Modeling and Simplifying Morse Complexes in Arbitrary Dimensions

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Abstract. Ascending and descending Morse complexes, defined by a scalar function f over a manifold domain M, decompose M into regions of influence of the critical points of f, thus representing the morphology of the scalar function f over M in a compact way. Here, we introduce two simplification operators on Morse complexes which work in arbitrary dimensions and we discuss their interpretation as n-dimensional Euler operators. We consider a dual representation of the two Morse complexes in terms of an incidence graph and we describe how our simplification operators affect the graph representation. This provides the basis for defining a multi-scale graph-based model of Morse complexes in arbitrary dimensions.

1 Introduction

The problem of representing morphological information extracted from discrete scalar fields is a relevant issue in several applications, such as terrain modeling and volume data analysis and visualization. The increasing availability of time-varying volume data sets and the need of extracting knowledge from such data sets makes it important to have a morphological representation of such fields. Time-varying volume data sets are often viewed as four-dimensional scalar fields and 4D models of such data sets based on hypercubic or simplicial meshes have been developed in the literature [3,27]. Morse theory offers a natural and intuitive way of analyzing the structure of a scalar field as well as of compactly representing a decomposition of its domain into meaningful regions associated with critical points of the field.

Discrete scalar fields are defined by a finite set of points in a domain D in $\mathbb{R}^n$. At each of these points a value of a scalar function f is given. Traditionally, discrete scalar fields are described by decomposing their domain into cells, on which an interpolating function is defined based on discrete function values given at the vertices of the cells. This geometry-based description provides an accurate representation of a scalar field, but it fails in capturing the morphological structure of the field, which is defined by its critical points and integral lines.

Based on Morse theory, subdivisions of a manifold M, induced by a function f defined over it, have been defined as suitable representations for analyzing the topology of M and the behavior of f over M. The ascending and descending Morse complexes are defined by considering the integral lines emanating from, or converging to, the critical points of f. The Morse-Smale complex describes the subdivision of M into parts, characterized by uniform flow of the gradient between two critical points of f. Here, we consider a representation, the incidence graph, that encodes both the ascending and descending Morse complexes which is dimension-independent, is based on encoding the incidence relations of the cells of the two complexes, and exploits the duality between the two complexes. The incidence graph can be effectively combined with a representation of the simplicial decomposition of the underlying domain M, in the discrete case.

Structural problems in Morse and Morse-Smale complexes, like over-segmentation in the presence of noise, or efficiency issues arising due to the very large size of the input data sets, can be faced and solved by defining simplification operators on those complexes and on their morphological representations. Morse and Morse-Smale complexes can be simplified by applying an operator called (general) *cancellation* of critical points, which eliminates pairs of critical points of f with consecutive index. A cancellation which does not involve a maximum or a minimum of f increases the number of pairs of cells in the Morse complexes which become incident to each other, and, counter-intuitively, it introduces new cells in the Morse-Smale complex. This motivated us to introduce two simplification operators on n-dimensional Morse complexes by imposing some constraints on a general cancellation. These operators do not enlarge the incidence relation on the Morse complexes, they do not introduce new cells in the Morse-Smale complex, and can be viewed as merging of cells in the Morse complexes. We show that the two simplification operators can be interpreted as Euler operators, and we show through an example how the general cancellation operator can be realized as a sequence of our elementary simplification operators. We discuss the effect of the two simplification operators on the incidence-based representation of the two Morse complexes. These simplification operators, together with their inverse refinement operators, are the basis for generating a dimension-independent multi-scale representation of Morse complexes, based on a hierarchy of incidence graphs. The inverse operators are not discussed here for brevity.

The remainder of the paper is organized as follows. In Section 2, we review some basic notions on Morse theory. In Section 3, we discuss some related work. In Section 4, we briefly describe the dual representation of the two Morse complexes. In Section 5, we define two simplification operators on Morse complexes, and in Section 6, we describe the effect of these operators on their incidence-based representation. Finally, in Section 7, we draw some concluding remarks.

2 Morse Theory

We review here some basic notions of Morse theory. For more details, see [20].

Let f be a C^2 real-valued function defined over a closed compact n-manifold M. A point p is a *critical point* of f if and only if the gradient $\nabla f = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})$ (in some local coordinate system around p) of f vanishes at p. Function f is a *Morse function* if all its critical points are non-degenerate (the Hessian matrix $Hess_p f$ of the second derivatives of f at p is non-singular). The number of negative eigenvalues of $Hess_p f$ is called the *index* of critical point p, and p is called an *i-saddle*. A 0-saddle, or an

n-saddle, is also called a *minimum*, or a *maximum*, respectively. An *integral line* of f is a maximal path which is everywhere tangent to the gradient of f. Each integral line connects two critical points of f, called its *origin* and *destination*. If m_i is the number of i-saddles of a Morse function f, and $\chi(M)$ is the Euler characteristic of M, then $\chi(M) = \sum_{i=0}^{n}(-1)^i m_i$.

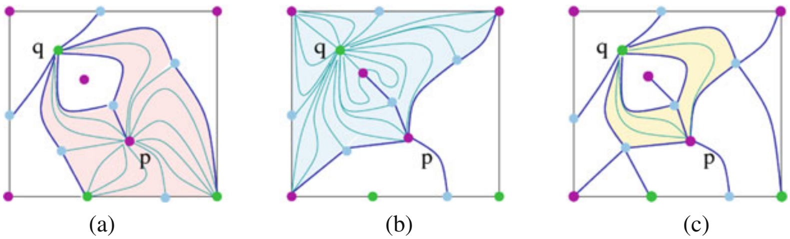


Fig. 1. (a) A 2D ascending Morse complex. The ascending cell of minimum p is shaded. (b) A dual descending Morse complex. The descending cell of maximum q is shaded. (c) Morse-Smale complex. 2-cells related to minimum p and maximum q are shaded.

Integral lines that converge to (originate from) a critical point p of index i form an i-cell ($(n-i)$-cell) called a *descending (ascending) cell* of p. The descending (ascending) cells decompose M into a Euclidean cell complex, called a *descending* (*ascending*) *Morse complex*, denoted as Γ_d (Γ_a). A Morse function f is called a *Morse-Smale function* if and only if the descending and the ascending cells intersect transversally. The connected components of the intersection of descending and ascending cells of a Morse-Smale function f decompose M into a *Morse-Smale complex*. We illustrate the correspondence between Morse and Morse-Smale complexes in Figure 1. If f is a Morse-Smale function, then the ascending complex Γ_a of f and the descending complex Γ_d of f are dual to each other. In this work, we restrict our consideration to Morse-Smale functions satisfying an additional condition: in the descending (and ascending) Morse complexes, each i-cell is bounded by at least one $(i-1)$-cell, $1 \leq i \leq n$, and each i-cell bounds at least one $(i+1)$-cell, $0 \leq i \leq n-1$.

3 Related Work

In this Section, we review the state of the art on morphological representation of scalar fields, focusing on algorithms which assume a discretization of the domain of the field as a manifold simplicial complex.

There have been two attempts in the literature to discretize Morse theory, either by developing its discrete version, called *Forman theory* [13], by considering functions defined on all cells, and not only on vertices, of a cell complex Γ, or by representing the combinatorial structure of Morse-Smale complexes in 2D and 3D by *quasi-Morse complexes* [11, 12]. This latter has been the basis for algorithms for computing discrete counterparts of the Morse-Smale complex.

The extraction of critical points of a scalar field f defined on a simplicial mesh has been investigated in 2D [2, 21], and in 3D [11, 14, 24–26], as a basis for computing Morse and Morse-Smale complexes. Algorithms for decomposing the domain D of f into an approximation of a Morse, or of a Morse-Smale complex in 2D can be classified as *boundary-based* [1, 5, 12, 22, 23], or *region-based* [6, 8, 18]. In [11], an algorithm for extracting the Morse-Smale complex from a tetrahedral mesh is proposed. In [8], a region-based dimension-independent algorithm is proposed, which subdivides the domain of f into an approximation of Morse complexes, by processing the points according to sorted function values. For a review of the work in this area, see [4].

One of the major issues that arise when computing a representation of a scalar field as a Morse, or as a Morse-Smale complex is the over-segmentation due to the presence of noise in the data sets. *Simplification algorithms* have been developed in order to eliminate less significant features from the Morse-Smale complex. Simplification is achieved by applying an operator, called *cancellation* of critical points. In 2D Morse-Smale complexes, cancellation operator has been investigated in [5, 12, 23, 28]. Cancellation operators on Morse and Morse-Smale complexes of a 3D scalar field have been investigated in [7, 15].

4 A Dual Representation for Morse Complexes

In this Section, we briefly describe a dual combinatorial representation for both the ascending and descending Morse complexes based on the *incidence graph* $G = (N, A)$ [10]. The nodes of G are in one-to-one correspondence with the critical points of f. We call a node of G representing an i-saddle of f a node at *level* i. Thus an i-level node in G represents an i-cell of the descending Morse complex Γ_d and an $(n-i)$-cell of the ascending Morse complex Γ_a. Values of the scalar field are attached to the nodes of the incidence graph as well as the level of the node. The direct incidence relations between cells in Γ_d (and thus also in Γ_a) are encoded as arcs. Arcs connect pairs of nodes which differ in level by 1 and represent integral lines connecting the corresponding critical points. An arc exists between an i-level node p and an $(i+1)$-level node q if and only if i-cell p is on the boundary of $(i+1)$-cell q in the descending complex Γ_d (and thus $(n-i)$-cell p is bounded by $(n-i-1)$-cell q in the ascending complex Γ_a). Each arc connecting an i-level node p to an $(i+1)$-level node q is labeled by the number of times the corresponding i-cell p and $(i+1)$-cell q in Γ_d are incident to each other [17]. Note that incidence graph can also be seen as a combinatorial representation of (the 1-skeleton of) a Morse-Smale complex.

In the discrete case, the underlying domain of the field is decomposed into a simplicial mesh, and for this a very compact representation is an indexed data structure with adjacencies, as shown in [9], which encodes only the vertices and the n-simplexes in the n-dimensional simplicial complex and each n-simplex is encoded as the list of the indexes of its $n+1$ vertices. Thus, the storage requirement is equal to n floats for each vertex of the mesh and $n+1$ integers for each n-simplex. We can obtain a combined morphological and geometrical representation by attaching to each n-level node p of the incidence graph, which corresponds to a maximum and to an n-cell p in the descending Morse complex, the set of n-simplexes whose union gives the n-cell p, and, symmetrically, to each 0-level node q of the incidence graph, which corresponds to a minimum and to an n-cell q in the ascending Morse complex, the set of n-simplexes whose union gives the n-cell q. A data structure for 3D Morse-Smale complexes described in [16] represents the connectivity of the complex as an incidence graph, and attaches geometrical information referring to the underlying simplicial decomposition to all nodes of the complex, thus encoding the geometry of 1- 2- and 3-cells in both the ascending and descending Morse complexes. This representation supports efficient traversal, but it is dimension-specific.

5 Removal and Contraction on Morse Complexes

The effect of a cancellation of a pair of critical points, as defined in Morse theory, has been investigated in 3D on Morse-Smale [15] and on Morse complexes [7]. An i-saddle p and an $(i+1)$-saddle q can be cancelled if there is a unique integral line connecting them. After a cancellation, each cell r which was on the boundary of $(i+1)$-cell q in Γ_d becomes incident to each cell t which was in the co-boundary of i-cell p in Γ_d. If a cancellation does not involve an extremum, the number of pairs of cells in the Morse complexes which become incident to each other increases, although the number of cells decreases by two, and the number of cells in the Morse-Smale complex increases, although the number of vertices decreases by two. In the current approaches, after a cancellation of a 1-saddle and a 2-saddle in 3D, additional cancellations of maxima and 2-saddles, or minima and 1-saddles, are applied to eliminate the new cells in the Morse-Smale complexes created by the cancellation [15].

Here, we define two operators, which we call *removal* and *contraction*. They are defined in arbitrary dimensions, and are obtained by imposing additional constraints on a cancellation. They do not enlarge the incidence relation on the Morse complexes, they do not introduce new cells in the Morse-Smale complex, and they can be viewed as merging of cells in the Morse complexes.

A *removal* (of index i) of an i-saddle σ_i and an $(i+1)$-saddle σ_{i+1} is defined if σ_i is connected by a unique integral line to

- $(i+1)$-saddle σ_{i+1} and exactly one $(i+1)$-saddle σ'_{i+1} different from σ_{i+1}, or
- exactly one $(i+1)$-saddle σ_{i+1}.

In the first case, a removal of σ_i and σ_{i+1} is denoted as $rem(\sigma_{i+1}, \sigma_i, \sigma'_{i+1})$, and in the second case as $rem(\sigma_{i+1}, \sigma_i, \emptyset)$. In both cases, a removal is specified only by σ_i and σ_{i+1}. We include σ'_{i+1} in the notation to emphasize the condition for the feasibility of the operator, and to highlight the cells merged by it.

After a removal $rem(\sigma_{i+1}, \sigma_i, \sigma'_{i+1})$ of an i-saddle σ_i and an $(i+1)$-saddle σ_{i+1}, i-cell σ_i is deleted in Γ_d, and $(i+1)$-cell σ_{i+1} is merged into $(i+1)$-cell σ'_{i+1}. All cells which were on the boundary of $(i+1)$-cell σ_{i+1} before a removal (except i-cell σ_i) are on the boundary of $(i+1)$-cell σ'_{i+1} after the removal. An example of the effect of a removal $rem(\gamma, \beta, \gamma')$ on a 3D descending Morse complex is illustrated in Figure 2 (a). After the removal, 1-cell β is deleted, and 2-cell γ is merged with the unique 2-cell γ' incident in β and different from γ.

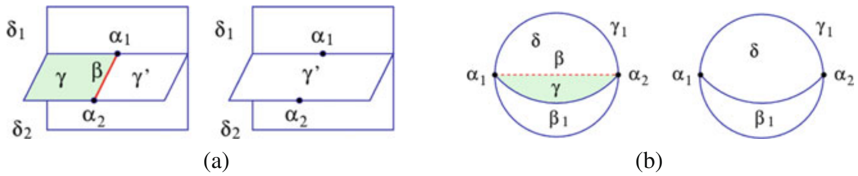


Fig. 2. Part of a 3D descending Morse complex before and after a removal (a) $rem(\gamma,\beta,\gamma')$, and (b) $rem(\gamma,\beta,\emptyset)$.

After a removal $rem(\sigma_{i+1},\sigma_i,\emptyset)$, i-cell σ_i and $(i+1)$-cell σ_{i+1} are deleted in Γ_d. An example of the effect of a removal $rem(\gamma,\beta,\emptyset)$ on a 3D descending Morse complex is illustrated in Figure 2 (b). 1-cell β is incident to exactly one 2-cell γ. 2-cell γ is bounded by 1-cells β and β_1. 3-cell δ is the only 3-cell in the co-boundary of γ. After the removal, cells β and γ are deleted, and the boundary and the co-boundary of all other cells remain unchanged.

Dually, a *contraction* (of index $i+1$) of an $(i+1)$-saddle σ_{i+1} and an i-saddle σ_i is defined if σ_{i+1} is connected by a unique integral line to

- i-saddle σ_i, and exactly one i-saddle σ_i' different from σ_i, or
- exactly one i-saddle σ_i.

In the first case, a contraction of σ_i and σ_{i+1} is denoted as $con(\sigma_i,\sigma_{i+1},\sigma_i')$, and in the second case as $con(\sigma_i,\sigma_{i+1},\emptyset)$.

The effect of a contraction of index i on Γ_d (Γ_a) is the same as the effect of a removal of index $n-i$ on Γ_a (Γ_d). After a contraction $con(\sigma_i,\sigma_{i+1},\sigma_i')$, $(i+1)$-cell σ_{i+1} in Γ_d is deleted, i-cell σ_i is merged into i-cell σ_i' and all cells which were in the co-boundary of i-cell σ_i before a contraction (except $(i+1)$-cell σ_{i+1}) are in the co-boundary of i-cell σ_i' after the contraction (i-cell σ_i is deleted, and each $(i+1)$-cell in the co-boundary of σ_i is extended to include a copy of $(i+1)$-cell σ_{i+1}). An example of the effect of a contraction $con(\beta,\gamma,\beta')$ on a 3D descending Morse complex is illustrated in Figure 3 (a). The two 1-cells β and β' are merged, and 2-cell γ is deleted.

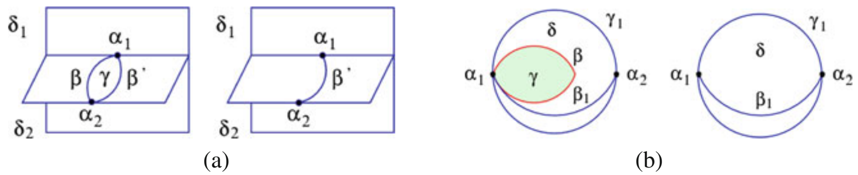


Fig. 3. Part of a 3D descending Morse complex before and after a contraction $con(\beta,\gamma,\beta')$ (a), and $con(\beta,\gamma,\emptyset)$ (b).

After a contraction $con(\sigma_i,\sigma_{i+1},\emptyset)$, $(i+1)$-cell σ_{i+1} and i-cell σ_i are deleted in Γ_d. An example of the effect of a contraction $con(\beta,\gamma,\emptyset)$ on a 3D descending Morse complex is illustrated in Figure 3 (b). 1-cell β is the only 1-cell on the boundary of

2-cell γ. The only 3-cell in the co-boundary of 2-cell γ is δ. After the contraction, 1-cell β and 2-cell γ are deleted.

It is possible to give a unifying definition of the two operators as a special case of a cancellation when one of the two canceled critical points satisfies certain valence conditions in the incidence graph. We define the two operators separately, due to the different (dual) geometric effect they have on the Morse complexes.

When $i = 0$ (and dually when $i = n$), a removal $rem(\sigma_1, \sigma_0, \sigma_1')$, which is defined when a 0-cell σ_0 is incident once to exactly two different 1-cells σ_1 and σ_1' in Γ_d, has the same geometric effect as a contraction $con(\sigma_0, \sigma_1, \sigma_0')$, where σ_0' is the other 0-cell on the boundary of 1-cell σ_1. For example, removal $rem(\beta, \alpha, \beta')$ illustrated in Figure 4 (a) can also be viewed as contraction $con(\alpha, \beta, \alpha')$. Note that the result of a removal or contraction may be a complex which does not satisfy the condition stated in Section 2, as illustrated in Figure 4 (b).

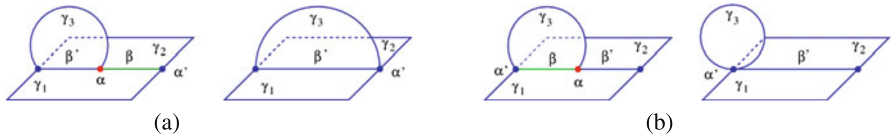


Fig. 4. Part of a 3D descending Morse complex before and after a removal $rem(\beta, \alpha, \beta')$ (or a contraction $con(\alpha, \beta, \alpha')$). γ_3 is a bubble-like 2-cell bounded by one 1-cell. (a) Operator is allowed. (b) Operator is not allowed.

Both removal and contraction can be interpreted as Euler operators, since they cancel a pair of cells of consecutive dimension in the two Morse complexes. Thus, Euler formula is satisfied after each simplification. The two operators are instances of the same Euler operator Kill_i-Cell_and_$(i+1)$-Cell in the descending complex, and to Kill_$(n-i)$-Cell_and_$(n-(i+1))$-Cell in the ascending complex.

We can show that the two operators form a basis of the set of operators for updating Morse complexes using the approach in [19]. Each operator c which simplifies Morse complexes in a topologically consistent manner maintains the Euler formula, and thus can be expressed as a vector $c = (c_0, c_1, .., c_n)$ with positive integer coordinates c_i, $0 \le i \le n$, in an n-dimensional discrete subspace V of an $(n+1)$-dimensional discrete space, defined by $c_0 - c_1 + .. + (-1)^n c_n = 0$. The ith coordinate c_i corresponds to the number of i-cells removed by operator c. A removal (and a contraction) b_i of an i-cell p and an $(i+1)$-cell q, $0 \le i \le n-1$, can be expressed as a vector $b_i = (a_0, a_1, .., a_n)$ in V, where $a_i = a_{i+1} = 1$, and $a_j = 0$, $j \ne i, i+1$. The set of all such vectors obviously forms a basis of subspace V. The above argument does not provide an algorithm for expressing an arbitrary operator c as a sequence of basis operators. As pointed out in [19], if $c = k_0 b_0 + k_1 b_1 + .. + k_{n-1} b_{n-1}$ then c may be expressed through $K_i \ge k_i$ simplifications b_i and $K_i - k_i$ refinements inverse to b_i. Moreover, the entities (cells in Morse complexes, or critical points of f) which are introduced by inverse operations are not restricted to belong to a fixed set of entities of the initial full-resolution model.

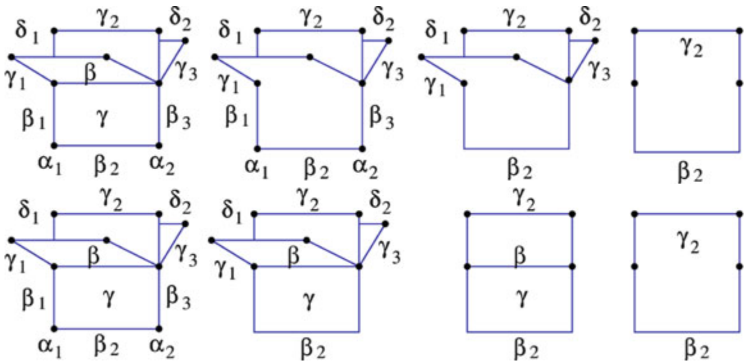
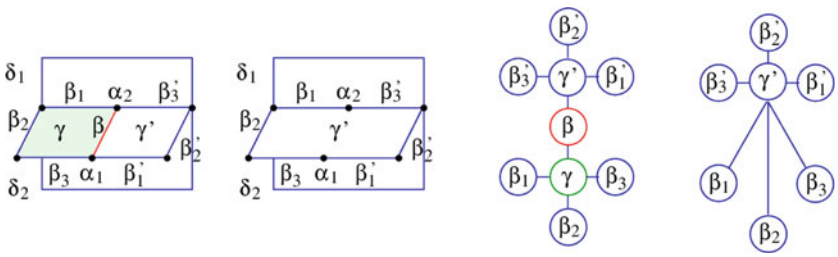


Fig. 5. A sequence of cancellations (top), and of removals and contractions (bottom) on a 3D descending Morse complex, which produce the same simplified Morse complex.

We illustrate the above argument on a simple example. After a cancellation of a 1-cell β and a 2-cell γ in a 3D descending Morse complex Γ_d shown in Figure 5 (top), each 1-cell β_i (and each 0-cell α_j) in Γ_d on the boundary of γ becomes incident to each 2-cell γ_k (and to each 3-cell δ_l) in the co-boundary of β. Each such pair of incident cells (e.g. δ_1 and α_1) induces a new cell in the Morse-Smale complex, which is eliminated by subsequent cancellations of 1-cells and 0-cells (e.g. α_1 and β_1), and of 2-cells and 3-cells (e.g. δ_1 and γ_1) [15]. Such a sequence of cancellations can be expressed as a (simpler) sequence of contractions (of 1-cells and 0-cells) and removals (of 2-cells and 3-cells), which do not have a side-effect of introducing new cells in the Morse-Smale complex, as illustrated in Figure 5 (bottom).

6 Removal and Contraction on the Incidence Graph

In this Section, we illustrate the effect of removals and contractions on the incidence graph and on the corresponding incidence-based representation.



(a) (b)

Fig. 6. Effect of a removal $rem(\gamma, \beta, \gamma')$ on a 3D descending Morse complex (a), and on a part of the corresponding incidence graph (b).

Before a removal $rem(\sigma_{i+1}, \sigma_i, \sigma'_{i+1})$, node σ_i in the incidence graph $G = (N, A)$ is connected through an arc in A to exactly two different nodes σ_{i+1} and σ'_{i+1} at level $i+1$ (the label of those arcs is 1), and to an arbitrary number of nodes at level $i-1$. The number of arcs incident to nodes σ_{i+1} or σ'_{i+1} may be arbitrary. For example, before a removal $rem(\gamma, \beta, \gamma')$ illustrated in Figure 6, node β at level 1 is connected to exactly two different nodes γ and γ' at level 2.

In terms of the incidence graph $G = (N, A)$, a removal $rem(\sigma_{i+1}, \sigma_i, \sigma'_{i+1})$ can be expressed as the deletion of the arcs connecting σ_i to lower-level nodes and of arcs connecting σ_{i+1} to higher-level nodes, and merging of nodes σ_i and σ_{i+1} into node σ'_{i+1} by contraction of arcs connecting σ_i to σ_{i+1} and to σ'_{i+1}. For each arc connecting $(i+1)$-level node σ_{i+1} to an i-level node $\tau \neq \sigma_i$, such that σ'_{i+1} and τ are connected through an arc in A, the label of the arc connecting σ'_{i+1} and τ is increased by the label of the arc connecting σ_{i+1} and τ.

In Figure 6, after a removal $rem(\gamma, \beta, \gamma')$, arcs connecting node β to 0-level nodes α_1 and α_2 (not illustrated in the Figure) and arcs connecting γ to 3-level nodes δ_1 and δ_2 (not illustrated in the Figure) are deleted. Nodes β and γ are merged into node γ', by contracting arcs connecting β to γ and γ', i.e., arcs connecting γ to β_1, β_2 and β_3 are replaced by arcs connecting γ' to β_1, β_2 and β_3.

In the example shown in Figure 7, before a removal $rem(\gamma, \beta, \gamma')$, 1-cell β_1 is incident to 2-cells γ and γ' once. After the removal, β_1 is incident twice to γ'.

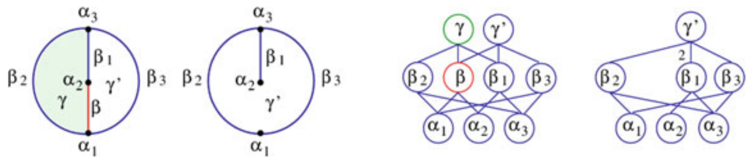


Fig. 7. Effect of a removal $rem(\gamma, \beta, \gamma')$ on a 2D descending Morse complex (a), and on a part of the incidence graph (b). The label of the arc connecting nodes β_1 and γ' is increased after the removal.

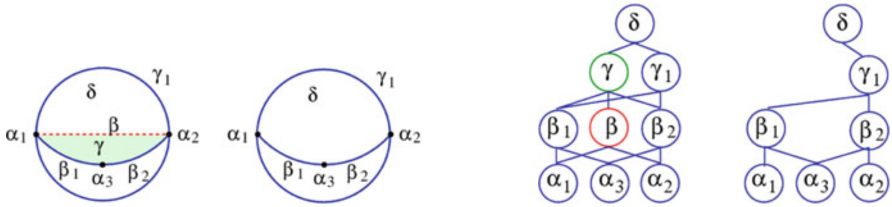


Fig. 8. Effect of a removal $rem(\gamma, \beta, \emptyset)$ on a 3D descending Morse complex (a), and on a part of the corresponding incidence graph (b).

Before a removal $rem(\sigma_{i+1}, \sigma_i, \emptyset)$ of the second kind, node σ_i at level i in the incidence graph is connected through an arc in A to exactly one node σ_{i+1} at level $i+1$. After the removal, nodes σ_i and σ_{i+1} together with all arcs incident in them are deleted, as illustrated in Figure 8.

A contraction $con(\sigma_i, \sigma_{i+1}, \sigma_i')$ is expressed in terms of the graph in a completely dual fashion. Its description is omitted here for brevity.

The effect on the incidence-based representation, that is on the combination of the incidence graph with the underlying simplicial decomposition of the domain, is restricted to the incidence graph when a simplification operator does not involve an extremum. When we perform a removal $rem(\sigma_n, \sigma_{n-1}, \sigma_n')$, then the partition of the n-simplexes of the underlying mesh into descending cells of maxima is updated by merging the set of n-simplexes forming the descending cell of σ_n into set of n-simplexes forming the descending cell of σ_n'. Dually, a contraction $con(\sigma_0, \sigma_1, \sigma_0')$ merges n-simplexes of the ascending cell of σ_0 with n-simplexes of the ascending cell of σ_0'.

7 Concluding Remarks

We have defined removal and contraction operators for simplifying n-dimensional Morse complexes. We have expressed them as Euler operators, thus providing a minimal set of atomic operators for updating Morse complexes. We have shown the effect of such simplification operators on the incidence graph representation of the two Morse complexes.

We are currently working on the definition of a multi-scale representation for Morse complexes, which will provide a combinatorial description of the ascending and descending Morse complexes at different levels of abstraction. This will be obtained by defining the inverse operators of removal and contraction operators, a dependency relation between such operators, and by using an incidence graph representation for the complexes. We are also planning to implement the simplification operators and the multi-scale model for 3D scalar fields. Our future work would be in the direction of combining a multi-scale morphological representation with a mesh-based multi-resolution representation of the field.

Acknowledgements

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