{weight}(E_1/N) = \mathrm{weight}(E/N) < m$. Hence by Lemma 10.1.11, there exists $\psi \in \Psi$ such that $\mathrm{Ker}(\psi)N = E_1$. Let $\psi_\lambda = \psi|_M$. By Lemma 10.1.3(d), $\mathrm{Ker}(\psi_\lambda)N = M_1$. Hence, ψ_λ is a solution of $\mathcal{E}_M$ independent of $\{\psi_\kappa \mid \kappa < \lambda\}$. This concludes the transfinite induction and proves, by Lemma 10.1.4, that $\{\psi_\lambda \mid \lambda < m\}$ is an independent set of solutions of $\mathcal{E}_M$. □

LEMMA 10.4.3: *Let M be a closed subgroup of a semi-free profinite group F of an infinite rank m. Suppose F/M is small. Then M is semi-free of rank m.*

Proof: Since $\mathrm{weight}(F/M) \leq \aleph_0$, the case where $m > \aleph_0$ is a special case of Lemma 10.4.2. Thus, we have only to prove the lemma under the additional assumption that $m = \aleph_0$.

By Proposition 10.2.4, it suffices to prove that every finite split embedding problem with a nontrivial kernel for M is solvable. Let $\mathcal{E}_M = (\mu\colon M \to G, \alpha\colon H \to G)$ be such an embedding problem.

Let $M_1 = \mathrm{Ker}(\mu)$. By [FrJ08, Lemma 1.2.5(c)], μ extends to an epimorphism $\varphi\colon E \to G$ for some open subgroup E of F containing M. In particular, $E_1 = \mathrm{Ker}(\varphi)$ is an open normal subgroup of E with $E_1 \cap M = M_1$ and $E_1 M = E$.

By Lemma 10.4.1, E is semi-free of rank $\aleph_0$. Hence, the finite split embedding problem $\mathcal{E}_E = (\varphi\colon E \to G,\ \alpha\colon H \to G)$ has an infinite independent family Ψ of solutions. Thus, $\mathcal{K} = \{\mathrm{Ker}(\psi) \mid \psi \in \Psi\}$ is an E_1-independent family of open subgroups of E_1 and each of them is normal in E. In particular, by Lemma 10.1.3, each proper open subgroup of E_1 contains at most one group belonging to $\mathcal{K}$.

Let $K \in \mathcal{K}$. Then $(KM : KM_1) = (M : M_1) = (E : E_1) = (KM : E_1 \cap KM)$, hence $KM_1 = E_1 \cap KM$. In addition, $(F : KM) \le (F : E)(E : K) = (F : E)|H|$. Since F/M is small, F has only finitely many open subgroups of the form KM with $K \in \mathcal{K}$.

Now assume the set $\mathcal{K}_0 = \{K \in \mathcal{K} \mid\ KM_1 < E_1\}$ is infinite. Then, by the preceding paragraph, there exist distinct $K_1, K_2 \in \mathcal{K}_0$ with $K_1M = K_2M$. By the preceding paragraph, $K_1M_1 = E_1 \cap K_1M = E_1 \cap K_2M = K_2M_1$. This contradicts the independency of K_1, K_2 in E_1. We conclude from that contradiction, that $\mathcal{K}_0$ is finite and choose $K \in \mathcal{K} \smallsetminus \mathcal{K}_0$. Let $\psi \in \Psi$ with $\mathrm{Ker}(\psi) = K$. Then $\psi|_M$ is a solution of $\mathcal{E}_M$. □

10.5 The Diamond Theorem

The diamond theorem for Hilbertian fields gives a convenient condition on a separable algebraic extension M of a Hilbertian field K to be Hilbertian. The condition requires the existence of Galois extensions M_1 and M_2 of K such that $M \not\subseteq M_1$, $M \not\subseteq M_2$, and $M \subseteq M_1M_2$ [FrJ08, Thm. 13.8.2]. In that case we say that M is **contained in a K-diamond**. An analog of that theorem holds for free profinite groups: Let F be a free profinite group of infinite rank m and M a closed subgroup of F that is contained in an F-diamond (Definition 10.5.2). Then M is free of rank m. The proofs of both theorems are similar, both utilize twists wreath products. It turns out that the same method applies also to semi-free groups.

We start with a technical lemma that emphasizes the incommutativity of twisted wreath products.

LEMMA 10.5.1 ([FrJ08, Lemma 13.7.4]): *Let $\alpha\colon G\mathrm{wr}_{G_0}A \to G$ be a twisted wreath product of finite groups, $H_1 \triangleleft G\mathrm{wr}_{G_0}A$, and $h_2 \in G\mathrm{wr}_{G_0}A$. Put $I = \mathrm{Ind}_{G_0}^{G}(A) = \mathrm{Ker}(\alpha)$ and $G_1 = \alpha(H_1)$. Suppose $A \neq 1$.*

(a) *Suppose $\alpha(h_2) \notin G_0$ and $(G_1G_0 : G_0) > 2$. Then there is an $h_1 \in H_1 \cap I$ with $h_1h_2 \neq h_2h_1$.*

(b) *Suppose $G_1 \not\le G_0$ and $\alpha(h_2) \notin G_1G_0$. Then there is an $h_1 \in H_1 \cap I$ with $h_1^{h_2} \notin \langle h_1\rangle^{h'}$ for all $h' \in \alpha^{-1}(G_1G_0)$. In particular, $h_1h_2 \neq h_2h_1$.*

Proof: Put $\sigma_2 = \alpha(h_2)$. Consider $\sigma_1 \in G_1$ and $g \in I$. By definition, there are $f_1, f_2 \in I$ with $\sigma_1 f_1 \in H_1$ and $h_2 = \sigma_2 f_2$. Put $h_1 = g^{\sigma_1 f_1} g^{-1}$. Then

$h_1 = [\sigma_1 f_1, g^{-1}] \in [H_1, I] \le H_1 \cap I$. For each $\tau \in G$

$$h_1(\tau) = \big((g^{\sigma_1})^{f_1}\big)(\tau) g(\tau)^{-1} = g(\sigma_1\tau)^{f_1(\tau)} g(\tau)^{-1}.$$

Hence, for all $\tau \in G$ and $f' \in I$ we have:

(1a) $h_1^{h_2}(1) = h_1^{\sigma_2 f_2}(1) = h_1(\sigma_2)^{f_2(1)} = g(\sigma_1\sigma_2)^{f_1(\sigma_2)f_2(1)} g(\sigma_2)^{-f_2(1)}$,

(1b) $h_1^{\tau f'}(1) = h_1(\tau)^{f'(1)} = g(\sigma_1\tau)^{f_1(\tau)f'(1)} g(\tau)^{-f'(1)}$, and

(1c) $h_1(1) = g(\sigma_1)^{f_1(1)} g(1)^{-1}$.

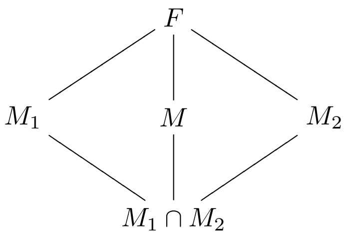
We apply (1) in the proofs of (a) and (b) to special elements σ_1 and g. Choose $a \in A$, $a \neq 1$.

Proof of (a): Since $(G_1G_0 : G_0) > 2$, there is a $\sigma_1 \in G_1$ with distinct cosets $\sigma_1^{-1}G_0, \sigma_2G_0, G_0$. Thus, none of the cosets $\sigma_1G_0, \sigma_2G_0, \sigma_1\sigma_2G_0$ is G_0. Therefore, by definition of I, there is a $g \in I$ with $g(\sigma_1) = g(\sigma_2) = g(\sigma_1\sigma_2) = 1$ and $g(1) = a$. By (1a), $h_1^{h_2}(1) = 1$. By (1c), $h_1(1) \neq 1$. Consequently, $h_1^{h_2} \neq h_1$, as desired.

Proof of (b): Since $H_1 \triangleleft G\mathrm{wr}_{G_0}A$, we have $G_1 \triangleleft G$, so $G_1^{\sigma_2} = G_1 \not\le G_0$, hence $G_1 \not\le G_0^{\sigma_2^{-1}}$. Hence, $G_1 \cap G_0$ and $G_1 \cap G_0^{\sigma_2^{-1}}$ are proper subgroups of G_1. Their union is a proper subset of G_1. Thus, there is an element $\sigma_1 \in G_1 \smallsetminus (G_0 \cup G_0^{\sigma_2^{-1}})$. It follows that $\sigma_2 \notin \sigma_1\sigma_2G_0$. By assumption, $\sigma_2 \notin G_1G_0$. Therefore, there is a $g \in I$ with $g(G_1G_0) = 1$, $g(\sigma_1\sigma_2) = 1$, and $g(\sigma_2) = a^{-1}$.

Consider $\tau \in G_1G_0$ and $f' \in I$. By (1a), $h_1^{h_2}(1) = a^{f_2(1)} \neq 1$. By (1b), $h_1^{\tau f'}(1) = 1$. Hence, $(h_1^k)^{\tau f'}(1) = 1$ for all integers k. It follows that $h_1^{h_2} \notin \langle h_1 \rangle^{h'}$ for all $h' \in \alpha^{-1}(G_1G_0)$. □

Definition 10.5.2: A closed subgroup M of a profinite group F is said to be **contained in an F-diamond** if F has closed normal subgroups M_1, M_2 such that $M_1 \cap M_2 \le M$, $M_1 \not\le M$, and $M_2 \not\le M$. The following diagram of profinite groups reveals why we have chosen that name:



□

THEOREM 10.5.3 (The diamond theorem for semi-free profinite groups): *Let F be a semi-free profinite group of infinite rank m. If a closed subgroup M of F is contained in an F-diamond, then M is semi-free of rank m.*

Proof: Let M_1, M_2 be as in Definition 10.5.2. We use Lemma 10.4.1 to assume $(F : M) = \infty$. Then we first prove the theorem under an additional assumption:

(2) Either $M_1M_2 = F$ or $(M_1M : M) > 2$.

The proof of the theorem in this case utilizes Proposition 10.3.4. It has two parts.

PART A: *Construction of L, F_0, and N.* We consider an open normal subgroup D of F, choose another open normal subgroup L of F in D, and set $F_0 = ML$. Let $G = F/L$ and φ: $F \to G$ the quotient map, $G_0 = \varphi(M) = F_0/L$, $G_1 = \varphi(M_1)$, and $G_2 = \varphi(M_2)$. Then
(3a) $G_1, G_2 \triangleleft G$.

Moreover, choosing L sufficiently small, the following holds:
(3b) $G_1, G_2 \not\leq G_0$ (use $M_1, M_2 \not\leq M$).
(3c) $(G : G_0) > 2$ (use $(F : M) = \infty$).
(3d) $G_1G_2 = G$ or $(G_1G_0 : G_0) > 2$ (use (2)).

This implies:
(4) $G_2 \not\leq G_1G_0$ or $(G_1G_0 : G_0) > 2$.

Indeed, suppose both $G_2 \leq G_1G_0$ and $G_1G_2 = G$. Then $G = G_1G_0$, so by (3c), $(G_1G_0 : G_0) > 2$.

Now let $N = L \cap M_1 \cap M_2$. Then $N \leq M$.

PART B: *An embedding problem.* Suppose G_0 acts on a nontrivial finite group $\bar{A}$ and set $H = G\mathrm{wr}_{G_0}\bar{A}$. Consider the embedding problem

$$(\varphi: F \to G, \alpha: H \to G) \tag{5}$$

where α is the quotient map. We have to prove that (5) has no solution that factors through F/N.

Assume ψ: $F \to H$ is an epimorphism with $\alpha \circ \psi = \varphi$ and $\psi(N) = 1$. For $i = 1, 2$ put $H_i = \psi(M_i)$. Then $H_i \triangleleft H$ and $\alpha(H_i) = \varphi(M_i) = G_i$.

We use (4) to find $h_1 \in H_1$ and $h_2 \in H_2$ with $\alpha(h_1) = 1$ and $[h_1, h_2] \neq 1$. First suppose $G_2 \not\leq G_1G_0$. Then there is an $h_2 \in H_2$ with $\alpha(h_2) \notin G_1G_0$. By (3b), $G_1 \not\leq G_0$, so Lemma 10.5.1(b) provides the required $h_1 \in H_1$. Now suppose $(G_1G_0 : G_0) > 2$. We use (3b) to find $h_2 \in H_2$ with $\alpha(h_2) \notin G_0$. Lemma 10.5.1(a) gives the required $h_1 \in H_1$.

Having chosen h_i, we choose $f_i \in M_i$ with $\psi(f_i) = h_i$. Then $\varphi(f_1) = \alpha(h_1) = 1$, so $f_1 \in L$. Then $[f_1, f_2] \in [L, M_2] \cap [M_1, M_2] \leq L \cap (M_1 \cap M_2) = N$. Therefore, $[h_1, h_2] = [\psi(f_1), \psi(f_2)] \in \psi(N) = 1$. This contradiction proves that ψ as above does not exist.

CONCLUSION OF THE PROOF: In the general case we use $M_1 \not\leq M$ to conclude that $(M_1M : M) \geq 2$. The case $(M_1M : M) > 2$ is covered by the special case proved above. Suppose $(M_1M : M) = 2$. Choose an open subgroup K_2 of F containing M but not M_1M. Then, $K_2 \cap M_1M = M$. Put $K = K_2M_1M$. Then $(K : K_2) = (M_1M : M) = 2$, hence $K_2 \triangleleft K$. Observe: $M_1K_2 = K$ and $K_2 \cap M_1 \leq K_2 \cap M_1M = M \leq K$. Furthermore, $K_2 \not\leq M$, because $(K_2 : M) = \infty$.

By Lemma 10.4.1, K is semi-free of rank m, so the first alternative of (2) applies with K replacing F and K_2 replacing M_2. Consequently, M is semi-free of rank m. □

THEOREM 10.5.4 (Bary-Soroker, Haran, and Harbater): *Let F be a semi-free profinite group of an infinite rank m and let M be a closed subgroup. Then M is semi-free of rank m in each of the following cases:*

(a) *M is an open subgroup of a closed subgroup M_0 of F and M_0 contains a closed normal subgroup N of F that is not contained in M (an analog of a theorem of Weissauer).*
(b) *M is a proper subgroup of finite index of a closed normal subgroup of F.*
(c) *$M \triangleleft F$ and F/M is Abelian (an analog of a theorem of Kuyk).*
(d) *$M \triangleleft F$, F/M is pronilpotent of order divisible by at least two prime numbers.*

Proof of (a): The case where M is open is taken care of by Lemma 10.4.1. Thus, we may assume that $(F:M)=\infty$ and choose an open normal subgroup M_1 of F with $M_1 \cap M_0 \le M$. In particular, $M_1 \not\le M$, $N \not\le M$, and $M_1 \cap N \le M$. It follows from the Diamond theorem 10.5.3 that M is semi-free of rank m.

Proof of (b): Take $N = M_0$ in (a).

Proof of (c): If $M = F$, there is nothing to prove. Otherwise, we choose $\sigma \in F \smallsetminus M$. Then $\langle M,\sigma\rangle/M$ is a nontrivial Abelian group. Hence, $\langle M,\sigma\rangle$ has a proper open subgroup L that contains M. By (a), L is semi-free of rank m. In addition, L/M is a procyclic group. Hence, by Proposition 10.4.3, M is semi-free of rank m.

Proof of (d): Since each Sylow subgroup of a pronilpotent group is normal, F has closed normal subgroups P_1, P_2 that properly contain M such that P_1/M is the p_1-Sylow subgroup of F/M, P_2/M is the p_2-Sylow subgroup of F/M, and $p_1 \ne p_2$. In particular, $P_1 \cap P_2 = M$. By the Diamond theorem 10.5.3, M is semi-free of rank m. □

Remark 10.5.5: One may add two more cases to the list of Theorem 10.5.4:

(e) M is a **sparse subgroup** of F. That is, for each positive integer n there exists an open subgroup K of F that contains M such that for each proper open subgroup L of K that contains M we have $[K:L] \ge n$.
(f) We have $(F:M) = \prod p^{\alpha(p)}$, where $\alpha(p) < \infty$ for each prime number p.

The reader may try to settle these cases by himself, applying Lemma 10.4.1 or consult [BHH10 Section 4]. □

Applying Proposition 10.1.14, the results we have proved so far about closed subgroups of semi-free profinite groups yield the corresponding results about closed subgroups of free profinite groups. This gives a new proof for results proved in [FrJ08, Section 25.4].

THEOREM 10.5.6: *Let F be a free profinite group of infinite rank m and let M be a closed subgroup of F. Then each of the following conditions on M suffices for M to be free of rank m:*
(a) *M is open in F.*
(b) weight$(F/M) < m$.
(c) *F/M is small.*
(d) *M is contained in an F-diamond.*
(e) *M is an open subgroup of a closed subgroup M_0 of F and M_0 contains a closed normal subgroup of F that is not contained in M.*
(f) *M is a proper subgroup of finite index of a closed normal subgroup of F.*
(g) *$M \triangleleft F$ and F/M is Abelian.*
(h) *$M \triangleleft F$ and F/M is pronilpotent of order divisible by at least two prime numbers.*
(i) *M is a sparse subgroup of F.*
(j) $(F : M) = \prod p^{\alpha(p)}$, *where $\alpha(p) < \infty$ for each prime number p.*

Proof: Since F is free, F is projective, so M is also projective [FrJ08, Prop. 22.4.7]. Thus, in order to prove that M is free of rank m, it suffices, by Proposition 10.1.14, to prove in each case that M is semi-free of rank m. By Proposition 10.1.13, F is semi-free. Therefore, (a) follows from Lemma 10.4.1, (b) follows from Lemma 10.4.2, (c) follows from Lemma 10.4.3, (d) follows from Theorem 10.5.3, (e), (f), (g), (h) follow from Theorem 10.5.4, and (i), (j) follow from Remark 10.5.5. □

Remark 10.5.7: $\mathcal{C}$-semi-free groups. Let $\mathcal{C}$ be a **Melnikov formation**, that is $\mathcal{C}$ is a family of finite groups closed under taking quotients, normal subgroups, and extensions. For example the family of all finite groups, the family of all finite p-groups, and the family of all finite solvable groups are Melnikov formations. A **$\mathcal{C}$-embedding problem** for a pro-$\mathcal{C}$ group F is an embedding problem $\mathcal{E} = (\varphi\colon F \to G,\ \alpha\colon H \to G)$ where H (hence also G) belongs to $\mathcal{C}$.

We say that a profinite group F of rank m is **$\mathcal{C}$-semi-free** if F is a pro-$\mathcal{C}$ group and each split $\mathcal{C}$-embedding problem with a nontrivial kernel has m independent solutions. □

The results about subgroups of semi-free groups yield the corresponding results about subgroups of $\mathcal{C}$-semi-free groups.

THEOREM 10.5.8: *Let $\mathcal{C}$ be a Melnikov formation of finite groups, F a $\mathcal{C}$-semi-free group of infinite rank m, and M a closed subgroup of F. Then M is $\mathcal{C}$-semi-free of rank m in each of the following cases:*
(a) *M is open in F.*
(b) weight$(F/M) < m$.
(c) *F/M is small.*
(d) *M is contained in an F-diamond.*
(e) *M is an open subgroup of a closed subgroup M_0 of F and M_0 contains a closed normal subgroup N of F not contained in M.*
(f) *M is a proper subgroup of finite index of a closed normal subgroup of F.*

(g) *$M \triangleleft F$ and F/M is Abelian.*
(h) *$M \triangleleft F$ and F/M is pronilpotent of order divisible by at least two prime numbers.*
(i) *M is a sparse subgroup of F.*
(j) *$(F : M) = \prod p^{\alpha(p)}$, where $\alpha(p) < \infty$ for each prime number p.*

Proof: Let $\mathcal{E} = (\varphi\colon M \to G, \alpha\colon H \to G)$ be a split $\mathcal{C}$-embedding problem with a nontrivial kernel. By [FrJ08, Prop. 17.4.8] there is a free profinite group $\hat{F}$ of rank m and an epimorphism $\psi\colon \hat{F} \to F$. For each closed subgroup L of M we let $\hat{L} = \psi^{-1}(L)$. In particular, $\hat{M}$ is a closed subgroup of $\hat{F}$ and $\hat{\mathcal{E}} = (\varphi \circ \psi|_{\hat{M}}\colon \hat{M} \to G, \alpha\colon H \to G)$ is a finite split embedding problem for $\hat{M}$. Moreover, $\mathrm{Ker}(\varphi \circ \psi|_{\hat{M}}) = \psi^{-1}(\mathrm{Ker}(\varphi))$.

We observe that in each of the cases (a)–(j), $\hat{M}$ satisfies the corresponding condition that M satisfies. Hence, by Theorem 10.5.6, the group $\hat{M}$ is semi-free of rank m. Hence, $\hat{\mathcal{E}}$ has m-independent solutions. The corresponding kernels $\hat{M}_i$ are normal subgroups of $\hat{M}$ such that $\hat{M}/\hat{M}_i$ are isomorphic to the $\mathcal{C}$-group H. It follows that those solutions define m-independent solutions to $\mathcal{E}$. Consequently, M is $\mathcal{C}$-semi-free of rank m. □

10.6 Quasi-Free Profinite Groups

If we drop the condition of the independence of the solutions from the definition of semi-free profinite group, we get the notion of "quasi-free profinite group" introduced in [HaS05]. In this section we introduce the latter notion and discuss its advantages and its limits.

Let G be a profinite group of infinite rank m. We say that G is **quasi-free** if every finite split embedding problem $\mathcal{E}$ for G with a nontrivial kernel has at least m solutions. In particular, every embedding problem of the form $(G \to 1, A \to 1)$, where A is a finite group is solvable, so A is a quotient of G. Unlike for semi-free profinite groups, we do not insist that the solutions will be independent. Still, by [FrJ08, Lemma 25.1.8], if G is projective and quasi-free of rank m, then G is free. By [RSZ07, Thm. 2.1], every open subgroup of a quasi-free profinite group is quasi-free. If G is quasi-free of rank m, then so is its commutator group [Hrb09, Thm. 2.4]. By [HaS05, Thm. 5.1], the absolute Galois group of the field $K((t_1, t_2))$ of formal power series in two variables over an arbitrary field K is quasi-free of rank equal to $\mathrm{card}(K)$. It follows that $\mathrm{Gal}(K((t_1, t_2))_{\mathrm{ab}})$ is quasi-free. If K is algebraically closed, then $\mathrm{Gal}(K((t_1, t_2))_{\mathrm{ab}})$ is projective [Hrb09, Thm. 4.4]. It follows that in this case $\mathrm{Gal}(K((t_1, t_2))_{\mathrm{ab}})$ is free of rank equal to $\mathrm{card}(K)$. Finally, Harbater proves that if K is ample, then $\mathrm{Gal}(K(x))$ is quasi-free [Hrb09, Thm. 3.4].

In view of Example 10.6.1 below, the latter result is weaker than Theorem 11.7.1 saying that $\mathrm{Gal}(K(x))$ is semi-free if K is ample. In addition, we have been able to prove the diamond theorem only for semi-free profinite groups but not for quasi-free profinite groups. Thus, using only the concept of quasi-free profinite groups, we would not be able to prove that the

K-radical extension of $K(x)$ (with K being PAC) that we have constructed in Theorem 11.7.6 has a free absolute Galois group. Finally, in contrast to semi-free groups (Theorem 10.5.8(b)), a closed subgroup N of a quasi-group G with weight$(G/N) <$ rank(N) need not be quasi-free, as Lemma 10.6.4 below demonstrates.

Example 10.6.1: (Bary-Soroker, Haran, Harbater) Example of a quasi-free profinite group that is not semi-free. Let X be a set of uncountable cardinality m and let $C = \prod_p \mathbb{Z}/p\mathbb{Z}$ be the direct product of all cyclic groups of prime order. For each $x \in X$ let C_x be an isomorphic copy of C. We consider the free product $E = \coprod^{*}_{x\in X} C_x$ in the sense of [BNW71]. Thus, each C_x is a closed subgroup of E and every family of homomorphisms $\psi_x\colon C_x \to \bar{C}$ into a finite group A, such that $\psi_x(C_x) = 1$ for all but finitely many $x \in X$, uniquely extends to a homomorphism $\psi\colon E \to \bar{C}$. Let $G = \coprod^{*}_{x\in X} C_x * \hat{F}_\omega$.

Claim A: *G is quasi-free of rank m.* The rank of $\coprod^{*}_{x\in X} C_x$ is m and the rank of $\hat{F}_\omega$ is $\aleph_0 < m$. Hence, rank$(G) = m$. Let

$$(1) \qquad (\varphi\colon G \to A,\ \alpha\colon B \to A)$$

be a finite split embedding problem with a nontrivial kernel and let $\alpha'\colon A \to B$ be its splitting. We need two auxiliary maps: First, there exists a nontrivial homomorphism $\pi\colon C \to \mathrm{Ker}(\alpha)$; namely, an epimorphism of C onto a subgroup of $\mathrm{Ker}(\alpha)$ of prime order. Secondly, there exists an epimorphism $\psi'\colon \hat{F}_\omega \to \alpha^{-1}(\varphi(\hat{F}_\omega))$ such that $\alpha\circ\psi'$ is the restriction of φ to $\hat{F}_\omega$ [FrJ08, Thm. 24.8.1]. In particular, $\psi'(\hat{F}_\omega)$ contains $\mathrm{Ker}(\alpha)$.

The definition of $\coprod^{*}_{x\in X} C_x$ gives a subset Y of X such that $X \smallsetminus Y$ is finite and $\varphi(C_x) = 1$ for every $x \in Y$. For every $y \in Y$ we define a homomorphism $\psi_y\colon G \to B$ in the following manner:

$$\begin{aligned} \psi_y|_{C_y} &= \pi \\ \psi_y|_{C_x} &= 1 \text{ if } x \in Y \text{ and } x \neq y \\ \psi_y|_{C_x} &= \alpha' \circ \varphi \text{ if } x \in X \smallsetminus Y \\ \psi_y|_{\hat{F}_\omega} &= \psi' \end{aligned}$$

Then, $\alpha\circ\psi_y = \varphi$. Since $\psi_y(G) \geq \psi'(\hat{F}_\omega) \geq \mathrm{Ker}(\alpha)$, the map ψ_y is a solution of (1).

Since $\psi_{y_1} \neq \psi_{y_2}$ for distinct $y_1, y_2 \in Y$, (1) has at least $|Y| = m$ distinct solutions. Thus, G is quasi-free of rank m.

Claim B: *G is not semi-free.* Consider the finite split embedding problem

$$(2) \qquad (\varphi\colon G \to 1,\ \mathbb{Z}/4\mathbb{Z} \to 1)$$

with the nontrivial kernel $\mathbb{Z}/4\mathbb{Z}$. Let Ψ be an independent set of solutions of (2). The map $\mathbb{Z}/4\mathbb{Z} \to 1$ decomposes into $\alpha\colon \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ and $\beta\colon \mathbb{Z}/2\mathbb{Z} \to 1$.

If $\psi_1, \psi_2 \in \Psi$ are independent, then $\alpha\circ\psi_1, \alpha\circ\psi_2$ are independent solutions of $(\varphi\colon G \to 1,\ \beta\colon \mathbb{Z}/2\mathbb{Z} \to 1)$ (Lemma 10.1.3(f)). In particular, $\alpha\circ\psi_1 \neq \alpha\circ\psi_2$. Thus, $\{\alpha\circ\psi \mid \psi\in\Psi\}$ has at least the cardinality of Ψ.

On the other hand, $\mathbb{Z}/4\mathbb{Z}$ is a 2-group and the 2-Sylow subgroup of C is of order 2. Hence, every $\psi\in\Psi$ maps each C_x into $\mathrm{Ker}(\alpha)$, the unique subgroup of $\mathbb{Z}/4\mathbb{Z}$ of order 2, so $\alpha\circ\psi$ is trivial on C_x. Therefore $\alpha\circ\psi$ is trivial on $\mathop{\ast}_{x\in X} C_x$. It follows that $\alpha\circ\psi$ is determined by its restriction to $\hat F_\omega$. But there are only $\aleph_0$ homomorphisms $\hat F_\omega \to \mathbb{Z}/4\mathbb{Z}$. Therefore, $\mathrm{card}(\Psi)\le\aleph_0$. □

In order to prove the last piece of information about the group G of Example 10.6.1, we need a basic lemma about free products of two profinite groups.

LEMMA 10.6.2: *Let A and B be profinite groups, $A*B$ their free product, and $\pi\colon A*B\to B$ the homomorphism defined by $\pi(a)=1$ for $a\in A$ and $\pi(b)=b$ for $b\in B$. Then, $A*B = B\ltimes \mathrm{Ker}(\pi)$ and $\mathrm{Ker}(\pi)=\langle A^b \mid b\in B\rangle$.*

Proof: Let $K=\langle A^b\mid b\in B\rangle$. Then K is a closed normal subgroup of $A*B$ and $K\le \mathrm{Ker}(\pi)$.

If $a_1,\ldots,a_n\in K$ and $b_1,\ldots,b_n\in B$, then

$$a_1b_1a_2b_2\cdots a_nb_n = b_1(a_1^{b_1}a_2)b_2\cdots a_nb_n$$

and $a_1^{b_1}a_2\in K$. Induction on n gives a $k\in K$ such that

$$(a_1^{b_1}a_2)b_2\cdots a_nb_n = b_2\cdots b_nk.$$

Hence, $a_1b_1\cdots a_nb_n = bk$ with $b=b_1\cdots b_n$.

Now let $g\in A*B$ and consider an open normal subgroup N of $A*B$. Since A and B generate $A*B$ there are $a_1,\ldots,a_n\in A$ and $b_1,\ldots,b_n\in B$ such that $g\equiv a_1b_1\cdots a_nb_n \mod N$. By the preceding paragraph, $g\in BKN$. Intersecting on all possible N gives $g\in BK$ [FrJ08, Lemma 1.2.2(b)]. Thus, $A*B=BK$.

If $g\in\mathrm{Ker}(\pi)$, then writing $g=bk$ with $b\in B$ and $k\in K$ and applying π we get that $1=b$, so $g=k\in K$. Therefore, $K=\mathrm{Ker}(\pi)$.

That $A*B=B\ltimes K$ follows now from the observation that $B\cap K=1$. □

Remark 10.6.3: It is further proved in [HJP09, Lemma 2.3] that $\mathrm{Ker}(\pi)$ in Lemma 10.6.2 is isomorphic to the free product $\mathop{\ast}_{b\in B} A^b$ in the sense of Melnikov [Mel90]. However, we do not need here that extra information. □

LEMMA 10.6.4: *Let $G=\mathop{\ast}_{x\in X} C_x * \hat F_\omega$ be as in Example 10.6.1. Suppose $m>\aleph_0$. Then G has a closed normal subgroup K such that* $\mathrm{weight}(G/K)<\mathrm{rank}(G)$ *but K is not quasi-free.*

Proof: Let K be the kernel of the projection of G onto $\hat F_\omega$, mapping each element of $E=\mathop{\ast}_{x\in X} C_x$ onto 1 and each element of $\hat F_\omega$ onto itself. Then

$G/K \cong \hat{F}_\omega$, so $\mathrm{weight}(G/K) = \mathrm{rank}(\hat{F}_\omega) = \aleph_0 < m = \mathrm{rank}(G)$. On the other hand, by Lemma 10.6.2, $K = \langle E^b \mid b \in \hat{F}_\omega \rangle$. Thus, K is generated by elements of prime order. Therefore, every finite quotient of K is generated by elements of prime order. In particular, $\mathbb{Z}/4\mathbb{Z}$ is not a quotient of K. Consequently, K is not quasi-free. □

Remark 10.6.5: Let K be an ample Hilbertian field. By Theorem 5.10.2(a), every finite split embedding problem for $\mathrm{Gal}(K)$ is solvable. Hence, if K is countable, or more generally, if $\mathrm{rank}(\mathrm{Gal}(K)) = \aleph_0$, then $\mathrm{Gal}(K)$ is semi-free (Proposition 10.2.4). It follows from Theorem 10.5.8 that $\mathrm{Gal}(K')$ is semi-free in each of the following cases:
(3a) K' is a finite separable extension of K,
(3b) K' is a **small separable algebraic extension** of K (i.e. for each n there are only finitely many extensions of K in K' of degree at most n),
(3c) K' is contained in a K-diamond,
(3d) K' is a finite proper separable extension of a Galois extension of K, and
(3e) K' is an Abelian extension of K. □

Following Remark 10.6.5, it is tempting to conjecture that $\mathrm{Gal}(K)$ is semi-free if K is ample and Hilbertian. However, as Example 10.6.7 shows, this is not the case.

Example 10.6.6: A projective non-semi-free profinite group N for which each finite embedding problem is solvable.

Let m be an uncountable cardinal number and set $F = \hat{F}_m$. By [FrJ08, Prop. 25.7.7], F has a closed normal subgroup N that has m N-independent open normal subgroups M with $N/M \cong \mathbb{Z}/p\mathbb{Z}$ for each prime number p but only $\aleph_0$ N-independent open normal subgroups M with $N/M \cong S$ for each non-Abelian simple finite group S. By [FrJ08, Lemma 25.7.1], N is not free. As a closed subgroup of a free profinite group, N is projective [FrJ08, Cor. 22.4.6]. Hence, by Proposition 10.1.14, N is not semi-free. We prove that every finite embedding problem

$$(4) \qquad (\varphi\colon N \to A,\ \alpha\colon B \to A)$$

for N is solvable. By induction on the order of $C = \mathrm{Ker}(\alpha)$, we may suppose that C is a minimal normal subgroup of B (see the proof of [FrJ08, Lemma 25.1.4]).

Let $N_1 = \mathrm{Ker}(\varphi)$. By [FrJ08, Lemma 1.2.5], F has an open normal subgroup F_0 such that $N \cap F_0 \le N_1$. Then $K = NF_0$ is an open normal subgroup of F and φ extends to a homomorphism $\kappa\colon K \to A$ by $\kappa(nf_0) = \varphi(n)$ for $f_0 \in F_0$ and $n \in N$.

By [FrJ08, Proposition 17.6.2], K is free of rank m. Hence, there exists an epimorphism $\theta\colon K \to B$ with $\alpha \circ \theta = \kappa$ [FrJ08, Lemma 25.1.2]. Set $K_1 = \mathrm{Ker}(\kappa)$ and $K_2 = \mathrm{Ker}(\theta)$. Then $N \cap K_1 = N_1$, $NK_1 = K$, $K/K_2 \cong B$, and $K_1/K_2 \cong C$. In particular, N_1 and K_2 are normal in K. Hence, N_1K_2/K_2 is a normal subgroup of K/K_2 which is contained in K_1/K_2. The latter group is minimal normal in K/K_2 (because C is minimal normal in B), so either $N_1K_2 = K_1$ or $N_1 \le K_2$.

CASE 1: $N_1K_2 = K_1$. Then $NK_2 = K$. Hence, $\theta(N) = \theta(K) = B$ and $\theta|_N: N \to B$ solves embedding problem (4).

CASE 2: $N_1 \le K_2$. Then $L = NK_2$ is normal in K, $L/K_2 \cong N/N_1 \cong A$, and $L \cap K_1 = K_2$. Thus,

$$B \cong K/K_2 \cong L/K_2 \times K_1/K_2 \cong A \times C.$$

Since C is a minimal normal subgroup of B, it is isomorphic to a direct product $\prod_{i=1}^r S_i$ of isomorphic copies of a single finite simple group S [FrJ08, Remark 16.8.4]. By assumption, N has infinitely many N-independent open normal subgroups M with $N/M \cong S$. Hence, by Lemma 10.1.12, N has open normal subgroups $M_1, \ldots, M_r$ with $N/M_i \cong S$, $i = 1, \ldots, r$, and $M_1, \ldots, M_r, N_1$ are N-independent. Let $M = \bigcap_{i=1}^r M_i$ and $N_2 = M \cap N_1$. Then $N/M \cong S^r \cong C$, $N/N_2 \cong N/M \times N/N_1 \cong C \times A \cong B$ and the quotient map $\gamma: N \to N/N_2$ solves embedding problem (4). □

Example 10.6.7: A Hilbertian ample field K with a non-semi-free absolute Galois group. Let N be the profinite group given by Example 10.6.6. In particular, N is projective. Hence, by Lubotzky-v.d.Dries, there exists a PAC field K with $\mathrm{Gal}(K) \cong N$ [FrJ08, Cor. 23.1.2]. In particular, K is ample (Example 5.6.1). Since every finite embedding problem for N is solvable, K is ω-free (Section 5.10). By Roquette, K is Hilbertian [FrJ08, Cor. 27.3.3]. Finally, by Example 10.6.6, $\mathrm{Gal}(K)$ is not semi-free. □

Remark 10.6.8: Although the absolute Galois group of an arbitrary Hilbertian ample field F need not be semi-free, there are many cases where $\mathrm{Gal}(F)$ is semi-free and F is uncountable. See Theorem 12.4.1 and Example 12.4.4. □

Remark 10.6.9: Non quasi-free fundamental groups. Let E be a function field of one variable over an algebraically closed field C of positive characteristic and S a finite nonempty set of prime divisors of E/C. Then $\mathrm{Gal}(E_S/E)$ is not quasi-free. Otherwise, since $\mathrm{Gal}(E_S/E)$ is projective (Theorem 9.5.7), [FrJ08, Lemma 25.1.8] will imply that $\mathrm{Gal}(E_S/E)$ is free. This will contradicts Proposition 9.9.4. □

Notes

The notions of independent subgroups of a profinite group and of twist fiber products of several finite groups is used in [BHH10] in order to improve the criterion developed by Haran in his proof of the diamond theorem for profinite groups. While [FrJ08, Prop. 24.14.1] which reconstruct that proof gives a criterion for a closed subgroup M of a profinite group F to have all finite split embedding problems solvable once the same holds for F, Lemma

10.3.3 gives a criterion for M to have independent solutions of those problems once F has independent solutions.

We have placed the group G on the left side of the twisted wreath product and A on its right side in order to be consistent with the placement of the factors in the semidirect product $G \ltimes \mathrm{Ind}_{G_0}^{G}(A)$. Note however that this is inconsistent with the notation we use in [FrJ08], where the same group is denoted by $A \mathrm{wr}_{G_0} G$.

Most of Sections 10.1 – 10.5 is a workout of some parts of [BHH10].

Example 10.6.1 is a workout of [BHH10, Prop. 6.1].

The concept "sparse subgroup" of a profinite group F is introduced in [BSo06] in order to prove the diamond theorem for free profinite groups of finite rank.