7 Statistical Quality Control & Reliability Tests

Statistical quality control and *reliability tests* are performed to estimate or demonstrate quality and reliability characteristics on the basis of data collected from sampling tests. *Estimation* leads to a *point or interval estimate* (marked with ˆ in this book), *demonstration* is a test of a given *hypothesis* on the unknown characteristic. Estimation and demonstration of an *unknown probability* is investigated in Section 7.1 for the case of a defective probability p and in Section 7.2.1 for some reliability figures. Procedures for availability estimation and demonstration for the case of continuous operation are given in Section 7.2.2. Estimation and demonstration of a *constant failure rate* λ (or *MTBF* for the case $MTBF = 1/\lambda$) are discussed in depth in Sections 7.2.3. The case of an *MTTR* is considered in Section 7.3. Basic models for *accelerated tests* are discussed in Section 7.4. *Goodness-of-fit tests* based on graphical and analytical procedures are summarized in Section 7.5. Some considerations on general reliability data analysis, with test on nonhomogeneous Poisson processes and trend tests, are given in Section 7.6. Models for reliability growth are introduced in Section 7.7. To simplify the notation, *sample* is used for *random sample* and the indices S, referring to system, is omitted in this chapter (*MTBF* instead of $MTBF_{S0}$ and λ or PA instead of λ_S or PA_S). Theoretical foundations for this chapter are in Appendix A8. Selected examples illustrate the practical aspects.

7.1 Statistical Quality Control

One of the main purposes of *statistical quality control* is to use *sampling tests* to *estimate* or *demonstrate* the *defective probability* p of a given item, to a required accuracy and often on the basis of tests by *attributes* (i. e., tests of type good / bad). However, considering p as an *unknown probability,* a broader field of applications can be covered by the same methods. Other tasks, such as *tests by variables* and *statistical processes control* [7.1-7.5], are not considered hereafter.

In this section, p will be considered as a *defective probability* (fraction of defective items). It will be assumed that p is the same for each element in the sample considered and that each sample element is statistically independent from each other. These assumptions presuppose that the lot is *homogeneous* and *much larger* than the sample. They allow the use of the *binomial distribution* (Appendix A6.10.7).

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7.1.1 Estimation of a Defective Probability *p*

Let n be the size of a (random) sample from a large homogeneous lot. If k defective items have been observed within the sample of size n, then (Eq. (A8.29))

$$\hat{p} = \frac{k}{n} \tag{7.1}$$

is the *maximum likelihood point estimate* of the defective probability p for an item in the lot under consideration. For a given *confidence level* $\gamma = 1-\beta_1-\beta_2$ ($0<\beta_1<1-\beta_2<1$), the *lower* $\hat{p}_l$ and *upper* $\hat{p}_u$ limit of the *confidence interval* of p can be obtained from

$$\sum_{i=k}^{n}\binom{n}{i}\hat{p}_l^{\,i}(1-\hat{p}_l)^{n-i}=\beta_2 \qquad \text{and} \qquad \sum_{i=0}^{k}\binom{n}{i}\hat{p}_u^{\,i}(1-\hat{p}_u)^{n-i}=\beta_1 \tag{7.2}$$

for $0<k<n$, and from

$$\hat{p}_l=0 \qquad \text{and} \qquad \hat{p}_u=1-\sqrt[n]{\beta_1} \qquad \text{for } k=0 \quad (\gamma=1-\beta_1), \tag{7.3}$$

or from

$$\hat{p}_l=\sqrt[n]{\beta_2} \qquad \text{and} \qquad \hat{p}_u=1 \qquad \text{for } k=n \quad (\gamma=1-\beta_2), \tag{7.4}$$

see Eqs. (A8.37) to (A8.40) and the remarks given there. β_1 is the risk that the true value of p is larger than $\hat{p}_u$ and β_2 the risk that the value of p is smaller than $\hat{p}_l$. The *confidence level* is nearly equal to (but not less than) $\gamma=1-\beta_1-\beta_2$. It can be considered as the relative frequency of cases in which the interval $[\hat{p}_l, \hat{p}_u]$ *overlaps* (covers) the true value of p, in an increasing series of repetitions of the experiment of taking a random sample of size n.

In many practical applications, a graphical determination of $\hat{p}_l$ and $\hat{p}_u$ is sufficient. The upper diagram in Fig. 7.1 can be used for $\beta_1=\beta_2=0.05$, the lower diagram for $\beta_1=\beta_2=0.1$ ($\gamma=0.9$ and $\gamma=0.8$, respectively). The continuous lines in Fig. 7.1 are the envelopes of staircase functions (k, n integer) given by Eq. (7.2). They converge rapidly, for $\min(np,\ n(1-p))\ge 5$, to the *confidence ellipses* (dashed lines in Fig. 7.1). Using the confidence ellipses (Eq. (A8.42)), $\hat{p}_l$ and $\hat{p}_u$ can be calculated from

$$\hat{p}_{u,l}=\frac{k+0.5\,b^2 \pm b\sqrt{k(1-k/n)+b^2/4}}{n+b^2}. \tag{7.5}$$

b is the $1-(1-\gamma)/2=(1+\gamma)/2$ quantile of the standard normal distribution $\Phi(t)$, given for some typical values of γ by (Table A9.1)

$\gamma=$	0.6	0.8	0.9	0.95	0.98	0.99
$b=$	0.84	1.28	1.64	1.96	2.33	2.58

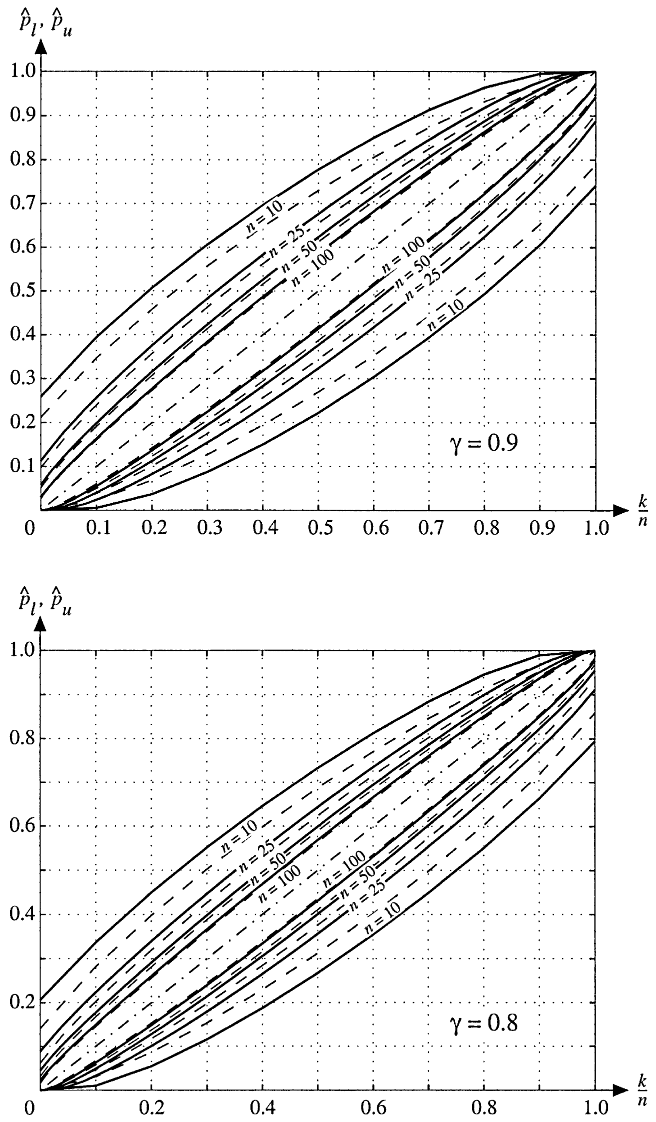


Figure 7.1 Confidence limits $\hat{p}_l$ and $\hat{p}_u$ for an unknown probability p (e.g. defective probability) as a function of the observed relative frequency k/n (n = sample size, k = observed events, γ = confidence level = $1-\beta_1-\beta_2$, here with $\beta_1=\beta_2$; continuous lines are the exact solution (Eqs. (7.2) - (7.4)), dashed the confidence ellipses (Eqs. (7.5), (A8.42), (A6.149))

Example: $n=25$, $k=5$ gives $\hat{p}=k/n=0.2$ and for $\gamma=0.9$ the confidence interval [0.08, 0.38] ([0.0823, 0.3754] using Eq. (7.2), and [0.1011, 0.3572] using Eq. (7.5))

The confidence limits $\hat{p}_l$ and $\hat{p}_u$ can also be used as *one-sided confidence intervals*. In this case (Eq. (A8.44)),

$$0 \le p \le \hat{p}_u \quad \text{(or simply } p \le \hat{p}_u\text{)}, \quad \text{with} \quad \gamma = 1 - \beta_1$$
$$\hat{p}_l \le p \le 1 \quad \text{(or simply } p \ge \hat{p}_l\text{)}, \quad \text{with} \quad \gamma = 1 - \beta_2. \tag{7.6}$$

Example 7.1

In a sample of size $n = 25$, exactly $k = 5$ items were found to be defective. Determine for the underlying defective probability p, (i) the point estimate, (ii) the interval estimate for $\gamma = 0.8$ ($\beta_1 = \beta_2 = 0.1$), (iii) the upper bound of p for a one sided confidence interval with $\gamma = 0.9$.

Solution

(i) Equation (7.1) yields the point estimate $\hat{p} = 5/25 = 0.2$. (ii) For the interval estimate, the lower part of Fig. 7.1 leads to the confidence interval [0.10, 0.34], [0.1006, 0.3397] using Eq. (7.2) and [0.1175, 0.3194] using Eq. (7.5). (iii) With $\gamma = 0.9$ it holds $p \le 0.34$.

Supplementary result: The upper part of Fig. 7.1, would lead to $p \le 0.38$ with $\gamma = 0.95$.

Note that the role of k/n and p can be *reversed* and Eq. (7.5) can be used to calculate the limits k_1 and k_2 of the number of observations k in n *independent trials* (e.g. the number k of defective items in a sample of size n) for *given* probability $\gamma = 1 - \beta_1 - \beta_2$ (with $\beta_1 = \beta_2$) and *known* values of p and n (Eq. (A8.45))

$$k_{2,1} = n\,p \pm b\sqrt{n\,p(1-p)}. \tag{7.7}$$

As in Eq. (7.5), the quantity b in Eq. (7.7) is the $(1+\gamma)/2$ quantile of the standard normal distribution (e. g. $b = 1.64$ for $\gamma = 0.9$, Table A9.1). For a graphical solution, Fig. 7.1 can be used by taking the ordinate p as known, and by reading k_1/n and k_2/n from the abscissa.

7.1.2 Simple Two-sided Sampling Plans for the Demonstration of a Defective Probability *p*

In the context of *acceptance testing*, the *demonstration* of a defective probability p is often required, instead of its estimation (Section 7.1.1). The main concern of this test is to check a *zero hypothesis* H_0: $p < p_0$ against an *alternative hypothesis* H_1: $p > p_1$ on the basis of the following agreement between producer and consumer:

> *The lot should be accepted with a probability nearly equal to (but not less than) $1-\alpha$ if the true (unknown) defective probability p is lower than p_0 but rejected with a probability nearly equal to (but not less than) $1-\beta$ if p is greater than p_1 (p_0, $p_1 > p_0$, and $0 < \alpha < 1-\beta < 1$ are given (fixed) values).*

p_0 is the *specified* defective probability and p_1 is the *maximum acceptable* defective probability. α is the allowed *producer's risk* (type I error), i.e., the probability of *rejecting a true hypothesis* H_0: $p < p_0$. β is the allowed *consumer's risk* (type II error), i.e., the probability of *accepting* the hypothesis H_0: $p < p_0$ when the alternative hypothesis H_1: $p > p_1$ is true. Verification of the agreement stated above is a problem of statistical hypothesis testing (Appendix A8.3) and can be performed, for instance, with a *simple two-sided sampling plan* or a *sequential test.* In both cases, the basic model is the sequence of *Bernoulli trials*, as introduced in Appendix A6.10.7.

7.1.2.1 Simple Two-sided Sampling Plans

The procedure (*test plan*) for the *simple two-sided sampling plan* is as follows (Appendix A8.3.1.1):

1. From p_0, p_1, α, and β, determine the smallest integers c and n which satisfy

$$\sum_{i=0}^{c} \binom{n}{i} p_0^i (1-p_0)^{n-i} \geq 1-\alpha \tag{7.8}$$

and

$$\sum_{i=0}^{c} \binom{n}{i} p_1^i (1-p_1)^{n-i} \leq \beta . \tag{7.9}$$

2. Take a sample of size n, determine the number k of defective items in the sample, and

   - reject H_0: $p < p_0$, if $k > c$
   - accept H_0: $p < p_0$, if $k \leq c$. (7.10)

The graph of Fig. 7.2 visualizes the validity of the above rule (see Appendix A8.3.1.1 for a proof). It satisfies the inequalities (7.8) and (7.9), and is known as *operating characteristic* (curve). For each value of p, it *gives the probability of having no more than c defective items in a sample of size n.* Since the operating characteristic (curve) as a function of p decreases monotonically, the risk for a false decision decreases for $p < p_0$ and $p > p_1$, respectively. It can be shown that the quantities c and np_0 depend only on α, β, and the ratio p_1 / p_0 (*discrimination ratio*). Table 7.3 (p. 315) gives c and np_0 for some important values of α, β and p_1 / p_0 for the case where the *Poisson approximation* (Eq. (A6.129)) applies.

Using the operating characteristic (curve), the *Average Outgoing Quality (AOQ)* can be calculated. *AOQ* represents the percentage of defective items that reach the customer, assuming that all rejected samples have been 100% inspected, and that the defective items have been replaced by good ones, and is given by

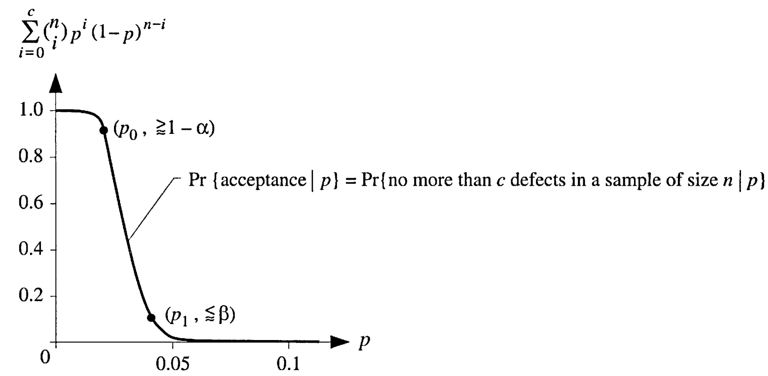


Figure 7.2 Operating characteristic (curve) as a function of the defective probability p for given (fixed) n and c ($p_0 = 2\%$, $p_1 = 4\%$, $\alpha \approx \beta \approx 0.1$; $n = 510$ and $c = 14$ as per Table 7.3)

$$AOQ = p\Pr\{\text{acceptance} \mid p\} = p\sum_{i=0}^{c}\binom{n}{i}p^i(1-p)^{n-i}. \tag{7.11}$$

The maximum value of AOQ is the *Average Outgoing Quality Limit* [7.4, 7.5].

Obtaining the solution of inequalities (7.8) and (7.9) is time-consuming. For small values of p_0 & p_1 (up to a few %), the *Poisson approximation* (Eq. (A6.129))

$$\binom{n}{i}p^i(1-p)^{n-i} \approx \frac{(n\,p)^i}{i!}e^{-np}$$

can be used. Introducing the Poisson approximation in Eqs. (7.8) and (7.9) leads to a Poisson distribution with parameters $m_1 = n\,p_1$ and $m_0 = n\,p_0$, which can be solved using a table of the χ^2 distribution (Table A9.2). Alternatively, the curves of Fig. 7.3 provide graphical solutions, sufficiently good for practical applications. Exact solutions are in Table 7.3 (p. 315).

Example 7.2

Determine the sample size n and the number of allowed defective items c to test the null hypothesis $H_0 : p < p_0 = 1\%$ against the alternative hypothesis $H_1 : p > p_1 = 2\%$ with producer and consumer risks $\alpha = \beta = 0.1$ (which means $\alpha \approx \beta \lesseqgtr 0.1$).

Solution

For $\alpha = \beta = 0.1$, Table A9.2 yields $\nu = 30$ (value of ν for which $t_{\nu,q_1} / t_{\nu,q_2} = 2$ with $q_1 \gtreqless 1-\alpha = 0.9$ and $q_2 \lesseqgtr \beta = 0.1$) and, with linear interpolation, F(20.4) $\approx$ 0.095 < β and F(40.8) $\approx$ 0.908 > $1-\alpha$ ($\nu = 28$ falls just short). Thus $c = \nu/2 - 1 = 14$ and $n = 20.4/(2\cdot 0.01) = 1020$. The values of c and n according to Table 7.3 would be $c = 14$ and $n = 10.17/0.01 = 1017$. Using the graph of Fig. 7.3 yields practically the same result: $c = 14$, $m_0 \approx 10.2$ and $m_1 \approx 20.4$ for $\alpha \approx \beta \approx 0.1$. Both the analytical and graphical methods require a solution by successive approximation (choice of c and check of conditions for α and β by considering the ratio p_1 / p_0).

7.1.2.2 Sequential Tests

The procedure for a *sequential test* is as follows (Appendix A8.3.1.2):

1. In a Cartesian coordinate system draw the *acceptance line* $y_1(n) = a\,n - b_1$ and the *rejection line* $y_2(n) = a\,n + b_2$, with

$$a = \frac{\ln\frac{1-p_0}{1-p_1}}{\ln\frac{p_1}{p_0} + \ln\frac{1-p_0}{1-p_1}}, \quad b_1 = \frac{\ln\frac{1-\alpha}{\beta}}{\ln\frac{p_1}{p_0} + \ln\frac{1-p_0}{1-p_1}}, \quad b_2 = \frac{\ln\frac{1-\beta}{\alpha}}{\ln\frac{p_1}{p_0} + \ln\frac{1-p_0}{1-p_1}}. \tag{7.12}$$

2. Select one item after another from the lot, test the item, enter the test result in the diagram drawn in step 1, and stop the test as soon as either the rejection or the acceptance line is crossed.

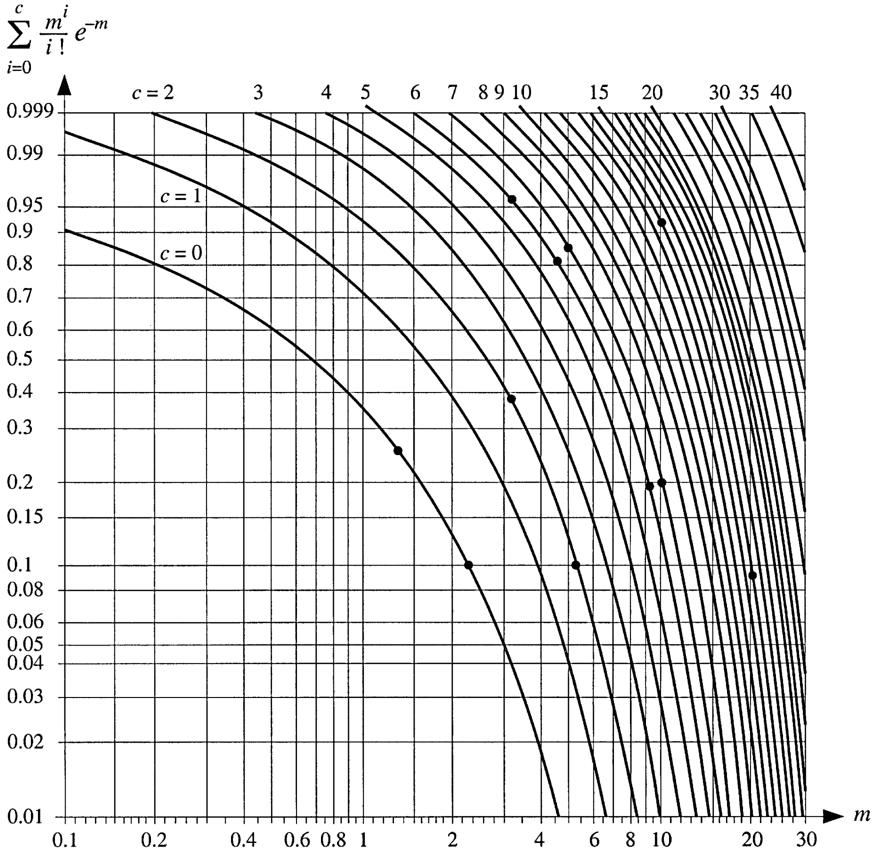


Figure 7.3 Poisson distribution (• results for Examples 7.2 ($c = 14$), 7.4 (7), 7.5 (0), 7.6 (2 & 0), 7.9 (6))

Figure A8.8 shows acceptance and rejection lines for $p_0 = 1\%$, $p_1 = 2\%$ and $\alpha \approx \beta \approx 0.2$. The advantage of the sequential test is that on *average* it requires a smaller sample size than the corresponding simple two-sided sampling plan (Ex. 7.10 or Fig. 7.8). A disadvantage is that the test duration (sample size) is random.

7.1.3 One-sided Sampling Plans for the Demonstration of a Defective Probability *p*

The two-sided sampling plans of Section 7.1.2 are fair in the sense that for $\alpha = \beta$, both producer and consumer run the *same risk* of making a false decision. In practical applications however, *one-sided sampling plans* are often used, i.e., only p_0 and α or p_1 and β are specified. In these cases, the operating characteristic (curve) is not completely defined. For every value of c ($c = 0, 1, \ldots$) a smallest n ($n = 1, 2, \ldots$) exists which satisfies inequality (7.8) for a given p_0 and α, or a largest n exists which satisfies inequality (7.9) for a given p_1 and β. It can be shown that operating characteristic (curves) become steeper as the value of c increases (see e.g. Figs. 7.4 or A8.9). Hence, for small values of c, the producer (if p_0 and α are given) or the consumer (if p_1 and β are given) *can be favored.* Figure 7.4 visualizes the reduction of the consumer risk ($\beta \approx 0.95$ for $p = 0.0065$) by increasing values of the defective probability p or values of c, see Fig 7.9 for a counterpart.

When only p_0 and α or p_1 and β are given, it is usual to set in these cases

$$p_0 = AQL \qquad \text{and} \qquad p_1 = LTPD, \tag{7.13}$$

respectively, where AQL is the *Acceptable Quality Level* and $LTPD$ is the *Lot Tolerance Percent Defective* (Eqs. (A8.79) to (A8.82)).

A large number of one-sided sampling plans for the demonstration of *AQL* values are given in national and *international standards* (*IEC 60410, ISO 2859, MIL-STD-105, DIN 40080* [7.3]). Many of these plans have been established empirically. The following remarks can be useful when evaluating such plans:

1. *AQL* values are given in %.
2. The values for n and c are in general obtained using the *Poisson approximation.*
3. Not all values of c are listed, the value of α often decreases with increasing c.
4. Sample size is related to lot size, and this relationship is empirical.
5. A distinction is made between reduced tests (level I), normal tests (level II) and tightened tests (level III); level II is normally used; transition from one level to another is often given empirically (e.g. transition from level II to level III is necessary if 2 out of 5 successive independent lots have been rejected and a return to level II follows if 5 successive independent lots are passed).
6. The value of α is not given explicitly (for $c = 0$, for example, α is approximately 0.05 for level I, 0.1 for level II, and 0.2 for level III).

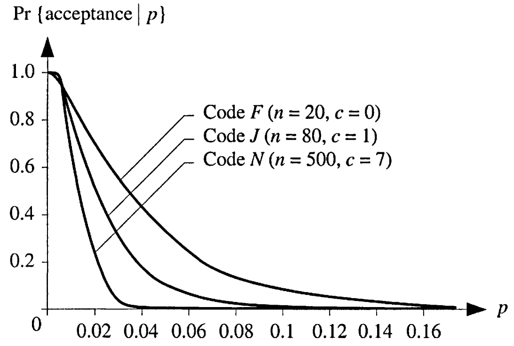


Figure 7.4 Operating characteristic (curve) for demonstration of an $AQL = 0.65\%$ with sample sizes $n = 20, 80$, and 500 as per Table 7.1 ($\alpha \approx 0.11$ for $c = 0$, ≈ 0.09 for $c = 1$, ≈ 0.03 for $c = 7$)

Table 7.1 presents some test procedures for AQL values from *IEC 60410* [7.3] and Fig. 7.4 shows the corresponding operating characteristic (curve) for $AQL = 0.65\%$ and sample size $n = 20, 80$, and 500.

Test procedures for demonstration of $LTPD$ values with given (fixed) customer risk β are for example in [3.12 (S-19500)]. They are often based on the Poisson approximation (Eq. (A6.129)) and can be easily established using a χ^2 - table (Appendix A9.2) or Fig. 7.3. For given β and $LTPD$, the values of n and c can be obtained taking in Fig. 7.3

$$\sum_{i=0}^{c} \frac{(m)^i}{i\,!} e^{-m} = \beta$$

and reading $m = np = n LTPD$ for $c = 0,1,2,...$ (Example: $\beta = 0.1$, $LTPD = 2\%$ yields $m = 3.9$ for $c = 1$, and from this $n = 3.9 / 0.02 = 195$; the procedure is thus: test 195 items and reject $LTPD = 2\%$ if more than 1 defect occur.)

In addition to the simple one-sided sampling plans described above, *multiple one-sided sampling plans* are often used to demonstrate AQL values. In a *double one-sided sampling plan*, the following procedure is used:

1. Take a first sample of size n_1 and accept definitely if no more than c_1 defects occur, but reject definitely if exactly or more than d_1 defects have occurred.
2. If after the first sample the number of defects is greater than c_1 but less than d_1, take a second sample of size n_2 and accept if there are totally (in the first and second sample) no more than c_2 defects; elsewhere reject.

The operating characteristic (curve) or acceptance probability for a *double one-sided sampling plan* can be calculated as

$$\Pr\{\text{acceptance} \mid p\} = \sum_{i=0}^{c_1} \binom{n_1}{i} p^i (1-p)^{n_1-i}$$

$$+ \sum_{i=c_1+1}^{d_1-1} [\binom{n_1}{i} p^i (1-p)^{n_1-i} \sum_{j=0}^{c_2-i} \binom{n_2}{j} p^j (1-p)^{n_2-j}]. \qquad (7.14)$$

Multiple one-sided sampling plans are also given in national and *international standards*, see for example *IEC 60410* [7.3] for the following double one-sided sampling plan to demonstrate $AQL = 1\%$

Sample Size	n_1	n_2	c_1	d_1	c_2
281 - 500	32	32	0	2	1
501 - 1,200	50	50	0	3	3
1,201 - 3,200	80	80	1	4	4
3,201 - 10,000	125	125	2	5	6

The advantage of multiple one-sided sampling plans is that on average they require smaller sample sizes than would be necessary for simple one-sided sampling plans. A disadvantage is that the test duration is not fixed in advance.

Table 7.1 Test procedures for AQL demonstration (test level II, from *IEC 60410* [7.3])

Code	Lot size	Sample size	*AQL* in %											
			0.04	0.065	0.10	0.15	0.25	0.40	0.65	1.0	1.5	2.5	4.0	6.5
	N	*n*	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*
A	2 - 8	2	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	0
B	9 - 15	3	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	0	↑
C	16 - 25	5	↓	↓	↓	↓	↓	↓	↓	↓	↓	0	↑	↓
D	26 - 50	8	↓	↓	↓	↓	↓	↓	↓	↓	0	↑	↓	1
E	51 - 90	13	↓	↓	↓	↓	↓	↓	↓	0	↑	↓	1	2
F	91 - 150	20	↓	↓	↓	↓	↓	↓	0	↑	↓	1	2	3
G	151 - 280	32	↓	↓	↓	↓	↓	0	↑	↓	1	2	3	5
H	281 - 500	50	↓	↓	↓	↓	0	↑	↓	1	2	3	5	7
J	501 - 1200	80	↓	↓	↓	0	↑	↓	1	2	3	5	7	10
K	1.2k - 3.2k	125	↓	↓	0	↑	↓	1	2	3	5	7	10	14
L	3.2k - 10k	200	↓	0	↑	↓	1	2	3	5	7	10	14	21
M	10k - 35k	315	0	↑	↓	1	2	3	5	7	10	14	21	↑
N	35k - 150k	500	↑	↓	1	2	3	5	7	10	14	21	↑	↑
P	150k - 500k	800	↓	1	2	3	5	7	10	14	21	↑	↑	↑
Q	over 500k	1250	1	2	3	5	7	10	14	21	↑	↑	↑	↑

Use the first sampling plan above for ↑ or below for ↓, c = number of allowed defects

7.2 Statistical Reliability Tests

Reliability tests are useful to evaluate the reliability *achieved* in a given item. Early initiation of such tests allows quick identification and *cost-effective correction* of weaknesses not discovered by reliability analyses. This supports a learning process, often related to a reliability growth program (Section 7.7). Since reliability tests are generally time-consuming and expensive, they must be coordinated with other tests. Test conditions should be as close as possible to those experienced in the field. As with quality control, a distinction is made between *estimation* and *demonstration* of a specific reliability figure. Section 7.2.1 uses results of Section 7.1 for reliability and availability testing for the case of a *given (fixed) mission*. In section 7.2.2 an unified method for availability estimation and demonstration for the case of continuous operation is introduced. Section 7.2.3 deals carefully with *estimation & demonstration* of a constant failure rate λ (or of $MTBF$ for the case $MTBF = 1/\lambda$). Furthermore, maintainability tests are considered in Section 7.3, accelerated tests in Section 7.4, goodness-of-fit tests in Section 7.5, general reliability data analysis and trend tests in Section 7.6, reliability growth in Section 7.7. To simplify notations, the indices S (for system) is omitted ($R, PA, MTBF, \lambda$ are used for R_{S0}, $PA_S, MTBF_{S0}, \lambda_S$).

7.2.1 Reliability & Availability Estimation and Demonstration for the Case of a given fixed Mission

Reliability (R) and availability (asymptotic & steady-state point and average availability $PA = AA$) are often defined as success probability for a given (fixed) mission. Their estimation and demonstration can thus be performed as for an unknown probability p (Section 7.1) by setting, for convenience,

$$p = 1 - R \qquad \text{or} \qquad p = 1 - PA = 1 - AA.$$

For a demonstration, the null hypothesis H_0: $p < p_0$ is converted to H_0: $R > R_0$ or H_0: $AA > AA_0$, which adheres better to the concept of reliability or availability. The same holds for any other reliability figure expressed as an unknown *probability p*.

The above considerations hold for a given (fixed) mission, repeated for reliability tests as n Bernoulli trials. However, for the case of continuous operation, estimation and demonstration of an availability can leads to a difficulty in defining the time points $t_1, t_2, \ldots, t_n$ at which the n observations according to Eqs. (7.2) - (7.4) or (7.8) - (7.10) have to be performed. The case of continuous operation is considered in Section 7.2.2 for availability and Section 7.2.3 for reliability. Examples 7.3 - 7.6 illustrate some cases of reliability tests for given fixed mission.

Example 7.3
In a reliability test 95 of 100 items pass. Give the confidence interval for R at $\gamma = 0.9$ ($\beta_1 = \beta_2$).
Solution
With $p = 1 - R$ and $\hat{R} = 0.95$ the confidence interval for p follows from Fig. 7.1 as [0.03, 0.10]. The confidence interval for R is then [0.9, 0.97]. (Eq. (7.5) leads to [0.901, 0.975] for R.)

Example 7.4

The reliability of a given subassembly was $R = 0.9$ and should have been improved through constructive measures. In a test of 100 subassemblies, 94 of them pass the test. Check with a type I error $\alpha = 20\%$ the hypothesis H_0: $R > 0.95$.

Solution

For $p_0 = 1 - R_0 = 0.05$, $\alpha = 20\%$, and $n = 100$, Eq. 7.8 delivers $c = 7$ (see also the graphical solution from Fig. 7.3 with $m = n\,p_0 = 5$ and acceptance probability $\geq 1 - \alpha = 0.8$, yielding $\alpha \approx 0.15$ for $m = 5$ and $c = 7$). As just $k = 6$ subassemblies have failed the test, the hypothesis H_0: $R > 0.95$ can be accepted (must not be rejected) at the level $1 - \alpha = 0.8$.

Supplementary result: Assuming as an alternative hypothesis H_1: $R < 0.90$, or $p > p_1 = 0.1$, the type II error β can be calculated from Eq. (7.9) with $c = 7$ & $n = 100$ or graphically from Fig. 7.3 with $m = n\,p_1 = 10$, yielding $\beta \approx 0.2$.

Example 7.5

Determine the minimum number of tests n that must be repeated to verify the hypothesis H_0: $R > R_1 = 0.95$ with a consumer risk $\beta = 0.1$. What is the allowed number of failures c?

Solution

The inequality (7.9) must be fulfilled with $p_1 = 1 - R_1 = 0.05$ and $\beta = 0.1$, n and c must thus satisfy

$$\sum_{i=0}^{c} \binom{n}{i} 0.05^i \cdot 0.95^{n-i} \lesssim 0.1 .$$

The number of tests n is a minimum for $c = 0$. From $0.95^n \lesssim 0.1$, it follows that $n = 45$, yielding $\beta \approx 0.099$ (calculation with the Poisson approximation (Eq. (7.12)) yields $n = 46$, graphical solution with Fig. 7.3 leads to $m \approx 2.3$ and then $n = m / p_1 \approx 46$).

Example 7.6

Continuing with Example 7.5, (i) find n for $c = 2$ and (ii) how large would the producer risk be for $c = 0$ and $c = 2$ if the true reliability were $R = 0.97$?

Solution

(i) From Eq. (7.9),

$$\sum_{i=0}^{2} \binom{n}{i} 0.05^i \cdot 0.95^{n-i} \lesssim 0.1$$

and thus $n = 105$ (Fig. 7.3 yields $m \approx 5.3$ and $n \approx 106$; from Table A9.2, $\nu = 6$, $t_{6,\,0.9} = 10.645$ and $n = 107$).

(ii) The producer risk is

$$\alpha = 1 - \sum_{i=0}^{c} \binom{n}{i} 0.03^i \cdot 0.97^{n-i},$$

hence, $\alpha \approx 0.75$ for $c = 0$ and $n = 45$, $\alpha \approx 0.61$ for $c = 2$ and $n = 105$ (Fig. 7.3, yields $\alpha \approx 0.75$ for $c = 0$ and $m = 1.35$, $\alpha \approx 0.62$ for $c = 2$ and $m = 3.15$; from Table A9.2, $\alpha \approx 0.73$ for $\nu = 2$ and $t_{2,\,\alpha} = 2.7$, $\alpha \approx 0.61$ for $\nu = 6$ and $t_{6,\,\alpha} = 6.3$ lin. int. (0.74 and 0.61 from [A9.1])).

7.2.2 Availability Estimation and Demonstration for the Case of Continuous Operation (asymptotic & steady-state)

Availability estimation & demonstration for a repairable item in continuous operation can be based on results given in Section 6.2 for the one-item repairable structure. Point estimate (with corresponding mean and variance) for the availability can be found for arbitrary distributions of failure-free and repair times (Section 7.2.2.3). However, interval estimation and demonstration tests can lead to some difficulties. An unified approach for estimating & demonstrating the asymptotic and steady-state point and average availability $PA = AA$ for the case of exponentially or Erlangian distributed failure-free and repair times is introduced in Appendices A8.2.2.4 & A8.3.1.4 (to simplify the notation, $PA = AA$ is used for $PA_S = AA_S$).

Sections 7.2.2.1 and 7.2.2.2 deal with this approach. Only the case of exponentially distributed failure-free and repair times, i. e., constant failure and repair rates ($\lambda(x) = \lambda$, $\mu(x) = \mu$) is considered here, extension to Erlangian distributions is easy. Point and average unavailability converge for this case rapidly ($1-\mathrm{PA}_{S0}(t)$ and $1-\mathrm{AA}_{S0}(t)$ in Table 6.3) to the asymptotic & steady-state value $\overline{PA} = 1 - PA = 1 - AA$ $= \lambda / (\lambda + \mu) \approx \lambda / \mu$. To simplify considerations, it will be assumed that the observed time interval $(0, t]$ is $\gg 1/\mu$, terminates at the conclusion of a repair, and exactly k (or n) failure-free times τ_i and corresponding repair times τ'_i have occurred (see Section 7.2.2.3 for other possibilities). Furthermore, considering $\lambda \ll \mu$,

$$\overline{PA}_a = \lambda / \mu \tag{7.15}$$

is estimated instead of $\overline{PA} = \lambda / (\lambda + \mu)$ (absolute error less than $(\lambda / \mu)^2$, see remark on p. 538). λ / μ is a probabilistic value of the asymptotic & steady-state unavailability and has his statistical counterpart in DT / UT, where DT and UT are the observed down and up times. The procedure given in Appendices A8.2.2.4 and A8.3.1.4 is based on the fact that the quantity $\mu \cdot DT / \lambda \cdot UT$ is distributed according to a *Fisher distribution* (***F***-distribution) with $\nu_1 = \nu_2 = 2k$ degrees of freedom. Section 7.2.2.1 deals with estimation and Section 7.2.2.2 with demonstration of $\overline{PA}$.

7.2.2.1 Availability Estimation

Having observed for an item good-as-new after each repair (Fig. 6.2), with constant failure & repair rates λ & $\mu \gg \lambda$, an operating time $UT = t_1 + \ldots + t_k$ and a repair time $DT = t'_1 + \ldots + t'_k$, the *maximum likelihood point estimate* for $\overline{PA}_a = \lambda / \mu$ is

$$\hat{\overline{PA}}_a = (\widehat{\lambda / \mu}) = DT / UT = (t'_1 + \ldots + t'_k) / (t_1 + \ldots + t_k), \tag{7.16}$$

DT / UT is biased, *unbiased* is $(1 - 1/k)\, DT / UT$, $k > 1$ (Example A8.10). $\overline{PA}_a = \lambda / \mu$ is an approximation for $\overline{PA} = \lambda / (\lambda + \mu)$, sufficiently good for practical applications (absolute error less than $(\lambda / \mu)^2$). For given $\beta_1, \beta_2, \gamma = 1 - \beta_1 - \beta_2$ $(0 < \beta_1 < 1 - \beta_2 < 1)$,

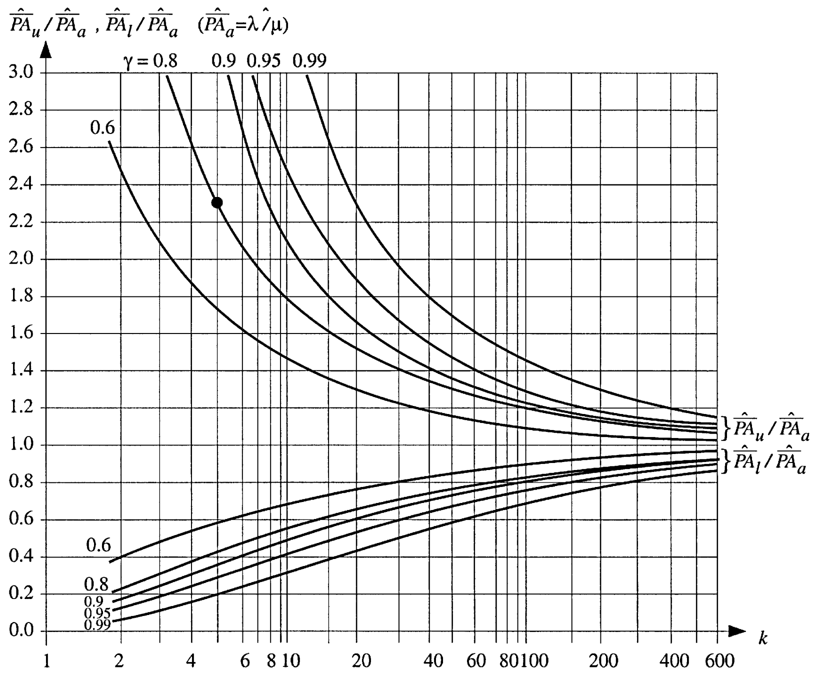


Figure 7.5 Confidence limits $\hat{\overline{PA}}_l / \hat{\overline{PA}}_a \approx \hat{\overline{PA}}_{a_l} / \hat{\overline{PA}}_a$ and $\hat{\overline{PA}}_u / \hat{\overline{PA}}_a \approx \hat{\overline{PA}}_{a_u} / \hat{\overline{PA}}_a$ (Eq. (7.17)) for an unknown asymptotic & steady-state unavailability $\overline{PA} = 1 - PA = 1 - AA$ ($\hat{\overline{PA}}_a = DT/UT$ = maximum likelihood estimate for λ/μ ($UT = t_1 + \ldots + t_k$, $DT = t_1' + \ldots + t_k'$); $\gamma = 1 - \beta_1 - \beta_2$ = confidence level (here $\beta_1 = \beta_2 = (1-\gamma)/2$)); • result for Example A8.8

lower and upper confidence limits for $\overline{PA}$ are (Eq. (A8.65))

$$\hat{\overline{PA}}_l \approx \hat{\overline{PA}}_{a_l} = \hat{\overline{PA}}_a / F_{2k,2k,1-\beta_2} \quad \text{and} \quad \hat{\overline{PA}}_u \approx \hat{\overline{PA}}_{a_u} = \hat{\overline{PA}}_a \cdot F_{2k,2k,1-\beta_1}, \tag{7.17}$$

where $F_{2k,2k,1-\beta_2}$ & $F_{2k,2k,1-\beta_1}$ are the $1-\beta_2$ & $1-\beta_1$ quantiles of the Fisher (F) distribution (Appendix A9.4, [A9.3 - A9.6]). Figure 7.5 gives these confidence limits for $\beta_1 = \beta_2 = (1-\gamma)/2$, useful for practical applications (Example A8.8). *One-sided confidence intervals* are

$$0 < \overline{PA} \le \hat{\overline{PA}}_u, \quad \text{with } \gamma = 1 - \beta_1 \quad \text{and} \quad \hat{\overline{PA}}_l \le \overline{PA} < 1, \quad \text{with } \gamma = 1 - \beta_2. \tag{7.18}$$

Corresponding values for the *availability* can be obtained using $PA = 1 - \overline{PA}$.

If failure free or repair times are Erlangian distributed (Eq. (A6.102)) with $\beta_\lambda = n_\lambda$ and $\beta_\mu = n_\mu$, $F_{2k,2k,1-\beta_2}$ and $F_{2k,2k,1-\beta_1}$ have to be replaced by $F_{2kn_\mu,2kn_\lambda,1-\beta_2}$ and $F_{2kn_\lambda,2kn_\mu,1-\beta_1}$, for unchanged *MTTF* & *MTTR* (Example A8.11). Results based only on the distribution of *DT* are not free of parameter (Section 7.2.2.3).

7.2.2.2 Availability Demonstration

In the context of an acceptance testing, *demonstration* of the asymptotic & steady-state point and average availability ($PA = AA$) is often required. The item is assumed as-good-as-new after each repair, and for practical applications it is useful to work with the unavailability $\overline{PA} = 1 - PA$. The main concern of this test is to check a *zero hypothesis* H_0: $\overline{PA} < \overline{PA}_0$ against an *alternative hypothesis* H_1: $\overline{PA} > \overline{PA}_1$ on the basis of the following agreement between producer and consumer:

> *The item should be accepted with a probability nearly equal to (but not less than) $1-\alpha$ if the true (unknown) unavailability $\overline{PA}$ is lower than $\overline{PA}_0$ but rejected with a probability nearly equal to (but not less than) $1-\beta$ if $\overline{PA}$ is greater than $\overline{PA}_1$ ($\overline{PA}_0$, $\overline{PA}_1 > \overline{PA}_0$, $0<\alpha<1-\beta<1$ are given (fixed) values).*

$\overline{PA}_0$ is the *specified* unavailability and $\overline{PA}_1$ is the *maximum acceptable* unavailability. α is the allowed *producer's risk* (type I error), i.e., the probability of *rejecting a true hypothesis* H_0: $\overline{PA} < \overline{PA}_0$. β is the allowed *consumer's risk* (type II error), i.e., the probability of *accepting* the hypothesis H_0: $\overline{PA} < \overline{PA}_0$ when the alternative hypothesis H_1: $\overline{PA} > \overline{PA}_1$ is true. Verification of the agreement stated above is a problem of statistical hypothesis testing (Appendix A8.3) and different approach are possible. In the following, the method introduced in Appendix A8.3.1.4 is given (comparison with other methods is in Section 7.2.2.3).

Assuming constant failure and repair rates $\lambda(x) = \lambda$ and $\mu(x) = \mu$, the procedure is as follows (see also [A8.29, A2.5 (IEC 61070)]):

1. For given (fixed) $\overline{PA}_0$, $\overline{PA}_1$, α, and β ($0<\alpha<1-\beta<1$), find the smallest integer n (1,2,...) which satisfy (Eq. (A8.91))

$$F_{2n,2n,1-\alpha} \cdot F_{2n,2n,1-\beta} \le \frac{\overline{PA}_1}{\overline{PA}_0} \cdot \frac{PA_0}{PA_1} = \frac{(1-PA_1)PA_0}{(1-PA_0)PA_1}, \qquad (7.19)$$

where $F_{2n,2n,1-\alpha}$ and $F_{2n,2n,1-\beta}$ are the $1-\alpha$ & $1-\beta$ quantiles of the F-distribution (Appendix A9.4), and compute the limiting value (Eq. (A8.92))

$$\delta = F_{2n,2n,1-\alpha}\, \overline{PA}_0 \,/\, PA_0 = F_{2n,2n,1-\alpha}\, (1-PA_0) \,/\, PA_0 . \qquad (7.20)$$

2. Observe n failure-free times $t_1, \ldots, t_n$ & corresponding repair times $t'_1, \ldots, t'_n$, and

$$\bullet \text{ reject } H_0: \overline{PA} < \overline{PA}_0, \quad \text{if} \quad \frac{t'_1 + \ldots + t'_n}{t_1 + \ldots + t_n} > \delta$$

$$\bullet \text{ accept } H_0: \overline{PA} < \overline{PA}_0, \quad \text{if} \quad \frac{t'_1 + \ldots + t'_n}{t_1 + \ldots + t_n} \le \delta . \qquad (7.21)$$

Table 7.2 gives n and δ for some values of $\overline{PA}_1 / \overline{PA}_0$ used in practical applications. It must be noted that the test duration is *not fixed in advance*. However, results for fixed time sample plans are not free of parameters (see e.g. the remark to Eq. (7.22)).

Table 7.2 Number n of failure-free times $\tau_1,\ldots,\tau_n$ & corresponding repair (restoration) times $\tau'_1,\ldots,\tau'_n$, and limiting value δ of the observed ratio $(t'_1+\ldots+t'_n)/(t_1+\ldots+t_n)$ to demonstrate $\overline{PA} < \overline{PA}_0$ against $\overline{PA} > \overline{PA}_1$ for various values of α (producer risk), β (consumer risk), and $\overline{PA}_1 / \overline{PA}_0$

	$\overline{PA}_1 / \overline{PA}_0 = 2$	$\overline{PA}_1 / \overline{PA}_0 = 4$	$\overline{PA}_1 / \overline{PA}_0 = 6$
$\alpha \approx \beta \lesseqgtr 0.1$	$n = 29$ $\delta = 1.41\,\overline{PA}_0 / PA_0$ $(PA_0 > 0.99)^*$	$n = 8$ $\delta = 1.93\,\overline{PA}_0 / PA_0$ $(PA_0 > 0.99)^*$	$n = 5$ $\delta = 2.32\,\overline{PA}_0 / PA_0$ $(PA_0 \ge 0.98)^*$
$\alpha \approx \beta \lesseqgtr 0.2$	$n = 13$ $\delta = 1.39\,\overline{PA}_0 / PA_0$ $(PA_0 \ge 0.99)^*$	$n = 4$ $\delta = 1.86\,\overline{PA}_0 / PA_0$ $(PA_0 > 0.98)^*$	$n = 3$ $\delta = 2.06\,\overline{PA}_0 / PA_0$ $(PA_0 > 0.99)^*$

*a lower n can be given (with corresponding δ as per Eq. (7.20)) for PA_0 smaller than the limit given

Corresponding values for the *availability* can be obtained using $PA = 1 - \overline{PA}$.

If failure free and/ or repair times are Erlangian distributed (Eq. (A6.102)) with $\beta_\lambda = n_\lambda$ and $\beta_\mu = n_\mu$, $F_{2n,2n,1-\alpha}$ and $F_{2n,2n,1-\beta}$ have to be replaced by $F_{2nn_\mu,2nn_\lambda,1-\alpha}$ and $F_{2nn_\lambda,2nn_\mu,1-\beta}$, for unchanged *MTTF* & *MTTR* (Example A8.11). Results based on distribution of *DT* (Eq. 7.2) are not parameter free (Section 7.2.2.3).

7.2.2.3 Further Availability Evaluation Methods (for Continuous Operation)

The approach introduced in Appendices A8.2.2.4 & A8.3.1.4 and given in Sections 7.2.2.1 & 7.2.2.2 yields to an exact solution based on the Fisher distribution for estimating and demonstrating an availability $PA = AA$, obtained by investigating DT/UT for exponentially or Erlangian distributed failure-free and repair times. Exponentially distributed failure-free times arise in many practical applications. The distribution of repair (restoration) times can often be approximated by an Erlang distribution (Eq. (A6.102) with $\beta > 3$). Generalization of the distribution of failure-free or repair times can lead to analytical difficulties. In the following some alternative approach for estimating and demonstrating an availability $PA = AA$ are briefly discussed and compared with the approach given in Sections 7.2.2.1 & 7.2.2.2 (item's behavior still described by an alternating renewal process (Fig. 6.2)).

A first possibility is to consider only the distribution of the down time *DT* (total repair or restoration time) in a given time interval $(0, t]$. At the given (fixed) time point t the item can be up or down, and Eq. (6.32) with $t - x$ instead of T_0 gives the distribution function of *DT* [A7.29 (1957)]. Moments of *DT* have been investigated in [A7.29 (1957)], mean and variance of the unavailability $\overline{PA} = 1 - PA = \mathrm{E}[DT/t]$ can thus be given for arbitrary distributions of failure-free and repair times. In particular, for the case of constant failure and repair rates ($\lambda(x) = \lambda$, $\mu(x) = \mu$) it holds that

$\Pr\{\text{total down time } DT \text{ in } (0, t] \le x \mid \text{new at } t=0\} =$

$$\Pr\{DT/t \le x/t \mid \text{new at } t=0\} = 1 - e^{-(\lambda(t-x)+\mu x)} \sum_{n=1}^{\infty} \Big(\frac{(\lambda(t-x))^n}{n!} \sum_{k=0}^{n-1} \frac{(\mu x)^k}{k!} \Big), \quad 0<x<t,$$

$$\lim_{t\to\infty} \mathrm{E}[DT/t] = \frac{\lambda}{\lambda+\mu} \approx \lambda/\mu, \quad \text{and} \quad \lim_{t\to\infty} \mathrm{Var}[DT/t] = \frac{2\lambda\mu}{t(\lambda+\mu)^3} \approx 2\lambda/t\mu^2. \qquad (7.22)$$

However, already for the case of constant failure and repair rates, results for interval estimation and demonstration test are not free of parameters (function of μ [A8.29] or λ [A8.18]). The use of the distribution of *DT*, or DT/t for fixed *t*, would bring the advantage of a test duration *t* fixed in advance, but results are not free of parameters and the method is thus of limited utility.

A second possibility is to assign to the state of the item an indicator $\zeta(t)$ taking values 1 for *item up* and 0 for *item down* (Boolean variable in Section 2.3.4). In this case it holds that $\mathrm{PA}(t) = \Pr\{\zeta(t)=1\}$, and thus $\mathrm{E}[\zeta(t)] = \mathrm{PA}(t)$ and $\mathrm{Var}[\zeta(t)] = \mathrm{E}[\zeta(t)^2] - \mathrm{E}^2[\zeta(t)] = \mathrm{PA}(t)(1-\mathrm{PA}(t))$ (Eq. (A6.118))). Investigation on $\mathrm{PA}(t)$ reduces to that on $\zeta(t)$, see e.g. [A7.4(1962)]. In particular, estimation and demonstration of $\mathrm{PA}(t)$ can be based on observations of $\zeta(t)$ at time points $t_1 < t_2 < \dots$. A basic problem here, is the choice of observation time points (randomly, at const. time intervals $\Delta = t_{i+1} - t_i$, or other). For the case of constant failure and repair rates (λ, μ), Eq. (6.20) yields $\mathrm{PA}(t) = \mathrm{PA}_{S0}(t) = \mu/(\lambda+\mu) + (\lambda/(\lambda+\mu))\, e^{-(\lambda+\mu)t}$ (item new at $t=0$). $\mathrm{PA}(t)$ convergence rapidly to $PA = AA = \mu/(\lambda+\mu) \approx 1-\lambda/\mu$. Furthermore, because of the *constant failure rate*, the joint availability is given by $\mathrm{JA}_{S0}(t, t+\Delta) = \mathrm{PA}_{S0}(t) \cdot \mathrm{PA}_{S0}(\Delta)$ (Eqs. (6.35)). Estimation and demonstration for the case of observations at constant time intervals Δ can thus be reduced to the case of an *unknown probability* $p = \overline{\mathrm{PA}}(\Delta) = 1-\mathrm{PA}(\Delta) \approx (1-e^{-\mu\Delta})\lambda/\mu \approx \lambda\Delta$ for $\Delta \ll 1/\mu$ or $p = \overline{PA} = \overline{AA} = \lambda/(\lambda+\mu) \approx \lambda/\mu$ for $\Delta \gg 1/\mu$ (Section 7.1).

A further possibility is to estimate and demonstrate λ and μ *separately* (Eqs. (7.28) - (7.30) and (7.33) - (7.35)) and put results in $\overline{PA} = \overline{AA} \approx \lambda/\mu$, or to consider the results of Section A8.2.2.3 yielding again to a Fisher distribution (Table A9.4). A refinement to an interval estimation of the form $\overline{PA} = \overline{AA} \approx \lambda/\mu \le \overline{PA}_u$ making use of the Chebyshev's inequality $\Pr\{|DT/t - \lambda/\mu| > \varepsilon\} \le 2\lambda/(t\mu^2\varepsilon^2) = \beta_1 = 1-\gamma$ (Eqs. (6.49), (A7.219), and (A7.220)), has been proposed recently [7.14], yielding $\Pr\{\overline{PA} \le \hat{\overline{PA}}_u = \hat\lambda/\hat\mu + \sqrt{2\hat\lambda/(t\hat\mu^2(1-\gamma))}\} \ge \gamma$ with $\hat\lambda$ & $\hat\mu$ as max. likelihood estimates for λ & μ and *t* as test time ((A7.220) follows from (6.33), not from (7.22)).

The different methods can basically be discussed by comparing Fig. 7.5 with Fig. 7.6 and Table 7.2 with Table 7.3. Analytical results based on the Fisher distribution yield broader confidence intervals and longer demonstration tests (this can be accepted, considering that λ *and* μ are unknown and that for high availability figures, higher $\hat{\overline{PA}}_u / \hat{\overline{PA}}_l$ or $\overline{PA}_1 / \overline{PA}_0$ can be agreed); the advantage being exact knowledge of the involved errors (β_1, β_2) or risks (α, β). However, for some aspects (test duration, possibility to verify maintainability with selected failures) it can become more appropriate to estimate and demonstrate λ and μ separately.

7.2.3 Estimation & Demonstration of a Constant Failure Rate λ (or of *MTBF* for the Case $MTBF = 1/\lambda$)

A constant (time independent) failure rate $\lambda(x) = \lambda$ occurs in many practical applications for nonrepairables items, as well as for repairable items *which are assumed as-good-as-new after repair* (x being the variable starting by $x = 0$ at the begin of the failure-free time considered, as for interarrival times). $\lambda(x) = \lambda$ implies that failure-free times are independent and exponentially distributed with the same parameter λ (Eq. (A6.81)). In this case, the reliability function is given by $R(x) = e^{-\lambda x}$ and for the mean time to failure, $MTTF = 1/\lambda$ holds for all failure-free times (Eq. (A6.84)). For the repairable case, *MTBF* (mean operating time between failures) is often used in practical applications instead of *MTTF*. However, *$MTBF = 1/\lambda$ holds only for the particular case $\lambda(x) = \lambda$*. To avoid misuses, *in this book MTBF is confined to the case $MTBF = 1/\lambda$*. A reason for the assumption of $\lambda(x) = \lambda$ is that, by neglecting repair times, *the flow of failures constitute a homogeneous Poisson process* (Appendix A7.2.5). This property characterizes exponentially distributed failure-free times and *highly simplifies investigations*.

This section deals with estimation and demonstration of a constant failure rate λ or of *MTBF* for the case $MTBF = 1/\lambda$ (see Appendix A8 for basic considerations and Sections 7.5 - 7.7 for further results. In particular, the case of a *given* (fixed) *cumulative operating time T* is considered, when repair times are neglected and individual failure-free times are assumed to be independent. Due to the relationship between exponentially distributed failure-free times and homogeneous Poisson process (Eq. (A7.39)) as well as the additive property of Poisson processes (Example 7.7),

the fixed cumulative operating time T can be partitioned in an arbitrary way from failure-free times of statistically independent and identical items,

see note to Table 7.3 for a rule. Following are some examples:

1. Operation of a single item that is immediately renewed after each failure (renewal time = 0); here, $T = t$ = calendar time = T_{test}.
2. Operation of n identical items, each of them being immediately renewed after each failure (renewal time = 0); here, $T = n\,t$ $(n = 1, 2, \ldots)$.

As stated above, in the case of a constant failure rate λ and immediate renewal, the failure process is a homogeneous Poisson process (HPP) with intensity λ (for $n = 1$) or $n\lambda$ (for $n > 1$) over the fixed time interval $(0, T = n\,t]$. Hence, the probability of k failures occurring within the cumulative operating time T is (Eq. (A7.41))

$$\Pr\{k \text{ failures within } T \mid \lambda\} = \frac{(\lambda T)^k}{k!} e^{-\lambda T}, \qquad k = 0, 1, 2, \ldots, \quad T = n\,t \text{ for } n > 1 .$$

Statistical procedures for estimation and demonstration of a failure rate λ can thus be based on the evaluation of the parameter $(m = \lambda T)$ of a Poisson distribution.

In addition to the case of a *given (fixed) cumulative operating time* T and immediate renewal (discussed above and investigated in the following Sections 7.2.3.1 - 7.2.3.3), for which the number k of failures in T is a sufficient statistic and $\hat{\lambda} = k / T$ is an unbiased estimate for λ, further possibilities are known. Assuming n identical items at $t = 0$ and labeling the *individual failure times* as $t_1 < t_2 < \ldots$, measured from $t = 0$, also following cases can occur in practical applications $(k > 1)$:

1. *Fixed number k of failures*, the test is stopped at the kth failure and failed items are *not renewed*; an unbiased point estimate of the failure rate λ is (Eq.(A8.35))

$$\hat{\lambda} = (k-1) / [n t_1 + (n-1)(t_2 - t_1) + \ldots + (n-k+1)(t_k - t_{k-1})]$$
$$= (k-1) / [t_1 + \ldots + t_k + (n-k) t_k]. \qquad (7.23)$$

2. *Fixed number k of failures*, the test is stopped at the kth failure and failed items are *instantaneously renewed*; an unbiased point estimate for λ is

$$\hat{\lambda} = (k-1) / (n t_1 + n(t_2 - t_1) + \ldots + n(t_k - t_{k-1}) = (k-1) / n t_k . \qquad (7.24)$$

3. *Fixed test time t*, failed items are *not renewed*; a biased point estimate of the failure rate λ (given k items have failed in $(0, t]$) is

$$\hat{\lambda} = k / [n t_1 + (n-1)(t_2 - t_1) + \ldots + (n-k)(t - t_k)] = k / [t_1 + \ldots + t_k + (n-k) t]. \qquad (7.25)$$

Example 7.7

An item with constant failure rate λ operates first for a fixed time T_1 and then for a fixed time T_2. Repair times are neglected. Give the probability that k failures will occur in $T = T_1 + T_2$.

Solution

The item's behavior within each of the time periods T_1 and T_2 can be described by a homogeneous Poisson process with intensity λ. From Eq. (A7.39) it follows that

$$\Pr\{i \text{ failures in the time period } T_1 \mid \lambda\} = \frac{(\lambda T_1)^i}{i!} e^{-\lambda T_1}$$

and, because of the *memoryless property* of the homogeneous Poisson process

$$\Pr\{k \text{ failures in } T = T_1 + T_2 \mid \lambda\} = \sum_{i=0}^{k} \frac{(\lambda T_1)^i}{i!} e^{-\lambda T_1} \cdot \frac{(\lambda T_2)^{k-i}}{(k-i)!} e^{-\lambda T_2}$$
$$= e^{-\lambda T} \lambda^k \sum_{i=0}^{k} \frac{T_1^i}{i!} \cdot \frac{T_2^{k-i}}{(k-i)!} = \frac{(\lambda T)^k}{k!} e^{-\lambda T}. \qquad (7.26)$$

The last part of Eq. (7.26) follows from the binomial expansion of $(T_1 + T_2)^k$. Eq. (7.26) shows that for λ *constant*, the *cumulative operating time* T can be partitioned in any *arbitrary way* (see note to Table 7.3 for a practical rule).

Supplementary result: The same procedure can be used to prove that the *sum of two independent homogeneous Poisson processes* with intensities λ_1 and λ_2 is a homogeneous *Poisson process* with intensity $\lambda_1 + \lambda_2$; in fact,

$$\Pr\{k \text{ failures in } (0,T] \mid \lambda_1,\lambda_2\}$$
$$= \sum_{i=0}^{k} \frac{(\lambda_1 T)^i}{i!} e^{-\lambda_1 T} \frac{(\lambda_2 T)^{k-i}}{(k-i)!} e^{-\lambda_2 T} = \frac{((\lambda_1+\lambda_2)T)^k}{k!} e^{-(\lambda_1+\lambda_2)T}. \tag{7.27}$$

This result can be extended to *nonhomogeneous Poisson processes.*

7.2.3.1 Estimation of a Constant Failure Rate λ +) (or of *MTBF* for the Case *MTBF* =1/λ)

Let us consider an item with a constant failure rate λ. If during the *given (fixed) cumulative operating time* T +) exactly k failures have occurred, the maximum likelihood *point estimate* for the unknown parameter λ follows as

$$\hat{\lambda} = \frac{k}{T}, \qquad k = 0, 1, 2, \ldots, \quad \mathrm{E}[\hat{\lambda}] = \lambda, \ \mathrm{Var}[\hat{\lambda}] = \lambda / T, \tag{7.28}$$

(Eq. (A8.46) or Example A6.21 ($m=\lambda T$), Eqs. (A6. 40) & (A6.46)). For a given *confidence level* $\gamma = 1-\beta_1-\beta_2$ $(0<\beta_1<1-\beta_2<1)$ and $k>0$, lower $\hat{\lambda}_l$ and upper $\hat{\lambda}_u$ limits of the *confidence interval* for λ can be obtained from (Eqs. (A8.47) - (A8.51))

$$\sum_{i=k}^{\infty} \frac{(\hat{\lambda}_l T)^i}{i!} e^{-\hat{\lambda}_l T} = \beta_2 \qquad \text{and} \qquad \sum_{i=0}^{k} \frac{(\hat{\lambda}_u T)^i}{i!} e^{-\hat{\lambda}_u T} = \beta_1, \tag{7.29}$$

or from

$$\hat{\lambda}_l = \frac{\chi^2_{2k,\beta_2}}{2T} \qquad \text{and} \qquad \hat{\lambda}_u = \frac{\chi^2_{2(k+1),1-\beta_1}}{2T}, \tag{7.30}$$

using the quantile of the χ^2-distribution (Table A9.2). For $k=0$, Eq. (A8.49) yields

$$\hat{\lambda}_l = 0 \qquad \text{and} \qquad \hat{\lambda}_u = \frac{\ln(1/\beta_1)}{T}, \qquad \text{with } \gamma = 1-\beta_1. \tag{7.31}$$

Figure 7.6 gives confidence limits $\hat{\lambda}_l / \hat{\lambda}$ and $\hat{\lambda}_u / \hat{\lambda}$ for $\hat{\lambda} = k/T$ and $\beta_1 = \beta_2 = (1-\gamma)/2$, useful for practical applications.

For the case $MTBF = 1/\lambda$, $M\hat{T}BF = T/k$, $k \ge 1$, is biased; unbiased for $\lambda T >> 1$ is $M\hat{T}BF = T/(k+1)$, yielding $\mathrm{E}[T/(k+1)] = (1-e^{-\lambda T})/\lambda$. For practical applications, $M\hat{T}BF_l \approx 1/\hat{\lambda}_u$ and $M\hat{T}BF_u \approx 1/\hat{\lambda}_l$ can often be used.

+) The case considered in Sections 7.2.3.1 to 7.2.3.3 corresponds to a sampling plan with n elements ($n=1,2,\ldots$) *with replacement* and k failures in the *given* (fixed) *time interval* $(0, T/n]$, *Type 1* (time) *censoring*; the underlying process is a homogeneous Poisson process with intensity $n\lambda$.

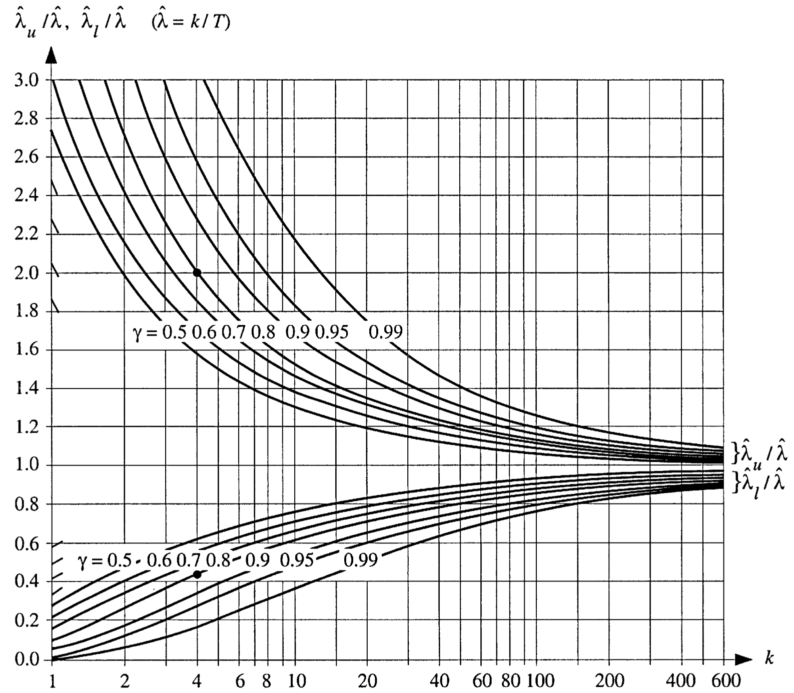


Figure 7.6 Confidence limits $\hat{\lambda}_l / \hat{\lambda}$, $\hat{\lambda}_u / \hat{\lambda}$ for an unknown constant failure rate λ per Eqs.7.28) & (7.29) (*T=given (fixed) cumulative* operating time (time censoring), k = number of failures during T, $\gamma = 1-\beta_1-\beta_2 =$ confidence level (here $\beta_1 = \beta_2 = (1-\gamma)/2$)); for the case $MTBF = 1/\lambda$, it holds $\hat{MTBF} = 1/\hat{\lambda}$ (unbiased for $k >> 1$) and $\hat{MTBF}_l \approx 1/\hat{\lambda}_u$, $\hat{MTBF}_u \approx 1/\hat{\lambda}_l$); • Examples 7.8, 7.13

Confidence limits $\hat{\lambda}_l$, $\hat{\lambda}_u$ can also be used to give *one-sided confidence intervals*:

$$\lambda \le \hat{\lambda}_u, \qquad \text{with} \quad \beta_2 = 0 \quad \text{and} \quad \gamma = 1-\beta_1,$$

or

$$\lambda \ge \hat{\lambda}_l, \qquad \text{with} \quad \beta_1 = 0 \quad \text{and} \quad \gamma = 1-\beta_2, \tag{7.32}$$

i.e., $MTBF \ge \hat{MTBF}_l \approx 1/\hat{\lambda}_u$ or $MTBF \le \hat{MTBF}_u \approx 1/\hat{\lambda}_l$ for the case $MTBF = 1/\lambda$.

Example 7.8

In testing a subassembly with constant failure rate λ, 4 failures occur during $T = 10^4$ cumulative operating hours. Find the confidence interval of λ for a confidence level $\gamma = 0.8$ ($\beta_1 = \beta_2 = 0.1$).

Solution

From Fig. 7.6 it follows that for $k=4$ and $\gamma = 0.8$, $\hat{\lambda}_l/\hat{\lambda} \approx 0.44$ and $\hat{\lambda}_u/\hat{\lambda} \approx 2$. With $T = 10^4$ h, $k = 4$, and $\hat{\lambda} = 4 \cdot 10^{-4}\,\text{h}^{-1}$, the confidence limits are $\hat{\lambda}_l \approx 1.7 \cdot 10^{-4}\,\text{h}^{-1}$ and $\hat{\lambda}_u \approx 8 \cdot 10^{-4}\,\text{h}^{-1}$.

Supplementary result: Corresponding one-sided conf. interval is $\lambda \le 8 \cdot 10^{-4}\,h^{-1}$ with $\gamma = 0.9$.

In the above considerations (Eqs. (7.28) - (7.32)), the *cumulative operating time* T was given (fixed), independent of the individual failure-free times and the number n of items involved (*Type I censoring*). The situation is different when *the number of failures* k is given (fixed), i.e., when the test is stopped at the occurrence of the kth failure (*Type II censoring*). Here, the cumulative operating time is a *random variable* (term $(k-1)/\hat{\lambda}$ of Eqs. (7.23) and (7.24)). Using the memoryless property of homogeneous Poisson processes, it can be shown that the quantities

$$n(t_i - t_{i-1}) \text{ for renewal, and } (n-i+1)(t_i - t_{i-1}) \text{ for no renewal}, \quad i=1,2,\ldots,k, \; t_0=0,$$

are independent observations of exponentially distributed random variables with parameters $n\lambda$ and $(n-i+1)\lambda$, respectively This is necessary and sufficient to prove that the $\hat{\lambda}$ given by Eqs. (7.23) and (7.24) are maximum likelihood estimates for λ. For confidence intervals, results of Appendix A8.2.2.3 can be used.

In some practical applications, *system's* failure rate confidence limits as a function of *component's* failure rate confidence limits is sought. Monte Carlo simulation can help. However, for a series system with n elements, *constant failure rates* $\lambda_1,\ldots,\lambda_n$, time censoring, and same observation time T, Eqs. (2.19), (7.28), and (7.27) yield $\hat{\lambda}_S = \hat{\lambda}_1 + \ldots + \hat{\lambda}_n$. Furthermore, for given fixed T, $2T\lambda_i$ (considered here as random variable, Appendix A8.2.2.2) has a χ^2 distribution with $2(k_i+1)$ degrees of freedom (Eq. (A8.48), Table A9.2); thus, $2T\lambda_S$ has a χ^2 distribution with $\sum 2(k_i+1)$ degrees of freedom. From this, it can be shown [7.17] that for λ_{i_u} (upper limit of the confidence interval) obtained from $\Pr\{2T\lambda_i \le 2T\hat{\lambda}_{i_u}\} = \Pr\{\lambda_i \le \hat{\lambda}_{i_u}\} \gtrapprox 0.8 = \gamma$ $(i=1,\ldots,n)$ it holds that $\Pr\{\lambda_S \le \hat{\lambda}_{1_u} + \ldots + \hat{\lambda}_{n_u}\} \ge \gamma$. Extension to different observation times T_i, series-parallel structures, or Erlangian distributed failure-free times is possible [7.17]. Estimation of λ/μ as approximation for an unavailability $\lambda/(\lambda+\mu)$ is given in Section 7.2.2.1.

7.2.3.2 Simple Two-sided Test for the Demonstration of a Constant Failure Rate λ (or of *MTBF* for the Case $MTBF = 1/\lambda$)

In the context of an acceptance test, demonstration of a constant failure rate λ (or of *MTBF* for the case $MTBF = 1/\lambda$) is often required, not merely its estimation as in Section 7.2.3.1. The main concern of this test is to check a zero hypothesis H_0: $\lambda < \lambda_0$ against an alternative hypothesis H_1: $\lambda > \lambda_1$, on the basis of the following agreement between producer and consumer:

> *Items should be accepted with a probability nearly equal to (but not less than)* $1-\alpha$, *if the true (unknown)* λ *is less than* λ_0, *but rejected with a probability nearly equal to (but not less than)* $1-\beta$, *if* λ *is greater than* λ_1 (λ_0, $\lambda_1 > \lambda_0$, *and* $0 < \alpha < 1-\beta < 1$ *are given (fixed) values*).

λ_0 is the *specified* λ and λ_1 is the *maximum acceptable* λ ($1/m_0$ and $1/m_1$ in *IEC 60605* [7.19] or $1/\theta_0$ and $1/\theta_1$ in *MIL-STD-781* [7.23] for the case $MTBF = 1/\lambda$).

α is the allowed *producer's risk* (type I error), i.e., the probability of rejecting a true hypothesis H_0: $\lambda<\lambda_0$. β is the allowed *consumer's risk* (type II error), i.e., the probability of accepting H_0 when the alternative hypothesis H_1: $\lambda > \lambda_1$ is true. Evaluation of the above agreement is a problem of statistical hypothesis testing (Appendix A8.3), and can be performed e. g. with a *simple two-sided test* or a *sequential test.*

With the *simple two-sided test* (also known as the *fixed length test*), the cumulative operating time T and the number of allowed failure c during T are fixed quantities. The procedure (test plan) follows in a way similar to that developed in Appendix A8.3.1.1 as:

1. From λ_0, λ_1, α, β determine the smallest integer c and the value T satisfying

$$\sum_{i=0}^{c} \frac{(\lambda_0 T)^i}{i!} e^{-\lambda_0 T} \geq 1-\alpha \tag{7.33}$$

and

$$\sum_{i=0}^{c} \frac{(\lambda_1 T)^i}{i!} e^{-\lambda_1 T} \leq \beta. \tag{7.34}$$

2. Perform a test with a total *cumulative operating time T*, determine the number of failures k during the test, and

   - reject H_0: $\lambda < \lambda_0$, if $k > c$
   - accept H_0: $\lambda < \lambda_0$, if $k \leq c$. (7.35)

For the case $MTBF = 1/\lambda$, the above procedure can be used to test H_0: $MTBF > MTBF_0$ against H_1: $MTBF < MTBF_1$, by replacing $\lambda_0 = 1/MTBF_0$ and $\lambda_1 = 1/MTBF_1$.

Example 7.9

Following conditions have been specified for the demonstration (acceptance test) of the constant (time independent) failure rate λ of an assembly: $\lambda_0 = 1/2000\,h$ (specified λ), $\lambda_1 = 1/1000\,h$ (minimum acceptable λ), producer risk $\alpha = 0.2$, consumer risk $\beta = 0.2$. Give: (i) the cumulative test time T and the allowed number of failures c during T; (ii) the probability of acceptance if the true failure rate λ were $1/3000\,h$.

Solution

(i) From Fig. 7.3, $c=6$ and $m \approx 4.6$ for Pr{acceptance} ≈ 0.82, $c=6$ and $m \approx 9.2$ for Pr{acceptance} ≈ 0.19 (see Example 7.2 for the procedure); thus $c=6$ and $T \approx 9200h$. These values agree well with those obtained from Table A9.2 ($\nu=14$), as given also in Table 7.3.
(ii) For $\lambda = 1/3000\,h$, $T = 9200\,h$, $c=6$

$$\Pr\{\text{acceptance} \mid \lambda = 1/3000\text{ h}\}$$
$$= \Pr\{\text{no more than 6 failures in } T = 9200\text{ h} \mid \lambda = 1/3000\text{ h}\} = \sum_{i=0}^{6} \frac{3.07^i}{i!} e^{-3.07} \approx 0.96,$$

see also Fig. 7.3 for $m = 3.07$ and $c = 6$.

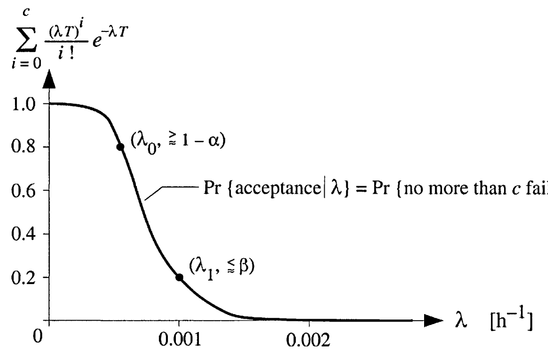


Figure 7.7 Operating characteristic (curve or acceptance probability curve) as a function of λ for fixed T and c ($\lambda_0 = 1/2000\text{h}$, $\lambda_1 = 1/1000\text{h}$, $\alpha \approx \beta \approx 0.2$; $T = 9200\,\text{h}$ and $c = 6$ as per Table 7.3; see also Fig. 7.3) (holds for $MTBF_0 = 2000\text{h}$ and $MTBF_1 = 1000\text{h}$, for the case $MTBF = 1/\lambda$)

The graph of Fig. 7.7 visualizes the validity of the above agreement between producer and consumer (customer). It satisfies the inequalities (7.33) and (7.34), and is known as *operating characteristic* (curve) or acceptance probability (curve). For each value of λ, it gives the probability of having not more than c failures during a cumulative operating time T. Since the operating characteristic (curve) as a function of λ is strictly decreasing (in this case), the risk for a false decision decreases for $\lambda < \lambda_0$ and $\lambda > \lambda_1$, respectively. It can be shown that the quantities c and $\lambda_0 T$ depend only on α, β, and the ratio λ_1 / λ_0 (*discrimination ratio*).

Table 7.3 gives c and $\lambda_0 T$ for some values of α, β and λ_1/λ_0 useful for practical applications. For the case $MTBF = 1/\lambda$, Table 7.3 holds for testing H_0: $MTBF > MTBF_0$ against H_1: $MTBF < MTBF_1$, by setting $\lambda_0 = 1/MTBF_0$ and $\lambda_1 = 1/MTBF_1$. Table 7.3 can also be used for the demonstration of an *unknown probability p* (Eqs. (7.8) and (7.9)) in the case where the *Poisson approximation* applies. A large number of test plans are in *international standards* [7.19 (61124)].

In addition to the simple two-sided test described above, a *sequential test* is often used (see Appendix A8.3.1.2 and Section 7.1.2.2 for basic considerations and Fig. 7.8 for an example). In this test, neither the cumulative operating time T, nor the number c of allowed failures during T are specified before the test begins. The number of failures is recorded as a function of the cumulative operating time (normalized to $1/\lambda_0$). As soon as the resulting staircase curve crosses the *acceptance line* or the *rejection line* the test is stopped. Sequential tests offer the advantage that on average the test duration is shorter than with simple two-sided tests. Using Eq. (7.12) with $p_0 = 1 - e^{-\lambda_0 \delta t}$, $p_1 = 1 - e^{-\lambda_1 \delta t}$, $n = T/\delta t$, and $\delta \to 0$ (continuous in time), the acceptance and rejection lines are obtained as

Table 7.3 Number of allowed failures c during the cumulative operating time T and value of $\lambda_0 T$ to demonstrate $\lambda < \lambda_0$ against $\lambda > \lambda_1$ for various values of α (producer risk), β (consumer risk), and λ_1 / λ_0 (can be used to test $MTBF < MTBF_0$ against $MTBF > MTBF_1$ for the case $MTBF = 1/\lambda$ or, using $\lambda_0 T = np_0$, to test $p < p_0$ against $p > p_1$ for an unknown probability p)

	$\lambda_1/\lambda_0 = 1.5$	$\lambda_1/\lambda_0 = 2$	$\lambda_1/\lambda_0 = 3$
$\alpha \approx \beta \lessapprox 0.1$	$c = 40$ $\lambda_0 T \approx 32.98$ $(\alpha \approx \beta \approx 0.098)$	$c = 14^*$ $\lambda_0 T \approx 10.17$ $(\alpha \approx \beta \approx 0.093)$	$c = 5$ $\lambda_0 T \approx 3.12$ $(\alpha \approx \beta \approx 0.096)$
$\alpha \approx \beta \lessapprox 0.2$	$c = 17$ $\lambda_0 T \approx 14.33$ $(\alpha \approx \beta \approx 0.197)$	$c = 6$ $\lambda_0 T \approx 4.62$ $(\alpha \approx \beta \approx 0.185)$	$c = 2$ $\lambda_0 T \approx 1.47$ $(\alpha \approx \beta \approx 0.184)$
$\alpha \approx \beta \lessapprox 0.3$	$c = 6$ $\lambda_0 T \approx 5.41$ $(\alpha \approx \beta \approx 0.2997)$	$c = 2$ $\lambda_0 T \approx 1.85$ $(\alpha \approx \beta \approx 0.284)$	$c = 1$ $\lambda_0 T \approx 0.92$ $(\alpha \approx \beta \approx 0.236)$

* $c = 13$ yields $\lambda_0 T = 9.48$ and $\alpha \approx \beta \approx 0.1003$; number of items under test $\approx T\lambda_0$, as a rule of thumb

- acceptance line: $y_1(x) = ax - b_1,$ (7.36)
- rejection line: $y_2(x) = ax + b_2,$ (7.37)

with $x = \lambda_0 T$, and

$$a = \frac{(\lambda_1/\lambda_0) - 1}{\ln(\lambda_1/\lambda_0)}, \qquad b_1 = \frac{\ln(1-\alpha)/\beta}{\ln(\lambda_1/\lambda_0)}, \qquad b_2 = \frac{\ln(1-\beta)/\alpha}{\ln(\lambda_1/\lambda_0)}. \tag{7.38}$$

Sequential tests used in practical applications are given in *international standards* [7.19 (61124)]. To limit testing effort, restrictions are often placed on the test duration and the number of allowed failures. Figure 7.8 shows two *truncated* sequential test plans for $\alpha \approx \beta \approx 0.2$ and $\lambda_1 / \lambda_0 = 1.5$ and 2, respectively. The lines defined by Eqs. (7.36) - (7.38) are shown dashed in Fig. 7.8a.

Example 7.10

Continuing with Example 7.9, give the expected test duration by assuming that the true λ equals λ_0 and a sequential test as per Fig. 7.8 is used.

Solution

From Fig. 7.8 with $\lambda_1 / \lambda_0 = 2$ it follows that $\mathrm{E}[\text{test duration} \mid \lambda = \lambda_0] \approx 2.4/\lambda_0 = 4800\,\text{h}$.

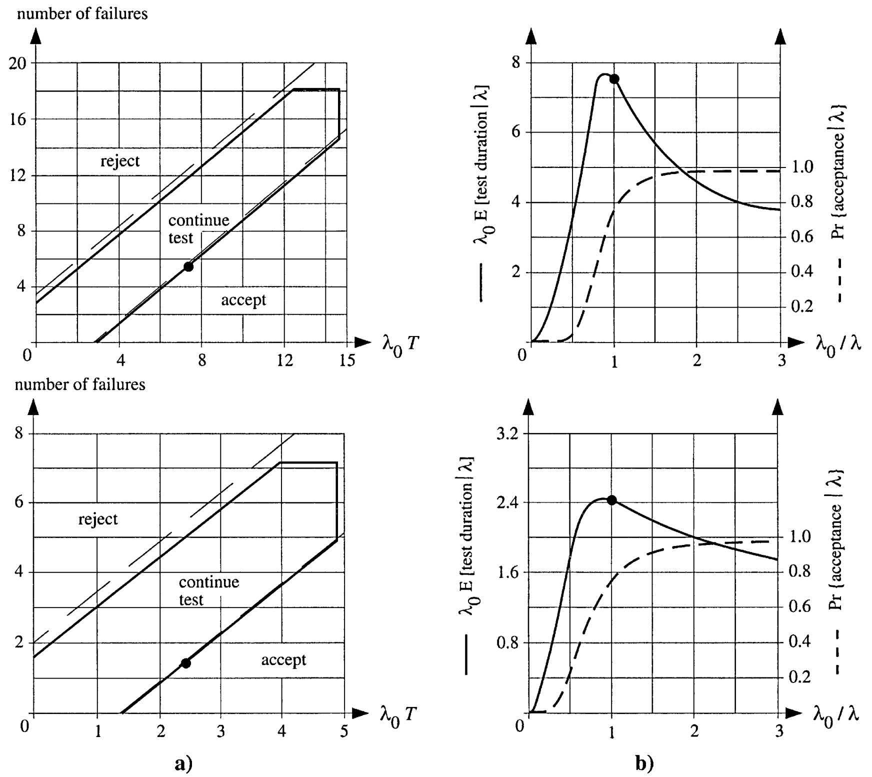


Figure 7.8 **a)** Sequential test plan to demonstrate $\lambda < \lambda_0$ against $\lambda > \lambda_1$ for $\alpha \approx \beta \approx 0.2$ and $\lambda_1 / \lambda_0 = 1.5$ (top), $\lambda_1 / \lambda_0 = 2$ (down), as per *IEC 61124* and *MIL-HDBK-781* [7.19, 7.23] (dashed on the left are the lines given by Eqs. (7.36) - (7.38)); **b)** Expected test duration until acceptance (continuous) and operating characteristic (curve) (dashed) as a function of λ_0 / λ (can be used to test $MTBF < MTBF_0$ against $MTBF > MTBF_1$, for the case $MTBF = 1/\lambda$)

7.2.3.3 Simple One-sided Test for the Demonstration of a Constant Failure Rate λ (or of *MTBF* for the Case $MTBF = 1/\lambda$)

Simple two-sided tests (Fig. 7.7) and sequential tests (Fig. 7.8) have the advantage that, for $\alpha = \beta$, producer and consumer run the same risk of making a false decision. However, in practical applications often only λ_0 and α or λ_1 and β, i.e. *simple one-sided tests*, are used. The considerations of Section 7.1.3 apply and care should be taken with small values of c, as operating with λ_0 and α (or λ_1 and β) the producer (or consumer) can be favored. Figure 7.9 shows the operating characteristic (curves) for various values of c as a function of λ for the demonstration of $\lambda < 1/1000\text{h}$ against $\lambda > 1/1000\text{h}$ with consumer risk $\beta \approx 0.2$ for $\lambda = 1/1000\text{h}$, and visualizes the reduction of producer's risk ($\alpha \approx 0.8$ for $\lambda = 1/1000\text{h}$) by decreasing λ, or increasing c (counterpart of Fig. 7.4).

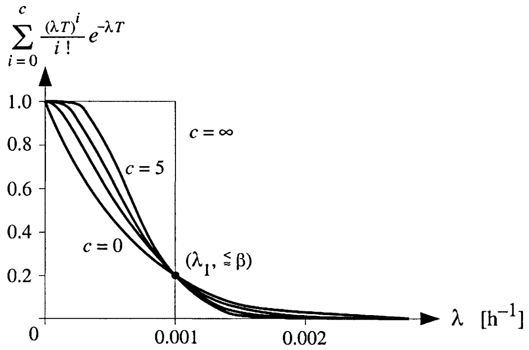


Figure 7.9 Operating characteristic (curves or acceptance probability curves) for $\lambda_1 = 1/1000\,\text{h}$, $\beta = 0.2$, and $c = 0$ ($T = 1610\,\text{h}$), $c = 1$ ($T = 2995\,\text{h}$), $c = 2$ ($T = 4280\,\text{h}$), $c = 5$ ($T = 7905\,\text{h}$), and $c = \infty$ ($T = \infty$) (holds for $MTBF_1 = 1000\text{h}$, for the case $MTBF = 1/\lambda$)

7.3 Statistical Maintainability Tests

Maintainability is generally expressed as a *probability*. In this case, results of Sections 7.1 and 7.2.1 can be used to estimate or demonstrate maintainability. However, estimation and demonstration of specific parameters, for instance *MTTR* (mean time to repair) is important for practical applications. If the underlying random values are exponentially distributed (constant repair rate μ), the results of Section 7.2.3 for a *constant failure rate* λ can be used. This section deals with the estimation and demonstration of an *MTTR* by assuming that repair time is *lognormally* distributed (for Erlangian distributed repair times, results of Section 7.2.3 can be used, considering Eqs. (A6.102) & (A6.103)). To simplify the notation, realizations (observations) of a repair time τ' will be denoted in this Section by $t_1, \ldots, t_n$ instead of $t'_1, \ldots, t'_n$.

7.3.1 Estimation of an *MTTR*

Let $t_1, \ldots, t_n$ be independent observations (realizations) of the repair time τ' of a given item. From Eqs. (A8.6) and (A8.10), the *empirical mean* and *variance* of τ' are given by

$$\hat{\mathrm{E}}[\tau'] = \frac{1}{n} \sum_{i=1}^{n} t_i, \tag{7.39}$$

$$\hat{\mathrm{Var}}[\tau'] = \frac{1}{n-1}\sum_{i=1}^{n}(t_i - \hat{\mathrm{E}}[\tau'])^2 = \frac{1}{n-1}[\sum_{i=1}^{n} t_i^2 - \frac{1}{n}(\sum_{i=1}^{n} t_i)^2]. \tag{7.40}$$

For these estimates it holds that $\mathrm{E}[\hat{\mathrm{E}}[\tau']] = \mathrm{E}[\tau'] = MTTR$, $\mathrm{Var}[\hat{\mathrm{E}}[\tau']] = \mathrm{Var}[\tau']/n$, and $\mathrm{E}[\hat{\mathrm{Var}}[\tau']] = \mathrm{Var}[\tau']$ (Appendix A8.1.2). As stated above, the repair time τ' can often be assumed lognormally distributed with distribution function (Eq. (A6.110))

$$\mathrm{F}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln(\lambda t)}{\sigma}} e^{-x^2/2} dx \tag{7.41}$$

and with mean and variance given by (Eqs. (A6.112) and (A6.113))

$$\mathrm{E}[\tau'] = MTTR = \frac{e^{\sigma^2/2}}{\lambda}, \qquad \mathrm{Var}[\tau'] = \frac{e^{2\sigma^2} - e^{\sigma^2}}{\lambda^2} = MTTR^2(e^{\sigma^2} - 1). \tag{7.42}$$

Form Eq. (7.41) one recognizes that $\ln\tau$ is normally distributed with mean $1/\ln\lambda$ and Variance σ^2. Using Eqs. (A8.24) and (A8.27), the *maximum likelihood* estimation of λ and σ^2 is obtained from

$$\hat{\lambda} = [\prod_{i=1}^{n} \frac{1}{t_i}]^{1/n} \qquad \text{and} \qquad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} [\ln(\hat{\lambda} t_i)]^2. \tag{7.43}$$

A *point estimate* for λ and σ can also be obtained by the method of quantiles. The idea is to substitute some particular quantiles with the corresponding empirical quantiles to obtain estimates for λ or σ. For $t = 1/\lambda$, $\ln(\lambda t) = 0$ and $\mathrm{F}(1/\lambda) = 0.5$, therefore, $1/\lambda$ is the 0.5 quantile (median) $t_{0.5}$ of the distribution function $\mathrm{F}(t)$ given by Eq. (7.41). From the empirical 0.5 quantile $\hat{t}_{0.5} = \inf(t\colon \hat{\mathrm{F}}_n(t) \ge 0.5)$ an estimate for λ follows as

$$\hat{\lambda} = \frac{1}{\hat{t}_{0.5}}. \tag{7.44}$$

Moreover, $t = e^{\sigma}/\lambda$ yields $\mathrm{F}(e^{\sigma}/\lambda) = 0.841$ (Table A9.1); thus $e^{\sigma}/\lambda = t_{0.841}$ is the 0.841 quantile of $\mathrm{F}(t)$ given by Eq. (7.41). Using $\lambda = 1/\hat{t}_{0.5}$ and $\sigma = \ln(\lambda t_{0.841}) = \ln(t_{0.841}/t_{0.5})$, an estimate for σ is obtained as

$$\hat{\sigma} = \ln(\hat{t}_{0.841}/\hat{t}_{0.5}). \tag{7.45}$$

Furthermore, considering $\mathrm{F}(e^{-\sigma}/\lambda) = 1 - 0.841 = 0.159$, i.e. $t_{0.159} = e^{-\sigma}/\lambda$, it follows that $e^{2\sigma} = \lambda t_{0.841}/\lambda t_{0.159}$ and thus Eq. (7.45) can be replaced by

$$\hat{\sigma} = \frac{1}{2}\ln(\hat{t}_{0.841}/\hat{t}_{0.159}). \tag{7.46}$$

The possibility of representing a lognormal distribution function as a straight line, to simplify interpretation of data, is discussed in Section 7.5.1 (Fig. 7.14, Appendix A9.8.1).

To obtain *interval estimates* for the parameters λ and σ, note that the logarithm of a lognormally distributed variable is normally distributed with mean $\ln(1/\lambda)$ and variance σ^2. Applying the transformation $t_i \rightarrow \ln t_i$ to the individual observations $t_1, \ldots, t_n$ and using the results known for the interval estimation of the parameters of a normal distribution [A6.1, A6.4], the confidence intervals

$$\left[\frac{n\,\hat{\sigma}^2}{\chi^2_{n-1,(1+\gamma)/2}}, \frac{n\,\hat{\sigma}^2}{\chi^2_{n-1,(1-\gamma)/2}}\right] \qquad (7.47)$$

for σ^2, and

$$[\hat{\lambda} e^{-\varepsilon}, \hat{\lambda} e^{\varepsilon}] \qquad \text{with} \qquad \varepsilon = \frac{\hat{\sigma}}{\sqrt{n-1}}\, t_{n-1,\,(1+\gamma)/2} \qquad (7.48)$$

for λ can be found with $\hat{\lambda}$ and $\hat{\sigma}$ as in Eq. (7.43). $\chi^2_{n-1,q}$ and $t_{n-1,q}$ are the q quantiles of the χ^2 and t-distribution with $n-1$ degrees of freedom, respectively (Tables A9.2 and A9.3).

Example 7.11

Let 1.1, 1.3, 1.6, 1.9, 2.0, 2.3, 2.4, 2.7, 3.1, and 4.2 h be 10 independent observations (realizations) of a lognormally distributed repair time. Give the maximum likelihood estimate and, for $\gamma = 0.9$, the confidence interval for the parameters λ and σ^2, as well as the maximum likelihood estimate for *MTTR*.

Solution

Equation (7.43) yields $\hat{\lambda} \approx 0.476\,\text{h}^{-1}$ and $\hat{\sigma}^2 \approx 0.146$ as maximum likelihood estimates for λ and σ^2. From Eq. (7.42), $\hat{MTTR} \approx e^{0.073} / 0.476\,\text{h}^{-1} \approx 2.26\,\text{h}$. Using Eqs. (7.47) and (7.48), as well as Tables A9.2 and A9.3, the confidence intervals are $[1.46/16.919, 1.46/3.325] \approx [0.086, 0.44]$ for σ^2 and $[0.476 e^{-0.127 \cdot 1.833}, 0.476 e^{0.127 \cdot 1.833}]\,\text{h}^{-1} \approx [0.38, 0.60]\,\text{h}^{-1}$ for λ, respectively.

7.3.2 Demonstration of an *MTTR*

The demonstration of an *MTTR* (in an acceptance test) will be investigated here by assuming that the repair time τ' is *lognormally* distributed with *known* σ^2 (method 1A of *MIL-STD-471* [7.23]). A rule is sought to test the null hypothesis H_0: $MTTR = MTTR_0$ against the alternative hypothesis H_1: $MTTR = MTTR_1$ for given type I error α and type II error β (Appendix A8.3). The procedure (test plan) is as follows:

1. From α and β ($0 < \alpha < 1 - \beta < 1$), determine the quantiles t_β and $t_{1-\alpha}$ of the standard normal distribution (Table A9.1)

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t_\beta} e^{-x^2/2}\, dx = \beta \qquad \text{and} \qquad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t_{1-\alpha}} e^{-x^2/2}\, dx = 1 - \alpha. \tag{7.49}$$

From $MTTR_0$ and $MTTR_1$, compute the sample size n (next highest integer)

$$n = \frac{(t_{1-\alpha}\, MTTR_0 - t_\beta\, MTTR_1)^2}{(MTTR_1 - MTTR_0)^2} (e^{\sigma^2} - 1). \tag{7.50}$$

2. Perform n independent repairs and record the observed repair times $t_1, \ldots, t_n$ (representative sample of repair times).

3. Compute $\hat{\mathrm{E}}[\tau']$ according to Eq. (7.39) and reject $H_0{:}\, MTTR = MTTR_0$ if

$$\hat{\mathrm{E}}[\tau'] > c = MTTR_0\, (1 + t_{1-\alpha} \sqrt{(e^{\sigma^2} - 1)/n}\,), \tag{7.51}$$

otherwise accept H_0.

The proof of the above rule implies a sample size $n > 10$, so that the quantity $\hat{\mathrm{E}}[\tau']$ can be assumed to have a normal distribution with mean $MTTR$ and variance $\mathrm{Var}[\tau']/n$ (Eqs. (A6.148), (A8.7), (A8.8)). Considering the type I and type II errors

$$\alpha = \Pr\{\hat{\mathrm{E}}[\tau'] > c \mid MTTR = MTTR_0\}, \qquad \beta = \Pr\{\hat{\mathrm{E}}[\tau'] \le c \mid MTTR = MTTR_1\},$$

and using Eqs. (A6.105) and (7.49), the relationship

$$c = MTTR_0 + t_{1-\alpha} \sqrt{\mathrm{Var}_0[\tau']/n} = MTTR_1 + t_\beta \sqrt{\mathrm{Var}_1[\tau']/n} \tag{7.52}$$

can be found, with $\mathrm{Var}_0[\tau'] = (e^{\sigma^2} - 1) MTTR_0^2$ for $t_{1-\alpha}$ and $\mathrm{Var}_1[\tau'] = (e^{\sigma^2} - 1) MTTR_1^2$ for t_β according to Eq. (7.42). The sample size n (Eq. (7.50)) follows then from Eq. (7.52) and the right hand side of Eq. (7.51) is equal to the constant c as per Eq. (7.52).

The *operating characteristic* (curve) can be calculated from

$$\Pr\{\text{acceptance} \mid MTTR\} = \Pr\{\hat{\mathrm{E}}[\tau'] \le c \mid MTTR\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-x^2/2} dx, \tag{7.53}$$

with

$$d = \frac{MTTR_0}{MTTR} t_{1-\alpha} - (1 - \frac{MTTR_0}{MTTR}) \sqrt{n/(e^{\sigma^2} - 1)}\,.$$

Replacing in d the quantity $n/(e^{\sigma^2} - 1)$ from Eq. (7.50) one recognizes that the operating characteristic (curve) is independent of σ^2 (rounding of n neglected).

Example 7.12

Give the rejection conditions (Eq. (7.51)) and the operating characteristic (curve) for the demonstration of $MTTR = MTTR_0 = 2\,\text{h}$ against $MTTR = MTTR_1 = 2.5\,\text{h}$ with $\alpha = \beta = 0.1$. and $\sigma^2 = 0.2$.

Solution

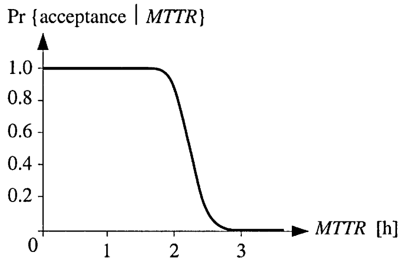
For $\alpha = \beta = 0.1$, Eq. (7.49) and Table A9.1 yield $t_{1-\alpha} = 1.28$ and $t_\beta = -1.28$. From Eq. (7.50) it follows that $n = 30$. The rejection condition is then given by

$$\sum_{i=1}^{30} t_i > 2\,\text{h}\,(1 + 1.28\sqrt{\frac{e^{0.2}-1}{30}}\,)30 = 66.6\,\text{h}.$$

From Eq. (7.53), the OC follows as

$$\Pr\{\text{acceptance} \mid MTTR\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-x^2/2}\, dx,$$

with $d \approx 25.84\,\text{h}/MTTR - 11.64$ (see graph).



7.4 Accelerated Testing

The failure rate λ of electronic components lies typically between 10^{-10} and $10^{-7}\,\text{h}^{-1}$, and that of assemblies in the range of 10^{-7} to $10^{-5}\,\text{h}^{-1}$. With such figures, cost and scheduling considerations demand the use of accelerated testing for λ estimation and demonstration, in particular if *reliable field data* are not available. An *accelerated test* is a test in which the applied stress is chosen to exceed that encountered in field operation, but still *below the technological limits.* This in order to shorten the time to failure of the item considered by avoiding an *alteration* of the involved *failure mechanism* (genuine acceleration). In accelerated tests, failure mechanisms are assumed to be activated selectively by increased stress. The quantitative relationship between degree of activation and extent of stress, i.e. the *acceleration factor A*, is determined via specific tests. Generally it is assumed that the stress will *not change the type* (family) *of the failure-free time distribution function* of the item under test, but only *modify the parameters.* In the following, this hypothesis is assumed to be valid; however, its verification should precede any statistical evaluation of data issued from accelerated tests.

Many electronic component *failure mechanisms* are activated through an increase in *temperature.* Calculating the acceleration factor A, the *Arrhenius model* can often be applied over a reasonably large temperature range (0 –150°C for ICs). The *Arrhenius model* is based on the Arrhenius rate law [3.43], which states that the rate ν of a simple (first-order) chemical reaction depends on temperature T as

$$\nu = \nu_0\, e^{-E_a/kT}. \tag{7.54}$$

E_a and ν_0 are parameters, k is the Boltzmann constant ($k = 8.6 \cdot 10^{-5}$ eV / K), and T the absolute temperature in Kelvin degrees. E_a is the *activation energy* and is expressed in eV. Assuming that the event considered (for example the diffusion between two liquids) occurs when the chemical reaction has reached a given *threshold*, and the reaction time dependence is given by a function $\mathrm{r}(t)$, then the relationship between the times t_1 and t_2 necessary to reach at two temperatures T_1 and T_2 a given level of the chemical reaction considered can be expressed as

$$\nu_1 \, \mathrm{r}(t_1) = \nu_2 \, \mathrm{r}(t_2).$$

Furthermore, assuming $\mathrm{r}(t) \sim t$, i.e. a *linear time dependence*, it follows that

$$\nu_1 t_1 = \nu_2 t_2 .$$

Substituting in Eq. (7.54) and rearranging, yields

$$\frac{t_1}{t_2} = \frac{\nu_2}{\nu_1} = e^{\frac{E_a}{k}\left(\frac{1}{T_1} - \frac{1}{T_2}\right)} .$$

By transferring this *deterministic* model to the mean times to failure $MTTF_1$ and $MTTF_2$ or to the constant failure rates λ_2 and λ_1 (using $MTTF = 1/\lambda$) of a given item at temperatures T_1 and T_2, it is possible to define an *acceleration factor A*

$$A = \frac{MTTF_1}{MTTF_2}, \qquad \text{or, for constant failure rate,} \quad A = \frac{\lambda_2}{\lambda_1}, \tag{7.55}$$

expressed by

$$A = e^{\frac{E_a}{k}\left(\frac{1}{T_1} - \frac{1}{T_2}\right)} \,{}^{+)} . \tag{7.56}$$

The right hand sides of Eq. (7.55) applies to the case of a constant (time independent but stress dependent) failure rate $\lambda(t) = \lambda$, for which $\mathrm{E}[\tau] = \sqrt{\mathrm{Var}[\tau]} = 1/\lambda$ holds (with τ as time to failure). Assuming that the left hand sides of Eq. (7.55) applies quite general (for time dependent failure rates) to *mean time to failure* ($E[\tau] = MTTF$) and *standard deviation* ($\sqrt{\mathrm{Var}[\tau]}$) as well, and that the *type of the distribution function is the same* at temperatures T_1 and T_2, it can be shown that for the distribution functions frequently used in reliability engineering (Table A6.1) the following holds for the parameters: $\lambda_2 = A\,\lambda_1$ for exponential, Gamma, Weibull, and lognormal; $\beta_2 = \beta_1$ for Gamma and Weibull; $\sigma_2 = \sigma_1$ for lognormal; $m_2 = m_1 / A$ & $\sigma_2 = \sigma_1 / A$ for normal distribution.[++)] This yields $\mathrm{F}_{\tau_1}(t) = \mathrm{F}_{\tau_2}(\frac{t}{A})$ and thus $\tau_1 = A\,\tau_2$,

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+) The case $T_2 = T_1 + \Delta T$ is discussed on p. 37.

++) The demonstration is straightforward for the exponential, Gamma, lognormal, and normal case; for Weibull, a quasi-analytic demonstration is possible using relations for $\Gamma(z+1)$ and $\Gamma(2z)$ (p. 558).

where τ_1 & τ_2 are the (random) *times to failure* at temperatures T_1 & T_2, with distribution functions $F_{\tau_1}(t)$ & $F_{\tau_2}(t)$ $(F_{\tau_1}(t)=F_{\tau_2}(\frac{t}{A}))$ belonging (per assumption) to the same family (case Vii in Example A6.18 and Eqs. (A6.40), (A6.46) with $C=A$).

Equation (7.56) can be reversed to give an estimate $\hat{E}_a$ for the activation energy E_a based on the failure rates $\hat{\lambda}_1$ and $\hat{\lambda}_2$ (or the mean times to failure $M\hat{T}TF_1$ and $M\hat{T}TF_2$) obtained empirically from two life tests at temperatures T_1 and T_2. However, at *least three tests* at T_1, T_2, and T_3 are necessary to verify the model.

The activation energy is highly dependent upon the particular *failure mechanism* involved (see Table 3.5 for some indicative figures). High E_a values lead to high acceleration factors. For ICs, global values of E_a lie between 0.3 and 0.7 eV (Table 3.5), values which could basically be obtained empirically from the curves of the failure rate as a function of the junction temperature. However, it must be noted that the Arrhenius model does not hold for all electronic devices and for any temperature range.

Figure 7.10 shows the acceleration factor A from Eq. (7.56) as a function of θ_2 in °C, for θ_1 = 35 and 55°C and with E_a as parameter ($\theta_i = T_i - 273$).

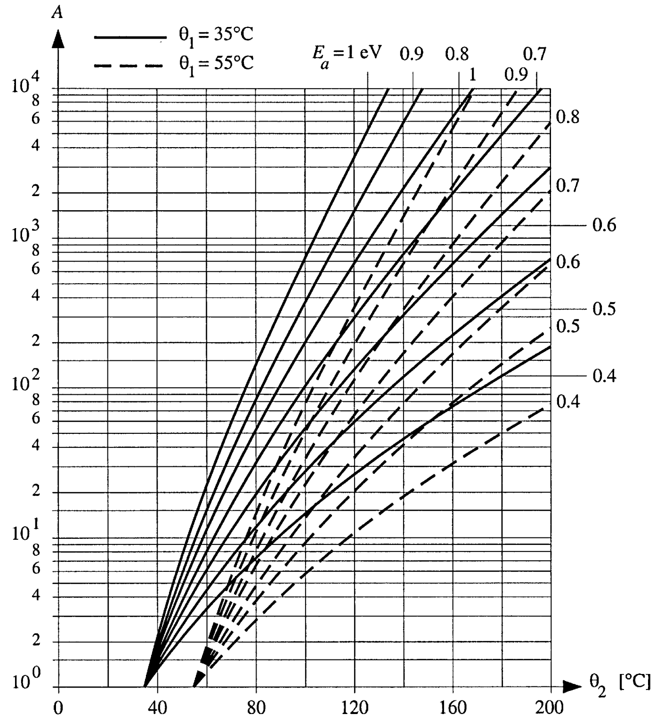


Figure 7.10 Acceleration factor A according to the Arrhenius model (Eq. (7.56)) as a function of θ_2 for $\theta_1 = 35$ and 55°C, and with E_a in eV as parameter ($\theta_i = T_i - 273$)

In particular for the case of a *constant* (time independent) *failure rate*, the acceleration factor A can be used as a *multiplicative factor* in the conversion of the *cumulative operating time* from stress T_2 to stress T_1 (Example 7.13, see also the remark to Eq. (7.55)). In practical applications, the acceleration factor A lies between 10 and some few hundreds, seldom > 1000 (Examples 7.13 & 7.14).

If the item under consideration exhibits *more than one dominant failure mechanism* or consists of series elements $E_1, \ldots, E_n$ having different failure mechanisms, the series reliability model (Sections 2.2.6.1 and 2.3.6) can often be used to calculate the *compound failure rate* $\lambda_S(T_2)$ at temperature T_2 by considering the failure rates $\lambda_i(T_1)$ and the acceleration factors A_i of the individual elements

$$\lambda_S(T_2) = \sum_{i=1}^{n} A_i \, \lambda_i(T_1). \tag{7.57}$$

Example 7.13

Four failures have occurred during 10^7 cumulative operating hours of a digital CMOS IC at a chip temperature of 130°C. Assuming $\theta_1 = 35°\text{C}$, a constant failure rate λ, and an activation energy $E_a = 0.4\,\text{eV}$, give the interval estimation of λ for $\gamma = 0.8$.

Solution

For $\theta_1 = 35°\text{C}$, $\theta_2 = 130°\text{C}$, and $E_a = 0.4\,\text{eV}$ it follows from Fig. 7.10 or Eq. (7.56) that $A \approx 35$. The cumulative operating time at 35°C is thus $T = 0.35 \cdot 10^9\,\text{h}$ and the point estimate for λ is $\hat{\lambda} = k/T \approx 11.4 \cdot 10^{-9}\,\text{h}^{-1}$. With $k = 4$ and $\gamma = 0.8$, it follows from Fig. 7.6 that $\hat{\lambda}_l / \hat{\lambda} \approx 0.43$ and $\hat{\lambda}_u / \hat{\lambda} \approx 2$; the confidence interval of λ is therefore $[4.9, 22.8] \cdot 10^{-9}\,\text{h}^{-1}$.

Example 7.14

A PCB contains 10 metal film resistors with stress factor $S = 0.1$ and $\lambda(25°\text{C}) = 0.2 \cdot 10^{-9}\,\text{h}^{-1}$, 5 ceramic capacitors (class 1) with $S = 0.4$ and $\lambda(25°\text{C}) = 0.8 \cdot 10^{-9}\,\text{h}^{-1}$, 2 electrolytic capacitors (Al wet) with $S = 0.6$ and $\lambda(25°\text{C}) = 6 \cdot 10^{-9}\,\text{h}^{-1}$, and 4 ceramic-packaged linear ICs with $\Delta\theta_{JA} = 10°\text{C}$ and $\lambda(35°\text{C}) = 20 \cdot 10^{-9}\,\text{h}^{-1}$. Neglecting the contribution of printed wiring and solder joints, give the failure rate of the PCB at a burn-in temperature θ_A of 80°C on the basis of failure rate relationships as given in Fig. 2.4.

Solution

The resistor and capacitor acceleration factors can be obtained from Fig. 2.4 as

resistor:	$A \approx 2.5/0.7 \approx 3.6$
ceramic capacitor (class 1):	$A \approx 4.2/0.5 \approx 8.4$
electrolytic capacitor (Al wet):	$A \approx 13.6/0.35 \approx 38.9$.

Using Eq. (2.4) for the ICs, it follows that $\lambda \sim \Pi_T$. With $\theta_J = 35°\text{C}$ and $90°\text{C}$, the acceleration factor for the linear ICs can then be obtained from Fig. 2.5 as $A \approx 7.5/0.8 \approx 9.4$. From Eq. (7.57), the failure rate of the PCB is then

$$\lambda(25°\text{C}) = (10 \cdot 0.2 + 5 \cdot 0.8 + 2 \cdot 6 + 4 \cdot 20) 10^{-9}\,\text{h}^{-1} \approx 100 \cdot 10^{-9}\,\text{h}^{-1}$$

$$\lambda(80°\text{C}) = (10 \cdot 0.2 \cdot 3.6 + 5 \cdot 0.8 \cdot 8.4 + 2 \cdot 6 \cdot 38.9 + 4 \cdot 20 \cdot 9.4) 10^{-9}\,\text{h}^{-1} \approx 1{,}260 \cdot 10^{-9}\,\text{h}^{-1} \approx 13 \cdot \lambda(25°\text{C}).$$

A further model for investigating the time scale reduction (time compression) resulting from an increase in temperature has been proposed by H. Eyring [3.43, 7.25]. The *Eyring model* defines the acceleration factor as

$$A = \frac{T_2}{T_1} e^{\frac{B}{k}(\frac{1}{T_1} - \frac{1}{T_2})} . \tag{7.58}$$

where B is not necessarily an activation energy. Eyring also suggests the following model, which considers the influences of temperature T and of a further stress X

$$A = \frac{T_2}{T_1} e^{\frac{B}{k}(\frac{1}{T_1} - \frac{1}{T_2})} e^{[X_1 (C + \frac{D}{k T_1}) - X_2 (C + \frac{D}{k T_2})]} . \tag{7.59}$$

Equation (7.59) is known as the *generalized Eyring model.* In this model, a function of the normalized variable $x = X / X_0$ can also be used instead of the quantity X itself (for example x^n, $1/x^n$, $\ln x^n$, $\ln(1/x^n)$). B is not necessarily an activation energy, C & D are constants. The generalized Eyring model led to accepted models, e. g. for *electromigration* (*Black*), *corrosion* (*Peck*), and *voltage* stress (*Kemeny*)

$$A = (\frac{j_2}{j_1})^m e^{\frac{E_a}{k}(\frac{1}{T_1} - \frac{1}{T_2})}, \quad A = (\frac{RH_2}{RH_1})^n e^{\frac{E_a}{k}(\frac{1}{T_1} - \frac{1}{T_2})}, \quad A = e^{(C_0 + \frac{E_a}{kT} + C_1 V / V_{max})}, \tag{7.60}$$

where j = current density, RH = relative humidity, and V = voltage, respectively (see also Eqs. (3.2) - (3.6) and Table 3.5). For failure mechanisms related to *mechanical fatigue*, Coffin-Manson simplified models [2.61, 2.72] (based on the inverse power law) can often be used, yielding for the number of cycles to failure

$$A = \frac{N_1}{N_2} = (\frac{\Delta T_2}{\Delta T_1})^{\beta_T} \quad \text{or} \quad A = \frac{N_1}{N_2} = (\frac{G_2}{G_1})^{\beta_M}, \tag{7.61}$$

where ΔT refers to thermal cycles and G refers to g_{rms} values in vibration tests ($0.5 < \beta_T < 0.8$ and $0.7 < \beta_M < 0.9$ often occur in practical applications). For damage accumulation, *Miner's hypothesis* of *independent* damage increments [3.53] can be used in some applications. Known for conductive filament formation in multilayer organic laminates is also the *Rudra's* model.

Critical remarks on accelerated tests are e. g. in [7.13, 7.15, 7.22]. Refinement of the above models is in progress, in particular for ULSI ICs with emphasis on:

1. New failure mechanisms in oxide and package, as well as new externally induced failure mechanisms.
2. Identification and analysis of causes for early failures or premature wearout.
3. Development of *physical models* for *failure mechanisms* and of *simplified models* for *reliability predictions* in practical applications.

Such efforts will give better *physical understanding* of the component's failure rate.

In addition to the accelerated tests discussed above, a rough estimate of component life time can often be obtained through *short-term tests* under extreme stresses (HALT, HAST, etc.). Examples are humidity testing of plastic-packaged ICs at high pressure and nearly 100% *RH*, or tests of ceramic-packaged ICs at up to 350°C. Experience shows that under high stress, life time is often lognormally distributed, thus with *strong time dependence* of the failure rate (Table A6.1). *Highly accelerated stress tests* (HAST) and *highly accelerated life tests* (HALT) can activate failure mechanisms which would not occur during normal operation, so care is necessary in extrapolating results to situations exhibiting lower stresses. Often, the purpose of such tests is to *force* (not only to activate) *failures*. They belong thus to the class of semi-destructive or destructive tests, often used at the qualification of prototype to investigate possible failure modes, mechanisms and/or technological limits. The same holds for *step-stress accelerated tests* (often used as life tests or in screening procedures), for which, accumulation of damage can be more complex as given e.g. by the Miner's hypothesis or in [7.20, 7.28]. A case-by-case investigation is mandatory for all this kind of tests.

7.5 Goodness-of-fit Tests

Let $t_1, \ldots, t_n$ be n independent observations of a random variable τ distributed according to $F(t)$, a rule is asked to test the null hypothesis H_0: $F(t) = F_0(t)$, for a given type I error α (probability of *rejecting* a true hypothesis H_0), against a general alternative hypothesis H_1: $F(t) \neq F_0(t)$. *Goodness-of-fit tests* deal with such testing of hypothesis and are often based on the *empirical distribution function* (EDF), see Appendices A8.3 for an introduction. This section shows the use of Kolmogorov-Smirnov and chi-square tests (see p. 548 for Cramér-von Mises tests). Trend tests are discussed in Section 7.6.

7.5.1 Kolmogorov-Smirnov Test

The *Kolmogorov-Smirnov test* (p. 548) is based on the convergence for $n \to \infty$ of the empirical distribution function (Eq. (A8.1))

$$\hat{F}_n(t) = \begin{cases} 0 & \text{for } t < t_{(1)} \\ \dfrac{i}{n} & \text{for } t_{(i)} \le t < t_{(i+1)} \\ 1 & \text{for } t \ge t_{(n)} \end{cases} \tag{7.62}$$

to the true distribution function, and compares the experimentally obtained $\hat{F}_n(t)$ with the given (postulated) $F_0(t)$. $F_0(t)$ is assumed here to be known and continuous, $t_{(1)}, \ldots, t_{(n)}$ are the ordered observations. The procedure is as follows:

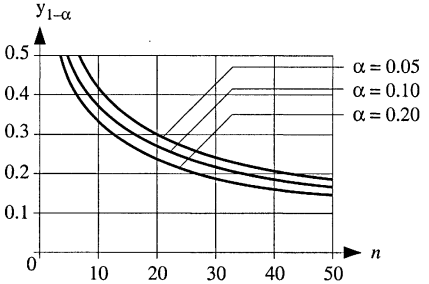


Figure 7.11 Largest deviation $y_{1-\alpha}$ between a postulated distribution function $F_0(t)$ and the corresponding empirical distribution function $\hat{F}_n(t)$ at the level $1-\alpha$ ($\Pr\{D_n \le y_{1-\alpha} \mid F_0(t) \text{ true}\} = 1-\alpha$)

1. Determine the largest deviation D_n between $\hat{F}_n(t)$ and $F_0(t)$

$$D_n = \sup_{-\infty < t < \infty} |\hat{F}_n(t) - F_0(t)|. \tag{7.63}$$

2. From the given type I error α and the sample size n, use Table A9.5 or Fig. 7.11 to determine the critical value $y_{1-\alpha}$.
3. Reject H_0: $F(t) = F_0(t)$ if $D_n > y_{1-\alpha}$; otherwise accept H_0.

This procedure can be easily combined with a *graphical evaluation of data.* For this purpose, $\hat{F}_n(t)$ and the band $F_0(t) \pm y_{1-\alpha}$ are drawn using a *probability chart* on which $F_0(t)$ can be represented by a straight line. If $\hat{F}_n(t)$ leaves the band $F_0(t) \pm y_{1-\alpha}$, the hypothesis H_0: $F(t) = F_0(t)$ is to be rejected (note that the band width is not constant when using a probability chart). Probability charts are discussed in Appendix A.8.1.3, examples are in Appendix A9.8 and Figs. 7.12 - 7.14. Example 7.15 (Fig. 7.12) shows a graphical evaluation of data for the case of a Weibull distribution, Example 7.16 (Fig. 7.13) investigates the distribution function of a population with *early failures* and a constant failure rate using a *Weibull probability chart*, and Example 7.17 (Fig. 7.14) uses the Kolmogorov-Smirnov test to check agreement with a lognormal distribution. If $F_0(t)$ is not completely known, a modification is necessary (Appendix A8.3.3).

Example 7.15

Accelerated life testing of a wet Al electrolytic capacitor leads following 13 ordered observations of lifetime: 59, 71, 153, 235, 347, 589, 837, 913, 1185, 1273, 1399, 1713, and 2567 h. (i) Draw the empirical distribution function of data on a Weibull probability chart. (ii) Assuming that the underlying distribution function is Weibull, determine $\hat{\lambda}$ and $\hat{\beta}$ graphically. (iii) The maximum likelihood estimation of λ & β yields $\hat{\beta} = 1.12$, calculate $\hat{\lambda}$ and compare results of (iii) with (ii).

Solution

(i) Figure 7.12 presents the empirical distribution function $\hat{F}_n(t)$ on Weibull probability paper.
(ii) The graphical determination of λ and β leads to (straight line (ii)) $\hat{\lambda} \approx 1/840\,\text{h}$ and $\hat{\beta} \approx 1.05$.
(iii) With $\hat{\beta} \approx 1.12$, Eq. (A8.31) yields $\hat{\lambda} \approx 1/908\,\text{h}$ (straight line (iii)) (see also Example A8.11).

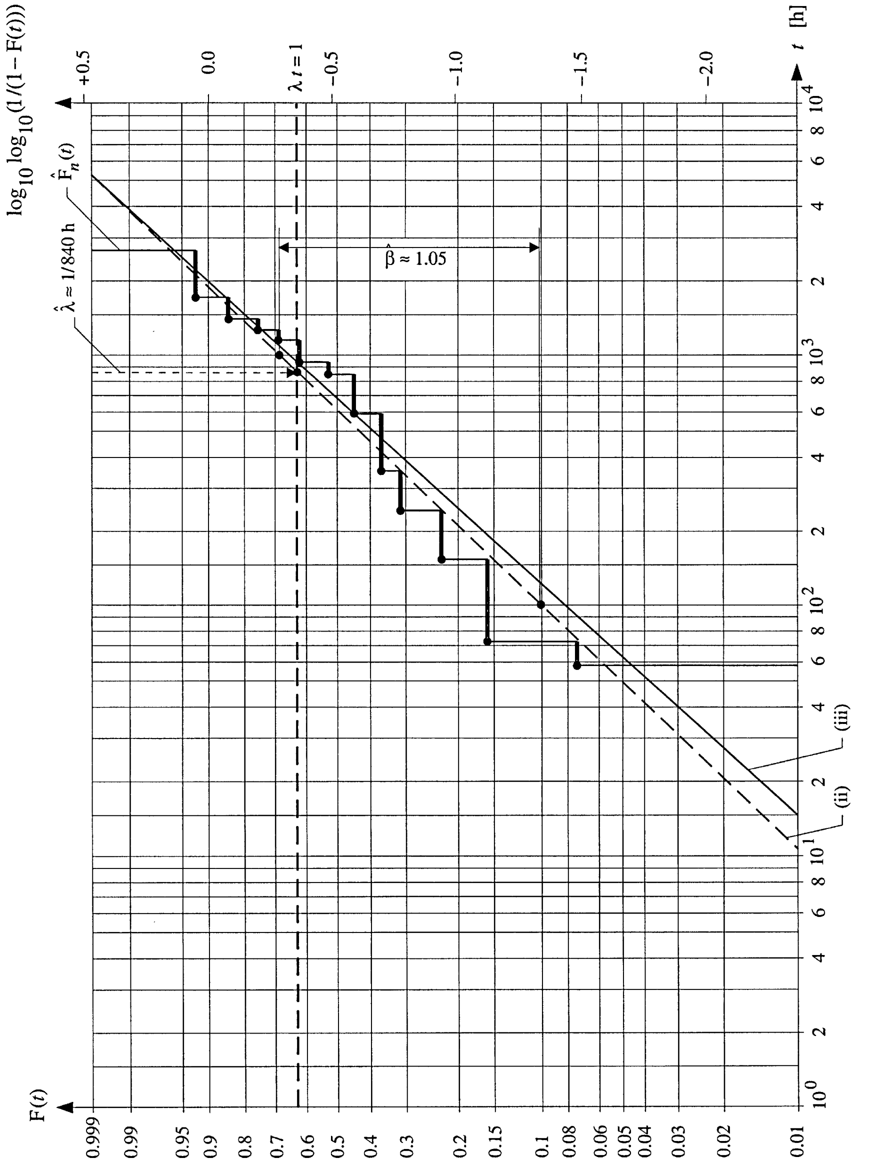


Figure 7.12 Empirical distribution function $\hat{F}_n(t)$ and estimated Weibull distribution functions ((ii) and (iii)) as per Example 7.15

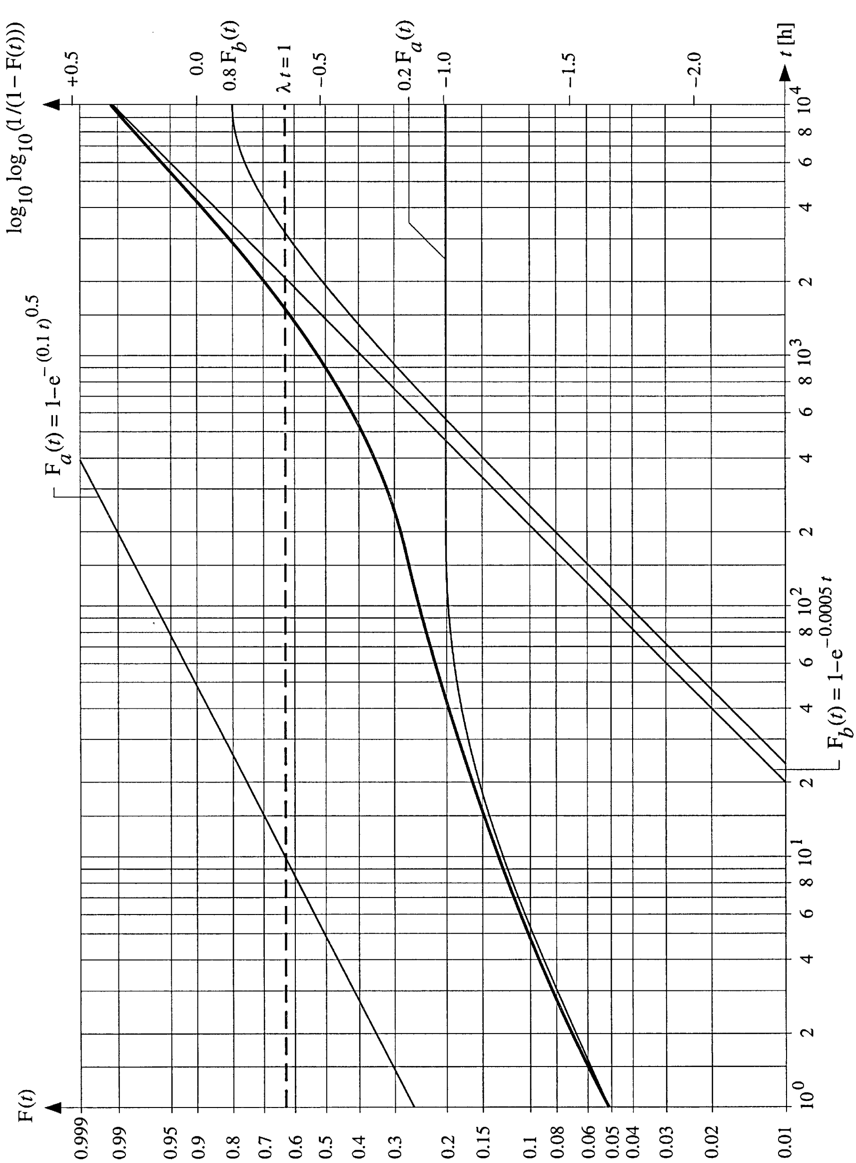


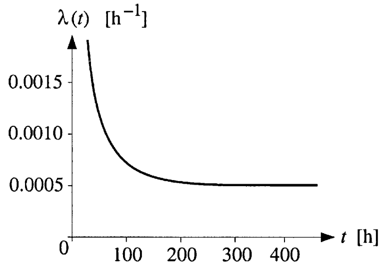
Figure 7.13 Shape of a weighted sum of a Weibull distribution $F_a(t)$ and an exponential distribution $F_b(t)$ as per Example 7.16, useful to detect (describe) *early failures* (similar for *wearout failures*)

Example 7.16

Investigate the *mixed* distribution function $F(t) = 0.2[1 - e^{-(0.1t)^{0.5}}] + 0.8[1 - e^{-0.0005t}]$ on a Weibull probability chart (describing a possible *early failure period*).

Solution

The weighted sum of a Weibull distribution ($\beta = 0.5$, $\lambda = 0.1\,h^{-1}$, and $MTTF = 20\,h$) with an exponential distribution ($\lambda = 0.0005\,h^{-1}$ and $MTTF = MTBF = 1/\lambda = 2000\,h$) represents the distribution function of a population of items with failure rate $\lambda(t) = [0.01(0.1t)^{-0.5} e^{-(0.1t)^{0.5}} + 0.0004 e^{-0.0005t}] / [0.2 e^{-(0.1t)^{0.5}} + 0.8 e^{-0.0005t}]$, i.e., with early failures up to about $t \approx 200\,h$, see graph ($\lambda(t)$ is practically constant at $0.0005\,h^{-1}$ for t between 300 h and 400,000 h, so that for $t > 300\,h$ a constant failure rate can be assumed for practical purposes). Figure 7.13 gives the function F(t) on a Weibull probability chart, showing the typical *s-shape*.



Example 7.17

Use the Kolmogorov-Smirnov test to verify with a type I error $\alpha = 0.2$, whether the repair time defined by the observations $t_1, \ldots, t_{10}$ of Example 7.11 are distributed according to a lognormal distribution function with parameters $\lambda = 0.5\,h^{-1}$ and $\sigma = 0.4$ (hypothesis H_0).

Solution

The lognormal distribution (Eq. (7.41)) with $\lambda = 0.5\,h^{-1}$ and $\sigma = 0.4$ is represented by a straight line on Fig. 7.14 ($F_0(t)$). With $\alpha = 0.2$ and $n = 10$, Table A9.5 or Fig. 7.11 yields $y_{1-\alpha} = 0.323$ and thus the band $F_0(t) \pm 0.323$. Since the empirical distribution function $\hat{F}_n(t)$ does not leave the band $F_0(t) \pm y_{1-\alpha}$, the hypothesis H_0 can be accepted.

7.5.2 Chi-square Test

The *chi-square test* (χ^2 *test*, pp. 549-552) can be used for *continuous* or *noncontinuous* $F_0(t)$. Furthermore, $F_0(t)$ need not to be completely known.

For $F_0(t)$ *completely known*, the procedure is as follows:

1. Partition the definition range of the random variable τ into k intervals (classes) $(a_1, a_2], (a_2, a_3], \ldots, (a_k, a_{k+1}]$; the choice of the classes must be made independently of the observations $t_1, \ldots, t_n$ (before test begin) and based on the rule: $n p_i \gtrapprox 5$, with p_i as per Eq. (7.64)).
2. Determine the number of observations k_i in each class $(a_i, a_{i+1}]$, $i = 1, \ldots, k$ (k_i = number of t_j with $a_i < t_j \le a_{i+1}$, $k_1 + \ldots + k_k = n$).
3. Assuming the hypothesis H_0, compute the expected number of observations for each class $(a_i, a_{i+1}]$

$$n\,p_i = n(F_0(a_{i+1}) - F_0(a_i)), \qquad i = 1, \ldots, k, \quad p_1 + \ldots + p_k = 1. \tag{7.64}$$

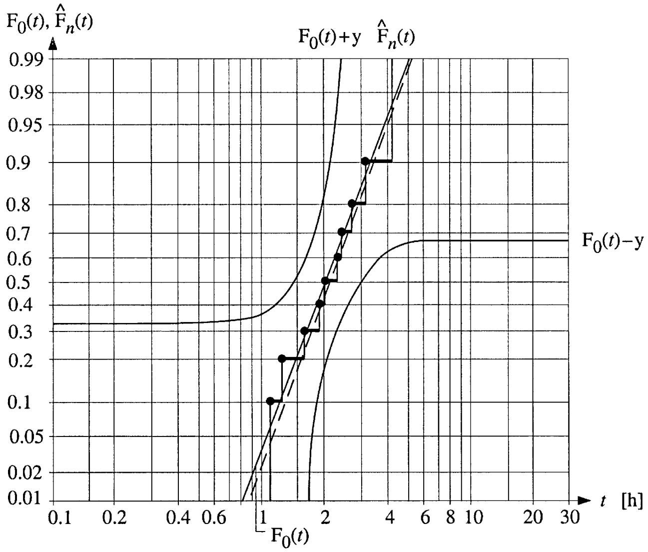


Figure 7.14 Kolmogorov-Smirnov test to check the repair time distribution as per Example 7.17 (the distribution function with $\hat{\lambda}$ and $\hat{\sigma}$ from Example 7.11 is shown dashed for information only)

4. Compute the statistic

$$X_n^2 = \sum_{i=1}^{k} \frac{(k_i - n\,p_i)^2}{n\,p_i} = \sum_{i=1}^{k} \frac{k_i^2}{n\,p_i} - n. \tag{7.65}$$

5. For a given type I error α, use Table A9.2 or Fig. 7.15 to determine the $(1-\alpha)$ quantile of the chi-square distribution with $k-1$ degrees of freedom $\chi^2_{k-1,1-\alpha}$.
6. Reject H_0: $\mathrm{F}(t) = \mathrm{F}_0(t)$ if $X_n^2 > \chi^2_{k-1,1-\alpha}$; otherwise accept H_0.

If $\mathrm{F}_0(t)$ is *not completely known* ($\mathrm{F}_0(t) = \mathrm{F}_0(t, \theta_1, \ldots, \theta_r)$, where $\theta_1, \ldots, \theta_r$ are unknown parameters, $r < k-1$), modify the above procedure after step 2 as follows:

3'. On the basis of the observations k_i in each class $(a_i, a_{i+1}]$, $i = 1, \ldots, k$ determine the maximum likelihood estimates for the parameters $\theta_1, \ldots, \theta_r$ from the following system of (r) algebraic equations

$$\sum_{i=1}^{k} \frac{k_i}{p_i(\theta_1, \ldots, \theta_r)} \cdot \frac{\partial p_i(\theta_1, \ldots, \theta_r)}{\partial \theta_j}\bigg|_{\theta_j = \hat{\theta}_j} = 0, \qquad j = 1, \ldots, r \tag{7.66}$$

with $p_i = \mathrm{F}_0(a_{i+1}, \theta_1, \ldots, \theta_r) - \mathrm{F}_0(a_i, \theta_1, \ldots, \theta_r) > 0$, $p_1 + \ldots + p_k = 1$, and for each class $(a_i, a_{i+1}]$ compute the expected number of observations

$$n\hat{p}_i = n[F_0(a_{i+1},\hat{\theta}_1,\ldots,\hat{\theta}_r) - F_0(a_i,\hat{\theta}_1,\ldots,\hat{\theta}_r)], \qquad i = 1,\ldots,k. \tag{7.67}$$

4'. Calculate the statistic

$$\hat{X}_n^2 = \sum_{i=1}^{k} \frac{(k_i - n\hat{p}_i)^2}{n\hat{p}_i} = \sum_{i=1}^{k} \frac{k_i^2}{n\hat{p}_i} - n. \tag{7.68}$$

5'. For given type I error α, use Table A9.2 or Fig. 7.15 to determine the $(1-\alpha)$ quantile of the χ^2distribution with $k-1-r$ degrees of freedom.

6'. Reject H_0: $F(t) = F_0(t)$ if $\hat{X}_n^2 > \chi^2_{k-1-r,1-\alpha}$; otherwise accept H_0.

Comparing the above two procedures, it can be noted that the number of degrees of freedom has been reduced from $k-1$ to $k-1-r$, where r is the number of parameters of $F_0(t)$ which have been estimated from the observations $t_1,\ldots,t_n$ using the *multinomial distribution* (Example A8.13, see Example 7.18 for an application).

Example 7.18

Let 160, 380, 620, 650, 680, 730, 750, 920, 1000, 1100, 1400, 1450, 1700, 2000, 2200, 2800, 3000, 4600, 4700, and 5000 h be 20 independent observations (realizations) of the failure-free time τ for a given assembly. Using the chi-square test for $\alpha = 0.1$ and the 4 classes (0, 500], (500, 1000], (1000, 2000], (2000, ∞), determine whether or not τ is exponentially distributed (hypothesis H_0 : $F(t) = 1 - e^{-\lambda t}$, λ unknown).

Solution

The given classes yield number of observations of $k_1 = 2$, $k_2 = 7$, $k_3 = 5$, and $k_4 = 6$. The point estimate of λ is then given by Eq. (7.66) with $p_i = e^{-\lambda a_i} - e^{-\lambda a_{i+1}}$, yielding for $\hat{\lambda}$ the numerical solution $\hat{\lambda} \approx 0.562 \cdot 10^{-3}\,h^{-1}$. Thus, the numbers of expected observations in each of the 4 classes are according to Eq. (7.67) $n\hat{p}_1 = 4.899$, $n\hat{p}_2 = 3.699$, $n\hat{p}_3 = 4.90$, and $n\hat{p}_4 = 6.499$. From Eq. (7.68) it follows that $\hat{X}_{20}^2 = 4.70$ and from Table A9.2, $\chi^2_{2,0.9} = 4.605$. The hypothesis H_0 : $F(t) = 1 - e^{-\lambda t}$ must be rejected since $\hat{X}_n^2 > \chi^2_{k-1-r,1-\alpha}$.

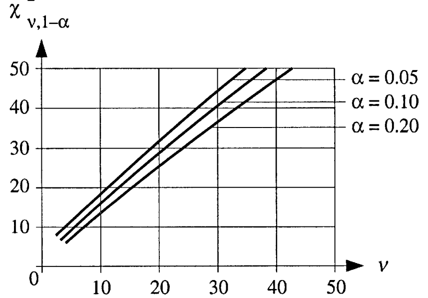


Figure 7.15 $(1-\alpha)$ quantile (α percentage point) of the chi-square distribution with ν degrees of freedom ($\chi^2_{\nu,1-\alpha}$, see also Table A9.2)

7.6 Statistical Analysis of General Reliability Data

7.6.1 General considerations

In Sections 7.2-7.5, data were issued from a sample of a random variable τ, i.e., they were n statistically independent realizations (observations) $t_1, ..., t_n$ of a random variable $\tau > 0$ distributed according to $F(t) = \Pr\{\tau \le t\}$ with $F(0) = 0$, and belonging to one of the following equivalent situations:

1. *Life times* $t_1, ..., t_n$ of n statistically identical and independent items, all starting at $t = 0$ when plotted on the time axis (e.g. as in Figs. 1.1, 7.12, 7.14).
2. Failure-free times separating successive failure occurrences of a repairable item (system) with negligible repair times and repaired (restored) as a *whole* to as-good-as-new at each repair; i.e., statistically identical and independent *interarrival times* with a common distribution function ($F(x)$), yielding a *renewal process.*

To this data structure belongs also the case considered in Example 7.19.

A basically different situation arises when the observations are arbitrary points on the time axis, i.e., when considering a general *point process.* To distinguish this case, the involved random variables are labeled $\tau_1^*, \tau_2^*, ...$, with $t_1^*, t_2^*, ...$ for the corresponding realizations ($t_1^* < t_2^* < ...$ is assumed). This situation occurs in reliability tests when *only the failed element* in a system is repaired to as-good-as-new, and there is *at least one element* in the system which has a *time dependent* failure rate. Failure-free times (interarrival times, by assuming negligible repair times) are in this case *neither independent nor equally distributed.* Considering failures at system level, only the case of a series system with *constant failure rates for all elements* ($\lambda_1, ..., \lambda_n$) leads (if repaired elements are as-good-as-new) to a *homogeneous Poisson process* (Appendix A7.2.5), for which interarrival times are statistically independent random variables with distribution function $F(x) = 1 - e^{-(\lambda_1 + ... + \lambda_n)x}$ (Eqs. (2.19), (7.27)). Shortcomings are known, see e.g. [6.1, 7.11, A7.30].

Example 7.19

Let $F(t)$ be the distribution function of the failure-free time of a given item. Suppose that at $t = 0$ an unknown number n of items are put into operation and that at the time t_0 exactly k item are failed (no replacement or repair has been done). Give a point estimate for n.

Solution

Setting $p = F(t_0)$, the number k of failures in $(0, t_0]$ is binomially distributed (Eq. (A6.120))

$$\Pr\{k \text{ failures in } (0, t_0]\} = p_k = \binom{n}{k} p^k (1-p)^{n-k}, \qquad \text{with} \quad p = F(t_0). \tag{7.69}$$

An estimate for n can be obtained using the maximum likelihood method, yielding (Eq. (A8.23)) $L = \binom{n}{k} p^k (1-p)^{n-k}$ and finally, with $\partial \ln L / \partial n = 0$ for $n = \hat{n}$ (n is the unknown parameter),

$$\hat{n} = k / p = k / F(t_0). \tag{7.70}$$

For Eq. (7.70), the approximation $\binom{n}{k} \approx (e^{-k} / k!)(n^n / (n-k)^{(n-k)}$ has been used (Stirling formula). The Poisson approximation $p_k \approx e^{-np}(np)^k / k!$ (Eq. (A6.129)) yields also $\hat{n} = k / p$.

Easy to investigate, when observing data on the time axis, are cases involving nonhomogeneous Poisson processes (Sections 7.6.2, 7.6.3, 7.7, Appendix A7.8.2). For more general situations, difficulties can arise (except for some general results valid for stationary point processes (Appendices A7.8.3 - A7.8.5)), and the following basic rule should apply:

If neither a Poisson process (homogeneous or nonhomogeneous) nor a renewal process can be assumed for the underlying point process, care is necessary in identifying possible models; in any case, *validation of model assumptions (from a physical and statistical point of view) should precede data analysis.*

The homogeneous Poisson process (HPP), introduced in Appendix A7.2.5 as particular case of a renewal process, is the simplest point process. It is *memoryless*, and tools for a statistical investigation are known. Nonhomogeneous Poisson processes (NHPPs) are *without aftereffect* (Appendix A7.8.2) and for investigation purposes they can be transformed into an HPP (Eq. (A7.200)). Investigation on renewal processes (Appendix A7.2) can be reduced to that of independent random variables with a common distribution function (cases 1 and 2 above). However, disregarding the last part of the above general rule can lead to mistakes, even in the case of renewal processes or independent realizations of a random variable τ. As an example, let us consider an item with *two independent failure mechanisms*, one appearing with constant failure rate $\lambda_0 = 10^{-3}\text{h}^{-1}$ and the second (wearout) with a shifted Weibull distribution $F(t) = 1 - e^{-(\lambda(t-\psi))^\beta}$ with $\lambda = 10^{-2}\text{h}^{-1}$, $\psi = 10^4\text{h}$, and $\beta = 3$ ($t > \psi$, $F(t) = 0$ for $t \le \psi$). As case 2 in Eq. (A6.34), the failure-free time τ has the distribution function $F(t) = 1 - e^{-\lambda_0 t}$ for $0 \le t \le \psi$ and $F(t) = 1 - e^{-\lambda_0 t} \cdot e^{-(\lambda(t-\psi))^\beta}$ for $t > \psi$ (failure rate $\lambda(t) = \lambda_0$ for $t \le \psi$ and $\lambda(t) = \lambda_0 + \beta\lambda^\beta(t-\psi)^{\beta-1}$ for $t > \psi$, similar to a series model with independent elements (Eq. (2.17)). If the presence of the above two failure mechanisms is not known and the test is stopped (censored) after 10^4h, the wrong conclusion can be drawn that the item has a constant failure rate of about 10^{-3}h^{-1}.

Investigation of cases involving general point processes is beyond the scope of this book (only some general results are given in Appendices A7.8.3 - A7.8.5). A large number of ad hoc procedures are known in the literature, but they often only apply to specific situations and their use needs a careful validation of the assumptions stated with the model.

After some considerations on tests for *nonhomogeneous Poisson* processes in Section 7.6.2, Sections 7.6.3.1 and 7.6.3.2 deal with *trend tests* to check the assumption *homogeneous Poisson process* versus *nonhomogeneous Poisson process with increasing* or *decreasing intensity*. A heuristic test to distinguish a homogeneous Poisson process from a general monotonic trend is discussed in Section 7.6.3.3. However, as stated in the above general rule, the validity of a model should be checked also on the basis of physical considerations on the item considered. This in particular for the property *without aftereffect*, characterizing Poisson processes.

7.6.2 Tests for Nonhomogeneous Poisson Processes

A nonhomogeneous Poisson process (NHPP) is a point processes which count function $\nu(t)$ has unit jumps, *independent increments* (in nonoverlapping intervals), and satisfies for any $b > a \geq 0$ (Appendix A7.8.2)

$$\Pr\{k \text{ events in } (a, b]\} = \frac{(M(b)-M(a))^k}{k!} e^{-(M(b)-M(a))}, \quad k=0,1,2,\ldots, \ 0 \leq a < b, \ M(0)=0. \tag{7.71}$$

For $a=0$ & $b=t$ it holds that $\Pr\{\nu(t)=k\} = (M(t))^k e^{-M(t)} / k!$. $M(t)$ is the *mean value function* of the NHPP, giving the expected number of points (events) in $(0, t]$

$$M(t) = E[\nu(t)], \qquad \nu(0)=0, \ M(0)=0. \tag{7.72}$$

Assuming M(*t*) derivable,

$$m(t) = dM(t) / dt \geq 0 \tag{7.73}$$

is the *intensity* of the NHPP and has for $\delta t \downarrow 0$ following interpretation (Eq. (A7.194))

$$\Pr\{\text{one event in } (t, t+\delta t]\} = m(t)\,\delta t + o(\delta t). \tag{7.74}$$

Because of independent increments, the number of events (failures) in a time interval $(t, t+\theta]$ (Eq. (7.71) with $a=t$ & $b=t+\theta$) and the rest waiting time to the next event from an arbitrary time point t

$$\Pr\{\tau_R(t) > x\} = \Pr\{\text{ no event in } (t, t+x]\} = e^{-(M(t+x)-M(t))}, \quad x \geq 0, \tag{7.75}$$

are *independent of the process development up to time t* (Eqs. (A7.195), (A7.196)). Thus, also the mean $E[\tau_R(t)]$ is independent of the process development up to time *t*, and given by (Eq. (A7.197))

$$E[\tau_R(t)] = \int_0^\infty e^{-(M(t+x)-M(t))} dx.$$

Furthermore, if $0 < \tau_1^* < \tau_2^* < \ldots$ are the occurrence times (arrival times) of the event considered (e.g. failures of a repairable system), measured from $t=0$, it holds for $m(t) > 0$ (M(*t*) derivable and strictly increasing) that the quantities

$$\psi_1^* = M(\tau_1^*) < \psi_2^* = M(\tau_2^*) < \ldots \tag{7.76}$$

are the occurrence times in a homogeneous Poisson processes with *intensity one* (Eq. (A7.200)). Moreover, *for given* (fixed) $t=T$ and $\nu(T)=n$, the occurrence times $0 < \tau_1^* < \ldots < \tau_n^* < T$ have the same distribution as if they where the *order statistic* of n independent identically distributed random variables with *density*

$$m(t) \,/\, M(T), \qquad 0 < t < T, \ m(t) > 0, \tag{7.77}$$

and distribution function $M(t) / M(T)$ on $(0, T)$ (Eq. (A7.205)).

Equation (7.74) gives the *unconditional* probability for *one event* in $(t, t+\delta t]$. Thus, $m(t)$ refers to the occurrence of *any one* of the events considered. It corresponds to the renewal density $h(t)$ and the *failure intensity* $z(t)$, but *differs basically* from the *failure rate* $\lambda(t)$ (see remark to Eq. (A7.24)).

Nonhomogeneous Poisson processes (NHPPs) are introduced in Appendix A7.8.2. Some examples are discussed in Section 7.7 with applications to reliability growth. Assuming that the underlying process is an NHPP, estimation of the model parameters (parameters of $m(t, \theta)$) can be performed using the maximum likelihood method on the basis of observed data $0 < t_1^* < t_2^* < \ldots < t_n^* < T$ (time censoring; t_1^*, t_2^*, ... are the observed values (realizations) of $\tau_1^*, \tau_2^*, \ldots$ and * is used to explicitly indicate that t_1^*, t_2^*, ... are points on the time axis and not independent realizations of a random variable τ (e.g. as in Figs. 1.1, 7.12, 7.14)). Considering Eqs. (7.71) and (7.74), the likelihood function follows as (Eq. (7.102))

$$L = e^{-M(T)} \prod_{i=1}^{n} m(t_i^*), \tag{7.78}$$

and delivers the maximum likelihood estimate $\hat{\theta}$ for the parameters θ of $m(t, \theta)$ by solving $\partial L / \partial \theta = 0$ for $\theta = \hat{\theta}$, where θ can be a vector (see e.g. Eq. (7.104) for the parameters α and β of the NHPP with $m(t) = \alpha \beta t^{\beta-1}$). Using the property stated by Eq. (7.76), statistical tests for exponential distribution or for homogeneous Poisson processes (Appendix A8.2.2.2 and Section 7.2.3) can be applied to NHPPs as well. Furthermore, using the property stated by Eq. (7.77), the goodness-of-fit tests introduced in Appendix A8.3.2 and Section 7.5 (Kolmogorov-Smirnov, chi-square, Cramér - von Mises) can be used to verify agreement of the observed data $t_1^*, \ldots, t_n^* < T$ with a *postulated* $M_0(t)$. For the Kolmogorov-Smirnov test, the procedure given in Section 7.5.1 applies with

$$\hat{F}_n(t) = \hat{\nu}(t) / \hat{\nu}(T) \tag{7.79}$$

and

$$F_0(t) = M_0(t) / M_0(T)), \tag{7.80}$$

where $\hat{\nu}(t)$ is the observed number of events in $(0, t]$.

More difficult is the situation when the assumption that the underlying model is an NHPP *must also be verified by a statistical data analysis*, for instance with a goodness-of-fit test. The problem in not completely solved. However, the property given by Eqs. (7.76) and (7.77) can be used for goodness-of-fit of the NHPP with incompletely specified (up to the parameters) mean function $M_0(t)$. The chi-square test holds with the procedure given in Section 7.5.2 and Appendix A8.3.3. For a first evaluation, the Kolmogorov-Smirnov test (and tests based on a quadrate statistic) can be used taking half (randomly selected) of the observations $t_1^*, \ldots, t_n^*$ to estimate the parameters and continuing with the whole sample the procedure given in Section 7.5.1 for the goodness-of-fit test [A8.11, A8.31].

7.6.3 Trend Tests

In reliability engineering one is often interested to test if there is a *monotonic trend* in the times between successive failures (interarrival times) of a repairable system with negligible repair (restoration) times, e.g. in order to detect the end of an *early failure period* or the begin of a *wearout period*. Such tests extend the tests for exponentiality or for homogeneous Poisson processes introduced in Section 7.2.3 (see also 7.5, A8.2.1, A8.2.2, A8.3.2, A8.3.3). If the underlying point process can be approximated by a *renewal process*, a *graphical approach* can be used in detecting the presence of trends, see e.g. Fig. 7.13 for the case of early failures. In the case of a nonhomogeneous Poisson process (NHPP), a trend is given by an increasing or decreasing intensity $m(t)$, e.g. $\beta > 1$ or $\beta < 1$ in Eq. (7.99). Trend tests can also be useful in investigating what kind of alternative hypothesis should be considered when an assumption is to be made about the statistical properties of a given data set. However, *trend tests* check in general a postulated hypothesis against a more or less *general alternative hypothesis*. Care is therefore necessary in drawing conclusions from this kind of statistical tests, and the basic rule given on p. 334 applies. In the following, some trend tests used in reliability data analysis are discussed, among them the Laplace test (see e.g. [A8.1] for greater details).

7.6.3.1 Tests of an HPP versus an NHPP with increasing intensity

The homogeneous Poisson process (HPP) is a point process which count function $\nu(t)$ has stationary, independent Poisson distributed increments (Eqs. (A7.41)). *Interarrival times* in an HPP are *independent* and distributed according to the *same exponential distribution* $F(x) = 1 - e^{-\lambda x}$ (occurrence times are Erlangian (Gamma) distributed). The parameter λ characterizes completely the HPP. *λ is at the same time the intensity of the HPP and the failure rate $\lambda(x)$ of all interarrival times*, x starting by 0 at each occurrence time of the event considered (e.g. failure of a repairable system with negligible repair (restoration) times). This *numerical equality* has been the cause for misinterpretations and misuses in practical applications, see e.g. [6.1, 7.11, A7.30]. The homogeneous Poisson process has been introduced in Appendix A7.2.5 as particular case of a renewal process. Considering $\nu(t)$ as the count function giving the number of events (failures) in $(0, t]$, Example A7.13 (Eq. (A7.213)) shows that:

> *For given (fixed) T and $\nu(T) = n$ (time censoring), the normalized arrival times $0 < \tau_1^* / T < \ldots < \tau_n^* / T$ of a homogeneous Poisson process (HPP) have the same distribution as if they where the order statistic of n independent identically uniformly distributed random variables on $(0,1)$.* (7.81)

Similar results hold for an *NHPP* (Eq. (A7.206)):

For given (fixed) T and $\nu(T)=n$ (time censoring), the normalized arrival times $0<\mathrm{M}(\tau_1^*)\,/\,\mathrm{M}(T)<\ldots<\mathrm{M}(\tau_n^*)\,/\,\mathrm{M}(T)<1$ of a nonhomogeneous Poisson process (NHPP) with mean value function $\mathrm{M}(t)$ have the same distribution as if they where the order statistic of n independent identically uniformly distributed random variables on (0,1). (7.82)

With the above transformations, properties of the uniform distribution can be used to support statistical tests on homogeneous and nonhomogeneous Poisson processes.

Let ω be an uniformly distributed random variable with density

$$\mathrm{f}_\omega(x)=1 \quad \text{on } (0,1), \qquad \mathrm{f}_\omega(x)=0 \text{ outside } (0,1), \tag{7.83}$$

and distribution function $\mathrm{F}_\omega(x)=x$ on $(0,1)$. Mean and variance of ω are given by (Eqs. (A6.37) and (A6.44))

$$\mathrm{E}[\omega]=1/2 \qquad \text{and} \qquad \mathrm{Var}[\omega]=1/12. \tag{7.84}$$

The sum $\omega_1+\ldots+\omega_n$ of n independent random variables ω has mean $n/2$ and variance $n/12$. The distribution function $\mathrm{F}_{\omega n}(x)$ of $\omega_1+\ldots+\omega_n$ is defined on $(0,n)$ and can be computed using Eq. (A7.12). $\mathrm{F}_{\omega n}(x)$ has been investigated in [A8.8], yielding to the conclusion that $\mathrm{F}_{\omega n}(x)$ rapidly approach a normal distribution as n increases. For practical applications one can assume that for *given* (fixed) T and $\nu(T)=n\geq 5$, the sum of the arrival times $0<\tau_1^*<\ldots<\tau_n^*<T$ of an HPP is distributed as

$$\Pr\{[(\sum_{i=1}^{n}\tau_i^*/T)-n/2]\,/\sqrt{n/12}\leq x\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-y^2/2}dy, \qquad -\infty<x<\infty, \tag{7.85}$$

(Eq. (A6.148)). Equation (7.85) can be used to test an HPP ($\mathrm{m}(t)=\lambda$) versus an NHPP with *increasing density* $\mathrm{m}(t)=d\mathrm{M}(t)/dt$. Using Eq. (7.85) and considering the observations (realizations) $t_1^*<t_2^*<\ldots<t_n^*<T$, the procedure is (Example 7.20):

1. Compute the statistic

$$[(\sum_{i=1}^{n}t_i^*/T)-n/2]\,/\sqrt{n/12}. \tag{7.86}$$

2. For given type I error α determine the critical value $t_{1-\alpha}$ ($1-\alpha$ quantile of the standard normal distribution, e.g. $t_{1-\alpha}=1.64$ for $\alpha=0.05$ (Tab. A9.1)).
3. Reject the hypothesis H_0: the underlying point process is an HPP, against H_1: the underlying process is an NHPP with *increasing density*, at $1-\alpha$ confidence, if $(\sum_{i=1}^{n}t_i^*/T-n/2)/\sqrt{n/12}>t_{1-\alpha}$; otherwise accept H_0. (7.87)

A test based on Eqs. (7.86) - (7.87) is called *Laplace test* and was first introduced by Laplace as a test of randnomness. From Eq. (7.87) one recognizes that $\sum t_i^*/T$ is a sufficient statistic (Appendix A8.2.1). It can be noted that $(\sum t_i^*/T-n/2)/\sqrt{n/12}$ tends to assume large values for H_0 false (i.e. for $\mathrm{m}(t)$ increasing). For $T=t_n^*$ (failure censoring), Eq. (7.86) holds with $n-1$ (see e.g. [A8.1]).

A further possibility to test an HPP ($m(t) = \lambda$) versus an NHPP with increasing density $m(t) = dM(t)/dt$ is to use the statistic

$$\sum_{i=1}^{n} \ln (T / t_i^*) . \tag{7.88}$$

Considering (Eq. (A7.213)) that for given (fixed) T and $\nu(T) = n$, the normalized arrival times $0 < \omega_1 = \tau_1^* / T < \ldots < \omega_n = \tau_n^* / T < 1$ of an HPP have the same distribution as if they where the order statistic of n independent identically uniformly distributed random variables on (0,1), and that (Example 7.21) $2 \sum_{i=1}^{n} \ln (T / \tau_i^*) = 2 \sum_{i=1}^{n} - \ln \omega_i$ has a χ^2 distribution (Eq. (A6.103)) with $2n$ degrees of freedom

$$F(x) = \Pr\{ 2 \sum_{i=1}^{n} \ln(T / \tau_i^*) \le x \} = \frac{1}{2^n (n-1)!} \int_0^x y^{n-1} e^{-y/2} dy, \tag{7.89}$$

the statistic given by Eq. (7.88) can be used to test an HPP ($m(t) = \lambda$) versus an NHPP with *increasing density* $m(t) = dM(t)/dt$. The corresponding test procedure is (Example 7.22):

1. Compute the statistic

$$2 \sum_{i=1}^{n} \ln(T / t_i^*). \tag{7.91}$$

2. For given type I error α determine the critical value $\chi^2_{2n, \alpha}$ (α quantile of the χ^2 distribution, e. g. $\chi^2_{2n, \alpha} = 7.96$ for $n = 8$ & $\alpha = 0.05$ (Table A9.2)).
3. Reject the hypothesis H_0: the underlying point process is an HPP, against H_1: the underlying process is an NHPP with *increasing density*, at $1-\alpha$ confidence, if $2 \sum_{i=1}^{n} \ln(T / t_i^*) < \chi^2_{2n, \alpha}$; otherwise accept H_0. (7.92)

From Eq. (7.92) one recognizes that $2 \sum \ln(T / t_i^*)$ is a sufficient statistic (Appendix A8.2.1). It can be noted that $2 \sum \ln(T / t_i^*)$ tends to assume small values for H_0 false (i.e. for $m(t)$ increasing). For $T = t_n^*$ (failure censoring), Eq. (7.91) hold with $n-1$ (see e.g. [A8.1]).

Example 7.20

In a reliability test, 8 failures have occurred in $T = 10{,}000$h and $t_1^* + \ldots + t_8^* = 43{,}000$h has been observed. Test with a risk $\alpha = 5\%$ (at 95% confidence), using the rule (7.87), the hypothesis H_0: the underlying point process is an HPP, against H_1: the underlying process is an NHPP with increasing density.

Solution

From Table A9.1 $t_{0.95} = 1.64 > (4.3 - 4) / 0.816 = 0.367$ and H_0 can not be rejected.

Example 7.21

Let the random variable ω be uniformly distributed on (0,1). Show that $\eta = -\ln(\omega)$ is distributed according to $F_\eta(t) = 1 - e^{-t}$ on $(0, \infty)$, and thus $2\sum_{i=1}^{n} -\ln(\omega_i) = 2\sum_{i=1}^{n} \eta_i = \chi^2_{2n}$.

Solution

Considering that for $0 < \omega < 1$, $-\ln(\omega)$ is a decreasing function defined on $(0, \infty)$, it follows that the events $\{\omega \le x\}$ and $\{\eta = -\ln(\omega) > -\ln(x)\}$ are equivalent. From this (see also Eq. (A6.31), $x = \Pr\{\omega \le x\} = \Pr\{\eta > -\ln(x)\}$ and thus, using $-\ln x = t$, one obtains $\Pr\{\eta > t\} = e^{-t}$ and finally

$$F_\eta(t) = \Pr\{\eta \le t\} = 1 - e^{-t}. \tag{7.90}$$

From Eqs. (A6.102) - (A6.104), $2\sum_{i=1}^{n} -\ln(\omega_i) = 2\sum_{i=1}^{n} \eta_i$ has a χ^2 distrib. with $2n$ degrees of freedom.

Example 7.22

In a reliability test, 8 failures have occurred in $T = 10{,}000$ h at 850, 1200, 2100, 3900, 4950, 5100 8300, 9050h. Test with a risk $\alpha = 5\%$ (at 95% confidence), using the rule (7.92), the hypothesis H_0: the underlying point process is an HPP, against the alternative hypothesis H_1: the underlying process is an NHPP with increasing density.

Solution

From Table A9.2, $\chi^2_{16,0.05} = 7.96 < 2(\ln(T/t_1^*) + \ldots + \ln(T/t_8^*)) = 17.5$ and H_0 can not be rejected.

7.6.3.2 Tests of an HPP versus an NHPP with decreasing intensity

Tests of a homogeneous Poisson process (HPP) versus a nonhomogeneous Poisson process (NHPP) with a *decreasing intensity* $m(t) = dM(t)/dt$ can be deduced from those for increasing intensity given in section 7.6.3.1. Equations (7.85) and (7.89) remain true. However, if the intensity is decreasing, most of the failures tend to occur before $T/2$ and test procedure for the Laplace test has to be changed in (Example 7.23):

1. Compute the statistic

$$[(\sum_{i=1}^{n} t_i^*/T) - n/2] \,/\, \sqrt{n/12}\,. \tag{7.93}$$

2. For given type I error α determine the critical value t_α (α quantile of the standard normal distribution, e.g. $t_\alpha = -1.64$ for $\alpha = 0.05$ (Tab. A9.1)).
3. Reject the hypothesis H_0: the underlying point process is an HPP, against H_1: the underlying process is an NHPP with *decreasing density*, at $1-\alpha$ confidence, if $(\sum_{i=1}^{n} t_i^*/T - n/2)/\sqrt{n/12} < t_\alpha$; otherwise accept H_0. (7.94)

From Eq. (7.93) one recognizes that $\sum t_i^*/T$ is a sufficient statistic (Appendix A8.2.1). It can be noted that $(\sum t_i^*/T - n/2)/\sqrt{n/12}$ tend to assume small values for H_0 false (i.e. for $m(t)$ decreasing). For $T = t_n^*$ (failure censoring), Eq. (7.93) holds with $n-1$ (see e.g. [A8.1]).

For the test according to the statistic (7.88), the test procedure is (Example 7.24):

1. Compute the statistic

$$2\sum_{i=1}^{n} \ln(T/t_i^*). \tag{7.95}$$

2. For given type I error α determine the critical value $\chi^2_{2n,1-\alpha}$ ($1-\alpha$ quantile of the χ^2 distribution, e. g. $\chi^2_{2n,1-\alpha}=26.3$ for $n=8$ & $\alpha=0.05$ (Table A9.2)).
3. Reject the hypothesis H_0: the underlying point process is an HPP, against H_1: the underlying process is an NHPP with *decreasing density*, at $1-\alpha$ confidence, if $2\sum_{i=1}^{n} \ln(T/t_i^*) > \chi^2_{2n,1-\alpha}$; otherwise accept H_0. (7.96)

From Eq. (7.95) one recognizes that $2\sum \ln(T/t_i^*)$ is a sufficient statistic (Appendix A8.2.1). It can be noted that $2\sum \ln(T/t_i^*)$ tend to assume large values for H_0 false (i.e. for $m(t)$ decreasing). For $T=t_n^*$ (failure censoring), Eq. (7.95) hold with $n-1$ (see e.g. [A8.1]).

7.6.3.3 Heuristic Tests to distinguish between HPP and General Monotonic Trend

In some applications, little information is available about the underlying point process describing failures occurrence of a complex repairable system. As in the previous sections, it will be assumed that repair times are neglected. What is sought is a test to identify a monotonic trend of the failure intensity against a constant failure intensity given by a homogeneous Poisson process (HPP).

Consider first, investigations based on successive *interarrival times*. Such an

Example 7.23

Continuing Example 7.20, test using the rule (7.94) and the data of Example 7.20, with a risk $\alpha=5\%$ (at 95% confidence), the hypothesis H_0: the underlying point process is an HPP, against the alternative hypothesis H_1: the underlying process is an NHPP with decreasing density.

Solution

From Table A9.1, $t_{0,05} = -1.64 < 0.367$ and H_0 can not be rejected.

Example 7.24

Continuing Example 7.22, test using the rule (7.96) and the data of Example 7.22, with a risk $\alpha=5\%$ (at 95% confidence), the hypothesis H_0: the underlying point process is an HPP, against the alternative hypothesis H_1: the underlying process is an NHPP with decreasing density.

Solution

From Table A9.2, $\chi^2_{16,0.95}=26.3 > 2(\ln(T/t_1^*)+\ldots+\ln(T/t_8^*))=17.5$ and H_0 can not be rejected.

investigation should be performed at the beginning of data analysis, also because it can quickly deliver a first information about a possible monotonic trend (e.g. interarrival times become more and more long or short). Moreover, if the underlying point process describing failures occurrence can be approximated by a *renewal process* (successive interarrival times are independent and identically distributed), procedures of Section 7.5 based on the empirical distribution function (EDF) have a *great intuitive appeal* and can be useful in testing for monotonic trends as well, see Examples 7.15 - 7.17 (Figs. 7.12 - 7.14). In particular, the *graphical approaches* given in Example 7.16 (Fig. 7.13) would allow the detection and quantification of an *early failure period.* The same would be for a *wearout period.* Similar considerations hold if the involved point process can be approximated by a nonhomogeneous Poisson process (NHPP), see Sections 7.6.1 - 7.6.3.2 and 7.7.

If a trend in successive interarrival times is recognized, but the underlying point process can not be approximated by a renewal process or an NHPP, a further possibility is to consider the observed failure time points $t_1^* < t_2^* < \dots$ directly. As shown in Appendix A7.8.5, a mean value function $\mathrm{Z}(t)=\mathrm{E}[\nu(t)]$ can be associated to each point process, where $\nu(t)$ is the count function giving the number of failures occurred in $(0,t]$ ($\mathrm{Z}_S(t)$ and $\nu_S(t)$ should be used for considerations at system level). From the observed failure time points (observed occurrence times) $t_1^* < t_2^* < \dots$, the empirical *mean value function* $\hat{\mathrm{Z}}(t)=\hat{\mathrm{E}}[\nu(t)]$ follows as (see e.g. also [7.24])

$$\hat{\mathrm{Z}}(t)=\hat{\mathrm{E}}[\nu(t)]=\begin{cases} 0 & \text{for } t < t_1^* \\ i & \text{for } t_i^* \le t < t_{i+1}^*, \qquad i=1,2,\dots . \end{cases} \tag{7.97}$$

The mean value function $\mathrm{Z}(t)$ corresponds to the renewal function $\mathrm{H}(t)$ in a renewal process (Eq. (A7.15)); $\mathrm{z}(t)=d\mathrm{Z}(t)/dt$ is the *failure intensity* and correspond to the renewal density $\mathrm{h}(t)$ in a renewal process (Eqs. (A7.18), (A7.24)). For a *homogeneous Poisson process,* $\mathrm{Z}(t)$ takes the form (Eq. (A7.42))

$$\mathrm{Z}(t)=\mathrm{E}[\nu(t)]=\lambda t. \tag{7.98}$$

Each deviation from a straight line $\mathrm{Z}(t)=\mathrm{a}\,t$ is thus an indication for a possible trend (besides statistical deviations). As shown in Example A7.1 (Fig. A7.2) for a renewal process, early failures or wearout gives a basically different shape of the underlying renewal function; a *convex shape* for the case of *early failures* and a *concave shape* for the case of *wearout.* This property can be used to recognize the presence of trends in a point process, by considering the shape of the associated empirical *mean value function* $\hat{\mathrm{Z}}(t)$ given by Eq. (7.97). Such a procedure *can help* in detecting possible trends, but remains a *rough evaluation* (see Fig. A7.2 for the case of a renewal process). *Care is thus necessary* when extrapolating results, e.g. about the failure rate value after the early failure period or the percentage of early failures.

7.7 Reliability Growth

At the prototype qualification tests, the reliability of complex equipment or systems can be less than expected. Disregarding any imprecision of data or model used in calculating the predicted reliability (Chapter 2), such a discrepancy is often the consequence of *weaknesses* (errors and flaws) during design or manufacturing. For instance, use of components or materials at their technological limits or with internal weaknesses, cooling problems, interface problems, EMC problems, transient phenomena, interference between hardware and software, assembling or soldering problems, damage during handling, transportation or testing, etc. Errors and flaws cause *defects* and *systematic failures*. Superimposed to these are *early failures* and failures with *constant failure rate* (wearout should not be present at this stage). A distinction between deterministic faults (defect and systematic failures) and random faults (early failures and failures with constant failure rate) is only possible with a *cause analysis*. Such an analysis is *necessary* to identify and eliminate *causes* of observed faults, i.e., *change* or *modification* (redesign) for defects and systematic failures, *screening* for *early failures*, and *repair* for *failures with constant failure rate*. Of course, *defects and systematic failures* can also be randomly distributed on the time axis, e.g. caused by a mission dependent time-limited overload, by software defects, or simply because of the system complexity. However, they still differ from failures, as they are basically *independent of operating time* (disregarding systematic failures which can appear only after a certain operating time, e.g. as for some cooling or software problems).

The aim of a *reliability growth program* is the *cost-effective* improvement of the item's reliability through successful correction / elimination of the *causes* of design or production weaknesses. Early failures should be precipitated with an appropriate screening (*environmental stress screening* (ESS)), see Section 8.2 for electronic components, Section 8.3 for electronic assemblies, and Section 8.4 for cost aspects. Considering that flaws found during reliability growth are in general *deterministic* (defects and systematic failures), reliability growth is performed during prototype qualification tests and *pilot production*, seldom for series-produced items (Fig. 7.16). Stresses during reliability growth are often higher than those expected in the field (as for ESS). Furthermore, the statistical methods used to investigate *reliability growth* are in general *basically different* from those given in Section 7.2 for standard reliability tests (e. g. to estimate or demonstrate a constant failure rate λ). This is because during the reliability growth program, design and / or production changes or modifications are introduced in the item(s) considered and statistical evaluation is *not restarted* after a change or modification.

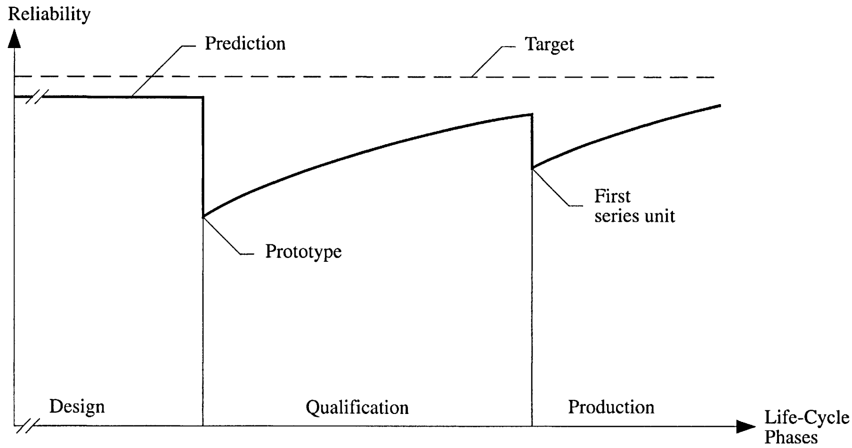


Figure 7.16 Qualitative visualization of a possible *reliability growth*

A large number of models have been proposed to describe reliability growth for hardware and software, see e. g. [5.88, 5.91, 7.31-7.47, A2.5 (61014 & 61164)], some of them on the basis of theoretical considerations. A practice oriented model, proposed by J.T. Duane [7.36] and refined as a statistical model by L.H. Crow [7.35 (1975)], known also as the *AMSAA model*, assumes that the *flow of events* (system failures) constitutes a *nonhomogeneous Poisson process* (NHPP) with *intensity*

$$\mathrm{m}(t) = \frac{d\mathrm{M}(t)}{dt} = \alpha\beta\, t^{\beta-1}, \qquad \alpha > 0, \;\; 0 < \beta < 1, \;\; t \ge 0, \qquad (7.99)$$

and *mean value function*

$$\mathrm{M}(t) = \alpha\, t^{\beta}, \qquad \alpha > 0, \;\; 0 < \beta < 1, \;\; t \ge 0. \qquad (7.100)$$

$\mathrm{M}(t)$ gives the expected number of failures in $(0, t]$. $\mathrm{m}(t)\delta t$ is the probability for one failure (any one) in $(t, t+\delta t]$ (Eq. (7.74)). It can be shown that for an NHPP, $\mathrm{m}(t)$ is equal to the failure rate $\lambda(t)$ of the *first occurrence time* (Eq. (A7.209). Comparing Eq. (7.99) with Eq. (A6.91) one recognizes that for the NHPP described by Eq. (7.99), the *first occurrence time* has a Weibull distribution. However, $\mathrm{m}(t)$ and $\lambda(t)$ are *fundamentally* different (see the remark on p. 370), and all others interarrival times *do not follow* a Weibull distribution and are *neither independent nor identically distributed.* Because of the distribution of the *first* occurrence time, the NHPP process described by Eq. (7.99) is often called *Weibull process*, causing great confusion. Also used is the term *power law process.* Nonhomogeneous Poisson processes are investigated in Appendix A7.8.2.

In the following it will be *assumed* that the underlying model is an NHPP. Verification of this assumption should also be based on *physical* considerations on the nature / causes of the defects and systematic failures involved, not only on statistical aspects. If the underlying process is an NHPP, estimation of the model parameters (α and β in the case of Eq. (7.99)) can easily be performed using observed data.

Let us consider first the *time censored* case (Type I censoring) and assume that up to the given (fixed) time T, n events have occurred at times $t_1^* < t_2^* < \ldots < t_n^* < T$. t_1^*, t_2^*,... are the realizations (observations) of the arrival times τ_1^*, τ_2^*,... and * indicates that t_1^*, t_2^*, ... are points on the time axis and not independent realizations of a random variable τ with a given (fixed) distribution function (e.g. as in Figs. 1.1. 7.12, 7.14). Considering the main property of an NHPP, i. e., that the number of events in nonoverlapping intervals are *independent* and distributed according to (Eq. (A7.195))

$$\Pr\{k \text{ events in } (a, b]\} = \frac{(\mathrm{M}(b)-\mathrm{M}(a))^k}{k!}\, e^{-(\mathrm{M}(b)-\mathrm{M}(a))}, \quad k=0,1,2,\ldots,\ 0 \le a < b,\ M(0)=0, \tag{A7.101}$$

and the interpretation of the intensity $\mathrm{m}(t)$ given by Eq. (7.74) or Eq. (A7.194), the following *likelihood function* (Eq. (A8.24)) can be found for the parameter estimation of the intensity $\mathrm{m}(t)$

$$\begin{aligned} \mathrm{L} &= \mathrm{m}(t_1^*)\, e^{-\mathrm{M}(t_1^*)}\mathrm{m}(t_2^*)\, e^{-(\mathrm{M}(t_2^*)-\mathrm{M}(t_1^*))} \ldots \mathrm{m}(t_n^*)\, e^{-(\mathrm{M}(t_n^*)-\mathrm{M}(t_{n-1}^*))} e^{-(\mathrm{M}(T)-\mathrm{M}(t_n^*))} \\ &= e^{-\mathrm{M}(T)} \prod_{i=1}^{n} \mathrm{m}(t_i^*) . \end{aligned} \tag{7.102}$$

Equation (7.102) considers no event ($k=0$ in Eq. (7.101)) in each of the non-overlapping intervals $(0, t_1^*), (t_1^*, t_2^*), \ldots, (t_n^*, T)$ and applies to an arbitrary NHPP. For the Duane model it follows that

$$\mathrm{L} = e^{-\mathrm{M}(T)} \prod_{i=1}^{n} \mathrm{m}(t_i^*) = \alpha^n \beta^n e^{-\alpha t^{\beta}} \prod_{i=1}^{n} t_i^{*\beta-1}, \tag{7.103}$$

or

$$\ln \mathrm{L} = n \ln(\alpha\beta) - \alpha t^{\beta} + (\beta - 1) \sum_{i=1}^{n} \ln(t_i^*) .$$

The *maximum likelihood estimates* $\hat{\alpha}$ and $\hat{\beta}$ of the parameters α and β are then obtained from

$$\left.\frac{\partial \ln \mathrm{L}}{\partial \alpha}\right|_{\alpha=\hat{\alpha}} = 0 \qquad \text{and} \qquad \left.\frac{\partial \ln \mathrm{L}}{\partial \beta}\right|_{\beta=\hat{\beta}} = 0,$$

yielding

$$\hat{\beta} = \frac{n}{\sum_{i=1}^{n} \ln(T / t_i^*)} \qquad \text{and} \qquad \hat{\alpha} = \frac{n}{T^{\hat{\beta}}} \,. \tag{7.104}$$

An estimate for the *intensity* of the underlying nonhomogeneous Poisson process is

$$\hat{m}(t) = \hat{\alpha}\, \hat{\beta}\, t^{\hat{\beta}-1}, \qquad 0 < t < T. \tag{7.105}$$

With known values for $\hat{\alpha}$ and $\hat{\beta}$, Eq. (7.105) can be used to extrapolate the *attainable intensity* if the reliability growth process were to be continued with the *same statistical properties* for a further time span Δ after T, yielding

$$\hat{m}(T+\Delta) = \hat{\alpha}\, \hat{\beta}\, (T+\Delta)^{\hat{\beta}-1}, \qquad \Delta > 0. \tag{7.106}$$

see Example 7.25 for a practical application.

In the case of *event censoring*, i.e., when the test is stopped at the occurrence of the nth event (Type II censoring), Eq. (7.104) holds with t_n^* instead of T and $n-1$ instead of n.

Interval estimation for the parameters α and β can be found, see e.g. [A8.1]. For *goodness-of-fit-tests* one can consider the property of nonhomogeneous Poisson processes that, for *given* (fixed) T and knowing that n events have been observed in $(0,T]$, i.e. for given T and $\nu(T)=n$, the occurrence times $0<\tau_1^*<\ldots<\tau_n^*<T$ have the same distribution as if they where the *order statistic* of n independent and identically distributed random variables with density $m(t) \,/\, M(T)$, on $(0,T)$ (Eq. (A7.205)). For example, the *Kolmogorov-Smirnov test* (Section 7.5) can be used with $\hat{F}_n(t) = \hat{\nu}(t) \,/\, \hat{\nu}(T)$ (Eq. (7.79)) and $F_0(t) = M_0(t) \,/\, M_0(T))$ (Eq. (7.80)), see also Appendices A7.8.2 and A8.3.2. Furthermore it holds that if $\tau_1^* < \tau_2^* < \ldots$ are the occurrence times of an NHPP, then $\psi_1^* = M(\tau_1^*) < \psi_2^* = M(\tau_2^*) < \ldots$ are the occurrence times in a *homogeneous Poisson process* (HPP) with *intensity one*

Example 7.25

During the reliability growth program of a complex equipment, the following data was gathered: $T = 1200\,\text{h}$, $n = 8$ and $\sum \ln(T/t_i^*) = 20$. Assuming that the underlying process can be described by a Duane model, estimate the intensity at $t = 1200\,\text{h}$ and the value attainable at $t + \Delta = 3000\,\text{h}$ if the reliability growth *would continue with the same statistical properties.*

Solution

With $T = 1200\,\text{h}$, $n = 8$ and $\sum \ln(T/t_i^*) = 20$, it follows from Eq. (7.104) that $\hat{\beta} = 0.4$ and $\hat{\alpha} \approx 0.47$. From Eq. (7.105), the estimate for the intensity leads to $\hat{m}(1200) \approx 2.67 \cdot 10^{-3}\,\text{h}^{-1}$ ($\hat{M}\,(1200) \approx 8$). The attainable intensity after an extension of the program for reliability growth by $1800\,\text{h}$ is given by Eq. (7.105) as $\hat{m}(3000) \approx 1.54 \cdot 10^{-3}\,\text{h}^{-1}$.

(Eq. (A7.200)). Results for independent and identically distributed random variables, for HPP or for exponential distribution function can thus be used. Important is also that the mean value of the random time $\tau_R(t)$ from an arbitrary (fixed) time point $t \ge 0$ to the next failure is *independent of the process development up to the time t* and is given by (Eq. (A7.197))

$$\mathrm{E}\,[\tau_R(t)] = \int_0^\infty \Pr\{\text{no event in } (t, t+x]\}\, dx = \int_0^\infty e^{-(\mathrm{M}(t+x)-\mathrm{M}(t))} dx, \qquad (7.107)$$

yielding, for instance,

$$\mathrm{E}\,[\tau_R(t)] = 1/\lambda, \qquad t \text{ given (fixed)}, x>0, \qquad (7.108)$$

for $\mathrm{M}(t+x) = \mathrm{M}(t) + \lambda x$, i.e. $\mathrm{m}(t+x) = \lambda$ for t given (fixed) and $x > 0$, and

$$\mathrm{E}\,[\tau_R(t)] = \Gamma(1 + 1/\beta)\ /\ \alpha^{1/\beta}, \qquad t \text{ given (fixed)}, x>0, \qquad (7.109)$$

for $\mathrm{M}(t+x) = \mathrm{M}(t) + \alpha x^\beta$ (Appendix A9.6 or Eq. (A6.92) with $\lambda = \alpha^{1/\beta}$).

The *Duane model* often applies to electronic, electromechanical, and mechanical equipment and systems. It can also be used to describe the occurrence of *software defects* (*dynamic defects*). However, other models have been discussed in the literature especially for software (Section 5.3.4). Among these, the *logarithmic Poisson model*, which assumes a *nonhomogeneous Poisson process* with intensity

$$\mathrm{m}(t) = \frac{1}{\delta + \gamma t} \qquad \text{or} \qquad \mathrm{m}(t) = \frac{\alpha + 1}{\beta + t}, \qquad 0<\alpha, \beta, \delta, \gamma < \infty, \ t \ge 0. \qquad (7.110)$$

For the logarithmic Poisson model, $\mathrm{m}(t)$ is monotonically decreasing with $\mathrm{m}(0) < \infty$ and $\mathrm{m}(\infty) = 0$. Considering $\mathrm{M}(0) = 0$, it follows that

$$\mathrm{M}(t) = \frac{\ln(1 + \gamma t/\delta)}{\gamma} \qquad \text{or} \qquad \mathrm{M}(t) = \ln\left(\frac{\beta + t}{\beta}\right)^{\alpha+1}. \qquad (7.111)$$

Models combining in a multiplicative way two possible mean value functions $\mathrm{M}(t)$ have been investigated in [7.33] by assuming

$$\mathrm{M}(t) = a \ln(1 + t/b) \cdot (1 - e^{-t/b}) \quad \text{and} \quad \mathrm{M}(t) = \alpha t^\beta \cdot [1 - (1 + t/\gamma) e^{-t/\gamma}], \qquad (7.112)$$

with $a, b, \alpha, \gamma > 0$, $0 < \beta < 1$, $t \ge 0$. In both cases, the intensity $\mathrm{m}(t)$ grows from 0 to a maximum, from which it goes to 0 with a shape similar to that of the models given by Eq. (7.110). The models described by Eqs. (7.100), (7.111), and (7.112) are based on *nonhomogeneous Poisson processes,* satisfying thus the properties discussed in Appendix A7.8.2 .

Although appealing, nonhomogeneous Poisson processes (NHPP) can not solve all reliability growth modeling problems, basically because of their intrinsic simplicity related to the assumption of *independent increments.* The consequence of this assumption, is that the NHPP is a process *without aftereffect* for which the waiting time to the next event from an arbitrary time point t is independent of the process development up to time t (Eq. (7.107) or Eq. (A7.197)). Furthermore, the first occurrence time τ_1^* characterizes the NHPP (Eq. (A7.209)). An NHPP can thus not necessarily be used to estimate the number of defects present in a software package, see e.g. [A7.30] for further comments.

In general, it is not possible to *fix a priori the model* to be used in a given situation. For hardware as well as for software, a *physical* motivation of the model, based on failure or defect (fault) *causes / mechanisms*, can help in such a choice. Having the *"best model"*, the next step should be to verify that assumptions made are compatibles with the model and after that to check the compatibility with data. Misuses or misinterpretations can occur, often because of dependencies between the involved random variables.