6 Reliability and Availability of Repairable Systems

Reliability and availability analysis of *repairable systems* is generally performed using stochastic processes, including *Markov*, *semi-Markov*, and *semi-regenerative processes*. The mathematical foundation of these processes is in Appendix A7. Equations used to investigate Markov and semi-Markov models are summarized in Table 6.2. This chapter investigates systematically most of the reliability models encountered in practical applications. Reliability figures at system level have indices Si (e.g. $MTTF_{Si}$), where *S stands for system* and i is *the state entered at* $t=0$ (Table 6.2). After Section 6.1 (introduction, assumptions, conclusions), Section 6.2 investigates the one-item structure under general conditions. Sections 6.3 - 6.6 deal extensively with series, parallel, and series-parallel structures. To unify models and simplify calculations, it is assumed that the system has *only one repair crew* and *no further failures occur at system down.* Starting from constant failure and repair rates between successive states (Markov processes), generalization is performed step by step (beginning with the repair rates) up to the case in which the process involved is *regenerative with a minimum number of regeneration states. Approximate expressions* for large series - parallel structures are investigated in Section 6.7. Sections 6.8 considers systems with *complex structure* for which a reliability block diagram often *does not exist.* On the basis of practical examples, preventive maintenance, imperfect switching, incomplete coverage, elements with more than two states, phased-mission systems, common cause failures, and general reconfigurable fault tolerant systems with reward & frequency / duration aspects are investigated. Basic considerations on *network reliability* are given in Section 6.8.8 and a general procedure for complex structures is in Section 6.8.9. Sections 6.9 introduces alternative investigation methods (dynamic FTA, BDD, event trees, Petri nets, computer-aided analysis), and gives a Monte Carlo approach useful for *rare events. Asymptotic* & *steady-state* is used as a synonym for *stationary* (pp. 490 & 501). Results are summarized in tables. Selected examples illustrate the practical aspects.

6.1 Introduction, General Assumptions, Conclusions

Investigation of the *time behavior of repairable systems* spans a very large class of *stochastic processes*, from simple Poisson process through Markov and semi-Markov processes up to sophisticated regenerative processes with only *one or just a few regeneration states.* Nonregenerative processes are rarely considered because

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of mathematical difficulties. Important for the choice of the class of processes to be used are the *distribution functions for the failure-free and repair times* involved. If failure and repair rates of all elements in the system are *constant* (time independent) *during the stay time in each state* (not necessarily at a state change, e.g. because of load sharing), the process involved is a (time-homogeneous) *Markov process* with a finite number of states, for which *stay time in each state* is *exponentially distributed.* The same holds if Erlang distributions occurs (supplementary states, Section 6.3.3). The possibility to transform a given stochastic process into a Markov process by introducing supplementary variables is not considered here. Generalization of the distribution functions for repair times leads to *semi-regenerative processes*, i.e., to processes with an *embedded semi-Markov process*. This holds in particular if the system has *only one repair crew*, since each termination of a repair is a renewal point (because of the constant failure rates). Arbitrary distributions of repair and failure-free times lead in general to *nonregenerative stochastic processes.*

Table 6.1 shows the processes used in reliability investigations of repairable systems, with their possibilities and limits. Appendix A7 introduces these processes with particular emphasis on reliability applications. All equations necessary for the reliability and availability calculation of systems described by (time-homogeneous) Markov processes and semi-Markov processes are summarized in Table 6.2.

Besides the assumption about the involved distribution functions for failure-free and repair times, reliability and availability calculation is largely influenced by the maintenance strategy, logistic support, type of redundancy, and dependence between elements. Existence of a reliability block diagram is assumed in Sections 6.2 - 6.7, not necessarily in Sections 6.8 and 6.9. Results are expressed as functions of time by solving appropriate systems of differential (or integral) equations, or given by the mean time to failure or the steady-state point availability at system level ($MTTF_{Si}$ or PA_S) by solving appropriate systems of *algebraic equations*. If the system has no redundancy, the reliability function is the same as in the nonrepairable case. In the presence of redundancy, it is generally assumed that redundant elements will be *repaired without operational interruption at system level. Reliability* investigations thus aim to find the occurrence of the *first system down*, whereas the *point availability* is the probability to find the system in an *up state* at a time t, independently of whether down states at system level have occurred before t.

In order to unify models and simplify calculations, the *following assumptions are made for analyses in Sections 6.2 - 6.6* (partly also in Sections 6.7 - 6.9).

1. *Continuous operation*: Each element of the system is in operating or reserve state, when not under repair or waiting for repair. (6.1)
2. *No further failures at system down* (no FF): At system down the system is repaired (restored) according to a given *maintenance strategy* to an up state at system level from which operation is continued, failures during a repair at system down are not considered. (6.2)
3. *Only one repair crew:* At system level only one repair crew is available,

repair is performed according to a stated strategy, e.g. first-in / first-out. (6.3)

4. *Redundancy*: Redundant elements are repaired without interruption of operation at system level; failure of redundant parts is immediately detected. (6.4)
5. *States*: Each *element in the reliability block diagram* has only two states (good or failed); after repair (restoration) it is *as-good-as-new*. (6.5)
6. *Independence*: Failure-free and repair times of each element are stochastically independent, >0, and continuous random variables with finite mean (*MTTF, MTTR*) and variance (*failure-free time* is used as a synonym for *failure-free operating time* and *repair* as a synonym for *restoration*). (6.6)
7. *Support*: Preventive maintenance is neglected; fault coverage, switching, and logistic support are ideal (repair time = restoration time = down time). (6.7)

The above assumptions holds for Sections 6.2-6.6, and apply in many practical situations. However, assumption (6.5) must be *critically verified*, in particular for the aspect *as-good-as-new,* when repaired elements contain parts with time dependent failure rate which have not been replaced by new ones. This assumption is valid if nonreplaced parts have *constant* (time independent) *failure rates,* and applies in this case at system level. At system level, reliability figures have indices Si (e.g. $MTTF_{Si}$), where S stands for system and i is the state entered at $t=0$ (Table 6.2). Assuming irreducible processes, *asymptotic & steady-state* is used for *stationary*.

Table 6.1 Stochastic processes used in reliability and availability analysis of repairable systems

Stochastic process	Can be used in modeling	Background	Difficulty
Renewal process	Spare parts provisioning in the case of arbitrary failure rates and negligible replacement or repair time (Poisson process for const. λ)	Renewal theory	Medium
Alternating renewal process	One-item repairable (renewable) structure with arbitrary failure and repair rates	Renewal theory	Medium
Markov process (MP) (finite state space, time-homogeneous)	Systems of arbitrary structure whose elements have constant failure and repair rates (λ_i, μ_i) *during the stay time (sojourn time) in every state* (not necessarily at a state change, e. g. because of load sharing)	Differential equations or Integral equations	Low
Semi-Markov process (SMP)	Some systems whose elements have constant or Erlangian failure rates (Erlang distributed failure-free times) and arbitrary repair rates	Integral equations	Medium
Semi-regenerative proc. (process with an embedded SMP, i.e. ≥ 2 reg. states)	Systems with only one repair crew, arbitrary structure, and whose elements have constant failure rates and arbitrary repair rates	Integral equations	High
Nonregenerative process	Systems of arbitrary structure whose elements have arbitrary failure and repair rates	Partial differential eqs. (case by case soluttion)	High to very high

Table 6.2 Relationships for the reliability, point availability & interval reliability of systems described by (time-homogeneous) Markov processes & semi-Markov processes (Appendices A7.5 - A7.6)

	Reliability	Point Availability	Interval Reliability
Semi - Markov Processes (SMP)	$\mathrm{R}_{Si}(t) = 1 - \mathrm{Q}_i(t) + \sum_{Z_j \in U, j \neq i} \int_0^t \mathrm{q}_{ij}(x) \mathrm{R}_{Sj}(t-x)\,dx, \quad Z_i \in U$ $MTTF_{Si} = T_i + \sum_{Z_j \in U, j \neq i} \mathcal{P}_{ij}\, MTTF_{Sj}, \quad Z_i \in U$ *with* $\mathrm{Q}_{ij}(x) = \Pr\{\tau_{ij} \le x \cap \tau_{ik} > \tau_{ij},\ k \neq j\} = \mathcal{P}_{ij} \mathrm{F}_{ij}(x)$, $\mathrm{q}_{ij}(x) = \frac{d\mathrm{Q}_{ij}(x)}{dx} = \mathcal{P}_{ij} \frac{d\mathrm{F}_{ij}(x)}{dx}, \quad \mathrm{Q}_{ii}(x) \equiv 0$, $\mathcal{P}_{ij} = \Pr\{\tau_{ik} > \tau_{ij},\ k \neq j\} = \mathrm{Q}_{ij}(\infty), \quad \mathcal{P}_{ii} \equiv 0$ $\mathrm{F}_{ij}(x) = \Pr\{\tau_{ij} \le x \mid \tau_{ik} > \tau_{ij},\ k \neq j\}$ (see also $\mathrm{PA}_{Si}(t)$ & $\mathrm{IR}_S(\theta)$)	$\mathrm{PA}_{Si}(t) = \sum_{Z_j \in U} \mathrm{P}_{ij}(t), \quad i = 0, \ldots, m$ $PA_S = \sum_{Z_j \in U} P_j = \sum_{Z_j \in U} \frac{T_j}{T_{jj}}$ *with* $\mathrm{P}_{ij}(t) = \delta_{ij}(1 - \mathrm{Q}_i(t)) + \sum_{k=0, k \neq i}^{m} \int_0^t \mathrm{q}_{ik}(x) \mathrm{P}_{kj}(t-x)\,dx$, $i, j = 0, \ldots, m, \quad \delta_{ii} = 1, \quad \delta_{ij} = 0 \text{ for } j \neq i$, $\mathrm{Q}_i(x) = \sum_{j=0, j \neq i}^{m} \mathrm{Q}_{ij}(x), \quad T_j = \int_0^\infty (1 - \mathrm{Q}_j(x))\,dx$ (see also $\mathrm{IR}_S(\theta)$)	Problem oriented calculation; for *constant failure rates* it follows in steady-state that $\mathrm{IR}_S(\theta) = \sum_{Z_j \in U} P_j \mathrm{R}_{Sj}(\theta) = \sum_{Z_j \in U} \frac{T_j}{T_{jj}} \mathrm{R}_{Sj}(\theta)$ *with* $P_j = \lim_{t \to \infty} \mathrm{P}_j(t) = \frac{T_j}{T_{jj}}, \quad T_{jj} = \frac{1}{\mathcal{P}_j} \sum_{k=0}^{m} \mathcal{P}_k T_k$, $T_j = \int_0^\infty (1 - \mathrm{Q}_j(x))\,dx$, and $\mathcal{P}_j$ from $\mathcal{P}_j = \sum_{i=0}^{m} \mathcal{P}_i \mathcal{P}_{ij}$ $\mathcal{P}_{ii} \equiv 0, \ \mathcal{P}_j > 0, \ \sum \mathcal{P}_j = 1$ (one eq. for $\mathcal{P}_j$, arbitrarily chosen, must be dropped and replaced by $\sum \mathcal{P}_j = 1$); $\mathcal{P}_j$ = stationary state prob. of embedded Markov chain
Time Homogeneous Markov Processes (method of integral equations)	$\mathrm{R}_{Si}(t) = e^{-\rho_i t} + \sum_{Z_j \in U, j \neq i} \int_0^t \rho_{ij} e^{-\rho_i x} \mathrm{R}_{Sj}(t-x)\,dx, \quad Z_i \in U$ $MTTF_{Si} = \frac{1}{\rho_i} + \sum_{Z_j \in U, j \neq i} \frac{\rho_{ij}}{\rho_i} MTTF_{Sj}, \quad Z_i \in U$ *with* ρ_{ij} = transition rate (see below), $\quad \rho_i = \sum_{j=0, j \neq i}^{m} \rho_{ij}$	$\mathrm{PA}_{Si}(t) = \sum_{Z_j \in U} \mathrm{P}_{ij}(t), \quad i = 0, \ldots, m$ $PA_S = \lim_{t \to \infty} \mathrm{PA}_{Si}(t) = \sum_{Z_j \in U} P_j$ (see $\mathrm{IR}_S(\theta)$ for P_j) *with* $\mathrm{P}_{ij}(t) = \delta_{ij} e^{-\rho_i t} + \sum_{k=0, k \neq i}^{m} \int_0^t \rho_{ik} e^{-\rho_i x} \mathrm{P}_{kj}(t-x)\,dx$ $i, j = 0, \ldots, m, \quad \delta_{ii} = 1, \quad \delta_{ij} = 0 \text{ for } j \neq i$	$\mathrm{IR}_{Si}(t, t+\theta) = \sum_{Z_j \in U} \mathrm{P}_{ij}(t) \mathrm{R}_{Sj}(\theta)$ $\mathrm{IR}_S(\theta) = \sum_{Z_j \in U} P_j \mathrm{R}_{Sj}(\theta)$ see below for P_j

Table 6.2 (cont.)

Time Homogeneous Markov Processes (method of differential equations)	$R_{Si}(t) = \sum_{Z_j \in U} P'_{ij}(t), \quad Z_i \in U$ $MTTF_{Si} = \frac{1}{\rho_i} + \sum_{Z_j \in U, j \neq i} \frac{\rho_{ij}}{\rho_i} MTTF_{Sj}, \quad Z_i \in U$ *with* $P'_{ij}(t) \equiv P'_j(t)$ and $P'_j(t)$ obtained from $\dot{P}'_j(t) = -\rho'_j P'_j(t) + \sum_{i=0, i \neq j}^{m} P'_i(t) \rho'_{ij}, \quad j = 0, \ldots, m$ $\rho'_{ij} = \rho_{ij}$ for $Z_i \in U$, $\rho'_{ij} = 0$ for $Z_i \in \overline{U}$, $\quad \rho'_j = \sum_{i=0, i \neq j}^{m} \rho'_{ji}$ $P'_i(0) = 1$, $P'_j(0) = 0$ for $j \neq i$, $Z_i \in U$	$PA_{Si}(t) = \sum_{Z_j \in U} P_{ij}(t), \quad i = 0, \ldots, m$ $PA_S = \lim_{t \to \infty} PA_{Si}(t) = \sum_{Z_j \in U} P_j$ (see $IR_S(\theta)$ for P_j) *with* $P_{ij}(t) \equiv P_j(t)$ and $P_j(t)$ obtained from $\dot{P}_j(t) = -\rho_j P_j(t) + \sum_{i=0, i \neq j}^{m} P_i(t) \rho_{ij}, \quad j = 0, \ldots, m$ $\rho_j = \sum_{i=0, i \neq j}^{m} \rho_{ji}$, $\quad P_i(0) = 1$, $P_j(0) = 0$ for $j \neq i$, $i = 0, \ldots, m$	$IR_{Si}(t, t+\theta) = \sum_{Z_j \in U} P_{ij}(t) R_{Sj}(\theta)$ $IR_S(\theta) = \sum_{Z_j \in U} P_j R_{Sj}(\theta)$ *with* P_j from $\rho_j P_j = \sum_{i=0, i \neq j}^{m} P_i \rho_{ij}, \quad j = 0, \ldots, m$ $P_j > 0$, $P_0 + \ldots + P_m = 1$ (one equation for P_j, arbitrarily chosen, must be dropped and replaced by $P_0 + \ldots + P_m = 1$)

$R_{Si}(t)$ = Pr{ system *up in* $(0, t]$ \| Z_i is entered at $t = 0$ },[+)] $Z_i \in U$; $\quad S$ stays for system; $\quad U$ = set of the *up states*, $\quad \overline{U}$ = set of the *down states*, $\quad U \cup \overline{U} = \{Z_0, \ldots, Z_m\}$

$MTTF_{Si}$ = E [system failure - free time \| Z_i is entered at $t = 0$] $= \int_0^\infty R_{Si}(t)\,dt = \tilde{R}_{Si}(0)$,[+)] $Z_i \in U$; $\quad \tilde{R}_{Si}(s) = \int_0^\infty R_{Si}(t) e^{-st} dt$ = Laplace transform of $R_{Si}(t)$

$PA_{Si}(t)$ = Pr{ system *up at t* \| Z_i is entered at $t = 0$ },[+)] $i = 0, \ldots, m$; $\quad PA_S$ = Pr{ system *up* at t in steady-state or for $t \to \infty$} $= AA_S$ = average av. in steady-state or for $t \to \infty$

$IR_{Si}(t, t+\theta)$ = Pr{ system *up in* $[t, t+\theta]$ \| Z_i is entered at $t = 0$ },[+)] $Z_i \in U$ (in general); $\quad IR_S(\theta)$ = Pr{ system *up in* $[t, t+\theta]$ in steady-state or for $t \to \infty$ }

T_i = mean stay (sojourn) time in Z_i ($= 1/\rho_i$ for Markov processes, $= \int_0^\infty (1 - Q_i(x))dx$ for SMP); $\quad T_{ii} = \frac{T_i}{P_i} = \frac{1}{\mathcal{P}_i} \sum_{k=0}^{m} \mathcal{P}_k T_k$ = mean recurrence time of Z_i

MUT_S[++)] $= PA_S / f_{udS}$ = system mean up time; $\quad MDT_S = (1 - PA_S)/f_{duS}$ = system mean down time; $\quad f_{udS} = f_{duS} = 1/(MUT_S + MDT_S) = \sum_{Z_i \in U, Z_j \in \overline{U}} P_i \rho_{ij}$ for Markov Prozesse

$P_{ij}(t)$ = Pr{ system in state Z_j at t \| Z_i is entered at $t = 0$ };[+)] $\quad P_j(t)$ = Pr{ system in state Z_j at t}, $\quad P_j = \lim_{t \to \infty} P_j(t) = P_j(t)$ in steady-state $= \lim_{t \to \infty} P_{ij}(t) = T_j / T_{jj}$

ρ_{ij} $= \lim_{\delta t \downarrow 0} \frac{1}{\delta t}$ Pr{ transition from Z_i to Z_j in $(t, t+\delta t]$ \| system in Z_i at t }, holds for Markov processes only ($\rho_{ij} = \rho_i \mathcal{P}_{ij}$, $\rho_{ii} \equiv 0$), t arbitrary

[+)] For Markov processes, "Z_i is entered at $t = 0$" can be replaced by "system in Z_i at $t = 0$". [++)] MUT_S is the mean time between a transition $\overline{U} \to U$ and the successive $U \to \overline{U}$ in steady-state or for $t \to \infty$. (Considering MUT_S and MDT_S, one recognizes that in steady-state, or for $t \to \infty$, a system beaves like a one-item structure ($MTTF = MUT_S$, $MTTR = MDT_S$); for practical applications, $MUT_S \approx MTTF_{SO}$; after a repair, the repaired *element is as-good-as-new* (for Markov processes, this holds for all other elements because of the constant failure rates); *repair* is used for *restoration*.)

Section 6.2 considers the one-item repairable structure under general assumptions, allowing a careful investigation of the *asymptotic and stationary behavior.* For the basic reliability structures encountered in practical applications (series, parallel, and series-parallel), investigations in Sections 6.3 - 6.6 begin by assuming *constant failure and repair rates* for every element in the reliability block diagram. Distributions of the repair times, and as far as possible of the failure-free times, are then generalized step by step up to the case in which the process involved remains regenerative with a *minimum number of regeneration states.* This, also to show capability & limits of the models involved. For large series-parallel structures, *approximate expressions* are developed in deep in Section 6.7. Procedures for investigating *repairable systems with complex structure* (for which a reliability block diagram often does not exist) are given in Section 6.8 on the basis of practical examples, including, among others, imperfect switching, incomplete coverage, more than two states, phased-mission systems, common cause failures, and fault tolerant reconfigurable systems with reward & frequency/duration aspects. It is shown that the tools developed in Appendix A7 (summarized in Tab. 6.2) can be used to solve many of the problems occurring in practical applications, on a *case-by-case basis* working with the *diagram of transition rates* or a *time schedule.* Alternative investigation methods, as well as computer-aided analysis is discussed in Section 6.9 and a Monte Carlo approach useful for rare events is given.

From the results of Sections 6.2 - 6.9, the following conclusions can be drawn:

1. As long as for each element in the reliability block diagram the condition $MTTR \ll MTTF$ holds, the *shape of the distribution function* of the repair time has small influence on the mean time to failure and on the steady-state availability at system level (see for instance Examples 6.8, 6.9, 6.10).
2. As a consequence of Point 1, it is preferable to start investigations by assuming *Markov models* (constant failure and repair rates for all elements, Table 6.2); in a second step, more appropriate distribution functions can be considered.
3. The assumption (6.2) of no further failure at system down *has no influence on the reliability function*; it allows a reduction of the state space and simplifies calculation of the availability and interval reliability (yielding good approximate values for the cases in which this assumption does not apply).
4. Already for moderately large systems, use of Markov models can become time-consuming (up to $e \cdot n!$ states for a reliability block diagram with n elements); *approximate expressions* are important, and the *macro-structures* introduced in Section 6.7 (Table 6.10) adheres well to many practical applications.
5. For large systems or complex structures, following possibilities are available:
   - work directly with the diagram of transition rates (Section 6.8),
   - calculation of the mean time to failure and of the steady-state availability at system level only (Table 6.2, Eqs. (A7.126), (A7.173), (A7.131), (A7.178)),
   - use of approximate expressions (Sections 6.7 and 6.9.7),
   - use of alternative methods or *Monte-Carlo simulation* (Section 6.9).

6.2 One-Item Structure

A *one-item structure* is an unit of arbitrary complexity, generally considered as an entity for investigations. Its reliability block diagram is a single element (Fig. 6.1). Considering that in practical applications a *repairable* one-item structure can have the complexity of a system, and also to use the same notation as in the following sections of this chapter, reliability figures are given with the indices S or $S0$ (e.g. PA_S, $R_{S0}(t)$, $MTTF_{S0}$), where S stands for *system* and 0 specifying *item new at* $t=0$ (S alone is used for arbitrary conditions at $t=0$ or for steady-state).

Under the assumptions (6.1) to (6.3) and (6.5) to (6.7), the repairable one-item structure is completely characterized by the distribution function of the *failure-free times* $\tau_0, \tau_1, \ldots$

$$F_A(x) = \Pr\{\tau_0 \le x\} \quad \text{and} \quad F(x) = \Pr\{\tau_i \le x\}, \qquad \begin{matrix} i=1,2,\ldots,\ x>0, \\ F_A(0)=F(0)=0, \end{matrix} \tag{6.8}$$

with densities

$$f_A(x) = \frac{dF_A(x)}{dx} \quad \text{and} \quad f(x) = \frac{dF(x)}{dx}, \tag{6.9}$$

the distribution function of the *repair times* $\tau_0', \tau_1', \ldots$

$$G_A(x) = \Pr\{\tau_0' \le x\} \quad \text{and} \quad G(x) = \Pr\{\tau_i' \le x\}, \qquad \begin{matrix} i=1,2,\ldots,\ x>0 \\ G(0)=G_A(0)=0, \end{matrix} \tag{6.10}$$

with densities

$$g_A(x) = \frac{dG_A(x)}{dx} \quad \text{and} \quad g(x) = \frac{dG(x)}{dx}, \tag{6.11}$$

and the probability p that the one-item structure is *up at* $t=0$

$$p = \Pr\{up \text{ at } t=0\} \tag{6.12}$$

or

$$1-p = \Pr\{down \text{ (i.e. under repair) at } t=0\},$$

respectively (τ_i & τ_i' are interarrival times, and x is used instead of t). The time behavior of the one-item structure can be investigated in this case with help of the *alternating renewal process* introduced in Appendix A7.3.

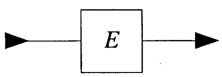


Figure 6.1 Reliability block diagram for a one-item structure

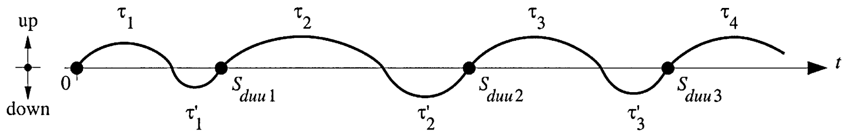


Figure 6.2 Possible time behavior for a repairable one-item structure new at $t = 0$ (repair times greatly exaggerated, alternating renewal process with renewal points 0, S_{duu1}, S_{duu2},... for a transition from down state to up state given that the item is up at $t = 0$ (marked by •))

Section 6.2.1 considers the one-item structure *new* at $t = 0$, i.e., the case $p = 1$ and $F_A(x) = F(x)$, with arbitrary $F(x)$ and $G(x)$. Generalization of the initial conditions at $t = 0$ (Sections 6.2.3) allows in Sections 6.2.4 and 6.2.5 a depth investigation of the *asymptotic* and *steady-state behavior.*

6.2.1 One-Item Structure New at Time $t = 0$

Figure 6.2 shows the time behavior of a one-item structure new at $t = 0$. $\tau_1, \tau_2, \ldots$ are the failure-free times. They are statistically independent and distributed according to $F(x)$ as per Eq. (6.8). Similarly, $\tau'_1, \tau'_2, \ldots$ are the repair times, distributed according to $G(x)$ as per Eq. (6.10). Considering assumption (6.5), the time points $0, S_{duu1}, \ldots$ are *renewal points* and constitute an ordinary *renewal process embedded* in the original alternating renewal process. Investigations of this Section are based on this property (S_{duu} means a transition from *down* (repair) to *up* (operating) starting *up* at $t = 0$).

6.2.1.1 Reliability Function

The *reliability function* $R_{S0}(t)$ gives the probability that the item operates failure free in $(0, t]$ given *item new at* $t = 0$

$$R_{S0}(t) = \Pr\{\text{up in } (0,t] \mid \text{new at } t = 0\}. \tag{6.13}$$

Considering Eqs. (2.7) and (6.8) it holds that

$$R_{S0}(t) = \Pr\{\tau_1 > t\} = 1 - F(t), \tag{6.14}$$

yielding $R_{S0}(t) = e^{-\lambda t}$ for the case of constant failure rate λ. The *mean time to failure* given *item new at* $t = 0$ follows from Eq. (A6.38)

$$MTTF_{S0} = \int_0^\infty R_{S0}(t)\, dt, \tag{6.15}$$

with upper limit of the integral T_L should the useful life of the item be limited to T_L ($R_{S0}(t)$ jumps to 0 at $t = T_L$). In the following, $T_L = \infty$ is assumed, yielding $MTTF_{S0} = 1/\lambda$ for the case of constant failure rate λ.

6.2.1.2 Point Availability

The *point availability* $PA_{S0}(t)$ gives the probability of finding the item operating at time t given *item new at* $t = 0$

$$PA_{S0}(t) = \Pr\{up \text{ at } t \mid \text{new at } t = 0\}. \tag{6.16}$$

For $PA_{S0}(t)$ it holds that

$$PA_{S0}(t) = 1 - F(t) + \int_0^t h_{duu}(x)(1 - F(t-x))\,dx. \tag{6.17}$$

$A(t)$ is often used instead of $PA_{S0}(t)$. Equation (6.17) is derived in Appendix A7.3 (Eq. (A7.56)) using the theorem of total probability. $1 - F(t)$ is the probability of no failure in $(0, t]$, $h_{duu}(x)dx$ gives the probability that any one of the renewal points $S_{duu1}, S_{duu2}, \ldots$ lies in $(x, x + dx]$, and $1 - F(t - x)$ is the probability that no further failure occurs in $(x, t]$. Using *Laplace transform* (Appendix A9.7) and considering Eq. (A7.50) with $F_A(x) = F(x)$, Eq. (6.17) yields

$$\tilde{PA}_{S0}(s) = \frac{1 - \tilde{f}(s)}{s(1 - \tilde{f}(s)\tilde{g}(s))}. \tag{6.18}$$

$\tilde{f}(s)$ and $\tilde{g}(s)$ are the Laplace transforms of the failure-free time and repair time densities, respectively (given by Eqs. (6.9) and (6.11)).

Example 6.1

a) Give the Laplace transform of the point availability $PA_{S0}(t)$ for the case of a *constant failure rate* λ $(\lambda(x) = \lambda)$.

b) Give the Laplace transform and the corresponding time function of the point availability for the case of *constant failure and repair rates* λ and μ $(\lambda(x) = \lambda$ and $\mu(x) = \mu)$.

Solution

a) With $F(x) = 1 - e^{-\lambda x}$ or $f(x) = \lambda e^{-\lambda x}$, Eq. (6.18) yields

$$\tilde{PA}_{S0}(s) = 1/(s + \lambda(1 - \tilde{g}(s))). \tag{6.19}$$

Supplementary results: $g(x) = \alpha(\alpha x)^{\beta-1} e^{-\alpha x} / \Gamma(\beta)$ (Eq. (A6.98)) yields

$$\tilde{PA}_{S0}(s) = \frac{(s+\alpha)^\beta}{(s+\lambda)(s+\alpha)^\beta - \lambda\alpha^\beta}.$$

b) With $f(x) = \lambda e^{-\lambda x}$ and $g(x) = \mu e^{-\mu x}$, Eq. (6.18) yields

$$\tilde{PA}_{S0}(s) = \frac{s+\mu}{s(s+\lambda+\mu)},$$

and thus (Table A9.7b)

$$PA_{S0}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} \approx (1-\lambda/\mu) + \frac{\lambda}{\mu} e^{-\mu t}. \qquad (6.20)$$

$PA_{S0}(t)$ converges rapidly, exponentially with a time constant

$1/(\lambda+\mu) \approx 1/\mu = MTTR,$

to the asymptotic value $\mu/(\lambda+\mu) \approx 1-\lambda/\mu$, see Section 6.2.4 for an extensive discussion.

$PA_{S0}(t)$ can also be obtained using *renewal process* arguments (Appendices A7.2, A7.3, A7.6). After the first repair the item is *as-good-as-new*. S_{duu1} is a *renewal point* and from this time point the process *restarts anew* as at $t=0$. Therefore

$$\Pr\{up \text{ at } t \mid S_{duu1} = x\} = PA_{S0}(t-x). \qquad (6.21)$$

Considering that the event

$\{up \text{ at } t\}$

occurs with exactly one of the following two mutually exclusive events

$\{\text{no failure in } (0, t]\}$

or

$\{S_{duu1} < t \cap up \text{ at } t\}$

it follows that

$$PA_{S0}(t) = 1 - F(t) + \int_0^t (f(x) * g(x))\, PA_{S0}(t-x)\, dx, \qquad (6.22)$$

where $f(x) * g(x)$ is the density of the sum $\tau_1 + \tau_1'$ (see Fig 6.2 and Eq. (A6.75)). The Laplace transform of $PA_{S0}(t)$ as per Eq. (6.22) is that given by Eq. (6.18).

6.2.1.3 Average Availability

The *average availability* $AA_{S0}(t)$ is defined as the expected proportion of time in which the item is operating in $(0, t]$ given *item new at* $t=0$

$$AA_{S0}(t) = \frac{1}{t} E[\text{total } up \text{ time in } (0, t] \mid \text{new at } t=0]. \qquad (6.23)$$

Considering $\text{PA}_{S0}(x)$ from Eq. (6.17), it holds that

$$\text{AA}_{S0}(t) = \frac{1}{t}\int_0^t \text{PA}_{S0}(x)\,dx. \tag{6.24}$$

Eq. (6.24) has a great intuitive appeal. It can be proved by considering that the time behavior of the repairable item can be described by a binary random function $\zeta(t)$ taking values 1 for *up* and 0 for *down*. From this, $\text{E}[\zeta(t)] = 0\cdot(1-\text{PA}_{S0}(t)) + 1\cdot\text{PA}_{S0}(t) = \text{PA}_{S0}(t)$ and, taking care of $\int_0^t \zeta(x)\,dx$ = total *up* time in $(0, t]$, it follows that (by Fubini's theorem [A6.6 (Vol. II)] and assuming existence of the integrals)

$$\text{AA}_{S0}(t) = \frac{1}{t}\,\text{E}[\int_0^t \zeta(x)dx] = \frac{1}{t}\int_0^t \text{E}[\zeta(x)]\,dx = \frac{1}{t}\int_0^t \text{PA}_{S0}(x)\,dx.$$

6.2.1.4 Interval Reliability

The *interval reliability* $\text{IR}_{S0}(t, t+\theta)$ gives the probability that the item operates failure free during an interval $[t, t+\theta]$ given *item new at* $t = 0$

$$\text{IR}_{S0}(t, t+\theta) = \Pr\{up \text{ in } [t, t+\theta] \mid \text{new at } t = 0\}. \tag{6.25}$$

The same method used to obtain Eq. (6.17) leads to

$$\text{IR}_{S0}(t, t+\theta) = 1 - \text{F}(t+\theta) + \int_0^t \text{h}_{duu}(x)(1-\text{F}(t+\theta-x))\,dx. \tag{6.26}$$

Example 6.2

Give the interval reliability $\text{IR}_{S0}(t, t+\theta)$ for the case of a *constant failure rate* λ $(\lambda(x) = \lambda)$.

Solution

With $\text{F}(x) = 1 - e^{-\lambda x}$ it follows that

$$\text{IR}_{S0}(t, t+\theta) = e^{-\lambda(t+\theta)} + \int_0^t \text{h}_{duu}(x)e^{-\lambda(t+\theta-x)}\,dx = [e^{-\lambda t} + \int_0^t \text{h}_{duu}(x)e^{-\lambda(t-x)}\,dx]\,e^{-\lambda\theta}.$$

Comparison with Eq. (6.17) for $\text{F}(x) = 1 - e^{\lambda x}$ yields

$$\text{IR}_{S0}(t, t+\theta) = \text{PA}_{S0}(t)\cdot e^{-\lambda\theta}. \tag{6.27}$$

It must be pointed out that the *product rule* in Eq. 6.27, expressing $\Pr\{up \text{ in } [t, t+\theta] \mid \text{new at } t = 0\} = \Pr\{up \text{ at } t \mid \text{new at } t = 0\} \cdot \Pr\{\text{no failure in } (t, t+\theta] \mid up \text{ at } t\}$, is valid *only* because of the *constant failure rate* λ (*memoryless property*, Eq. (2.14)); in the general case, the second term is $\Pr\{\text{no failure in } (t, t+\theta] \mid (up \text{ at } t \cap \text{new at } t = 0)\}$, which differs from $\Pr\{\text{no failure in } (t, t+\theta] \mid up \text{ at } t\}$ (see also Example A7.2).

6.2.1.5 Special Kinds of Availability

In addition to the point and average availability (Sections 6.2.1.2 and 6.2.1.3), there are several other kinds of availability useful for practical applications [6.5 (1973)]:

1. *Mission Availability*: The mission availability $MA_{S0}(T_o, t_o)$ gives the probability that in a mission of total operating time (*total up time*) T_o each failure can be repaired within a time span t_o, given *item new at* $t = 0$

$$MA_{S0}(T_o, t_o) = \Pr\{\text{each individual failure occuring in a mission with total operating time } T_o \text{ can be repaired in a time} \le t_o \mid \text{new at } t=0\}. \qquad (6.28)$$

Mission availability is important in applications where *interruptions of length* $\le t_o$ can be accepted. Its computation considers all cases with $n = 0, 1, \ldots$ failures, taking care that at the end of the mission the item is operating (to reach the given (fixed) operating time T_o).+) Thus, for *given* $T_o > 0$ and t_o,

$$MA_{S0}(T_o, t_o) = 1 - F(T_o) + \sum_{n=1}^{\infty} (F_n(T_o) - F_{n+1}(T_o))\,(G(t_o))^n \qquad (6.29)$$

holds. $F_n(T_o) - F_{n+1}(T_o)$ is the probability for n failures during the total operating time T_o (Eq. (A7.14)); $(G(t_o))^n$ is the probability that all n repair times will be shorter than t_o. For *constant failure rate* λ it holds that $F_n(T_o) - F_{n+1}(T_o) = (\lambda T_o)^n e^{-\lambda T_o} / n!$ and thus

$$MA_{S0}(T_o, t_o) = e^{-\lambda T_o (1 - G(t_o))}. \qquad (6.30)$$

2. *Work-Mission Availability*: The work-mission availability $WMA_{S0}(T_o, x)$ gives the probability that the *sum* of the repair times for all failures occurring in a mission of total operating time (*total up time*) T_o is $\le x$, given *item new at* $t = 0$

$$WMA_{S0}(T_o, x) = \Pr\{\text{sum of the repair times for all failures occurring in a mission of total operating time } T_o \text{ is } \le x \mid \text{new at } t=0\}. \qquad (6.31)$$

Similarly as for Eq. (6.29) it follows that for *given* (fixed) $T_o > 0$ and $x > 0$ +)

$$WMA_{S0}(T_o, x) = 1 - F(T_o) + \sum_{n=1}^{\infty} (F_n(T_o) - F_{n+1}(T_o))\, G_n(x), \text{ ++)} \qquad (6.32)$$

where $G_n(x)$ is the distribution function of the sum of n repair times with distribution $G(x)$ (Eq. (A7.13)). As for the *mission availability*, the item is up at the end of the mission (to reach the given (fixed) operating time T_o). For constant failure and repair rates (λ, μ), Eq. (6.32) yields (see also Eq. (A7.219))

+) An unlimited number n of repair is assumed here, see e.g. Section 4.6 (p. 140) for n limited.
++) See e.g. p. 514 for a possible application of Eq. (6.32) to a cumulative damage model.

$$\text{WMA}_{S0}(T_o, x) = 1 - e^{-(\lambda T_0 + \mu x)} \sum_{n=1}^{\infty} \left[\frac{(\lambda T_0)^n}{n!} \sum_{k=0}^{n-1} \frac{(\mu x)^k}{k!} \right], \qquad \begin{array}{l} T_0 > 0 \text{ given}, \ x > 0, \\ \text{WMA}_{S0}(T_0, 0) = e^{-\lambda T_0}. \end{array} \tag{6.33}$$

Defining DT as total down time and $UT = t - DT$ as total up time in $(0,t]$, one can recognize that for given fixed t, $\text{WMA}_{S0}(t-x, x) = \Pr\{DT \text{ in } (0,t] \le x\}$ holds for an item described by Fig. 6.2 ($t > 0$, $0 < x \le t$). However, the item can now be up or down at t, and the situation differs from that defined by Eq. (6.31). The function $\text{WMA}_{S0}(t-x, x)$ has been investigated in [A7.29(57)]. In particular, a closed analytical expression for $\text{WMA}_{S0}(t-x, x)$ is given for constant failure and repair rates (λ, μ), and it is shown that the distribution of DT converges for $t \to \infty$ to a normal distribution with mean $t\lambda/(\lambda+\mu) \approx t\lambda/\mu$ and variance $t 2\lambda\mu/(\lambda+\mu)^3 \approx t 2\lambda/\mu^2$. It can be noted, that for the interpretation described by Eq. (6.32), mean and variance of the total repair time are given exactly by $T_0 \lambda/\mu$ and $T_0 2\lambda/\mu^2$, respectively (Eq. (A7.220)).

3. *Joint Availability*: The joint availability $\text{JA}_{S0}(t, t+\theta)$ gives the probability of finding the item operating at the time points t and $t+\theta$, given *item new at* $t=0$ (θ is given (fixed), see e.g. [6.15 (1999), 6.28] for stochastic demand)

$$\text{JA}_{S0}(t, t+\theta) = \Pr\{(\text{up at } t \cap \text{up at } t+\theta) \mid \text{new at } t=0\}. \tag{6.34}$$

For the case of *constant failure rate* $\lambda(x) = \lambda$, Eq. (6.27) yields

$$\text{JA}_{S0}(t, t+\theta) = \text{PA}_{S0}(t) \cdot \text{PA}_{S0}(\theta). \tag{6.35}$$

For arbitrary failure rate, one has to consider that $\{up \text{ at } t \cap up \text{ at } t+\theta \mid \text{new at } t=0\}$ occurs with one of the following 2 mutually exclusive events (Appendix A7.3)

$$\{up \text{ in } [t, t+\theta] \mid \text{new at } t=0\}$$

or

$$\{(up \text{ at } t \cap \text{next failure occurs before } t+\theta \cap up \text{ at } t+\theta) \mid \text{new at } t=0\}.$$

The probability for the first event is the interval reliability $\text{IR}_{S0}(t, t+\theta)$ given by Eq. (6.26). For the second event, it is necessary to consider the distribution function of the *forward recurrence time in the up state* $\tau_{Ru}(t)$. As shown in Fig. 6.3, $\tau_{Ru}(t)$ can only be defined if the item is up at time t, hence

$$\Pr\{\tau_{Ru}(t) > x \mid \text{new at } t=0\} = \Pr\{up \text{ in } (t, t+x] \mid (up \text{ at } t \cap \text{new at } t=0)\}$$

and thus, as for Example A7.2 and considering Eqs. (6.16) and (6.25),

$$\Pr\{\tau_{Ru}(t) > x \mid \text{new at } t=0\} = \frac{\Pr\{up \text{ in } [t, t+x] \mid \text{new at } t=0\}}{\Pr\{up \text{ at } t \mid \text{new at } t=0\}} = \frac{\text{IR}_{S0}(t, t+x)}{\text{PA}_{S0}(t)} = 1 - \text{F}_{\tau_{Ru}}(x). \tag{6.36}$$

For constant failure rate $\lambda(x) = \lambda$ one has $1 - F_{\tau_{Ru}}(x) = e^{-\lambda x}$, as per Eq. (6.27). Considering Eq. (6.36) it follows that

$$\begin{aligned} JA_{S0}(t, t+\theta) &= IR_{S0}(t, t+\theta) + PA_{S0}(t) \int_0^\theta f_{\tau_{Ru}}(x) PA_{S1}(\theta - x) dx \\ &= IR_{S0}(t, t+\theta) - \int_0^\theta \frac{\partial IR_{S0}(t, t+x)}{\partial x} PA_{S1}(\theta - x) dx \,, \end{aligned} \tag{6.37}$$

where $PA_{S1}(t) = \Pr\{up \text{ at } t \mid \text{a repair begins at } t = 0\}$ is given by

$$PA_{S1}(t) = \int_0^t h_{dud}(x)(1 - F(t - x)) dx \,, \tag{6.38}$$

with $h_{dud}(t) = g(t) + g(t) * f(t) * g(t) + g(t) * f(t) * g(t) * f(t) * g(t) + \dots$ (Eq. (A7.50)). $JA_{S0}(t, t+\theta)$ can also be obtained in a similar way to $PA_{S0}(t)$ in Eq. (6.17), by considering the alternating renewal process starting up at the time t with $\tau_{Ru}(t)$ distributed according to $F_{\tau_{Ru}}(x)$ as per Eq. (6.36). This leads to

$$JA_{S0}(t, t+\theta) = IR_{S0}(t, t+\theta) + \int_0^\theta h'_{duu}(x)(1 - F(\theta - x)) dx, \tag{6.39}$$

with $h'_{duu}(x) = f'_{\tau_{Ru}}(x) * g(x) + f'_{\tau_{Ru}}(x) * g(x) * f(x) * g(x) + \dots$, see Eq. (A7.50), and $f'_{\tau_{Ru}}(x) = PA_{S0}(t) f_{\tau_{Ru}}(x) = PA_{S0}(t) dF_{\tau_{Ru}}(x) / dx = -\partial IR_{S0}(t, t+x) / \partial x$, see Eqs. (6.36) and (6.37). Similarly as for $\tau_{Ru}(t)$, the distribution function for the *forward recurrence time in the down state* $\tau_{Rd}(t)$ is given by (Fig. 6.3)

$$\Pr\{\tau_{Rd}(t) \le x \mid \text{new at } t=0\} = 1 - \int_0^t h_{udu}(y)(1 - G(t+x-y)) dy \,/ (1 - PA_{S0}(t)), \tag{6.40}$$

with $h_{udu}(t) = f(t) + f(t) * g(t) * f(t) + \dots$ (Eq. (A7.50)). For constant failure rate $\lambda(x) = \lambda$, Eq. (6.37) or (6.39) leads to Eq. (6.35), by considering Eq. (6.19).

Other kinds of availability are possible. For instance, availability by omitting down times for repair shorter than a given fixed or random time Δ has been investigated recently in [6.48], yielding for the case of fixed Δ to $\lim_{t \to \infty} PA_\Delta(t) = 1 - \frac{\lambda}{\lambda + \mu}(1 + \mu\Delta)\, e^{-\mu\Delta}$.

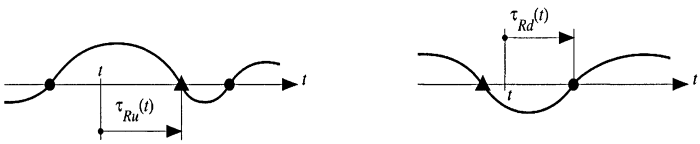


Figure 6.3 Forward recurrence times $\tau_{Ru}(t)$ and $\tau_{Rd}(t)$ in an alternating renewal process

6.2.2 One-Item Structure New at Time $t = 0$ and with Constant Failure Rate λ

In many practical applications, a *constant failure rate* λ can be assumed. In this case, the expressions of Section 6.2.1 can be simplified making use of the *memoryless property* given by the constant failure rate. Table 6.3 summarizes the results for the cases of constant failure rate (λ) and constant or arbitrary repair rate (μ or $\mu(x) = g(x)/(1-G(x))$). Approximations in Table 6.3 are valid for $\lambda << \mu$ and $t > 10/\mu = 10\,MTTR$. For points 3 in Table 6.3 it can be noted that $AA_{S0}(0) = 1$, as for $PA_{S0}(0)$, and that the convergence of $AA_{S0}(t)$ toward $AA_S = PA_S$ is slower than that of $PA_{S0}(t)$. The product rule for $IR_{S0}(t, t+\theta)$ and $JA_{S0}(t, t+\theta)$ is valid because of the constant failure rate λ.

Table 6.3 Results for a repairable one-item structure new at $t = 0$ and with *constant failure rate* λ

	Repair rate: arbitrary ($\mu(x)$)	Repair rate: constant (μ)$^{+)}$	Remarks, Assumptions
1. Reliability function $R_{S0}(t)$	$e^{-\lambda t}$	$e^{-\lambda t}$	$R_{S0}(t) = \Pr\{up \text{ in } (0, t] \mid \text{new at } t = 0\}$
2. Point availability $PA_{S0}(t)$	$e^{-\lambda t} + \int_0^t h_{duu}(x)\, e^{-\lambda(t-x)}dx$	$\frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$ $\approx \mu/(\lambda+\mu) \approx 1 - \lambda/\mu$	$PA_{S0}(t) = \Pr\{up \text{ at } t \mid \text{new at } t = 0\}$, $h_{duu} = f*g + f*g*f*g + \ldots$
3. Average availability $AA_{S0}(t)$	$\frac{1}{t}\int_0^t PA_{S0}(x)dx$	$\frac{\mu}{\lambda+\mu} + \frac{\lambda(1-e^{-(\lambda+\mu)t})}{t(\lambda+\mu)^2}$ $\approx \mu/(\lambda+\mu) + \lambda/t\mu^2$	$AA_{S0}(t) = \text{E[total } up \text{ time in } (0, t] \mid \text{new at } t = 0\}/t$
4. Interval reliability $IR_{S0}(t, t+\theta)$	$PA_{S0}(t)\, e^{-\lambda\theta}$	$\frac{\mu e^{-\lambda\theta}}{\lambda+\mu} + \frac{\lambda e^{-(\lambda+\mu)t-\lambda\theta}}{\lambda+\mu}$	$IR_{S0}(t, t+\theta) = \Pr\{up \text{ in } [t, t+\theta] \mid \text{new at } t = 0\}$
5. Joint availability $JA_{S0}(t, t+\theta)$	$PA_{S0}(t)PA_{S0}(\theta)$	$PA_{S0}(t)PA_{S0}(\theta)$	$JA_{S0}(t, t+\theta) = \Pr\{up \text{ at } t \cap up \text{ at } t+\theta \mid \text{new at } t = 0\}$, $PA_{S0}(x)$ as in point 2
6. Mission availability $MA_{S0}(T_o, t_f)$	$e^{-\lambda T_o(1-G(t_f))}$	$e^{-\lambda T_o e^{-\mu t_f}}$	$MA_{S0}(T_o, t_f) = \Pr\{\text{each failure in a mission with total operating time } T_o \text{ can be repaired in a time} \le t_f \mid \text{new at } t = 0\}$

$^{+)}$ Markov process; up = operating state; approximations valid for $\lambda << \mu$ and $t > 10/\mu = 10\,MTTR$

6.2.3 One-Item Structure with Arbitrary Conditions at $t = 0$

Generalization of the initial conditions at time $t = 0$, i.e., the introduction of p, $F_A(x)$ and $G_A(x)$as defined by Eqs. (6.12), (6.8), and (6.10), leads to a time behavior of the one-item repairable structure described by Fig. A7.3 and to the following results:

1. *Reliability function* $R_S(t)$

$$R_S(t) = \Pr\{up \text{ in } (0, t] \mid up \text{ at } t = 0\} = 1 - F_A(t). \tag{6.41}$$

Equation (6.41) follows from $\Pr\{up \text{ in } [0,t]\} = \Pr\{up \text{ at } t = 0 \cap \Pr\{up \text{ in } (0,t]\}$ $= \Pr\{up \text{ at } t = 0\} \cdot \Pr\{up \text{ in} (0, t] \mid up \text{ at } t = 0\} = p \cdot (1 - F_A(t) = p \cdot R_S(t)$.

2. *Point availability* $PA_S(t)$

$$PA_S(t) = \Pr\{up \text{ at } t\} = p[1 - F_A(t) + \int_0^t h_{duu}(x)(1 - F(t-x))\,dx] + (1-p)\int_0^t h_{dud}(x)(1 - F(t-x))\,dx, \tag{6.42}$$

with $h_{duu}(t) = f_A(t) * g(t) + f_A(t) * g(t) * f(t) * g(t) + \ldots$ and $h_{dud}(t) = g_A(t) + g_A(t) * f(t) * g(t) + g_A(t) * f(t) * g(t) * f(t) * g(t) + \ldots$ (see also Eq. (A7.50)).

3. *Average availability* $AA_S(t)$

$$AA_S(t) = \frac{1}{t}\, E[\text{total } up \text{ time in } (0, t]] = \frac{1}{t}\int_0^t PA_S(x)\,dx. \tag{6.43}$$

4. *Interval reliability* $IR_S(t, t+\theta)$

$$IR_S(t, t+\theta) = \Pr\{\text{up in } [t, t+\theta]\} = p[1 - F_A(t+\theta) + \int_0^t h_{duu}(x)(1 - F(t+\theta-x))\,dx] + (1-p)\int_0^t h_{dud}(x)(1 - F(t+\theta-x))\,dx. \tag{6.44}$$

5. *Joint availability* $JA_S(t, t+\theta)$

$$JA_S(t, t+\theta) = \Pr\{up \text{ at } t \cap up \text{ at } t+\theta\} = IR_S(t, t+\theta) - \int_0^\theta \frac{\partial IR_S(t, t+x)}{\partial x} PA_{S1}(\theta - x)\,dx, \tag{6.45}$$

with $IR_S(t, t+\theta)$ from Eq. (6.44) and $PA_{S1}(t)$ from Eq. (6.38).

6. *Forward recurrence times* ($\tau_{Ru}(t)$ and $\tau_{Rd}(t)$ as in Fig. 6.3)

$$\Pr\{\tau_{Ru}(t) \le x\} = 1 - \mathrm{IR}_S(t, t+x) \,/\, \mathrm{PA}_S(t), \qquad (6.46)$$

with $\mathrm{IR}_S(t, t+x)$ according to Eq. (6.44) and $\mathrm{PA}_S(t)$ from Eq. (6.42), and

$$\Pr\{\tau_{Rd}(t) \le x\} = 1 - \frac{\Pr\{down \text{ in } [t, t+x]\}}{1 - \mathrm{PA}_S(t)}, \qquad (6.47)$$

where

$$\Pr\{down \text{ in } [t, t+x]\} = p\int_0^t \mathrm{h}_{udu}(y)(1 - \mathrm{G}(t+x-y))\,dy$$
$$+ (1-p)[1 - \mathrm{G}_A(t+x) + \int_0^t \mathrm{h}_{udd}(y)(1 - \mathrm{G}(t+x-y))\,dy],$$

with $\mathrm{h}_{udu}(t) = \mathrm{f}_A(t) + \mathrm{f}_A(t) * \mathrm{g}(t) * \mathrm{f}(t) + \mathrm{f}_A(t) * \mathrm{g}(t) * \mathrm{f}(t) * \mathrm{g}(t) * \mathrm{f}(t) + \ldots$ and $\mathrm{h}_{udd}(t) = \mathrm{g}_A(t) * \mathrm{f}(t) + \mathrm{g}_A(t) * \mathrm{f}(t) * \mathrm{g}(t) * \mathrm{f}(t) + \ldots$

Expressions for mission availability and work-mission availability are generally only used for items new at time $t = 0$ (see [6.5 (1973)] for a generalization.

6.2.4 Asymptotic Behavior

As $t \to \infty$ expressions for the point availability, average availability, interval reliability, joint availability, and distribution function of the forward recurrence time (Eqs. (6.42)-(6.47)) converge to quantities which are independent of t and initial conditions at $t = 0$. Using the *key renewal theorem* (Eq. (A7.29)) it follows that

$$\lim_{t\to\infty} \mathrm{PA}_S(t) = PA_S = \frac{MTTF}{MTTF + MTTR}, \qquad (6.48)$$

$$\lim_{t\to\infty} \mathrm{AA}_S(t) = AA_S = \frac{MTTF}{MTTF + MTTR} = PA_S, \qquad (6.49)$$

$$\lim_{t\to\infty} \mathrm{IR}_S(t, t+\theta) = \mathrm{IR}_S(\theta) = \frac{1}{MTTF + MTTR} \int_\theta^\infty (1 - \mathrm{F}(y))\,dy, \qquad (6.50)$$

$$\lim_{t\to\infty} \mathrm{JA}_S(t, t+\theta) = \mathrm{JA}_S(\theta) = \frac{MTTF}{MTTF + MTTR}\, \mathrm{PA}_{S0_e}(\theta), \qquad (6.51)$$

$$\lim_{t\to\infty} \Pr\{\tau_{Ru}(t) \le x\} = \frac{1}{MTTF} \int_0^x (1 - \mathrm{F}(y))\,dy, \qquad (6.52)$$

$$\lim_{t\to\infty} \Pr\{\tau_{Rd}(t) \le x\} = \frac{1}{MTTR} \int_0^x (1 - \mathrm{G}(y))\,dy, \qquad (6.53)$$

where $MTTF = E[\tau_i]$, $MTTR = E[\tau_i']$, $i = 1, 2, \ldots$, and $PA_{0e}(\theta)$ is the point availability according to Eq. (6.42) with $p = 1$ and $F_A(t)$ from Eq. (6.57) or Eq. (6.52). In practical applications, PA and AA (or PA_S and AA_S for system oriented values) are often referred as *availability* and denoted by A. The use of $PA_S = AA_S = (MTBF - MTTR) / MTBF$ is to avoid, because it implies $MTBF = MTTF + MTTR$.

Example 6.3

Show that for a repairable one-item structure in continuous operation, the limit

$$\lim_{t\to\infty} PA_S(t) = PA_S = \frac{MTTF}{MTTF + MTTR}$$

is valid for any distribution function $F(x)$ of the failure-free time and $G(x)$ of the repair time, if $MTTF < \infty$, $MTTR < \infty$, and the densities $f(x)$ and $g(x)$ go to 0 as $x \to \infty$.

Solution

Using the *renewal density theorem* Eq. (A7.31) it follows that

$$\lim_{t\to\infty} h_{duu}(t) = \lim_{t\to\infty} h_{dud}(t) = \frac{1}{MTTF + MTTR}.$$

Furthermore, applying the *key renewal theorem* Eq.(A.7.29) to $PA_S(t)$ given by Eq. (6.42) yields

$$\lim_{t\to\infty} PA_S(t) = p\left(1 - 1 + \frac{\int_0^\infty (1 - F(x))dx}{MTTF + MTTR}\right) + (1 - p)\frac{\int_0^\infty (1 - F(x))dx}{MTTF + MTTR}$$
$$= p\frac{MTTF}{MTTF + MTTR} + (1 - p)\frac{MTTF}{MTTF + MTTR} = \frac{MTTF}{MTTF + MTTR}.$$

The limit $MTTF / (MTTF + MTTR)$ can also be obtained from the final value theorem of the Laplace transform (Table A9.7), considering for $s \to 0$

$$\tilde{f}(s) = 1 - s\ MTTF + o(s) \approx 1 - s\ MTTF$$

and

$$\tilde{g}(s) = 1 - s\ MTTR + o(s) \approx 1 - s\ MTTR. \qquad (6.54)$$

with $o(s)$ as per Eq. (A7.89). When considering $\tilde{g}(\lambda)$ for availability calculations, the approximation given by Eq. (6.54) often leads to $PA_S = 1$, already by simple redundancy structures. In these cases, Eq. (6.113) has to be used.

In the case of *constant* failure & repair rates $\lambda(x) = \lambda$ and $\mu(x) = \mu$, Eq. (6.42) yields

$$PA_S(t) = \frac{\mu}{\lambda + \mu} + \left(p - \frac{\mu}{\lambda + \mu}\right)e^{-(\lambda+\mu)t}. \qquad (6.55)$$

Thus, for this important case, the convergence of $PA_S(t)$ toward $PA_S = \mu / (\lambda + \mu)$ is *exponential* with a time constant $1 / (\lambda + \mu) < 1/\mu = MTTR$. In particular, for

$p = 1$, i.e. for $\mathrm{PA}_S(0) = 1$ and $\mathrm{PA}_S(t) \equiv \mathrm{PA}_{S0}(t)$, it follows that

$$\mathrm{PA}_{S0}(t) - PA_S = \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} \le \frac{\lambda}{\mu} e^{-\mu t} = \lambda MTTR\, e^{-t/MTTR}. \qquad (6.56)$$

Generalizing the distribution function $\mathrm{G}(x)$ of the repair time and/or $\mathrm{F}(x)$ of the failure-free time, $\mathrm{PA}_{S0}(t)$ can oscillate damped (as in general for the renewal density $\mathrm{h}(t)$ given by Eq. (A7.18)). However, for *constant failure rate* λ and providing $\lambda\, MTTR$ sufficiently small and some rather weak conditions on the density $\mathrm{g}(x)$, lower and upper bounds for $\mathrm{PA}_{S0}(t)$ can be found [6.25]

$$\mathrm{PA}_{S0}(t) \ge \frac{1}{1+\lambda MTTR} - c_l \frac{\lambda MTTR}{1+\lambda MTTR} e^{-(\lambda+1/MTTR)t}, \qquad t \ge 0$$

and

$$\mathrm{PA}_{S0}(t) \le \frac{1}{1+\lambda MTTR} + c_u \frac{\lambda MTTR}{1+\lambda MTTR} e^{-(\lambda+1/MTTR)t}, \qquad t \ge 0.$$

$c_l = 1$ holds for many practical applications ($\lambda\, MTTR << 0.1$). Sufficient conditions for $c_u = 1$ are given in [6.25]. However, conditions on c_u are less important as on c_l, since $\mathrm{PA}_{S0}(t) \le 1$ is always true. The case of a *gamma distribution* with density $\mathrm{g}(x) = \alpha^\beta x^{\beta-1} e^{-\alpha x} / \Gamma(\beta)$, mean β/α, and shape parameter $\beta \ge 3$, leads for instance to $|\, \mathrm{PA}_{S0}(t) - PA_S \,| \le \lambda MTTR\, e^{-t/MTTR}$ at least for $t \ge 3\,MTTR = 3\beta/\alpha$.

6.2.5 Steady-State Behavior

For

$$p = \frac{MTTF}{MTTF + MTTR}, \qquad \mathrm{F}_A(x) = \frac{1}{MTTF}\int_0^x (1-\mathrm{F}(y))\,dy, \qquad \mathrm{G}_A(x) = \frac{1}{MTTR}\int_0^x (1-\mathrm{G}(y))\,dy \qquad (6.57)$$

the *alternating renewal process* describing the time behavior of a *one-item repairable structure* is *stationary* (in *steady-state*), see Appendix A7.3. With p, $\mathrm{F}_A(t)$, and $\mathrm{G}_A(t)$ as per Eq. (6.57), the expressions for the point availability (6.42), average availability (6.43), interval reliability (6.44), joint availability (6.45), and the distribution functions of the forward recurrence time (6.46) and (6.47) take the values given by Eqs. (6.48) – (6.53) for all $t \ge 0$, see Example 6.4 for the point availability PA_S. This relationship between asymptotic & steady-state (stationary) behavior is important in practical applications because it allows the following interpretation (see also the remark on pp. 464 & 469):

A one-item repairable structure is in a steady-state (stationary behavior) if it began operating at the time $t = -\infty$ and will be considered only for $t \ge 0$, the time $t = 0$ being an arbitrary time point.

Table 6.4 Results for a repairable one-item in asymptotic & steady-state (stationary) behavior

	Failure and repair rates: Arbitrary	Failure and repair rates: Constant+)	Remarks, assumptions
1. Pr{*up* at $t = 0$} (p)	$\frac{MTTF}{MTTF + MTTR}$	$\frac{\mu}{\lambda + \mu}$	$MTTF = E[\tau_i]$, $i \geq 1$; $MTTR = E[\tau'_i]$, $i \geq 1$
2. Distribution of τ_0 ($F_A(x) = \Pr\{\tau_0 \leq x\}$)	$\frac{1}{MTTF}\int_0^t (1 - F(x))dx$	$1 - e^{-\lambda t}$	$F_A(x)$ is also the distribution function of $\tau_{Ru}(t)$ as in Fig. 6.3 ($F_A(x) = \Pr\{\tau_{Ru}(t) \leq x\}$)
3. Distribution of τ'_0 ($G_A(x) = \Pr\{\tau'_0 \leq x\}$)	$\frac{1}{MTTR}\int_0^t (1 - G(x))dx$	$1 - e^{-\mu t}$	$G_A(x)$ is also the distribution function of $\tau_{Rd}(t)$ as in Fig. 6.3 ($G_A(x) = \Pr\{\tau_{Rd}(t) \leq x\}$)
4. Renewal densities $h_{du}(t)$ and $h_{ud}(t)$	$\frac{1}{MTTF + MTTR}$	$\frac{\lambda\mu}{\lambda + \mu}$	$h_{du}(t) = p\,h_{duu}(t) + (1-p)h_{dud}(t)$, $h_{ud}(t) = p\,h_{udu}(t) + (1-p)h_{udd}(t)$, p as in point 1 $\rightarrow$ $h_{du}(t) = h_{ud}(t)$
5. Point availability (PA_S)	$\frac{MTTF}{MTTF + MTTR}$	$\frac{\mu}{\lambda + \mu}$	$PA_S = \Pr\{up \text{ at } t\}$, $t \geq 0$
6. Average availability (AA_S)	$\frac{MTTF}{MTTF + MTTR}$	$\frac{\mu}{\lambda + \mu}$	$AA_S = \frac{1}{t}$ E[total up time in $(0, t]$], $t > 0$
7. Interval reliability ($IR_S(\theta)$)	$\frac{\int_\theta^\infty (1 - F(x))dx}{MTTF + MTTR}$	$\frac{\mu}{\lambda + \mu} e^{-\lambda\theta}$	$IR_S(\theta) = \Pr\{up \text{ in } [t, t + \theta]\}$, $t \geq 0$
8. Joint availability ($JA_S(\theta)$)	$\frac{MTTF \cdot PA_{S0e}(\theta)}{MTTF + MTTR}$	$\frac{\mu}{\lambda + \mu}\left(\frac{\mu}{\lambda + \mu} + \frac{\lambda e^{-(\lambda+\mu)\theta}}{(\lambda + \mu)}\right)$	$JA_S(\theta) = \Pr(up \text{ at } t \cap up \text{ at } t + \theta\}$, $PA_{S0e}(\theta) = PA_S(\theta)$ as per Eq. (6.42) with $p = 1$ and $F_A(t)$ as in point 2

+) Markov process; λ, μ = failure, repair rate; *up* = operating state; $h_{ud}(t)$, $h_{du}(t)$ = failure, repair frequency

For *constant failure rate* λ and *repair rate* μ, the convergence of $PA_{S0}(t)$ to PA_S is *exponential with time constant* $\approx 1/\mu = MTTR$ as per Eqs. (6.55). Extrapolating the results of Section 6.2.4, one can assume that for practical applications, the function $PA_{S0}(t)$ is *captured* at least for some $t > t_0 > 0$ in the band $|PA_{S0}(t) - PA_S| \approx \lambda\, MTTR\, e^{-t/MTTR}$ when generalizing the distribution function of repair times. Thus,

for practical purposes one can assume that after a time $t \approx 10\, MTTR$, the point availability $PA_{S0}(t)$ has reached its steady-state (stationary) value $PA_S = AA_S$

(this, considering $e^{-10} \approx 5 \cdot 10^{-5}$ and $\lambda / \mu \leq 10^{-2}$, see Tab. 6.3). Important results for the steady-state behavior of a repairable one-item structure are given in Table 6.4.

Example 6.4

Show that for a repairable one-item structure in steady-state, i.e. with p, $F_A(x)$, and $G_A(x)$ as per Eq. (6.57), the point availability is $PA_S(t) = PA_S = MTTF / (MTTF + MTTR)$ for all $t \geq 0$.

Solution

Applying the Laplace transform to Eq. (6.42) and using Eqs. (A7.50) and (6.57) yields

$$\tilde{PA}_S(s) = \frac{MTTF}{MTTF + MTTR}\left(\frac{1}{s} - \frac{1-\tilde{f}(s)}{s^2\, MTTF} + \frac{\frac{1-\tilde{f}(s)}{s\, MTTF}\,\tilde{g}(s)}{1-\tilde{f}(s)\tilde{g}(s)} \cdot \frac{1-\tilde{f}(s)}{s}\right)$$

$$+\frac{MTTR}{MTTF + MTTR} \cdot \frac{\frac{1-\tilde{g}(s)}{s\, MTTR}}{1-\tilde{f}(s)\tilde{g}(s)} \cdot \frac{1-\tilde{f}(s)}{s},$$

and finally

$$\tilde{PA}_S(s) = \frac{MTTF}{MTTF + MTTR}\left(\frac{1}{s} - \frac{1-\tilde{f}(s)}{s^2\, MTTF}\right) + \frac{[1-\tilde{f}(s)][\tilde{g}(s) - \tilde{f}(s)\tilde{g}(s) + 1 - \tilde{g}(s)]}{s^2\,(MTTF + MTTR)[1-\tilde{f}(s)\tilde{g}(s)]},$$

from which

$$\tilde{PA}_S(s) = \frac{MTTF}{MTTF + MTTR} \cdot \frac{1}{s},$$

and thus $PA_S(t) = PA_S$ for all $t \geq 0$.

6.3 Systems without Redundancy

The reliability block diagram of a *system without redundancy* consists of the series connection of all its elements E_1 to E_n, see Fig. 6.4. Each element E_i in Fig. 6.4 is characterized by the distribution functions $F_i(x)$ for the failure-free time and $G_i(x)$ for the repair time.

6.3.1 Series Structure with Constant Failure and Repair Rates for Each Element

In this section, *constant* failure and repair rates are assumed, i.e.

$$F_i(x) = 1 - e^{-\lambda_i x}, \qquad x > 0,\ F_i(0) = 0,, \tag{6.58}$$

and

$$G_i(x) = 1 - e^{-\mu_i x}, \qquad x > 0,\ G_i(0) = 0, \tag{6.59}$$

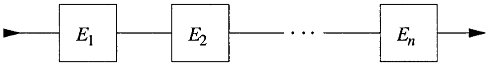


Figure 6.4 Reliability block diagram for a system without redundancy (series structure)

holds for $i = 1, \dots, n$. Because of Eqs. (6.58) and (6.59), the stochastic behavior of the system is described by a (time-homogeneous) *Markov process*. Let Z_0 be the system up state and Z_i the state in which element E_i is down. Taking assumption (6.2) into account, i. e., neglecting further failures during a repair at system level (in short: *no further failures at system down*), the corresponding *diagram of transition probabilities* in $(t, t + \delta t]$ is given in Fig. 6.5. Equations of Table 6.2 can be used to obtain the expressions for the reliability function, point availability and interval reliability. With $U = \{Z_0\}$, $\overline{U} = \{Z_1, \dots, Z_n\}$ and the transition rates according to Fig. 6.5, the *reliability function* (see Table 6.2 for notation) follows from

$$\mathrm{R}_{S0}(t) = e^{-\lambda_S t}, \qquad \text{with} \qquad \lambda_S = \sum_{i=1}^{n} \lambda_i, \tag{6.60}$$

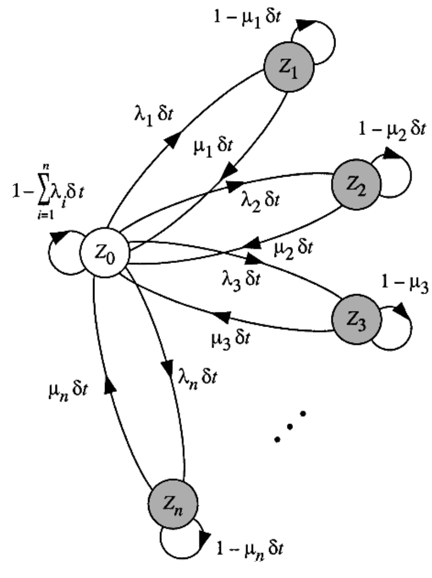


Figure 6.5 Diagram of the transition probabilities in $(t, t + \delta t]$ for a repairable series structure (constant failure & repair rates λ_i, μ_i, ideal failure detection & switch, one repair crew, no further failures at system down, Z_0 down state, arbitrary t, $\delta t \downarrow 0$, Markov process)

and thus, for the *mean time to failure*,

$$MTTF_{S0} = \frac{1}{\lambda_S}. \tag{6.61}$$

The *point availability* is given by

$$PA_{S0}(t) = P_{00}(t), \tag{6.62}$$

with $P_{00}(t)$ from (Table 6.2)

$$P_{00}(t) = e^{-\lambda_S t} + \sum_{i=1}^{n} \int_0^t \lambda_i e^{-\lambda_S x} P_{i0}(t-x)\, dx$$

$$P_{i0}(t) = \int_0^t \mu_i e^{-\mu_i x} P_{00}(t-x)\, dx, \qquad i = 1, \ldots, n. \tag{6.63}$$

The solution Eq. (6.63) leads to the following Laplace transform (Table A9.7) for $PA_{S0}(t)$

$$\tilde{PA}_{S0}(s) = \frac{1}{s(1+\sum_{i=1}^{n} \frac{\lambda_i}{s+\mu_i})}. \tag{6.64}$$

From Eq. (6.64) there follows the *asymptotic & steady-state* value of the *point* and *average availability* $PA_S = AA_S = \lim_{s \to 0} s\,\tilde{PA}_S(s)$

$$PA_S = AA_S = \frac{1}{1+\sum_{i=1}^{n} \frac{\lambda_i}{\mu_i}} \approx 1 - \sum_{i=1}^{n} \frac{\lambda_i}{\mu_i}. \tag{6.65}$$

Because of the *constant* failure rate of all elements, the *interval reliability* can be directly obtained from Eq. (6.27) by

$$IR_{S0}(t, t+\theta) = PA_{S0}(t)\, e^{-\lambda_S \theta}, \tag{6.66}$$

with the asymptotic & steady-state value

$$IR_S(\theta) = PA_S\, e^{-\lambda_S \theta}, \tag{6.67}$$

where

$$\lambda_S = \sum_{i=1}^{n} \lambda_i.$$

6.3.2 Series Structure with Constant Failure Rate and Arbitrary Repair Rate for Each Element

Generalization of the repair time distribution functions $G_i(x)$, with densities $g_i(x)$ and $G_i(0) = 0$, leads to a *semi-Markov process* with state space $Z_0, \ldots, Z_n$, as in Fig. 6.5 (this because of Assumption (6.2) of no further failures at system down). The *reliability function* and the *mean time to failure* are still given by Eqs. (6.60) and (6.61). For the *point availability* let us first calculate the *semi-Markov transition probabilities* $Q_{ij}(x)$ using Table 6.2

$$Q_{0i}(x) = \Pr\{\tau_{0i} \le x \cap \tau_{0k} > \tau_{0i}, \quad k \ne i\}$$
$$= \int_0^x \lambda_i e^{-\lambda_i y} \prod_{k \ne i} e^{-\lambda_k y} dy = \frac{\lambda_i}{\lambda_S}(1 - e^{-\lambda_S x})$$
$$Q_{i0}(x) = G_i(x), \qquad i = 1, \ldots, n. \tag{6.68}$$

The system of integral Equations for the *transition probabilities* (conditional state probabilities) $P_{ij}(t)$ follows then from Table 6.2

$$P_{00}(t) = e^{-\lambda_S t} + \sum_{i=1}^{n} \int_0^x \lambda_i e^{-\lambda_S x} P_{i0}(t-x)\, dx,$$
$$P_{i0}(t) = \int_0^t g_i(x) P_{00}(t-x)\, dx, \qquad i = 1, \ldots, n. \tag{6.69}$$

For the Laplace transform of the *point availability* $PA_{S0}(t) = P_{00}(t)$ one obtains finally from Eq. (6.69)

$$\tilde{PA}_{S0}(s) = \frac{1}{s + \lambda_S - \sum_{i=1}^{n} \lambda_i \tilde{g}_i(s)} = \frac{1}{s + \sum_{i=1}^{n} \lambda_i (1 - \tilde{g}_i(s))}, \tag{6.70}$$

from which, the *asymptotic & steady-state value* of the *point* and *average availability*

$$PA_S = AA_S = \frac{1}{1 + \sum_{i=1}^{n} \lambda_i MTTR_i}, \tag{6.71}$$

with $\lim_{s \to 0} (1 - \tilde{g}(s)) \approx s\, MTTR$, as per Eq. (6.54), and (Eq. (A6.38))

$$MTTR_i = \int_0^\infty (1 - G_i(t))\, dt. \tag{6.72}$$

The *interval reliability* can be calculated either from Eq. (6.66) with $PA_{S0}(t)$ from Eq. (6.70) or from Eq. (6.67) with PA_S from Eq. (6.71).

Example 6.5

A system consists of elements E_1 to E_4 which are necessary for the fulfillment of the required function (series structure). Let the failure rates $\lambda_1 = 10^{-3}\text{h}^{-1}$, $\lambda_2 = 0.5\cdot 10^{-3}\text{h}^{-1}$, $\lambda_3 = 10^{-4}\text{h}^{-1}$, $\lambda_4 = 2\cdot 10^{-3}\text{h}^{-1}$ be constant and assume that for all elements the repair time is lognormally distributed with parameters $\lambda = 0.5\text{h}^{-1}$ and $\sigma = 0.6$. The system has only one repair crew and no further failure can occur at system down (failures during repair are neglected). Give the reliability function for a mission of duration $t = 168\text{h}$, the mean time to failure, the asymptotic & steady-state values of the point and average availability, and he asymptotic & steady-state values of the interval reliability for $\theta = 12\text{h}$.

Solution

The system failure rate is $\lambda_S = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 36\cdot 10^{-4}\text{h}^{-1}$, according to Eq. (6.60). The reliability function follows as $R_{S0}(t) = e^{-0.0036t}$, from which $R_{S0}(168\text{h}) \approx 0.55$. The mean time to failure is $MTTF_{S0} = 1/\lambda_S \approx 278\text{h}$. The mean time to repair is obtained from Table A6.2 as $E[\tau'] = (e^{\sigma^2/2})/\lambda = MTTR \approx 2.4\text{h}$. For the asymptotic & steady-state values of the point and average availability as well as for the interval reliability for $\theta = 12\text{h}$ it follows from Eqs. (6.71) and (6.67) that $PA_S = AA_S = 1/(1+36\cdot 10^{-4}\cdot 2.4) \approx 0.991$ and $IR_S(12) \approx 0.991\cdot e^{-0.0036\cdot 12} \approx 0.95$.

6.3.3 Series Structure with Arbitrary Failure and Repair Rates for Each Element

Generalization of repair and failure-free time distribution functions leads to a *nonregenerative stochastic process*. This model can be investigated using supplementary variables, or by approximating the distribution functions of the failure-free time in such a way that the involved stochastic process can be reduced to a regenerative process. Using for the approximation an *Erlang distribution function* leads to a semi-Markov process. As an example, let us consider the case of a two-element series structure (E_1, E_2) and assume that the repair times are arbitrary, with densities $g_1(x)$ and $g_2(x)$, and the failure-free times have densities

$$f_1(x) = \lambda_1^2\, x\, e^{-\lambda_1 x}, \qquad x \ge 0, \tag{6.73}$$

and

$$f_2(x) = \lambda_2\, e^{-\lambda_2 x}, \qquad x \ge 0. \tag{6.74}$$

Equation (6.73) is the density of the sum of two exponentially distributed random time intervals with density $\lambda_1 e^{-\lambda_1 x}$. Under these assumptions, the two-element series structure corresponds to a *1-out-of-2 standby redundancy* with constant failure rate λ_1, in series with an element with constant failure rate λ_2. Figure 6.6 gives the equivalent reliability block diagram and the corresponding *state transition diagram*. This diagram only *visualizes* the possible transitions and *can not be considered* as a diagram of the transition probabilities in $(t, t+\delta t]$. Z_0 is the system up state, $Z_{1'}$ and $Z_{2'}$ are *supplementary states* necessary for calculation only.

For the *semi-Markov transition probabilities* $Q_{ij}(x)$ one obtains (Table 6.2)

$$Q_{01'}(x) = Q_{1'1}(x) = \frac{\lambda_1}{\lambda_1 + \lambda_2}(1 - e^{-(\lambda_1+\lambda_2)x})$$

$$Q_{02}(x) = Q_{1'2'}(x) = \frac{\lambda_2}{\lambda_1 + \lambda_2}(1 - e^{-(\lambda_1+\lambda_2)x})$$

$$Q_{20}(x) = Q_{2'1'}(x) = \int_0^x g_2(y)\,dy$$

$$Q_{10}(x) = \int_0^x g_1(y)\,dy. \qquad (6.75)$$

From Eq. (6.75) it follows that (Table 6.2 and Eq. (6.54))

$$R_{S0}(t) = (1 + \lambda_1 t)\, e^{-(\lambda_1+\lambda_2)t}, \qquad (6.76)$$

$$MTTF_{S0} = \frac{2\lambda_1 + \lambda_2}{(\lambda_1 + \lambda_2)^2}, \qquad (6.77)$$

$$\tilde{PA}_{S0}(s) = \tilde{P}_{00}(s) + \tilde{P}_{01'}(s) = \frac{[s + \lambda_1 + \lambda_2(1 - \tilde{g}_2(s))] + \lambda_1}{[s + \lambda_1 + \lambda_2(1 - \tilde{g}_2(s))]^2 - \lambda_1^2\, \tilde{g}_1(s)}, \qquad (6.78)$$

$$PA_S = AA_S = \frac{2}{2 + 2\lambda_2\, MTTR_2 + \lambda_1\, MTTR_1}, \qquad (6.79)$$

$$IR_S(\theta) = \frac{(2 + \lambda_1 \theta)\, e^{-(\lambda_1+\lambda_2)\theta}}{2 + 2\lambda_2\, MTTR_2 + \lambda_1\, MTTR_1}. \qquad (6.80)$$

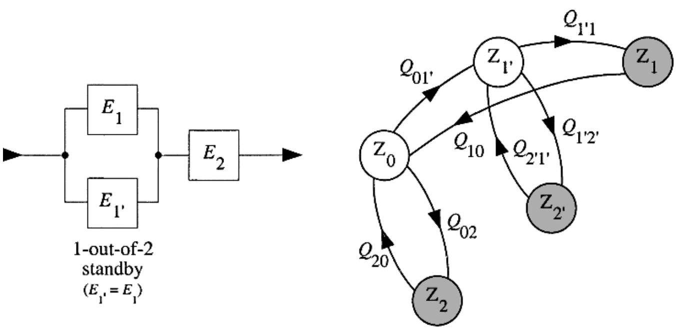


Figure 6.6 Equivalent reliability block diagram and *state transition diagram* for a two series element system (E_1 and E_2) with arbitrarily distributed repair times, constant failure rate for E_2, and Erlangian ($n = 2$) distributed failure-free time for E_1 (ideal failure detection & switch, one repair crew, no further failures at system down, Z_1, Z_2, $Z_{2'}$ down states, semi-Markov process)

The *interval reliability* $IR_{S0}(t, t+\theta)$ can be obtained from

$$IR_{S0}(t, t+\theta) = P_{00}(t)R_{S0}(\theta) + P_{01'}(t)R_{S1'}(\theta),$$

with $R_{S1'}(\theta) = e^{-(\lambda_1+\lambda_2)\theta}$, because of the constant failure rates λ_1 and λ_2.

Important results for repairable series structures are summarized in Table 6.5. Asymptotic results for the case of arbitrary failure and repair rates are investigated e.g. in [2.34(1975)] yielding $AA_S = PA_S = 1/(1+\sum_{i=1}^{n} MTTR_i / MTTF_i)$ for the asymptotic & steady-state value of the point and average availability (Point 4 of Table 6.5). $AA_S = PA_S = 1/(1+\sum_{i=1}^{n} MTTR_i / MTTF_i)$ follows also in a way similar to the development of Eq. (4.6).

Table 6.5 Results for a repairable system without redundancy (elements $E_1, \ldots, E_n$ in series), ideal failure detection & switch, one repair crew, no further failures at system down

Quantity	Expression	Remarks, assumptions
1. Reliability function ($R_{S0}(t)$)	$\prod_{i=1}^{n} R_i(t)$	Independent elements (up to system failure)
2. Mean time to system failure ($MTTF_{S0}$)	$\int_0^\infty R_{S0}(t)\,dt$	$R_i(t) = e^{-\lambda_i t} \to R_{S0}(t) = e^{-\lambda_S t}$ and $MTTF_{S0} = 1/\lambda_S$ with $\lambda_S = \lambda_1 + \ldots + \lambda_n$
3. System failure rate up to system failure ($\lambda_S(t)$)	$\sum_{i=1}^{n} \lambda_i(t)$	Independent elements (up to system failure)
4. Asymptotic & steady-state value of the point availability & average availability ($PA_S = AA_S$)	a) $\dfrac{1}{1+\sum_{i=1}^{n} \frac{\lambda_i}{\mu_i}} \approx 1 - \sum_{i=1}^{n} \frac{\lambda_i}{\mu_i}$ +) b) $\dfrac{1}{1+\sum_{i=1}^{n} \lambda_i MTTR_i} \approx 1 - \sum_{i=1}^{n} \lambda_i MTTR_i$ c) $\dfrac{1}{1+\lambda_2 MTTR_2 + \lambda_1 MTTR_1/2}$	At system down, no further failures can occur: a) Constant failure rate λ_i and constant repair rate μ_i for element E_i $(i=1,\ldots,n)$ b) Constant failure rate λ_i and arbitrary repair rate $\mu_i(t)$ with $MTTR_i$ = mean time to repair for element E_i $(i=1,\ldots,n)$ c) 2-element series structure with failure rates $\lambda_1^2 t/(1+\lambda_1 t)$ for E_1 and λ_2 for E_2
5. Asymptotic & steady-state value of the interval reliability ($IR_S(\theta)$)	$PA_S e^{-\lambda_S \theta}$	Each element has constant failure rate λ_i, $\lambda_S = \lambda_1 + \ldots + \lambda_n$

+) ***Supplementary results:*** If *n repair crews* were available, $PA_S = \prod_i (1/(1+\lambda_i/\mu_i)) \approx 1 - \sum_i \lambda_i/\mu_i$

6.4 1-out-of-2 Redundancy

The *1-out-of-2 redundancy*, also known as *1-out-of-2: G,* is the simplest redundant structure arising in practical applications. It consists of two elements E_1 and E_2, one of which is in the operating state and the other in reserve. When a failure occurs, one element is repaired while the other continues operation. The system is down when an element fails while the other one is being repaired. Assuming ideal *switching* and *failure detection*, the reliability block diagram is a parallel connection of elements E_1 and E_2, see Fig. 6.7.

Investigations are based on assumptions (6.1) - (6.7). This implies in particular, that the repair of a redundant element *begins at failure occurrence and is performed without interruption of operation at system level.* The distribution functions of the repair times, and of the failure-free times are generalized step by step, beginning with exponential distribution, up to the case in which the process involved has only *one regeneration state* (Section 6.4.3). Influence of preventive maintenance, switching, incomplete coverage, common cause failures are considered in Sections 6.8.

6.4.1 1-out-of-2 Redundancy with Constant Failure and Repair Rates for Each Element

Because of the constant failure and repair rates, the time behavior of the 1-out-of-2 redundancy can be described by a (time-homogeneous) *Markov process.* The number of states is 3 if elements E_1 and E_2 are identical (Figs. 6.8 or A7.4) and 5 if they are different (Fig. 6.9, see footnote on p. 479), the diagrams of transition probabilities in $(t, t+\delta t]$ are given in Figs. 6.8 or A7.4 and 6.9, respectively.

Let us consider the case of identical elements E_1 and E_2 (see Example 6.6 for different elements) and assume as distribution function of the failure-free time

$$\mathrm{F}(x) = 1 - e^{-\lambda x}, \qquad x > 0, \ \mathrm{F}(0) = 0, \tag{6.81}$$

in the *operating state* and

$$\mathrm{F}_r(x) = 1 - e^{-\lambda_r x}, \qquad x > 0, \ \mathrm{F}_r(0) = 0, \tag{6.82}$$

in the *reserve state.* This includes *active* (parallel) redundancy for $\lambda_r = \lambda$, *warm* redundancy for $\lambda_r < \lambda$, and *standby* redundancy for $\lambda_r \equiv 0$. Repair times are assumed

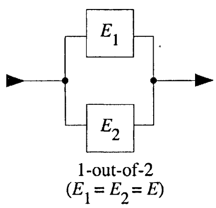


Figure 6.7 1-out-of-2 redundancy reliability block diagram (ideal failure detection and switch)

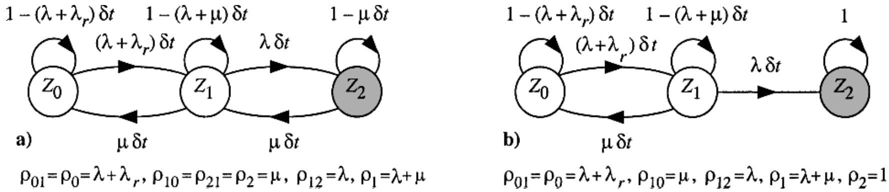


Figure 6.8 Diagrams of the transition probabilities in $(t, t+\delta t]$ for a repairable 1-out-of-2 warm redundancy (2 identical elements, constant failure & repair rates (λ,λ_r,μ), ideal failure detection & switch, one repair crew, Z_2 down state, arbitrary t, $\delta t\downarrow 0$, Markov process): **a)** For the point availability; **b)** For the reliability function

to be independent of failure-free times and distributed according to

$$G(x) = 1-e^{-\mu x}, \qquad x>0,\ G(0)=0. \tag{6.83}$$

Refinements are in Examples 6.6 (different elements) and 6.7 (travel time). For more general situations (particular load sharing, more repair crews, failure and / or repair rates changing at a state transition, etc.), *birth and death processes* (Appendix A7.5.5) can often be used. For all these cases, investigations are generally performed using the *method of differential equations* (Table 6.2 and Appendix A7.5.3.1). Figure 6.8 gives the *diagram of transition probabilities* in $(t, t+\delta t]$ for the point availability (Fig. 6.8a) and the reliability function (Fig. 6.8b), respectively.

Considering the memoryless property of exponential distributions (Eq. (A6.87)), the system behavior at times t and $t+\delta t$ can be described by following *difference equations* for the *state probabilities* $P_i(t) \equiv \Pr\{\text{process in } Z_i \text{ at } t\}$, $i=0,1,2$ (Fig. 6.8a)

$$\begin{aligned}
P_0(t+\delta t) &= P_0(t)(1-(\lambda+\lambda_r)\delta t) + P_1(t)\mu\delta t\\
P_1(t+\delta t) &= P_1(t)(1-(\lambda+\mu)\delta t) + P_0(t)(\lambda+\lambda_r)\delta t + P_2(t)\mu\delta t\\
P_2(t+\delta t) &= P_2(t)(1-\mu\delta t) + P_1(t)\lambda\delta t.
\end{aligned}$$

For $\delta t\downarrow 0$, it follows that

$$\begin{aligned}
\dot{P}_0(t) &= -(\lambda+\lambda_r)P_0(t) + \mu P_1(t)\\
\dot{P}_1(t) &= -(\lambda+\mu)P_1(t) + (\lambda+\lambda_r)P_0(t) + \mu P_2(t)\\
\dot{P}_2(t) &= -\mu P_2(t) + \lambda P_1(t).
\end{aligned} \tag{6.84}$$

The system of differential equations (6.84) can also be obtained directly from Table 6.2 and Fig. 6.8a. Its solution leads to the state probabilities $P_i(t)$, $i=0,1,2$. Assuming as *initial conditions* at $t=0$, $P_0(0)=1$ and $P_1(0)=P_2(0)=0$, the above state probabilities are identical to the *transition probabilities* $P_{0i}(t)$, $i=0,1,2$, i.e., $P_{00}(t)\equiv P_0(t)$, $P_{01}(t)\equiv P_1(t)$, and $P_{02}(t)\equiv P_2(t)$. The *point availability* $PA_{S0}(t)$ is then given by (see Table 6.2 for notation)

$$PA_{S0}(t) = P_{00}(t) + P_{01}(t). \tag{6.85}$$

$\mathrm{PA}_{S1}(t)$ or $\mathrm{PA}_{S2}(t)$ could have been determined for suitable initial conditions. From Eq. (6.85) it follows for the Laplace transform of $\mathrm{PA}_{S0}(t)$ that

$$\tilde{\mathrm{P}}\mathrm{A}_{S0}(s) = \tilde{\mathrm{P}}_{00}(s) + \tilde{\mathrm{P}}_{01}(s) = \frac{((s+\mu)^2 + s\lambda)) + (s+\mu)(\lambda+\lambda_r)}{s[(s+\lambda+\lambda_r)(s+\lambda+\mu)+\mu(s+\mu)]}, \qquad (6.86)$$

and thus for $t \to \infty$

$$\lim_{t\to\infty}\mathrm{PA}_{S0}(t) = PA_S = P_0 + P_1 = \frac{\mu^2 + \mu(\lambda+\lambda_r)}{(\lambda+\lambda_r)(\lambda+\mu)+\mu^2} \approx 1 - \frac{\lambda(\lambda+\lambda_r)}{\mu^2}, \qquad (6.87)$$

with $P_i = \lim_{t\to\infty}\mathrm{P}_i(t) = \lim_{t\to\infty}\mathrm{P}_{ji}(t)$, $i,j=0,1,2$ (Eq. (A7.129)). If $\mathrm{PA}_{S0}(t) = PA_S$ for all $t \geq 0$, then PA_S is also the point and *average availability* (AA_S) in *steady-state*. Obviously, $P_2 = 1 - PA_S$. Investigation of $\mathrm{PA}_{S0}(t)$ for $\lambda_r = \lambda$ leads to (Eq. (6.86))

$$\mathrm{PA}_{S0}(t) = PA_S + 2\lambda^2(a_2 e^{a_1 t} - a_1 e^{a_2 t}) \,/\, a_1 a_2 (a_2 - a_1),$$

with

$$a_{1,2} = -\mu(1 + 3\lambda/2\mu) \pm \mu\sqrt{\lambda/\mu + (\lambda/2\mu)^2} \approx -\mu(1 \mp \sqrt{\lambda/\mu}),$$

and PA_S from Eq.(6.87). It can be noted that $a_1 a_2 = \mu^2 + 2\lambda\mu + 2\lambda^2$ yielding $\mathrm{PA}_{S0}(0) = 1$, $d\mathrm{PA}_{S0}(t)/dt = 0$ at $t=0$ and thus $\mathrm{PA}_{S0}(t) = 1$ for some t,+) and $a_{1,2} \to -\mu$ for $\lambda \to 0$. From these results, and considering $\lambda \ll \mu$, following approximation can be used for practical applications ($e^{a_1 t} \approx e^{a_2 t} \approx e^{-\mu t}$, $a_1 a_2 \approx \mu^2$)

$$\mathrm{PA}_{S0}(t) \approx PA_S + (1 - PA_S)\, e^{-\mu t}, ^{+)} \qquad t > 0, \ \mathrm{PA}_{S0}(0) = 1. \qquad (6.88)$$

Equation (6.88) is similar to Eq. (6.20). It holds also for $0 \leq \lambda_r \leq \lambda$ and is an important result in developing, together with Eq. (6.94), approximate expressions for large series-parallel systems, based on *macro-structures* (Section 6.7, Table 6.10).

To calculate the *reliability function* it is necessary to consider that the 1-out-of-2 redundancy will operate failure free in $(0, t]$ *only* if in this time interval the *down state at system level* (state Z_2) will *not be visited.* To recognize if Z_2 has been entered before t it is sufficient to make Z_2 *absorbing* (Fig. 6.8b). In this case, if Z_2 is entered the process *remains there indefinitely.* Thus *the probability of being in Z_2 at t is the probability of having entered Z_2 before the time t*, i. e. the unreliability $1 - \mathrm{R}_S(t)$. To avoid ambiguities, the *state probabilities* in Fig. 6.8b are marked by an *apostrophe* (prime). The procedure is similar to that for Eq. (6.84) and leads to

$$\dot{\mathrm{P}}_0'(t) = -(\lambda+\lambda_r)\mathrm{P}_0'(t) + \mu \mathrm{P}_1'(t)$$
$$\dot{\mathrm{P}}_1'(t) = -(\lambda+\mu)\mathrm{P}_1'(t) + (\lambda+\lambda_r)\mathrm{P}_0'(t)$$
$$\dot{\mathrm{P}}_2'(t) = \lambda\, \mathrm{P}_1'(t), \qquad (6.89)$$

and to the corresponding state probabilities $\mathrm{P}_0'(t)$, $\mathrm{P}_1'(t)$, and $\mathrm{P}_2'(t)$. With the *initial*

+) More precisely, for $t \downarrow 0$ it holds that $\mathrm{PA}_{S0}(t) \approx 1 - \lambda^2 t^2$ (using $e^x \approx 1 + x + x^2/2$).

conditions at $t=0$, $P_0'(0)=1$ and $P_1'(0)=P_2'(0)=0$, the state probabilities $P_0'(t)$, $P_1'(t)$ and $P_2'(t)$ are identical to the *transition probabilities* $P_{00}'(t) \equiv P_0'(t)$, $P_{01}'(t) \equiv P_1'(t)$ and $P_{02}'(t) \equiv P_2'(t)$. The *reliability function* is then given by (Table 6.2 for notation)

$$R_{S0}(t) = P_{00}'(t) + P_{01}'(t). \tag{6.90}$$

Equation (6.90) yields following Laplace transform for $R_{S0}(t)$

$$\tilde{R}_{S0}(s) = \frac{(s+\lambda+\mu)+(\lambda+\lambda_r)}{(s+\lambda+\lambda_r)(s+\lambda)+s\mu}, \tag{6.91}$$

from which the *mean time to failure* ($MTTF_{S0} = \tilde{R}_{S0}(0)$, Eq. (2.61)) follows as

$$MTTF_{S0} = \frac{2\lambda+\lambda_r+\mu}{\lambda(\lambda+\lambda_r)} \approx \frac{\mu}{\lambda(\lambda+\lambda_r)}. \tag{6.92}$$

Investigation of $R_{S0}(t)$ for $\lambda_r = \lambda$ leads to (Eq. (6.91))

$$R_{S0}(t) = (r_2 e^{r_1 t} - r_1 e^{r_2 t}) \,/\, (r_2 - r_1),$$

with

$$r_{1,2} = -[(3\lambda+\mu)/2] \pm \sqrt{((3\lambda+\mu)/2)^2 - 2\lambda^2}.$$

For $\lambda << \mu$, it follows that $r_1 \approx 0$ and $r_2 \approx -\mu$, yielding

$$R_{S0}(t) \approx e^{r_1 t}.^{+)} \tag{6.93}$$

Using $\sqrt{1-\varepsilon} \approx 1-\varepsilon/2$ for $2r_1 = -(3\lambda+\mu)(1-\sqrt{1-8\lambda^2/(3\lambda+\mu)^2})$ leads to $r_1 \approx -2\lambda^2/(3\lambda+\mu)$. $R_{S0}(t)$ can thus be *approximated by a decreasing exponential function* with time constant $MTTF_{S0} \approx (3\lambda+\mu)/2\lambda^2$. +) Considering $\lambda << \mu$, extension to a warm redundancy $0 \le \lambda_r \le \lambda$ leads to

$$R_{S0}(t) \approx e^{-\lambda_S t}, \quad t>0,\ R_{S0}(0)=1, \quad \lambda_S = \frac{1}{MTTF_{S0}} = \frac{\lambda(\lambda+\lambda_r)}{2\lambda+\lambda_r+\mu} \approx \frac{\lambda(\lambda+\lambda_r)}{\mu}. \tag{6.94}$$

Similarly as for $PA_{S0}(t)$, $dR_{S0}(t)/dt = 0$ at $t=0$ and thus $R_{S0}(t)=1$ for some t.+) Concluding the above investigations, also validated by numerical computation, results of Eqs. (6.88) & (6.94) show that:

> *For $\lambda, \lambda_r << \mu$, a repairable 1-out-of-2 warm redundancy with constant failure rates λ, λ_r, constant repair rate μ, and one repair crew behaves approximately like a one-item structure with constant failure rate $\lambda_S \approx \lambda(\lambda+\lambda_r)/\mu$ and repair rate $\mu_S \approx \mu$; result on which the macro structures method (Tab. 6.10) can be based* ($\mu_S \approx 2\mu$ for two repair crews (Table 6.9)).

+) More precisely, for $t \downarrow 0$ it holds that $R_{S0}(t) \approx 1 - r_1 r_2 t^2/2 \approx 1 - \lambda^2 t^2 \approx PA_{S0}(t)$.

Using Eqs.(A7.141), (A7.142), (6.86), the *system mean up time* MUT_S follows as

$$MUT_S = \frac{PA_S}{f_{udS}} = \frac{P_0 + P_1}{0 \cdot P_0 + \lambda P_1} = \frac{\mu^2 + \mu(\lambda + \lambda_r)}{\lambda\mu(\lambda + \lambda_r)} = \frac{\mu + \lambda + \lambda_r}{\lambda(\lambda + \lambda_r)} \approx MTTF_{S0}. \quad (6.95)$$

Because of the *memoryless property* of the (time-homogeneous) Markov process, the *interval reliability* follows directly from the *transition probabilities* $P_{ij}(t)$ and the reliability functions $R_{Si}(t)$, see Table 6.2. Assuming $P_0(0) = 1$ yields

$$IR_{S0}(t, t+\theta) = P_{00}(t) R_{S0}(\theta) + P_{01}(t) R_{S1}(\theta),$$

with $P_{00}(t)$, $P_{01}(t)$ as in Eq. (6.85). The asymptotic & steady-state value follows as

$$IR_S(\theta) = P_0 R_{S0}(\theta) + P_1 R_{S1}(\theta) = \frac{\mu^2 R_{S0}(\theta) + \mu(\lambda + \lambda_r) R_{S1}(\theta)}{(\lambda + \lambda_r)(\lambda + \mu) + \mu^2} \approx R_{S0}(\theta). \quad (6.96)$$

Further results for a 1-out-of-2 redundancy are in Sections 6.8.3 (imperfect switching), 6.8.4 (incomplete coverage), and 6.8.7 (common cause failures).

To compare the effectiveness of calculation methods, let us now express the reliability function, point availability, and interval reliability using the *method of integral equations* (Appendix A7.5.3.2). Using Eq. (A7.102) and Fig. 6.8a yields

$$Q_{01}(x) = \Pr\{\tau_{01} \le x\} = 1 - \Pr\{\tau_{01} > x\} = 1 - e^{-\lambda x} e^{-\lambda_r x} = 1 - e^{-(\lambda + \lambda_r)x}$$

$$Q_{10}(x) = \Pr\{\tau_{10} \le x \cap \tau_{12} > \tau_{10}\} = \int_0^x \mu e^{-\mu y} e^{-\lambda y} dy = \frac{\mu}{\lambda + \mu}(1 - e^{-(\lambda + \mu)x})$$

$$Q_{12}(x) = \Pr\{\tau_{12} \le x \cap \tau_{10} > \tau_{12}\} = \int_0^x \lambda e^{-\lambda y} e^{-\mu y} dy = \frac{\lambda}{\lambda + \mu}(1 - e^{-(\lambda + \mu)x})$$

$$Q_{21}(x) = \Pr\{\tau_{21} \le x\} = 1 - e^{-\mu x}.$$

From Table 6.2 it follows then that

$$R_{S0}(t) = e^{-(\lambda + \lambda_r)t} + \int_0^t (\lambda + \lambda_r) e^{-(\lambda + \lambda_r)x} R_{S1}(t - x) dx$$

$$R_{S1}(t) = e^{-(\lambda + \mu)t} + \int_0^t \mu e^{-(\lambda + \mu)x} R_{S0}(t - x) dx, \quad (6.97)$$

for the *reliability functions* $R_{S0}(t)$ and $R_{S1}(t)$, as well as

$$P_{00}(t) = e^{-(\lambda + \lambda_r)t} + \int_0^t (\lambda + \lambda_r) e^{-(\lambda + \lambda_r)x} P_{10}(t - x) dx, \qquad P_{20}(t) = \int_0^t \mu e^{-\mu x} P_{10}(t - x) dx,$$

$$P_{10}(t) = \int_0^t \mu e^{-(\lambda + \mu)x} P_{00}(t - x) dx + \int_0^t \lambda e^{-(\lambda + \mu)x} P_{20}(t - x) dx.$$

Table 6.6 Reliability function $R_{S0}(t)$, mean time to failure $MTTF_{S0}$, steady-state availability $PA_S = AA_S$, and interval reliability $IR_S(\theta)$ for a repairable 1-out-of-2 redundancy with identical elements (Fig. 6.7, constant failure & repair rates λ, λ_r, μ, ideal failure detection & switch, one repair crew, Markov process; approximations valid for $(\lambda+\lambda_r) \ll \mu$)

	Standby ($\lambda_r \equiv 0$)	Warm ($\lambda_r < \lambda$)	Active ($\lambda_r = \lambda$)
$R_{S0}(t)^*$	$\approx e^{-\frac{\lambda^2 t}{2\lambda+\mu}}$	$\approx e^{-\frac{\lambda(\lambda+\lambda_r)t}{2\lambda+\lambda_r+\mu}}$	$\approx e^{-\frac{2\lambda^2 t}{3\lambda+\mu}}$
$MTTF_{S0}{}^*$	$\frac{2\lambda+\mu}{\lambda^2} \approx \frac{\mu}{\lambda^2}$	$\frac{2\lambda+\lambda_r+\mu}{\lambda(\lambda+\lambda_r)} \approx \frac{\mu}{\lambda(\lambda+\lambda_r)}$	$\frac{3\lambda+\mu}{2\lambda^2} \approx \frac{\mu}{2\lambda^2}$
$PA_S = AA_S{}^{**}$	$\frac{\mu(\lambda+\mu)}{\lambda(\lambda+\mu)+\mu^2}$ $\approx 1-(\lambda/\mu)^2$	$\frac{\mu(\lambda+\lambda_r+\mu)}{(\lambda+\lambda_r)(\lambda+\mu)+\mu^2}$ $\approx 1-\lambda(\lambda+\lambda_r)/\mu^2$	$\frac{\mu(2\lambda+\mu)}{2\lambda(\lambda+\mu)+\mu^2}$ $\approx 1-2(\lambda/\mu)^2$
$PA_{S0}(t)^{***}$	$\approx PA_S + (1-PA_S)e^{-\mu t}$	$\approx PA_S + (1-PA_S)e^{-\mu t}$	$\approx PA_S + (1-PA_S)e^{-\mu t}$
$IR_S(\theta)^{**}$	$\approx R_{S0}(\theta)$	$\approx R_{S0}(\theta)$	$\approx R_{S0}(\theta)$

* new at $t = 0$; ** asymptotic & steady-state value (for practical applications, convergence of $PA_S(t)$ to PA_S and of $IR_S(t, t+\theta)$ to $IR_S(\theta)$ is good after $t \approx 10/\mu = 10\, MTTR$, see also pp. 196 - 198)

Supplementary results: See Example 6.6 for *two different elements* and Table 6.9 for two different elements and *two repair crews* (active redundancy); assuming in Fig. 6.8a $Z_2 \to Z_0$ with μ_g instead of $Z_2 \to Z_1$ with μ yields $PA_S{=}AA_S \approx 1-2\lambda^2/\mu\mu_g$ (active redundancy)

and

$$P_{01}(t) = \int_0^t (\lambda+\lambda_r)e^{-(\lambda+\lambda_r)x} P_{11}(t-x)dx, \qquad P_{21}(t) = \int_0^t \mu e^{-\mu x} P_{11}(t-x)dx,$$

$$P_{11}(t) = e^{-(\lambda+\mu)t} + \int_0^t \mu e^{-(\lambda+\mu)x} P_{01}(t-x)dx + \int_0^t \lambda e^{-(\lambda+\mu)x} P_{21}(t-x)dx. \qquad (6.98)$$

for the transition probabilities. The solution of Eqs. (6.97) yields, in particular, Eq. (6.91) and the solution of Eqs. (6.98) yields, in particular, Eq. (6.86). Equations (6.97) and (6.98) show how the use of integral equations leads to a quicker solution than differential equations for arbitrary initial conditions at $t = 0$.

Table 6.6 summarizes the main results of Section 6.4.1. It gives *approximate expressions* valid for $\lambda \ll \mu$ and distinguishes between the cases of active ($\lambda_r = \lambda$), warm ($\lambda_r < \lambda$), and standby redundancy ($\lambda_r \equiv 0$).

From Table 6.6, the improvement in $MTTF_{S0}$ through repair, without interruption of operation at system level (by repair of a redundant element), is given as lower and upper bounds by

$$\lambda\, MTTF_{S0} \approx \qquad \underset{active}{\frac{\mu}{2\lambda} = \frac{MTBF}{2MTTR}} \qquad \underset{standby}{\frac{\mu}{\lambda} = \frac{MTBF}{MTTR}}$$

Investigation of the *unavailability* in steady-state $1 - PA_S$ leads to

$$1 - PA_S = 1 - AA_S \approx \qquad \underset{active}{2\left(\frac{\lambda}{\mu}\right)^2 = 2\left(\frac{MTTR}{MTBF}\right)^2} \qquad \underset{standby}{\left(\frac{\lambda}{\mu}\right)^2 = \left(\frac{MTTR}{MTBF}\right)^2}$$

The above results can easily be extended to cover situations in which failure or repair rates are modified at *state changes* (e.g. because of *load sharing*, *differences within the element*, *repair priority*, etc.). These cases, simply modify the transition rates on the diagram of transition probabilities in $(t, t+\delta t]$, see for instance Figs. 2.12 and A7.4 - A7.6.

Example 6.6

Give the mean time to failure $MTTF_{S0}$ and the asymptotic & steady-state value of the point availability PA_S for a 1-out-of-2 active redundancy with two *different elements* E_1 and E_2, constant failure rates λ_1, λ_2, and constant repair rates μ_1, μ_2 (one repair crew).

Solution

Figure 6.9 gives the reliability block diagram and the diagram of transition probabilities in $(t, t+\delta t]$. $MTTF_{S0}$ and PA_S can be calculated from appropriate systems of algebraic equations. According to Table 6.2 and considering Fig. 6.9 it follows for the *mean time to failure* that

$$MTTF_{S0} = (1 + \lambda_1 MTTF_{S1} + \lambda_2 MTTF_{S2}) \,/\, (\lambda_1 + \lambda_2)$$

$$MTTF_{S1} = (1 + \mu_1 MTTF_{S0}) \,/\, (\lambda_2 + \mu_1), \quad MTTF_{S2} = (1 + \mu_2 MTTF_{S0}) \,/\, (\lambda_1 + \mu_2),$$

which leads to

$$MTTF_{S0} = \frac{(\lambda_1 + \mu_2)(\lambda_2 + \mu_1) + \lambda_1(\lambda_1 + \mu_2) + \lambda_2(\lambda_2 + \mu_1)}{\lambda_1 \lambda_2 (\lambda_1 + \lambda_2 + \mu_1 + \mu_2)}, \tag{6.99}$$

and in particular for $\lambda_1 \ll \mu_1$ and $\lambda_2 \ll \mu_2$,

$$MTTF_{S0} \approx \mu_1 \mu_2 \,/\, (\lambda_1 \lambda_2 (\mu_1 + \mu_2)). \tag{6.100}$$

As for Eq. (6.93), the *reliability function* can be expressed by

$$R_{S0}(t) \approx e^{-\lambda_S t} \quad \text{with} \quad \lambda_S = \frac{1}{MTTF_{S0}} \approx \frac{\lambda_1 \lambda_2 (\mu_1 + \mu_2)}{\mu_1 \mu_2} = \lambda_1 \lambda_2 \left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right). \tag{6.101}$$

For the asymptotic & steady-state value of the point availability and average availability $PA_S = AA_S = P_0 + P_1 + P_2$ holds with P_0, P_1, and P_2 as solution of (Table 6.2)

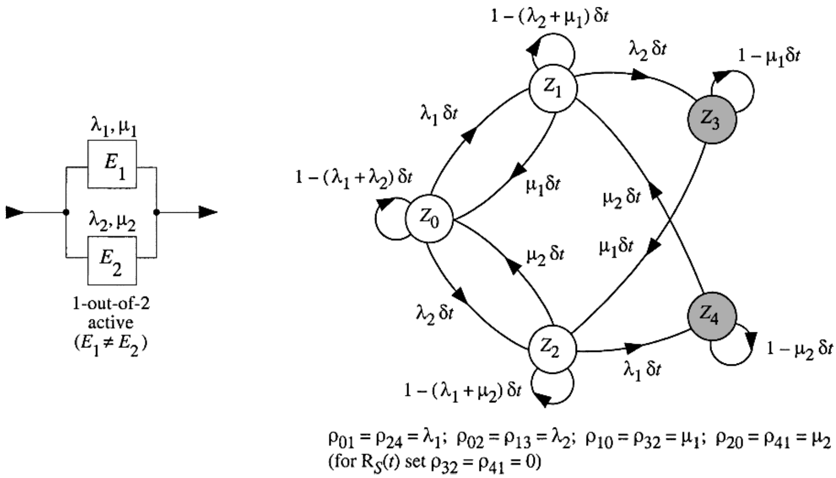


Figure 6.9 Reliability block diagram and diagram of transition probabilities in $(t, t+\delta t]$ for a repairable 1-out-of-2 active redundancy with different elements (const. failure & repair rates $(\lambda_1, \lambda_2, \mu_1, \mu_2)$, ideal failure detection & switch, one repair crew, Z_3, Z_4 down states, arbitr. t, $\delta t \downarrow 0$, Markov proc.)

$$(\lambda_1+\lambda_2)P_0 = \mu_1 P_1 + \mu_2 P_2, \quad (\lambda_2+\mu_1)P_1 = \lambda_1 P_0 + \mu_2 P_4,$$
$$(\lambda_1+\mu_2)P_2 = \lambda_2 P_0 + \mu_1 P_3, \quad \mu_1 P_3 = \lambda_2 P_1, \quad \mu_2 P_4 = \lambda_1 P_2.$$

One (arbitrarily chosen) of the five equations must be dropped and replaced by $P_0 + P_1 + P_2 + P_3 + P_4 = 1$. The solution yields P_0 through P_4, from which

$$PA_S = AA_S = \frac{1}{1+\dfrac{\lambda_1\lambda_2[\mu_1^2+\mu_2^2+(\lambda_1+\lambda_2)(\mu_1+\mu_2)]}{\mu_1\mu_2[\mu_1\mu_2+(\lambda_1+\lambda_2)(\lambda_1+\lambda_2+\mu_1+\mu_2)]}}, \tag{6.102}$$

yielding, for $\lambda_1 << \mu_1$ and $\lambda_2 << \mu_2$,

$$PA_S = AA_S \approx 1 - \frac{\lambda_1\lambda_2}{\mu_1^2\mu_2^2}(\mu_1^2+\mu_2^2) = 1 - \frac{\lambda_1}{\mu_1}\cdot\frac{\lambda_2}{\mu_2}\cdot\left(\frac{\mu_1}{\mu_2}+\frac{\mu_2}{\mu_1}\right). \tag{6.103}$$

With $\lambda_1=\lambda_2=\lambda$ and $\mu_1=\mu_2=\mu$, Eqs. (6.99) & (6.103) become Eqs. (6.92) & (6.88) with $\lambda_r=\lambda$.

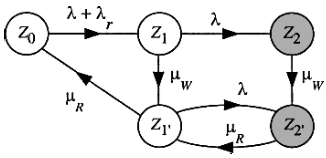
Example 6.7

As a refinement of the case investigated with Fig. 6.8 assume that to the repair time, distributed according to $G(x)=1-e^{-\mu_R x}$, a *wait time for travel* distributed according to $W(x)=1-e^{-\mu_W x}$ has to be added to the repair time for a failure occurred when both units are up (one operating, the other in reserve state). Repairs for failures occurred during the travel time or a repair do not need to wait for a further travel time. As before, the system has only one repair crew. Investigate the mean time to failure $MTTF_{S0}$ and the steady sate availability $PA_S = AA_S$.

Solution

The system behavior can be described by a 5 states Markov process (graph). $MTTF_{S0}$ follows as solution of (Table 6.2, $M_i = MTTF_{Si}$): $M_1 = (1+\mu_W M_{1'}) / (\lambda+\mu_W)$, $M_{1'} = (1+\mu_R M_0) / (\lambda+\mu_R)$, $M_0 = M_1 + 1 / (\lambda+\lambda_r)$; yielding

$$MTTF_{S0} = \frac{1}{\lambda} + \frac{(\lambda+\mu_R)(\lambda+\mu_W)}{\lambda(\lambda+\lambda_r)(\lambda+\mu_R+\mu_W)} \approx \frac{1/(1/\mu_R+1/\mu_W)}{\lambda(\lambda+\lambda_r)}. \quad (6.103)$$



$PA_S = AA_S$ follows as solution of (Table 6.2): $\mu_W P_2 = \lambda P_1$, $(\lambda+\lambda_r)P_0 = \mu_R P_{1'}$, $(\lambda+\mu_W)P_1 = (\lambda+\lambda_r)P_0$, $(\lambda+\mu_R)P_{1'} = \mu_W P_1 + \mu_R P_{2'}$, $P_0 + P_1 + P_{1'} + P_2 + P_{2'} = 1$, yielding

$$PA_S = AA_S = P_0 + P_1 + P_{1'} = \frac{1}{1 + \dfrac{\lambda(\lambda+\lambda_r)[\lambda+\mu_R+\mu_W+\mu_R^2/\mu_W]}{\mu_R[(\lambda+\lambda_r)(\lambda+\mu_W+\mu_R)+(\lambda+\mu_W)\mu_R]}}$$

$$\approx 1 - \frac{\lambda(\lambda+\lambda_r)}{\mu_R^2}\left(1 + \frac{\mu_R}{\mu_W} + \frac{\mu_R^2}{\mu_W^2}\right). \quad (6.104)$$

For $\mu_W = \infty$ and $\mu_R = \mu$, Eqs. (6.103) & (6.104) yield Eqs. (6.92) & (6.87).

Supplementary results: Addition of a travel time to each repair has no practical significance. Generalization of distribution functions for repair and travel time lead to a 4 states semi-regenerative process with 3 reg. states (Fig. A7.12).

6.4.2 1-out-of-2 Redundancy with Constant Failure Rate and Arbitrary Repair Rate

Consider now a 1-out-of-2 *warm redundancy* with 2 identical elements E_1 and E_2, failure-free times distributed according to Eqs. (6.81) and (6.82), and repair time with mean $MTTR < \infty$, distributed according to an arbitrary distribution function $G(x)$ with $G(0) = 0$ and density $g(x)$. The time behavior of this system can be described by a process with states Z_0, Z_1, and Z_2. Because of the arbitrary repair rate, only states Z_0 and Z_1 are *regeneration states*. These states constitute a semi-Markov process *embedded* in the original *semi-regenerative process* (Fig. A.7.11). The *semi-Markov transition probabilities* $Q_{ij}(x)$ are given by Eq. (A7.183). Setting these quantities in the equations of Table 6.2 (SMP), by considering $Q_0(x) = Q_{01}(x)$ and $Q_1(x) = Q_{10}(x) + Q'_{12}(x)$ with $Q'_{12}(x)$ as per Eq. (A7.184), it follows for the *reliability functions* $R_{Si}(t)$

$$R_{S0}(t) = e^{-(\lambda+\lambda_r)t} + \int_0^t (\lambda+\lambda_r)\, e^{-(\lambda+\lambda_r)x} R_{S1}(t-x)\, dx$$

$$R_{S1}(t) = e^{-\lambda t}(1 - G(t)) + \int_0^t g(x)\, e^{-\lambda x} R_{S0}(t-x)\, dx, \quad (6.105)$$

and for the *transition probabilities* $P_{ij}(t)$ of the embedded semi-Markov process

$$P_{00}(t) = e^{-(\lambda+\lambda_r)t} + \int_0^t (\lambda+\lambda_r)e^{-(\lambda+\lambda_r)x} P_{10}(t-x)\,dx$$

$$P_{10}(t) = \int_0^t g(x)e^{-\lambda x} P_{00}(t-x)\,dx + \int_0^t g(x)(1-e^{-\lambda x})P_{10}(t-x)\,dx$$

$$P_{01}(t) = \int_0^t (\lambda+\lambda_r)e^{-(\lambda+\lambda_r)x} P_{11}(t-x)\,dx$$

$$P_{11}(t) = (1-G(t))e^{-\lambda t} + \int_0^t g(x)e^{-\lambda x} P_{01}(t-x)\,dx + \int_0^t g(x)(1-e^{-\lambda x})P_{11}(t-x)\,dx. \qquad (6.106)$$

The solution of Eq. (6.105) leads to

$$\tilde{R}_{S0}(s) = \frac{s+\lambda+(\lambda+\lambda_r)(1-\tilde{g}(s+\lambda))}{(s+\lambda)[(s+(\lambda+\lambda_r)(1-\tilde{g}(s+\lambda))]} \qquad (6.107)$$

and (with $MTTF_{S0} = \tilde{R}_{S0}(0)$, Eq. (2.61))

$$MTTF_{S0} = \frac{\lambda+(\lambda+\lambda_r)(1-\tilde{g}(\lambda))}{\lambda(\lambda+\lambda_r)(1-\tilde{g}(\lambda))} \approx \frac{1}{(\lambda+\lambda_r)(1-\tilde{g}(\lambda))}, \qquad (6.108)$$

The Laplace transform of the *point availability* $PA_{S0}(t) = P_{00}(t) + P_{01}(t)$ follows as a solution of Eq. (6.106)

$$\tilde{PA}_{S0}(s) = \frac{(s+\lambda)(1-\tilde{g}(s)) + \lambda_r(1-\tilde{g}(s+\lambda)) + \lambda + s\,\tilde{g}(s+\lambda)}{(s+\lambda)[(s+\lambda+\lambda_r)(1-\tilde{g}(s)) + s\,\tilde{g}(s+\lambda)]}, \qquad (6.109)$$

and leads to the asymptotic & steady-state value of the *point availability* PA_S and *average availability* AA_S (with $\lim_{s\to 0}(1-\tilde{g}(s)) = s\cdot MTTR + o(s)$ as per Eq. (6.54))

$$PA_S = AA_S = (\lambda+\lambda_r(1-\tilde{g}(\lambda)) \;/\; (\lambda(\lambda+\lambda_r)MTTR + \lambda\tilde{g}(\lambda)), \qquad (6.110)$$

where

$$MTTR = \int_0^\infty x\,g(x)\,dx = \int_0^\infty (1-G(x))\,dx \qquad (6.111)$$

and $\tilde{g}(\lambda)$ is the Laplace transform of the density $g(t)$ for $s=\lambda$, see Examples 6.8 & 6.9 for the approximation of $\tilde{g}(\lambda)$. Calculation of the *interval reliability* is difficult because state Z_1 is regenerative only at its occurrence point (Fig. A7.11). However, for $\lambda\,MTTR \ll 1$. $\tilde{g}(\lambda)\to 1$ and the asymptotic value of the state probability for Z_1 ($P_1 = \lim_{t\to\infty} P_{01}(t)$) becomes very small with respect to that for Z_0 ($P_0 = \lim_{t\to\infty} P_{00}(t)$). For the asymptotic & steady-state value of the *interval reliability* it holds then that

$$IR_S(\theta) \approx P_0\,R_{S0}(\theta) = \lambda\tilde{g}(\lambda)\,R_{S0}(\theta) \;/\; (\lambda(\lambda+\lambda_r)\,MTTR + \lambda\,\tilde{g}(\lambda)). \qquad (6.112)$$

In practical applications, $\lambda\,MTTR < 0.01$ and Eq. (6.112) yields $IR_S(\theta) \approx R_{S0}(\theta)$.

Example 6.8

Let the density $g(x)$ of the repair time τ' of a system with constant failure rate $\lambda > 0$ be continuous and assume furthermore that $\lambda E[\tau'] = \lambda\, MTTR << 1$ and $\lambda\sqrt{Var[\tau']} << 1$. Investigate the quantity $\tilde{g}(\lambda)$ for $\lambda \to 0$.

Solution

For $\lambda \to 0$, $\lambda\, MTTR << 1$, $\lambda\sqrt{Var[\tau']} << 1$, the 3 first terms of the series expansion of $e^{-\lambda t}$ lead to

$$\tilde{g}(\lambda) = \int_0^\infty g(t) e^{-\lambda t} dt \approx \int_0^\infty g(t)(1 - \lambda t + \frac{(\lambda t)^2}{2}) dt = 1 - \lambda E[\tau'] + E[\tau'^2]\, \lambda^2 / 2 .$$

From this, follows the *approximate expression*

$$\tilde{g}(\lambda) \approx 1 - \lambda\, MTTR + \lambda^2 (MTTR^2 + Var[\tau']) / 2. \tag{6.113}$$

In many practical applications,

$$\tilde{g}(\lambda) \approx 1 - \lambda\, MTTR \tag{6.114}$$

is a sufficiently good approximation, however not in calculating steady-state availability (Eq. (6.114) would give for Eq. (6.110) $PA_S = 1$, thus Eq. (6.113) has to be used).

Supplementary results: $g(x) = \mu e^{-\mu x}$ leads to $\tilde{g}(\lambda) = \frac{\mu}{\lambda + \mu} \approx 1 - \frac{\lambda}{\mu} + (\frac{\lambda}{\mu})^2$, which agree with Eq. (6.113) considering $MTTR = 1/\mu$ and $Var[\tau'] = 1/\mu^2$.

Example 6.9

In a 1-out-of-2 warm redundancy with identical elements E_1 and E_2 let the failure rates λ in the operating state and λ_r in the reserve state be constant. For the repair time let us assume that it is distributed according to $G(x) = 1 - e^{-\mu'(x-\psi)}$ for $x > \psi$ and $G(t) = 0$ for $x \le \psi$, with $MTTR \equiv 1/\mu > \psi$. Assuming $\lambda\psi << 1$, investigate the influence of ψ on the mean time to failure $MTTF_{S0}$ and on the asymptotic & steady-state value of the point availability PA_S.

Solution

With

$$\tilde{g}(\lambda) = \int_\psi^\infty \mu' e^{-\mu'(t-\psi) - \lambda t} dt = \frac{\mu'}{\lambda + \mu'} e^{-\lambda\psi} \approx \frac{\mu'}{\lambda + \mu'}(1 - \lambda\psi)$$

and considering $MTTR = \int_0^\infty t\, g(t)\, dt = \int_\psi^\infty t\mu' e^{-\mu'(t-\psi)} dt = \psi + \frac{1}{\mu'} \equiv \frac{1}{\mu}$, i.e., $\mu' = \mu / (1 - \mu\psi)$ and thus $\tilde{g}(\lambda) \approx \mu(1 - \lambda\psi) / (\lambda + \mu(1 - \lambda\psi))$, Eq. (6.108) (left-hand equality) and Eq. (6.110) lead to the *approximate expressions*

$$MTTF_{S0,\, \psi>0} \approx \frac{2\lambda + \lambda_r + \mu(1 - \lambda\psi)}{\lambda(\lambda + \lambda_r)}$$

and

$$PA_{S,\, \psi>0} \approx \frac{\mu(\lambda + \lambda_r + \mu(1 - \lambda\psi))}{(\lambda + \lambda_r)(\lambda + \mu(1 - \lambda\psi)) + \mu^2(1 - \lambda\psi)} \approx 1 - \frac{\lambda(\lambda + \lambda_r)(1 - \mu\psi))}{\mu(\lambda + \lambda_r + \mu(1 - \lambda\psi))} .$$

On the other hand, $\psi = 0$ leads to $1 - \tilde{g}(\lambda) = \lambda / (\lambda + \mu)$ and thus (Eqs. (6.92) and (6.87))

$$MTTF_{S0,\, \psi=0} = \frac{2\lambda + \lambda_r + \mu}{\lambda(\lambda + \lambda_r)} \quad \text{and} \quad PA_{S,\, \psi=0} = \frac{\mu(\lambda + \lambda_r + \mu)}{(\lambda + \mu)(\lambda + \lambda_r) + \mu^2} .$$

Assuming $\mu >> \lambda, \lambda_r$ yields (considering $\lambda\psi < \lambda/\mu << 1$)

$$\frac{MTTF_{S0,\, \psi>0}}{MTTF_{S0,\, \psi=0}} \approx 1 - \lambda\psi \quad \text{and} \quad \frac{PA_{S,\, \psi>0}}{PA_{S,\, \psi=0}} \approx 1 + \lambda\psi \frac{\lambda + \lambda_r}{\mu} \approx 1. \tag{6.115}$$

Equation (6.115) allows the conclusion to be made that:

For $\lambda\, MTTR << 1$, the shape of the distribution function of the repair time has (as long as MTTR is unchanged) a small influence on results at system level, in particular on the mean time to failure $MTTF_{S0}$ and on the asymptotic & steady-state value of the point availability PA_S of a 1-out-of-2 redundancy.

Example 6.10 shows a numerical comparison. This result can be extended to complex structures.

Example 6.10

A 1-out-of-2 parallel redundancy with identical elements E_1 and E_2 has failure rate $\lambda = 10^{-2}\,h^{-1}$ and lognormally distributed repair times with mean $MTTR = 2.4h$ and variance $0.6h^2$ (Eqs. (A6.112), (A6.113) with $\lambda = 0.438h^{-1}$, $\sigma = 0.315$). Compute the mean time to failure $MTTF_{S0}$ and the asymptotic & steady-state point and average availability PA_S with *approximate expressions*: (i) $\tilde{g}(\lambda)$ from Eq. (6.114); (ii) $\tilde{g}(\lambda)$ from Eq. (6.113); (iii) $g(t) = \mu' e^{-\mu'(t-\psi)}$, $t \geq \psi$, $\psi = 1.3h$, $1/\mu' = 1.1\,h$, $1/\mu = 2.4\,h$ (Eq. (4.2)); (iv) $g(t) = \mu e^{-\mu t}$ and $1/\mu = 2.4\,h$.

Solution

(i) With $\tilde{g}(\lambda) = 0.976$ it follows (Eq. (6.108)) that $MTTF_{S0} \approx 2183h$ and (Eq. (6.110)) $PA_S = 1$. (ii) With $\tilde{g}(\lambda) \approx 0.9763$ it follows (Eq. (6.108)) that $MTTF_{S0} \approx 2211\,h$ and (Eq. (6.110)) $PA_S \approx 0.9994$. (iii) Example 6.9 yields $MTTF_{S0,\,\psi=1.3\,h} \approx 2206h$ and $PA_{S,\,\psi=1.3\,h} \approx 0.9995$. (iv) From Eqs. (6.92) and (6.87) it follows that $MTTF_{S0} \approx 2233h$ and $PA_S \approx 0.9989$.

Supplementary results: Numerical computation with the lognormal distribution ($MTTR = 2.4h$, $Var[\tau'] = 0.6h^2$) yields $MTTF_{S0} \approx 2186h$ and $PA_S \approx 0.9995$. For a failure rate $\lambda = 10^{-3}h^{-1}$, results were: 209'333h, 1; 209'611h, 0.999997; 209'563h, 0.999995; 209'833, 0.999989; 209'513h, 0.999994.

6.4.3 1-out-of-2 Redundancy with Constant Failure Rate only in the Reserve State and Arbitrary Repair Rates

Generalization of repair and failure rates for a 1-out-of-2 redundancy leads to a *nonregenerative stochastic process*. However, in many practical applications it can be assumed that *the failure rate in reserve state is constant.* If this holds, and the 1-out-of-2 redundancy has only one repair crew, then the process involved is regenerative with exactly *one regeneration state* [6.5 (1975)].

To see this, consider a 1-out-of-2 warm redundancy, satisfying assumptions (6.1) - (6.7), with failure-free times distributed according to $F(x)$ in operating state and $V(x) = 1 - e^{-\lambda_r x}$ in reserve state, and repair times distributed according to $G(x)$ for repair of failures in operating state and $W(x)$ for repair of failures in reserve state ($F(0) = V(0) = G(0) = W(0) = 0$, densities $f(x), v(x), g(x), w(x) \to 0$ for $x \to \infty$, means and variances $< \infty$). Figure 6.10a shows a possible time schedule and Fig. 6.10b gives the *state transition diagram* of the involved stochastic process.

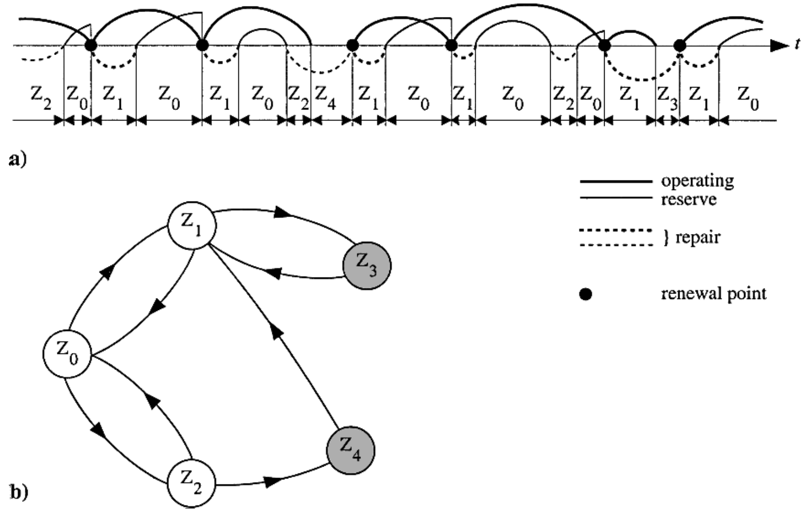


Figure 6.10 Repairable 1-out-of-2 warm redundancy with constant failure rate λ_r in reserve state, arbitrary failure rate in operating state, arbitrary repair rates, ideal failure detection & switch, one repair crew, Z_3 & Z_4 down states): **a)** Possible time schedule (repair times greatly exaggerated); **b)** state transition diagram to visualize state transitions (only Z_1 is a *regeneration state*)

States Z_0, Z_1, Z_2 are up states. State Z_1 is the only *regeneration state* present here. At its occurrence, *a failure-free time of the operating element and a repair time for a failure in the operating state are started* (Fig. 6.10a). The occurrence of Z_1 is a renewal (regeneration) point and brings the process to a situation of *total independence from the previous development.* It is therefore sufficient to investigate the time behavior from $t=0$ up to the first regeneration point and between *two consecutive regeneration points* (Appendix A7.7).

Let us consider first the case in which the regeneration state Z_1 is *entered* at $t=0$ (S_{RP0}) and let S_{RP1} be the first renewal point after $t=0$. The *reliability function* $\text{R}_{S1}(t)$ is given by (see Table 6.2 for notations)

$$\text{R}_{S1}(t) = 1 - \text{F}(t) + \int_0^t \text{u}_1(x)\,\text{R}_{S1}(t-x)\,dx, \tag{6.116}$$

with

$$1-\text{F}(t) = \Pr\{\text{failure-free operating time of the operating element (new at } t=0) > t \mid Z_1 \text{ entered at } t=0\}$$

and

$$\int_0^t \text{u}_1(x)\text{R}_{S1}(t-x)dx = \Pr\{(S_{RP1} \le t \cap \text{system not failed in } (0, S_{RP1}] \cap up \text{ in } (S_{RP1}, t]) \mid Z_1 \text{ entered at } t=0\}.$$

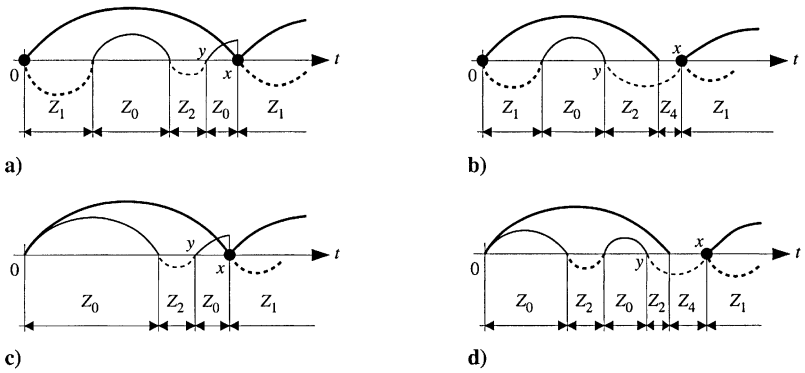


Figure 6.11 Possible time schedules for the 1-out-of-2 redundancy according to Fig. 6.10 for the cases in which state Z_1 (a, b) or state Z_0 *with both elements new* (c, d) is entered at $t=0$

The *first renewal point* S_{RP1} occurs at the time x (i.e. within the interval $(x, x+dx]$) only if at this time *the operating element fails and the reserve element is ready to enter the operating state*. The quantity $u_1(x)$, defined as (Eq. (A6.12))

$$u_1(x) = \lim_{\delta x \downarrow 0} \frac{1}{\delta x} \Pr\{(x < S_{RP1} \le x + \delta x \cap \text{system not failed in } (0, x]) \mid Z_1 \text{ entered at } t=0\},$$

follows from (Fig. 6.11a)

$$u_1(x) = f(x) \mathrm{PA}_d(x), \tag{6.117}$$

with

$$\mathrm{PA}_d(x) = \Pr\{\text{reserve element } up \text{ at time } x \mid Z_1 \text{ entered at } t=0\}$$

$$= \int_0^x h'_{dud}(y) e^{-\lambda_r (x-y)} dy \tag{6.118}$$

and

$$h'_{dud}(y) = g(y) + g(y) * v(y) * w(y) + g(y) * v(y) * w(y) * v(y) * w(y) + \dots . \tag{6.119}$$

The *point availability* is given by

$$\mathrm{PA}_{S1}(t) = 1 - F(t) + \int_0^t u_1(x) \mathrm{PA}_{S1}(t-x)\, dx + \int_0^t u_2(x) \mathrm{PA}_{S1}(t-x)\, dx, \tag{6.120}$$

with $1 - F(t)$ as for Eq. (6.116),

$$\int_0^t u_1(x) PA_{S1}(t-x)\, dx = \Pr\{(S_{RP1} \le t \cap \text{system not failed in } (0, S_{RP1}] \cap up \text{ at } t \mid Z_1 \text{ entered at } t=0\},$$

and

$$\int_0^t u_2(x) PA_{S1}(t-x)\, dx = \Pr\{(S_{RP1} \le t \cap \text{system failed in } (0, S_{RP1}] \cap up \text{ at } t \mid Z_1 \text{ entered at } t=0\}.$$

The quantity $u_2(x)$, defined as

$$u_2(x) = \lim_{\delta x \downarrow 0} \frac{1}{\delta x} \Pr\{(x < S_{RP1} \le x + \delta x \cap \text{system failed in } (0, x]) \mid Z_1 \text{ entered at } t = 0\},$$

follows from (Fig. 6.11b)

$$u_2(x) = g(x)F(x) + \int_0^x h'_{udd}(y)\, w(x-y)(F(x) - F(y))\, dy \tag{6.121}$$

with

$$h'_{udd}(y) = g(y) * v(y) + g(y) * v(y) * w(y) * v(y) + \dots . \tag{6.122}$$

One can recognizes that $u_1(x)+u_2(x)$ is the density of the *interval times separating consecutive renewal points* $0 \equiv S_{RP0}, S_{RP1}, S_{RP2}, \dots$, i.e., successive occurrence times of state Z_1 of the *embedded renewal process.*

Consider now the case in which at $t=0$ the state Z_0 *with both elements new* is *entered.* The *reliability function* $R_{S0}(t)$ is given by

$$R_{S0}(t) = 1 - F(t) + \int_0^t u_3(x) R_{S1}(t-x)\, dx, \tag{6.123}$$

with (Fig. 6.11c)

$$u_3(x) = \lim_{\delta x \downarrow 0} \frac{1}{\delta x} \Pr\{(x < S_{RP1} \le x + \delta x \cap \text{system not failed in } (0, x]) \mid Z_0 \textit{ with both elements new} \text{ is entered at } t=0\} = f(x) PA_0(x), \tag{6.124}$$

where

$$PA_0(x) = \Pr\{\text{reserve element } up \text{ at time } x \mid Z_0 \textit{ with both elem. new} \text{ is entered at } t=0\}$$

$$= e^{-\lambda_r x} + \int_0^x h'_{duu}(y)\, e^{-\lambda_r (x-y)} dy, \tag{6.125}$$

with

$$h'_{duu}(y) = v(y) * w(y) + v(y) * w(y) * v(y) * w(y) + \dots . \tag{6.126}$$

The *point availability* $PA_{S0}(t)$ is given by

$$PA_{S0}(t) = 1 - F(t) + \int_0^t u_3(x) PA_{S1}(t-x)\,dx + \int_0^t u_4(x) PA_{S1}(t-x)\,dx, \quad (6.127)$$

with (Fig. 6.11d)

$$u_4(x) = \lim_{\delta x \downarrow 0} \frac{1}{\delta x} \Pr\{(x < S_{RP1} \le x + \delta x \cap \text{system failed in } (0,x]) \mid Z_0 \text{ with both elements new is entered at } t=0\} = \int_0^x h'_{udu}(y)\,w(x-y)(F(x)-F(y))\,dy \quad (6.128)$$

and

$$h'_{udu}(y) = v(y) + v(y) * w(y) * v(y) + v(y) * w(y) * v(y) * w(y) * v(y) + \dots . \quad (6.129)$$

One can recognizes that $u_3(x) + u_4(x)$ is the density of the random time from $t=0$, when Z_0 *is entered with both elements new*, to the first renewal point S_{RP1} (first occurrence of Z_1) of the *embedded renewal process* with density $u_1(x) + u_2(x)$ for the time intervals separating successive renewal points ($S_{RP(i+1)} - S_{RPi}$, $i = 1, 2, \dots$).

Equations (6.116), (6.120), (6.123), (6.127) can be solved using Laplace transforms (LT). However, analytical difficulties can arise when calculating LT for $F(x)$, $G(x)$, $W(x)$, $u_1(x)$, $u_2(x)$, $u_3(x)$, $u_4(x)$ or at the inversion of final equations. Easier is the calculation of the *mean time to failure* $MTTF_{S0} = \tilde{R}_{S0}(0)$ (Eqs. (2.59), (2.61)) and of the *asymptotic & steady-state point and average availability* $PA_S = AA_S = \lim_{s \to 0} s\tilde{PA}_S(s)$, for which the following expressions can be found using LT (see Eqs. (6.123) & (6.116) for $MTTF_{S0}$ and Eqs. (6.120) & (6.127) for PA_S, and consider $\lim_{s \to 0}(1 - \tilde{f}(s))/s = MTTF$)

$$MTTF_{S0} = MTTF\,\left[1 + \frac{\int_0^\infty u_3(x)\,dx}{1 - \int_0^\infty u_1(x)\,dx}\right], \quad (6.130)$$

and

$$\lim_{t \to \infty} PA_{S0}(t) = \lim_{t \to \infty} PA_{S1}(t) = PA_S = AA_S = \frac{MTTF}{\int_0^\infty x(u_1(x) + u_2(x))\,dx}, \quad (6.131)$$

with

$$MTTF = \int_0^\infty (1 - F(x))\,dx. \quad (6.132)$$

Eq. (6.131) considers that PA_S exists and that $u_1(x) + u_2(x)$ is the density of a random variable with finite mean, and thus $\int_0^\infty (u_1(x) + u_2(x))\,dx = 1$; similarly for $u_3(x) + u_4(x)$.

It must be pointed out that $R_{S0}(t)$ and $PA_{S0}(t)$ apply only to the case in which at $t = 0$ *both elements are new* (Fig. 6.11 c & d). Situations with arbitrary initial conditions at $t = 0$ (e.g. entering state Z_0 with the operating element not new or entering state Z_2) are not considered here because their computation requires the knowledge of the time spent in the operating state before $t = 0$.

The model investigated in this section has as special cases that of Section 6.4.2 ($F(x) = 1 - e^{-\lambda x}$, $W(x) = G(x)$) and the 1-out-of-2 *standby redundancy* with identical elements and arbitrarily distributed failure-free and repair times (Example 6.11).

Table 6.7 summarizes the results for the 1-out-of-2 redundancy with arbitrary repair rates, and failure rates as general as possible within a regenerative process.

Example 6.11

Using the results of Section 6.4.3, give the expressions for the reliability function $R_{S0}(t)$ and the point availability $PA_{S0}(t)$ for a 1-out-of-2 *standby redundancy* with 2 identical elements, failure-free time distributed according to $F(x)$, with density $f(x)$, and repair time distributed according to $G(x)$ with density $g(x)$.

Solution

For a standby redundancy, $u_1(x) = f(x)G(x)$, $u_2(x) = g(x)F(x)$, $u_3(x) = f(x)$, and $u_4(x) \equiv 0$ (Eqs. (6.117), (6.121), (6.124), and (6.128)). From this, the expressions for $R_{S0}(t)$, $R_{S1}(t)$, $PA_{S0}(t)$, and $PA_{S1}(t)$ can be given. The Laplace transforms of $R_{S0}(t)$ and $PA_{S0}(t)$ are

$$\tilde{R}_{S0}(s) = \frac{1 - \tilde{f}(s)}{s} + \frac{\tilde{f}(s)(1 - \tilde{f}(s))}{s(1 - \tilde{u}_1(s))}, \qquad (6.133)$$

$$\tilde{PA}_{S0}(s) = \frac{1 - \tilde{f}(s)}{s} + \frac{\tilde{f}(s)(1 - \tilde{f}(s))}{s[1 - (\tilde{u}_1(s) + \tilde{u}_2(s))]}, \qquad (6.134)$$

with

$$\tilde{u}_1(s) = \int_0^\infty f(t)G(t)e^{-st}\,dt \qquad \text{and} \qquad \tilde{u}_2(s) = \int_0^\infty g(t)F(t)e^{-st}\,dt$$

The mean time to failure $MTTF_{S0}$ follows from Eq. (6.133) as $MTTF_{S0} = \tilde{R}_{S0}(0)$, or directly from Eq. (6.130),

$$MTTF_{S0} = MTTF + \frac{MTTF}{1 - \int_0^\infty f(x)\,G(x)\,dx}. \qquad (6.135)$$

The asymptotic & steady-state value of the point and average availability $PA_S = AA_S$ follows from Eq. (6.134) as $PA_S = AA_S = \lim_{s \to 0} s\,\tilde{PA}_S(s)$, or directly form Eq. (6.131),

$$PA_S = AA_S = \frac{MTTF}{\int_0^\infty x\,d(F(x)G(x))}. \qquad (6.136)$$

Table 6.7 Mean time to failure $MTTF_{S0}$, asymptotic & steady-state point and average availability $PA_S = AA_S$, and interval reliability $IR_S(\theta)$ for a repairable 1-out-of-2 redundancy with two identical elements (Fig. 6.10, arbitrary repair rates, failure rates as general as possible within a semi-regenerative process, ideal failure detection & switch, one repair crew)

			Standby ($\lambda_r \equiv 0$)	Warm ($\lambda_r < \lambda$)		Active ($\lambda_r = \lambda$)
Element E_1 and E_2	Distribution of the failure-free times	OS	$F(x)$	$1-e^{-\lambda x}$	$F(x)$	$1-e^{-\lambda x}$
		RS	-	$1-e^{-\lambda_r x}$	$1-e^{-\lambda_r x}$	$1-e^{-\lambda x}$
	Distribution of the repair times	OS	$G(x)$	$G(x)$	$G(x)$	$G(x)$
		RS	–	$G(x)$	$W(x)$	$G(x)$
	Mean of the failure-free times		$MTTF = \int_0^\infty (1-F(x))dx$	$\frac{1}{\lambda}$ or $\frac{1}{\lambda_r}$	$MTTF$ or $\frac{1}{\lambda_r}$	$\frac{1}{\lambda}$
	Mean of the repair times		$MTTR = \int_0^\infty (1-G(x))dx$	$MTTR$	$MTTR$ or $MTTR_W$	$MTTR$
1-out-of-2 redundancy	Mean time to failure ($MTTF_{S0}$)		$MTTF + \frac{MTTF}{1-\int_0^\infty f(x)G(x)\,dx}$	$\frac{1}{\lambda}+\frac{1}{(\lambda+\lambda_r)(1-\tilde{g}(\lambda))}$ $\approx \frac{1}{\lambda}(1+\frac{1}{(\lambda+\lambda_r)MTTR})$	$\frac{MTTF + MTTF\int_0^\infty u_3(x)dx}{1-\int_0^\infty u_1(x)dx}$	$\frac{1}{\lambda}+\frac{1}{2\lambda(1-\tilde{g}(\lambda))}$ $\approx \frac{1}{\lambda}+\frac{1}{2\lambda^2 MTTR}$
	Point & average availability ($PA_S = AA_S$)*		$\frac{MTTF}{\int_0^\infty x\,d(F(x)G(x))}$	$\frac{\lambda+\lambda_r(1-\tilde{g}(\lambda))}{\lambda(\lambda+\lambda_r)MTTR+\lambda\tilde{g}(\lambda)}$	$\frac{MTTF}{\int_0^\infty x(u_1(x)+u_2(x))\,dx}$	$\frac{2-\tilde{g}(\lambda)}{2\lambda MTTR+\tilde{g}(\lambda)}$
	Interval reliability ($IR_S(\theta)$)*		$\approx R_{S0}(\theta)$	$\approx R_{S0}(\theta)$	$\approx R_{S0}(\theta)$	$\approx R_{S0}(\theta)$

$u_1(x)$, $u_2(x)$, $u_3(x)$ as per Eqs. (6.117), (6.121), and (6.124); OS = operating state, RS = reserve state

* asymptotic & steady-state value

6.5 *k*-out-of-*n* Redundancy

A *k-out-of-n redundancy,* also known as *k-out-of-n: G,* consists of *n* often identical elements, of which *k* are necessary for the required function and *n* - *k* are in reserve state (or repair). Assuming ideal failure detection and switching, the reliability block diagram is as given in Fig. 6.12. Investigations in this Section assume

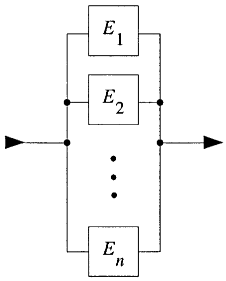


Figure 6.12 k-out-of-n redundancy reliability block diagram (ideal failure detection & switch)

identical elements $E_1, \ldots, E_n$, only one repair crew, and *no further failures at system down* (failures during repair at system level are neglected, as per assumption (6.2)). Section 6.5.1 considers the case of *warm redundancy* with constant failure rate λ in the operation state and $\lambda_r < \lambda$ in the *reserve state* as well as constant repair rate μ. This case includes *active redundancy* ($\lambda_r = \lambda$) and *standby redundancy* ($\lambda_r \equiv 0$). An extension to cover other situations in which the failure rate is modified at state changes (e. g. for *load sharing*) is possible using the equations for the *birth and death process* developed in Appendix A7.5.5 (see also Section 2.3.5). Section 6.5.2 investigates a k-out-of-n active redundancy with constant failure rate and arbitrary repair rate. The influence of series elements (including *switching elements*) is considered in Sections 6.6 - 6.7. Imperfect switching, incomplete coverage, and common cause failures are investigated in Section 6.8.

6.5.1 *k*-out-of-*n* Warm Redundancy with Identical Elements and Constant Failure and Repair Rates

Assuming constant failure and repair rates, the time behavior of the k-out-of-n redundancy with identical elements can be investigated using a *birth and death process* (Appendix A7.5.5). Figure 6.13 gives the corresponding diagram of transition probabilities in $(t, t + \delta t]$. From Fig. 6.13 and Table 6.2, the following system of differential equations can be established for the state probabilities $\mathrm{P}_j(t) = \Pr\{\text{in state } Z_j \text{ at } t\}$ of a k-out-of-n warm redundancy with one repair crew and no further failures at system down (constant failure rates λ & λ_r and repair rate μ)

$$\dot{\mathrm{P}}_0(t) = -\nu_0 \mathrm{P}_0(t) + \mu \mathrm{P}_1(t)$$
$$\dot{\mathrm{P}}_j(t) = \nu_{j-1} \mathrm{P}_{j-1}(t) - (\nu_j + \mu) \mathrm{P}_j(t) + \mu \mathrm{P}_{j+1}(t), \qquad j = 1, \ldots, n-k,$$
$$\dot{\mathrm{P}}_{n-k+1}(t) = \nu_{n-k} \mathrm{P}_{n-k}(t) - \mu \mathrm{P}_{n-k+1}(t), \qquad (6.137)$$

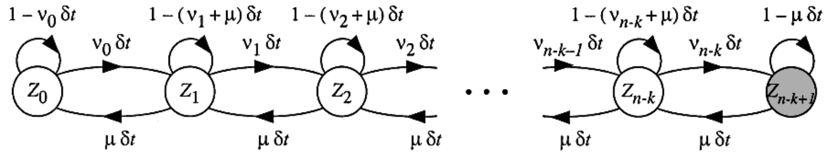


Figure 6.13 Diagram of transition probabilities in $(t, t+\delta t]$ for a repairable k-out-of-n *warm* redundancy (n identical elements, const. failure & repair rates, ideal failure detection & switch, one repair crew, no further failures at system down (Z_{n-k+1} down state, arbitrary t, $\delta t \downarrow 0$, birth & death proc.)

with

$$\nu_j = k\lambda + (n-k-j)\lambda_r, \qquad j = 0, \ldots, n-k. \tag{6.138}$$

For the investigation of more general situations (arbitrary load sharing, more than one repair crew, or other cases in which failure and / or repair rates change at a state transition) one can use the *birth and death process* introduced in Appendix A7.5.5. The solution of the system (6.137) with the *initial conditions* at $t = 0$, $\mathrm{P}_i(0) = 1$ and $\mathrm{P}_j(0) = 0$ for $j \neq i$, yields the *point availability* (see Table 6.2 for notations)

$$\mathrm{PA}_{Si}(t) = \sum_{j=0}^{n-k} \mathrm{P}_{ij}(t), \tag{6.139}$$

with $\mathrm{P}_{ij}(t) \equiv \mathrm{P}_j(t)$ from Eq. (6.137) with $\mathrm{P}_i(0) = 1$. In many practical applications, only the *asymptotic & steady-state* value of the point availability PA_S is required. This can be obtained by setting $\dot{\mathrm{P}}_j(t) = 0$ and $\mathrm{P}_j(t) = P_j$ $(j = 0, \ldots, n-k+1)$ in Eq. (6.137). The solution is (Appendix A7.5.5)

$$PA_S = \sum_{j=0}^{n-k} P_j = 1 - P_{n-k+1}, \quad \text{with } P_j = \frac{\pi_j}{\sum_{i=0}^{n-k+1} \pi_i}, \quad \pi_i = \frac{\nu_0 \cdots \nu_{i-1}}{\mu^i}, \quad \pi_0 = 1. \tag{6.140}$$

PA_S is also the asymptotic & steady-state value of the *average availability* AA_S. As shown in Example A 7.11 (Eq. (A7.157)), for $2\nu_j < \mu$ it holds that

$$P_j \geq \sum_{i=j+1}^{n-k+1} P_i, \qquad j = 0, \ldots, n-k.$$

From this, the following *bounds for* PA_S can be used in many practical applications (assuming $2\nu_j < \mu$, $j = 0, \ldots, n\text{-}k$) to obtain an *approximate expression* for PA_S

$$\sum_{j=0}^{i} P_j \leq PA_S < P_i + \sum_{j=0}^{i} P_j, \qquad i = 0, \ldots, n-k. \tag{6.141}$$

The *reliability function* follows from Table 6.2 and Fig. 6.13

$$\mathrm{R}_{S0}(t) = e^{-\nu_0 t} + \int_0^t \nu_0 e^{-\nu_0 x} \mathrm{R}_{S1}(t-x)\,dx$$

$$\mathrm{R}_{Sj}(t) = e^{-(\nu_j+\mu)t} + \int_0^t [\nu_j \mathrm{R}_{Sj+1}(t-x) + \mu \mathrm{R}_{Sj-1}(t-x)]\, e^{-(\nu_j+\mu)x} dx, \qquad j=1, \ldots, n-k-1,$$

$$\mathrm{R}_{Sn-k}(t) = e^{-(\nu_{n-k}+\mu)t} + \int_0^t \mu \mathrm{R}_{Sn-k-1}(t-x) e^{-(\nu_{n-k}+\mu)x} dx, \qquad (6.142)$$

with ν_i as in Eq. (6.138). Similar results hold for the *mean time to failure*

$$MTTF_{S0} = MTTF_{S1} + 1/\nu_0$$

$$MTTF_{Sj} = (1 + \nu_j\, MTTF_{Sj+1} + \mu\, MTTF_{Sj-1})/(\nu_j + \mu), \qquad j = 1, \ldots, n-k-1,$$

$$MTTF_{Sn-k} = (1 + \mu\, MTTF_{Sn-k-1})/(\nu_{n-k} + \mu). \qquad (6.143)$$

The solution of Eqs. (6.142) and (6.143), shows that $\mathrm{R}_{Si}(t)$ and $MTTF_{Si}$ *depend* on $n-k$ only. This leads for $n-k=1$ to

$$\tilde{\mathrm{R}}_{S0_1}(s) = \frac{s + \nu_0 + \nu_1 + \mu}{(s+\nu_0)(s+\nu_1) + s\mu}, \qquad MTTF_{S0_1} = \frac{\nu_0 + \nu_1 + \mu}{\nu_0 \nu_1} \approx \frac{\mu}{\nu_0 \nu_1}, \qquad (6.144)$$

and for $n-k=2$ to

$$\tilde{\mathrm{R}}_{S0_2}(s) = \frac{(s+\nu_0+\nu_1+\mu)(s+\nu_2+\mu) + \nu_1(\nu_0-\mu)}{s(s+\nu_0+\nu_1+\mu)(s+\nu_2+\mu) + \nu_0\nu_1\nu_2 + s\nu_1(\nu_0-\mu)}$$

$$MTTF_{S0_2} = \frac{\nu_2(\nu_0+\nu_1+\mu) + \mu(\nu_0+\mu) + \nu_0\nu_1}{\nu_0\nu_1\nu_2} \approx \frac{\mu^2}{\nu_0\nu_1\nu_2}. \qquad (6.145)$$

This property holds for the *point availability* PA_S as well, see Table 6.8 for results.

Because of the constant failure rate, the *interval reliability* follows directly from

$$\mathrm{IR}_{Si}(t, t+\theta) = \sum_{j=0}^{n-k} \mathrm{P}_{ij}(t)\, \mathrm{R}_{Sj}(\theta), \qquad i = 0, \ldots, n-k \qquad (6.146)$$

with $\mathrm{P}_{ij}(t)$ as in Eq. (6.139) and $\mathrm{R}_{Si}(\theta)$ from Eq. (6.142) with $t=\theta$. The asymptotic & steady-state value is then given by

$$\mathrm{IR}_S(\theta) = \sum_{j=0}^{n-k} P_j\, \mathrm{R}_{Sj}(\theta), \qquad (6.147)$$

with P_j from Eq. (6.140). Table 6.8 summarizes the main results for a k-out-of-n warm redundancy with constant failure and repair rates.

Table 6.8 Mean time to failure $MTTF_{S0}$, asymptotic & steady-state point and average availability $PA_S = AA_S$, and interval reliability $IR_S(\theta)$ for a repairable *k*-out-of-*n* *warm* redundancy with *n* identical elements (constant failure & repair rates λ, λ_r, μ ($\lambda_r < \lambda$ reserve state, $\lambda_r \equiv 0$ standby), ideal failure detection & switch, one repair crew, no further failures at system down, Markov proc.)

		Mean time to failure ($MTTF_{S0}$)	Asymptotic & steady-state point and average availability ($PA_S = AA_S$)	Interval reliability ($IR_S(\theta)$)*
$n-k=1$	gen. case	$\frac{\nu_0+\nu_1+\mu}{\nu_0\,\nu_1} \approx \frac{\mu}{\nu_0\,\nu_1}$	$\frac{\nu_0\,\mu+\mu^2}{\nu_0\,\nu_1+\nu_0\,\mu+\mu^2} \approx 1-\frac{\nu_0\,\nu_1}{\mu^2}$	$\approx R_{S0}(\theta)$
$n-k=1$	$n=2$ $k=1$	$\frac{2\lambda+\lambda_r+\mu}{\lambda(\lambda+\lambda_r)} \approx \frac{\mu}{\lambda(\lambda+\lambda_r)}$	$\frac{\mu(\lambda+\lambda_r+\mu)}{(\lambda+\lambda_r)(\lambda+\mu)+\mu^2} \approx 1-\frac{\lambda(\lambda+\lambda_r)}{\mu^2}$	$\approx R_{S0}(\theta)$
$n-k=1$	$n=3$ $k=2$	$\frac{4\lambda+\lambda_r+\mu}{2\lambda(2\lambda+\lambda_r)} \approx \frac{\mu}{2\lambda(2\lambda+\lambda_r)}$	$\frac{\mu(2\lambda+\lambda_r+\mu)}{(2\lambda+\lambda_r)(2\lambda+\mu)+\mu^2} \approx 1-\frac{2\lambda(2\lambda+\lambda_r)}{\mu^2}$	$\approx R_{S0}(\theta)$
$n-k=2$	gen. case	$\frac{\nu_2(\nu_0+\nu_1+\mu)}{\nu_0\,\nu_1\,\nu_2} + \frac{\mu(\nu_0+\mu)+\nu_0\,\nu_1}{\nu_0\,\nu_1\,\nu_2} \approx \frac{\mu^2}{\nu_0\,\nu_1\,\nu_2}$	$\frac{\nu_0\,\nu_1\,\mu+\nu_0\,\mu^2+\mu^3}{\nu_0\nu_1\,\nu_2+\nu_0\,\nu_1\,\mu+\nu_0\,\mu^2+\mu^3} \approx 1-\frac{\nu_0\,\nu_1\,\nu_2}{\mu^3}$	$\approx R_{S0}(\theta)$
$n-k=2$	$n=3$ $k=1$	$\approx \frac{\mu^2}{\lambda(\lambda+\lambda_r)(\lambda+2\lambda_r)}$	$\approx 1-\frac{\lambda(\lambda+\lambda_r)(\lambda+2\lambda_r)}{\mu^3}$	$\approx R_{S0}(\theta)$
$n-k=2$	$n=5$ $k=3$	$\approx \frac{\mu^2}{3\lambda(3\lambda+\lambda_r)(3\lambda+2\lambda_r)}$	$\approx 1-\frac{3\lambda(3\lambda+\lambda_r)(3\lambda+2\lambda_r)}{\mu^3}$	$\approx R_{S0}(\theta)$
$n-k$ arbitrary		$\approx \frac{\mu^{n-k}}{\nu_0\cdots\nu_{n-k}}$	$\approx 1-\frac{\nu_0\ldots\nu_{n-k}}{\mu^{n-k+1}} \approx 1-\frac{1/\mu}{MTTF_{S0}} = 1-\frac{MTTR_S}{MTTF_{S0}}$	$\approx R_{S0}(\theta)$

$\nu_i = k\lambda + (n-k-i)\lambda_r$, $i=0, \ldots, n-k$; λ, λ_r = failure rates ($\lambda_r = \lambda \rightarrow$ *active red.* $\Rightarrow \nu_0 \cdots \nu_{n-k} = \lambda^{n-k+1} n!/(k-1)!$, $\lambda_r \equiv 0 \rightarrow$ *standby redundancy* $\Rightarrow \nu_0 \cdots \nu_{n-k} = (k\lambda)^{n-k+1}$); μ = repair rate ($\mu = 1/MTTR_S$ because of only one repair crew); $R_{S0}(\theta)$ from Eq. (6.142); * see [6.5 (1985)] for exact solutions

Assuming, for *comparative investigations* with results of Table 6.8, *n repair crews* (one for each element), following *approximate expressions* can be found for *active redundancy* (*totally independent elements,* Table 6.9 or e.g. [6.27, 6.44])

$$MTTF_{S0} \approx \frac{1}{k\lambda\binom{n}{k}} (\mu/\lambda)^{n-k}, \qquad n \text{ repair crews, active red., } \lambda/\mu << 1 \qquad (6.148)$$

$$PA_S \approx 1 - \frac{k}{n-k+1}\binom{n}{k}(\lambda/\mu)^{n-k+1} = 1 - \frac{1}{(n-k+1)\,\mu\, MTTF_{S0}},$$

and for *standby redundancy* (see e.g. [6.44])

$$MTTF_{S0} \approx \frac{(n-k)!\,\mu^{n-k}}{(k\lambda)^{n-k+1}}, \qquad n \text{ repair crews, standby red., } \lambda/\mu << 1 \quad (6.149)$$

$$PA_S \approx 1 - \frac{(k\lambda/\mu)^{n-k+1}}{(n-k+1)!} = 1 - \frac{1}{(n-k+1)\,\mu\, MTTF_{S0}}.$$

As for Eq. (A7.189), PA_S in Eq. (6.148) and Eq. (6.149) can be expressed as $PA_S \approx 1 - MTTR_S / MTTF_S$ with $MTTR_S = 1/(n-k+1)\mu$ and $MTTF_S = MTTF_{S0}$ (see also Table 6.8, row $n-k$ arbitrary). Comparing results of Eq. (6.148) with those of Table 6.8 for $\lambda_r = \lambda$, one recognizes that $MTTF_{S0_{IE}} / MTTF_{S0_{MS}} \approx (n-k)!$ and $\overline{PA}_{S_{IE}} / \overline{PA}_{S_{MS}} \approx 1/(n-k+1)!$, with $\overline{PA}_S = 1 - PA_S$; where *IE* stands for independent elements (Eq. (6.148) or Table 6.9) and *MS* for macro-structure (Tables 6.8 or 6.10).

6.5.2 *k*-out-of-*n* Active Redundancy with Identical Elements, Constant Failure Rate, and Arbitrary Repair Rate

Generalization of the repair rate (by conserving constant failure rates (λ, λ_r), only one repair crew, and no further failure at system down), leads to stochastic processes with basically $n-k+1$ *regeneration and* $n-k$ *not regeneration states* (Z_0, Z_1 & Z_2 in Fig. A7.11 for $n-k=1$ and $Z_0, Z_1, Z_{2'}$ & Z_2, Z_3 in Fig. A7.13 for $n-k=2$). As an example let us consider a 2-out-of-3 *active* redundancy, i.e. a *majority redundancy*, with 3 identical elements, failure rate λ and repair time distributed according to $G(x)$ with $G(0)=0$ and density $g(x)$. Because of the assumption of *no further failure at system down*, results of Section 6.4.2 for the 1-out-of-2 warm redundancy can be used for $n-k=1$ by setting $k\lambda$ instead of λ (see Tab. 6.8 as well as Eq. (A7.183) for $n-k=1$ and Eqs. (A7.186) for $n-k=2$). For the 2-out-of-3 *active* redundancy one has to set 2λ instead of λ and λ instead of λ_r in Eqs. (6.108) & (6.110) to obtain Eqs. (6.152) & (6.155). However, in order to show the utility of representative time schedules, an alternative derivation is given below.

Using Fig. 6.14a, the following integral equation can be established for the *reliability function* $R_{S0}(t)$ (see Table 6.2 for notations)

$$R_{S0}(t) = e^{-3\lambda t} + \int_0^t 3\lambda e^{-3\lambda x} e^{-2\lambda(t-x)}(1 - G(t-x))\,dx + \int_0^t\int_0^y 3\lambda e^{-3\lambda x} g(y-x) e^{-2\lambda(y-x)} R_{S0}(t-y))\,dx\,dy. \quad (6.150)$$

The Laplace transform of $R_{S0}(t)$ follows as

$$\tilde{R}_{S0}(s) = \frac{s + 5\lambda - 3\lambda\tilde{g}(s+2\lambda)}{(s+2\lambda)(s+3\lambda) - 3\lambda(s+2\lambda)\tilde{g}(s+2\lambda)}, \tag{6.151}$$

and the *mean time to failure* as

$$MTTF_{S0} = \frac{5 - 3\tilde{g}(2\lambda)}{6\lambda(1-\tilde{g}(2\lambda))}. \tag{6.152}$$

For the *point availability*, Fig. 6.14b yields

$$PA_{S0}(t) = e^{-3\lambda t} + \int_0^t 3\lambda e^{-3\lambda x} PA_{S1}(t-x)dx$$

$$PA_{S1}(t) = e^{-2\lambda t}(1-G(t)) + \int_0^t g(x)e^{-2\lambda x} PA_{S0}(t-x)dx + \int_0^t g(x)(1-e^{-2\lambda x})PA_{S1}(t-x)dx \tag{6.153}$$

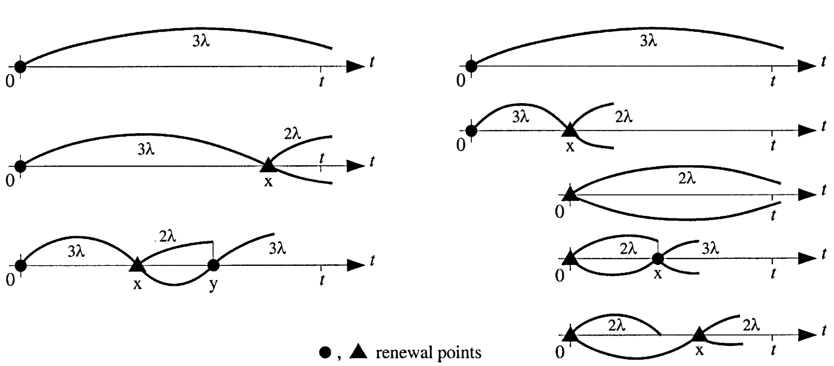
from which,

$$\tilde{PA}_{S0}(s) = \frac{(s+2\lambda)[1+\tilde{g}(s+2\lambda)-\tilde{g}(s)] + 3\lambda(1-\tilde{g}(s+2\lambda))}{s(s+2\lambda)[1+\tilde{g}(s+2\lambda)-\tilde{g}(s)] + 3\lambda(s+2\lambda)(1-\tilde{g}(s))}. \tag{6.154}$$

Asymptotic & steady-state value of the point and average availability follows from

$$PA_S = AA_S = \frac{3-\tilde{g}(2\lambda)}{2\tilde{g}(2\lambda) + 6\lambda\, MTTR}, \tag{6.155}$$

by considering $\lim_{s\to 0}(1-\tilde{g}(s)) = s \cdot MTTR + o(s)$ as per Eq. (6.54). For the approximation of $\tilde{g}(2\lambda)$, Eq. (6.113) must be used. For the asymptotic & steady-state value of the *interval reliability*, Eq. (6.112) can be used in most applications. Generalization of failure and repair rates leads to *nonregenerative stochastic processes*.



a) Calculation of $R_{S0}(t)$ b) Calculation of $PA_{S0}(t)$

Figure 6.14 Possible time schedule for a repairable 2-out-of-3 active redundancy (const. failure rate, arbitrary repair rate, ideal failure detection & switch, only one repair crew, no further failures at system down, repair times exaggerated)

6.6 Simple Series - Parallel Structures

A *series-parallel structure* is an arbitrary combination of series and parallel models, see Table 2.1 for some examples. Such a structure is generally investigated on a case-by-case basis using the methods of Sections 6.3 – 6.5. If the time behavior can be described by a Markov or semi-Markov process, Table 6.2 can be used to establish equations for the reliability function, point availability, and interval reliability (inclusive mean time to failure and asymptotic & steady-state values).

As a first example, let us consider a repairable *1-out-of-2 active redundancy* with elements $E_1 = E_2 = E$ in series with a *switching element* E_v. The failure rates λ and λ_v as well as the repair rates μ and μ_v are constant (time independent). The system has only one repair crew, *repair priority* on E_v (a repair on E_1 or E_2 is stopped as soon as a failure of E_v occurs, see Example 6.12 for the case of no priority), and *no further failures at system down* (failures during a repair at system level are neglected). Figure 6.15 gives the reliability block diagram and the diagram of transition probabilities in $(t, t+\delta t]$. The *reliability function* can be calculated using Table 6.2, or directly by considering that for a series structure the reliability at system level is still the product of the reliability of the elements

$$\mathrm{R}_{S0}(t) = \mathrm{R}_{S0_{1\text{-out-of-}2}}(t)\, e^{-\lambda_v t}. \tag{6.156}$$

Because of the term $e^{-\lambda_v t}$, the Laplace transform of $\mathrm{R}_{S0}(t)$ follows directly from the Laplace transform of the reliability function for the 1-out-of-2 parallel redundancy $\mathrm{R}_{S0_{1\text{-out-of-}2}}$, by replacing s with $s + \lambda_v$ (Table A9.7)

$$\tilde{\mathrm{R}}_{S0}(s) = \frac{s + 3\lambda + \lambda_v + \mu}{(s + 2\lambda + \lambda_v)(s + \lambda + \lambda_v) + (s + \lambda_v)\mu}. \tag{6.157}$$

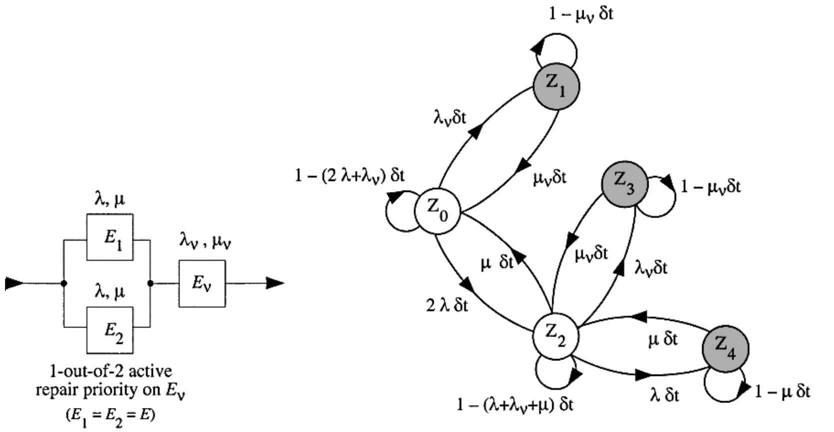
The *mean time to failure* $MTTF_{S0}$ follows from $MTTF_{S0} = \tilde{\mathrm{R}}_{S0}(0)$

$$MTTF_{S0} = \frac{3\lambda + \lambda_v + \mu}{(2\lambda + \lambda_v)(\lambda + \lambda_v) + \mu\lambda_v} = \frac{1}{\lambda_v + 2\lambda^2/(3\lambda + \lambda_v + \mu)} \le \frac{1}{\lambda_v}. \tag{6.158}$$

The last part of Eq. (6.158) clearly shows the effect of the series element E_v. The *asymptotic & steady-state* value of the *point* and *average availability* $PA_S = AA_S$ is obtained as solution of following system of algebraic equations, see Fig. 6.15 and Table 6.2,

$$P_0 = \frac{(\mu_v P_1 + \mu P_2)}{2\lambda + \lambda_v}, \qquad P_1 = \frac{\lambda_v}{\mu_v} P_0, \qquad P_3 = \frac{\lambda_v}{\mu_v} P_2,$$

$$P_2 = \frac{1}{\lambda + \lambda_v + \mu}(\mu_v P_3 + \mu P_4 + 2\lambda P_0), \qquad P_4 = \frac{\lambda}{\mu} P_2. \tag{6.159}$$



$\rho_{01} = \rho_{23} = \lambda_v;\ \rho_{02} = 2\lambda;\ \rho_{10} = \rho_{32} = \mu_v;\ \rho_{20} = \rho_{42} = \mu;\ \rho_{24} = \lambda;$
(for $R_S(t)$ set $\rho_{10} = \rho_{32} = \rho_{42} = 0$)

Figure 6.15 Reliability block diagram and diagram of transition probabilities in $(t, t+\delta t]$ for a repairable 1-out-of-2 active redundancy with a switch element ($E_1 = E_2 = E$, constant failure and repair rates $(\lambda, \lambda_v, \mu, \mu_v)$, ideal failure detection & switch, one repair crew, repair priority on E_v, no further failures at system down, Z_1, Z_3, Z_4 down states, arbitrary t, $\delta t \downarrow 0$, Markov process)

Note: The diagram of transition probabilities would have 8 states for the case of totally independent elements ($E_1 \neq E_2$, 3 repair crews), 9 states for the case as in Fig. A7.6c, and 16 states (p. 224) for $E_1 \neq E_2$, one repair crew and repair as per first-in first-out (see also the footnote on p. 479)

For the solution of the system given by Eq. (6.159), one (arbitrarily chosen) equation must be dropped and replaced by $P_0 + P_1 + P_2 + P_3 + P_4 = 1$. The solution yields P_0 through P_4, from which (assuming $2\lambda < \mu$ for the last inequality)

$$PA_S = AA_S = P_0 + P_2 = \frac{\mu^2 \mu_v + 2\lambda\mu\mu_v}{\mu^2\mu_v + 2\lambda\mu\mu_v + 2\lambda(\lambda\mu_v + \lambda_v\mu) + \mu^2\lambda_v}$$
$$= 1 / [\, 1 + \lambda_v / \mu_v + 2(\lambda/\mu)^2 / (1 + 2\lambda/\mu)] \approx 1 - \frac{\lambda_v}{\mu_v} - 2(\frac{\lambda}{\mu})^2 (1 - \frac{2\lambda}{\mu}). \quad (6.160)$$

As for the mean time to failure (Eq. (6.158)), the last part of Eq. (6.160) shows the influence of the series element E_v. For the asymptotic & steady-state value of the *interval reliability* one obtains (Table 6.2)

$$\mathrm{IR}_S(\theta) = P_0 \mathrm{R}_{S0}(\theta) + P_2 \mathrm{R}_{S2}(\theta) \approx P_0 \mathrm{R}_{S0}(\theta) \approx \mathrm{R}_{S0}(\theta). \quad (6.161)$$

Example 6.12

Give the reliability function and the asymptotic & steady-state value of the point and average availability for a 1-out-of-2 active redundancy in series with a switching element, as in Fig. 6.15, but *without repair priority* on the switching element.

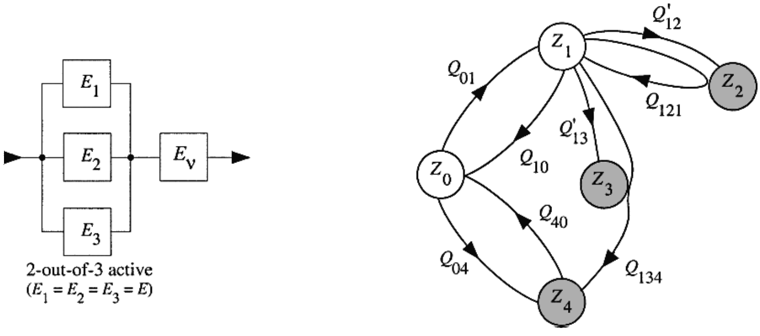


Figure 6.16 Reliability block diagram and state transition diagram for a 2-out-of-3 majority redundancy (constant failure rates λ for E and λ_v for E_v, repair time distributed according to G(x) with density g(x), ideal failure detection & switch, one repair crew, no repair priority, no further failures at system down, Z_2, Z_3, Z_4 down states, Z_0, Z_1, Z_4 constitute an embedded semi-Markov process)

Solution

The diagram of transition probabilities in $(t, t+\delta t]$ of Fig. 6.15 can be used by changing the transition from state Z_3 to state Z_2 to one from Z_3 to Z_1 and μ_v in μ. The reliability function is still given by Eq. (6.156), then states Z_1, Z_3, and Z_4 are absorbing states for reliability calculations. For the asymptotic & steady-state value of the point and average availability $PA_S = AA_S$, Eq. (6.159) is modified to

$$P_0 = \frac{(\mu_v P_1 + \mu P_2)}{2\lambda + \lambda_v}, \quad P_1 = \frac{\lambda_v}{\mu_v} P_0 + \frac{\mu}{\mu_v} P_3, \quad P_3 = \frac{\lambda_v}{\mu} P_2, \quad P_2 = \frac{\mu P_4 + 2\lambda P_0}{\lambda + \lambda_v + \mu}, \quad P_4 = \frac{\lambda}{\mu} P_2,$$

and the solution yields (considering $P_1 + ... + P_4 = 1$ and assuming $(3\lambda + \lambda_v) < \mu$ for the inequality)

$$PA_S = AA_S = \frac{1}{1 + \frac{\lambda_v}{\mu_v} + \frac{2\lambda(\lambda+\lambda_v)/\mu^2}{1+(2\lambda+\lambda_v)/\mu}} \approx 1 - \frac{\lambda_v}{\mu_v} - \frac{2\lambda(\lambda+\lambda_v)/\mu^2}{1+(2\lambda+\lambda_v)/\mu}$$

$$\approx 1 - \frac{\lambda_v}{\mu_v} - \frac{2\lambda^2}{\mu^2}(1 - \frac{2\lambda}{\mu}) - \frac{2\lambda\lambda_v}{\mu^2}(1 - \frac{3\lambda+\lambda_v}{\mu}) \le 1 - \frac{\lambda_v}{\mu_v}. \qquad (6.162)$$

Comparison of Eq. (6.160) with Eq. (6.162) shows the advantage ($\approx 2\lambda\lambda_v / \mu^2$) of the repair priority on E_v on the availability $PA_S = AA_S$.

As a second example let us consider a 2-out-of-3 *majority redundancy* (2-out-of-3 active redundancy in series with a voter E_v) with arbitrary repair rate Assumptions (6.1) - (6.7) also hold here, in particular (6.2), i.e. *no further failures at system down*. The system has constant failure rates, λ for the three redundant elements and λ_v for the series element E_v, and repair time distributed according to G(x) with G(0) = 0 and density g(x). Figure 6.16 shows the corresponding reliability diagram and the *state transition diagram*. Z_0 and Z_1 are up states. Z_0, Z_1

and Z_4 are *regeneration states* and constitute a *semi-Markov process embedded* in the original process. This property will be used for the investigations. From Fig. 6.16 and Table 6.2 there follows for the *semi-Markov transition probabilities* $Q_{01}(x)$, $Q_{10}(x)$, $Q_{04}(x)$, $Q_{40}(x)$, $Q_{121}(x)$, $Q_{134}(x)$ (similar as for Figs. A7.11 - A7.13)

$$Q_{01}(x) = \Pr\{\tau_{01} \le x \cap \tau_{04} > \tau_{01}\} = \int_0^x 3\lambda e^{-3\lambda y} e^{-\lambda_\nu y} dy = \frac{3\lambda(1 - e^{-(3\lambda+\lambda_\nu)x})}{3\lambda + \lambda_\nu}$$

$$Q_{10}(x) = \Pr\{\tau_{10} \le x \cap (\tau_{12} > \tau_{10} \cap \tau_{13} > \tau_{10})\} = \int_0^x g(y) e^{-(2\lambda+\lambda_\nu)y} dy$$

$$= G(x) e^{-(2\lambda+\lambda_\nu)x} + \int_0^x (2\lambda + \lambda_\nu) e^{-(2\lambda+\lambda_\nu)y} G(y) dy$$

$$Q_{121}(x) = \Pr\{\tau_{121} \le x\}$$

$$= \int_0^x g(y) \int_0^y 2\lambda e^{-(2\lambda+\lambda_\nu)z} dz\, dy = \int_0^x g(y) \frac{2\lambda}{2\lambda + \lambda_\nu} (1 - e^{-(2\lambda+\lambda_\nu)y}) dy$$

$$Q_{134}(x) = \Pr\{\tau_{134} \le x = \int_0^x g(y) \int_0^y \lambda_\nu e^{-(2\lambda+\lambda_\nu)z} dz\, dy = \frac{\lambda_\nu}{2\lambda} Q_{121}(x)$$

$$Q_{04}(x) = \Pr\{\tau_{04} \le x \cap \tau_{01} > \tau_{04}\} = \int_0^x \lambda_\nu e^{-(\lambda_\nu+3\lambda)y} dy = \frac{\lambda_\nu}{3\lambda} Q_{01}(x)$$

$$Q_{40}(x) = \Pr\{\tau_{40} \le x\} = G(x). \tag{6.163}$$

$Q_{121}(x)$ is used to calculate the *point availability*. It accounts for the process returning from state Z_2 to state Z_1 and that Z_2 is *not a regeneration state* (probability for the transition $Z_1 \to Z_2 \to Z_1$, see also Fig. A7.11a), similarly for $Q_{134}(x)$. $Q'_{12}(x)$ and $Q'_{13}(x)$ as given in Fig 6.16 are *not* semi-Markov transition probabilities (Z_2 and Z_3 are *not regeneration states*). However,

$$Q'_{12}(x) = \Pr\{\tau_{12} \le x \cap (\tau_{13} > \tau_{12} \cap \tau_{10} > \tau_{12})\} = \int_0^x 2\lambda e^{-2\lambda y} e^{-\lambda_\nu y} (1 - G(y)) dy$$

$$Q'_{13}(x) = \Pr\{\tau_{13} \le x \cap (\tau_{12} > \tau_{13} \cap \tau_{10} > \tau_{13})\} = \frac{\lambda_\nu}{2\lambda} Q'_{12}(x)$$

yields an equivalent $Q_1(x) = Q_{10}(x) + Q'_{12}(x) + Q'_{13}(x)$ useful for the calculation of the *reliability function.* Considering that Z_0 and Z_1 are up states and *regeneration states*, as well as the above expressions, the following system of integral equations can be established for the *reliability functions* $R_{S0}(t)$ & $R_{S1}(t)$, as per Eq. (A7.172),

$$R_{S0}(t) = e^{-(3\lambda+\lambda_\nu)t} + \int_0^t 3\lambda e^{-(3\lambda+\lambda_\nu)x} R_{S1}(t-x)\, dx$$

$$R_{S1}(t) = e^{-(2\lambda+\lambda_\nu)t}(1 - G(t)) + \int_0^t g(x) e^{-(2\lambda+\lambda_\nu)x} R_{S0}(t-x)\, dx. \tag{6.164}$$

The system of equations (6.164) for $R_{S0}(t)$ & $R_{S1}(t)$ has a great intuitive appeal and could have been written without the use of $Q_{ij}(x)$. Its solution yields

$$\tilde{R}_{S0}(s) = \frac{s + 5\lambda + \lambda_v - 3\lambda\tilde{g}(s + 2\lambda + \lambda_v)}{(s + 2\lambda + \lambda_v)[s + \lambda_v + 3\lambda(1 - \tilde{g}(s + 2\lambda + \lambda_v))]} \tag{6.165}$$

and

$$MTTF_{S0} = \frac{5\lambda + \lambda_v - 3\lambda\tilde{g}(2\lambda + \lambda_v)}{(2\lambda + \lambda_v)[\lambda_v + 3\lambda(1 - \tilde{g}(2\lambda + \lambda_v))]} . \tag{6.166}$$

$\tilde{R}_{S0}(s)$ and $MTTF_{S0}$ could have been obtained as for Eq. (6.157) by setting $s = s + \lambda_v$ in Eq (6.151). For the *point availability*, calculation of the transition probabilities $P_{ij}(t)$ with Table 6.2 (or Eq. (A7.169) and Eq. (6.163) leads to

$$P_{00}(t) = e^{-(3\lambda+\lambda_v)t} + \int_0^t 3\lambda e^{-(3\lambda+\lambda_v)x} P_{10}(t-x)dx + \int_0^t \lambda_v e^{-(3\lambda+\lambda_v)x} P_{40}(t-x)dx$$

$$P_{10}(t) = \int_0^t g(x)e^{-(2\lambda+\lambda_v)x} P_{00}(t-x)dx$$

$$+\int_0^t \frac{2\lambda}{2\lambda+\lambda_v}(1 - e^{-(2\lambda+\lambda_v)x})g(x)P_{10}(t-x)dx$$

$$+\int_0^t \frac{\lambda_v}{2\lambda+\lambda_v}(1 - e^{-(2\lambda+\lambda_v)x})g(x)P_{40}(t-x)dx$$

$$P_{40}(t) = \int_0^t g(x)P_{00}(t-x)dx, \tag{6.167}$$

and

$$P_{01}(t) = \int_0^t 3\lambda e^{-(3\lambda+\lambda_v)x} P_{11}(t-x)dx + \int_0^t \lambda_v e^{-(3\lambda+\lambda_v)x} P_{41}(t-x)dx$$

$$P_{11}(t) = e^{-(2\lambda+\lambda_v)t}(1 - G(t)) + \int_0^t g(x)e^{-(2\lambda+\lambda_v)x} P_{01}(t-x)dx$$

$$+\int_0^t \frac{1}{2\lambda+\lambda_v}(1 - e^{-(2\lambda+\lambda_v)x})g(x)[2\lambda P_{11}(t-x) + \lambda_v P_{41}(t-x)]dx$$

$$P_{41}(t) = \int_0^t g(x)P_{01}(t-x)dx. \tag{6.168}$$

From Eqs. (6.167) and (6.168) it follows the *point availability* $PA_{S0}(t) = P_{00}(t) + P_{01}(t)$and from this (using Laplace transform) the asymptotic & steady-state *value*

$$PA_S = AA_S = \frac{2\lambda + \lambda_\nu + \lambda(1 - \tilde{g}(2\lambda + \lambda_\nu))}{(2\lambda + \lambda_\nu)[1 + (3\lambda + \lambda_\nu)MTTR] + \lambda(\lambda_\nu MTTR - 2)(1 - \tilde{g}(2\lambda + \lambda_\nu))}, \tag{6.169}$$

with $MTTR$ as per Eq. (6.111). For the asymptotic & steady-state value of the *interval reliability*, the following *approximate expression* can be used for practical applications (Eq. (6.112))

$$\mathrm{IR}_S(\theta) \approx P_0 \mathrm{R}_{S0}(\theta) = \frac{[(2\lambda + \lambda_\nu) - 2\lambda(1 - \tilde{g}(2\lambda + \lambda_\nu))]\mathrm{R}_{S0}(\theta)}{(2\lambda + \lambda_\nu)[1 + (3\lambda + \lambda_\nu)MTTR] + \lambda(\lambda_\nu MTTR - 2)(1 - \tilde{g}(2\lambda + \lambda_\nu))}. \tag{6.170}$$

In Eq. (6.170) it holds that $P_0 = \lim_{t\to\infty} \mathrm{P}_{00}(t)$, with $\mathrm{P}_{00}(t)$ from Eqs. (6.167). For $\tilde{g}(2\lambda + \lambda_\nu) \approx 1$, $\mathrm{IR}_S(\theta) \approx \mathrm{R}_{S0}(\theta)$ can be used.

Example 6.13

(i) Give using Eqs. (6.166) and (6.169) the mean time to failure $MTTF_{S0}$ and the asymptotic & steady-state point and average availability $PA_S = AA_S$ for the case of a constant repair rate μ.
(ii) Compare for the case of constant repair rate the true value of the interval reliability $\mathrm{IR}_S(\theta)$ with the approximate expression given by Eq. (6.170).

Solution

(i) With $G(x) = 1 - e^{-\mu x}$ it follows that $\tilde{g}(2\lambda + \lambda_\nu) = \mu/(2\lambda + \lambda_\nu + \mu)$ and thus from Eq. (6.166)

$$MTTF_{S0} = \frac{5\lambda + \lambda_\nu + \mu}{(3\lambda + \lambda_\nu)(2\lambda + \lambda_\nu) + \mu\lambda_\nu} = \frac{1}{\lambda_\nu + 6\lambda^2 / (5\lambda + \lambda_\nu + \mu)} \approx \frac{1}{\lambda_\nu + 6\lambda^2/\mu} \le \frac{1}{\lambda_\nu}, \tag{6.171}$$

and from Eq. (6.169)

$$PA_S = AA_S = \frac{\mu(3\lambda + \lambda_\nu + \mu)}{(3\lambda + \lambda_\nu + \mu)(\lambda_\nu + \mu) + 3\lambda(2\lambda + \lambda_\nu)} = \frac{1}{1 + \frac{\lambda_\nu}{\mu} + \frac{3\lambda(2\lambda + \lambda_\nu)}{\mu(\mu + 3\lambda + \lambda_\nu)}}$$
$$\approx 1 - \frac{\lambda_\nu}{\mu} - \frac{3\lambda(2\lambda + \lambda_\nu)}{\mu^2}. \tag{6.172}$$

(ii) With $\mathrm{P}_{00}(t)$ and $\mathrm{P}_{01}(t)$ from Eqs. (6.167) & (6.168) it follows for the asymptotic & steady-state value of the interval reliability (Table 6.2) that

$$\mathrm{IR}_S(\theta) = \frac{\mu(\lambda_\nu + \mu)\mathrm{R}_{S0}(\theta) + 3\lambda\mu\mathrm{R}_{S1}(\theta)}{(3\lambda + \lambda_\nu + \mu)(\lambda_\nu + \mu) + 3\lambda(2\lambda + \lambda_\nu)}. \tag{6.173}$$

The *approximate expression* according to Eq. (6.170) yields

$$\mathrm{IR}_S(\theta) \approx \frac{\mu(\lambda_\nu + \mu)\mathrm{R}_{S0}(\theta)}{(3\lambda + \lambda_\nu + \mu)(\lambda_\nu + \mu) + 3\lambda(2\lambda + \lambda_\nu)},$$

i.e., the same value as per Eq. (6.173) for $3\lambda << \mu$ and considering $\mathrm{R}_{S1}(\theta) \le \mathrm{R}_{S0}(\theta)$.

To give a better feeling for the mutual influence of the different parameters involved, Figs. 6.17 and 6.18 compare the mean time to failure $MTTF_{S0}$ and the asymptotic & steady-state unavailability $1-PA_S$ of some basic *series - parallel structures*. The equations are taken from Table 6.10 which summarizes results of Sections 6.2 to 6.6 for constant failure and repair rates. Comparison with Figs. 2.8 & 2.9 (nonrepairable case) confirms that the most important gain is obtained by the first step (structure b), and shows that the influence of series elements is much greater in the repairable than in the nonrepairable case. Referring to the structures a), b), and c) of Figs. 6.17 and 6.18 the following *design rule* can be formulated:

The failure rate of the series element in a repairable 1-out-of-2 active redundancy should not be greater than 1% (0.2% for $\mu / \lambda_1 > 500$) of the failure rate of the redundant elements, i.e., with respect to Fig. 6.17,

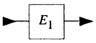
$$\lambda_2 < 0.01\lambda_1 \text{ in general, and } \lambda_2 < 0.002\lambda_1 \text{ for } \mu/\lambda_1 > 500. \qquad (6.174)$$

6.7 Approximate Expressions for Large Series - Parallel Structures

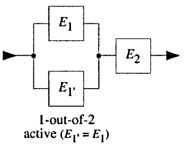
6.7.1 Introduction

Reliability and availability calculation of *large series - parallel structures* rapidly becomes time-consuming, even if *constant failure rate* λ_i and *repair rate* μ_i is assumed for each element E_i of the reliability block diagram and only mean time to failure $MTTF_{S0}$ or steady-state availability $PA_S = AA_S$ is required. This is because of the large number of states involved, which for a reliability block diagram with n elements can reach $1+\sum_{i=1}^{n}\prod_{k=n-i+1}^{n} k = n!\sum_{i=0}^{n} 1/i! \approx e \cdot n!$ by n different elements and repair as per first-in first-out (see e.g. Notes to Figs. 6.15 and 6.20). 2^n states holds for nonrepairable systems or for repairable system with totally independent elements (Point 1 below). Use of *approximate expressions* becomes thus important. Besides the assumption of *one repair crew and no further failure at system down* (Sections 6.2 - 6.6, partly 6.7 & 6.8), given below as Point 3, further assumptions yielding approximate expressions for system reliability and availability are possible for the case of constant failure rate λ_i and constant repair rate $\mu_i >> \lambda_i$ for each element E_i. Here some examples:

1. *Totally independent elements*: If each element of the reliability block diagram operates independently from every other (active redundancy, independent elements, one repair crew for each element), series - parallel structures can be reduced to one-item structures, which are themselves successively integrated into further series - parallel structures up to the system level. To each of the

a) 

$$MTTF_{S0a} = 1/\lambda_1, \qquad 1 - PA_{Sa} \approx \lambda_1/\mu$$

b) 

$$MTTF_{S0b} \approx \frac{\mu}{2\lambda_1^2 + \mu\lambda_2}, \qquad 1 - PA_{Sb} \approx \frac{\lambda_2}{\mu} + \frac{2(\lambda_1/\mu)^2}{1 + 2\lambda_1/\mu}$$

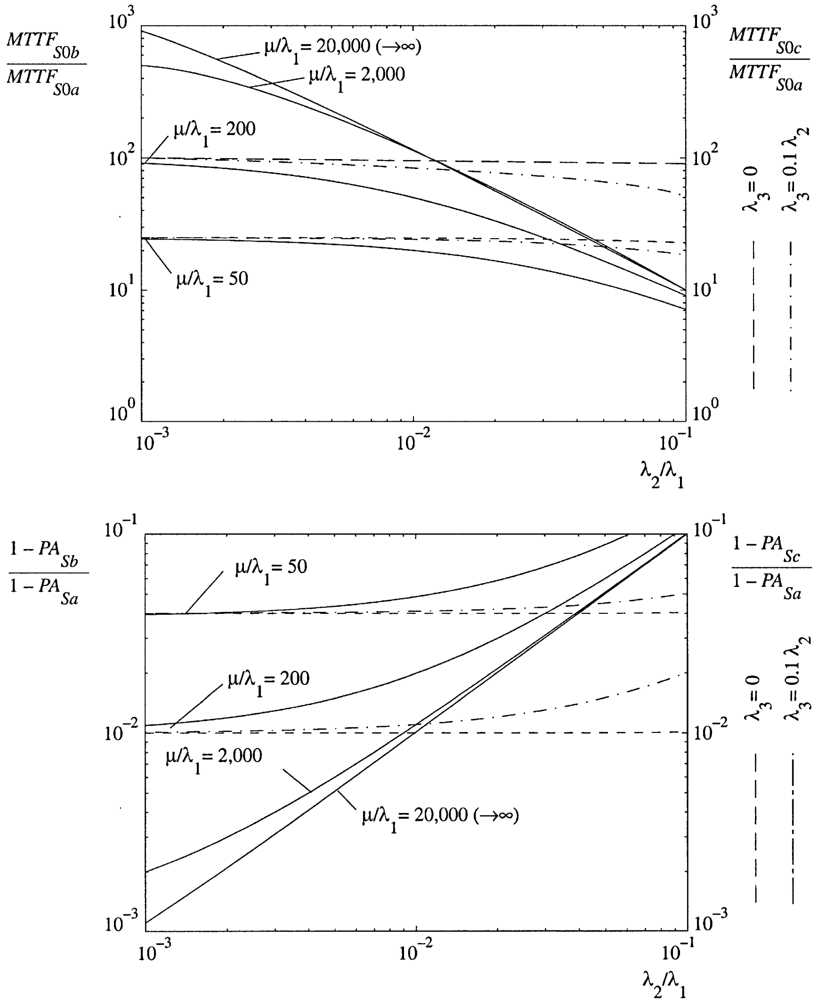
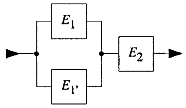
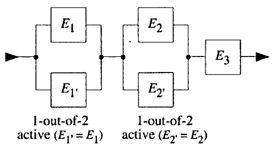


Figure 6.17 Comparison between a one-item structure and a 1-out-of-2 active redundancy with a series element (constant failure & repair rates ($\lambda_1, \lambda_2, \mu$), ideal failure detection & switch, one repair crew, repair priority on E_2, no further failure at system down, Markov process; λ_1 remains the same in both structures; Eqs. according Table 6.10; on the right, $MTTF_{S0c} / MTTF_{S0a}$ and $(1 - PA_{Sc})/(1 - PA_{Sa})$ with $MTTF_{S0c}$ and $1 - PA_{Sc}$ from Fig. 6.18; see also Fig. 2.8)

b) 

$$MTTF_{S0b} \approx \frac{\mu}{2\lambda_1^2 + \mu\lambda_2}, \qquad 1 - PA_{Sb} \approx \frac{\lambda_2}{\mu} + \frac{2(\lambda_1/\mu)^2}{1+2\lambda_1/\mu}$$

c) 

$$MTTF_{S0c} \approx \frac{\mu}{2\lambda_1^2 + 2\lambda_2^2 + \mu\lambda_3}, \qquad 1 - PA_{Sc} \approx \frac{2\lambda_2^2}{\mu^2} + \frac{\lambda_3}{\mu} + \frac{2(\lambda_1/\mu)^2}{1+2\lambda_1/\mu}$$

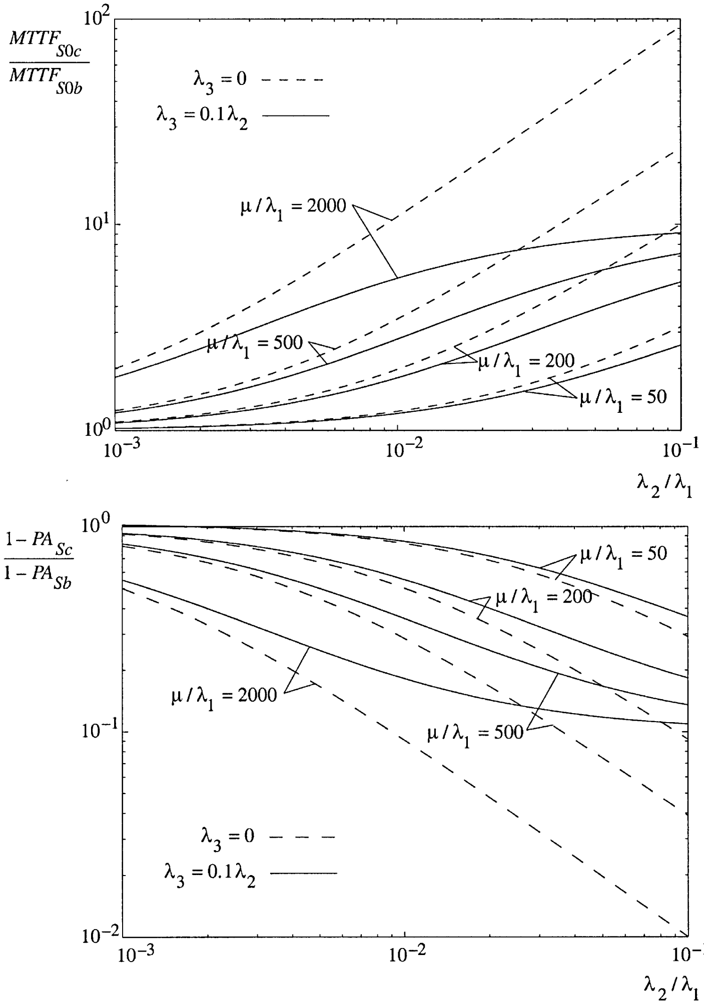


Figure 6.18 Comparison between basic *series - parallel structures* (active redundancy, constant failure & repair rates ($\lambda_1, \lambda_2, \lambda_3, \mu$), ideal failure detection & switch, one repair crew, repair priority on E_3, no further failure at system down, Markov process; λ_1 and λ_2 remain the same in both structures; equations according to Table 6.10; see also Fig. 6.17 and Fig. 2.9)

one-item structure obtained, the mean time to failure $MTTF_{S0}$ and steady-state availability PA_S, calculated for the underlying series-parallel structure, are used to calculate an equivalent $MTTR_S$ from $PA_S = MTTF_S / (MTTF_S + MTTR_S)$ using $MTTF_S = MTTF_{S0}$. To simplify calculations, and considering the comment given to Eq. (6.94), p. 197, *constant failure rate* $\lambda_S = 1 / MTTF_{S0}$ and *constant repair rate* $\mu_S = 1 / MTTR_S$ are assumed for each of the one-item structures obtained. Table 6.9 (p. 230) summarizes basic series-parallel structures based on totally independent elements (see Section 6.7.2 for an example). +)

2. *Macro-structures.* A macro-structure is a series, parallel, or simple series-parallel structure which is considered as a one-item structure for calculations at higher levels (integrated into further macro-structures up to system level) [6.5 (1991)]. It satisfies Assumptions (6.1) - (6.7), in particular one repair crew for each macro-structure and no further failures during a repair at the macro-structure level. The procedure is similar to that of point 1 above (see also the remarks to Eqs. (4.37) and (6.94)). Table 6.10 (p. 231) summarizes basic macro-structures (investigated in Sections 6.2 - 6.6) useful for practical applications, see Section 6.7.2 for an example. +)

3. *One repair crew and no further failures at system down*: Assumptions (6.3) and (6.2), valid for all models investigated in Sections 6.3 - 6.6, applies in many practical applications. No further failures at system down means that failures during a repair at system level are neglected. This assumption has *no influence on the reliability function* at system level and its *influence on the availability is limited* if $\lambda_i << \mu_i$ can be assumed for each element E_i.

4. *Cutting states*: Removing the states with more than k failures from the diagram of transition probabilities in $(t, t+\delta t]$ (or the state transition diagram) produces in general an important reduction of the state diagram. The choice of k (often $k=2$) is based on the required precision. An upper bound of the error for the asymptotic & steady-state value of the point and average availability $PA_S = AA_S$ (based on the mapping of states with k failures at the system level in state Z_k of a birth & death process and using Eq. (A7.157) ($P_k \geq \sum_{i=k+1}^{n} P_i$) has been introduced in [2.50 (1992)].

5. *Clustering of states*: Grouping of elements in the reliability block diagram or of states in the diagram of transition probabilities in $(t, t+\delta t]$ produces in general an important reduction of the number of states in the state diagram.

Combination of the above methods is possible. In any case, *series elements must be grouped before every analysis* (see Section 6.3 and the second row of Table 6.10).

Considering that the steady-state probability for states with more than one failure decreases rapidly as the number of failures increases ($\sim \lambda / \mu$ for each failure, see e.g.

+) Methods 1 & 2 apply in particular for const. failure & repair rates for each element, yielding approximately *constant failure & repair rates* (λ_S, μ_S) for the reduced structure (Eqs. (6.88), (6.94)).

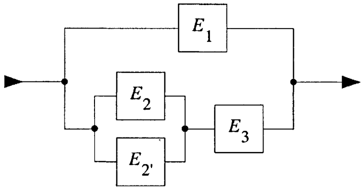


Figure 6.19 Basic reliability block diagram for an uninterruptible electrical power supply (UPS)

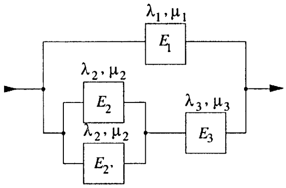
pp. 235 and 265 and the corresponding Figs. 6.20 and 6.34), all methods given above yield good *approximate expressions* for $MTTF_{S0}$ and PA_S in practical applications. However, referring to the *unavailability* $1 - PA_S$, method 1 above can deliver lower values, for instance a factor 2 with an order of magnitude $(\lambda / \mu)^2$ for a 1-out-of-2 active redundancy (compare Tables 6.9 & 6.10). Analytical comparison of the above methods is difficult, in general. Numerical investigations show a close convergence of the results given by the different methods, as illustrated for instance in Section 6.7.2 (p. 235) for a practical example with very low values for μ / λ.

6.7.2 Application to a Practical Example

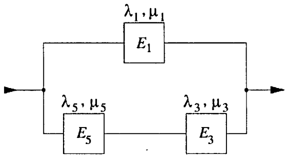
To illustrate how methods 1 to 3 of Section 6.7.1 work, let us consider the system with a reliability block diagram as in Fig. 6.19, and assume system new at $t = 0$, active redundancy, constant failure rates λ_1 to λ_3, constant repair rates μ_1 to μ_3, repair priority E_1, E_3, E_2 [6.5 (1988)]. Except for some series elements (to be considered separately in a final step), the reliability block diagram of Fig. 6.19 describes an uninterruptible power supply (UPS) used for instance to buffer electrical power network failures in computer systems (E_1 being the power network).+) Although limited to 4 elements, the stochastic process describing the system of Fig. 6.19 would contain up to 65 states (pp. 224, 233) if the assumption of *no further failure at system down* were dropped. Assuming no further failure at system down, the state space is reduced to 12 states (Fig. 6.20, p. 233). In the following, the mean time to failure ($MTTF_{S0}$) and the asymptotic & steady-state point and average availability ($PA_S = AA_S$) of the system given by Fig. 6.19 is investigated using method 1 (Table 6.9), method 2 (Table 6.10), and method 3 (Table 6.2) of Section 6.7.1. For a numerical comparison, results are given on p. 235 (also for method 4 and for the exact solution obtained by dropping the assumption of no further failure at system down), showing that all methods used deliver good approximate expressions.

+) A refinement to include the battery discharge has been investigated recently [6.47 (2002)].

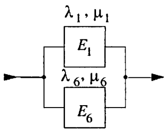
Method 1 of Section 6.7.1 yields, using Table 6.9,



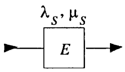
System



$$\lambda_5 \approx \frac{2\lambda_2^2}{\mu_2}, \qquad \mu_5 = 2\mu_2, \tag{6.175}$$



$$\lambda_6 \approx \lambda_3 + \lambda_5, \qquad \mu_6 \approx \frac{\lambda_5 + \lambda_3}{\lambda_5 / \mu_5 + \lambda_3 / \mu_3}, \tag{6.176}$$



$$\lambda_S \approx \frac{\lambda_1 \lambda_6 (\mu_1 + \mu_6)}{\mu_1 \mu_6}, \qquad \mu_S \approx \mu_1 + \mu_6 . \tag{6.177}$$

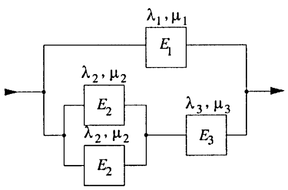
From Eqs. (6.175) – (6.177) it follows that

$$1 / MTTF_{S0} \equiv \lambda_S \approx \lambda_1 \left[\frac{\lambda_3}{\mu_1} + \frac{2\lambda_2^2}{\mu_1 \mu_2} + \frac{\lambda_3}{\mu_3} + \left(\frac{\lambda_2}{\mu_2}\right)^2 \right] \tag{6.178}$$

and

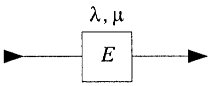
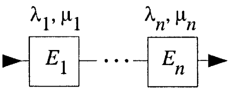
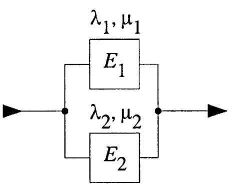
$$PA_S \approx 1 - \frac{\lambda_S}{\mu_S} \approx 1 - \frac{\lambda_1}{\mu_1} \left[\frac{\lambda_3}{\mu_3} + \left(\frac{\lambda_2}{\mu_2}\right)^2 \right]. \tag{6.179}$$

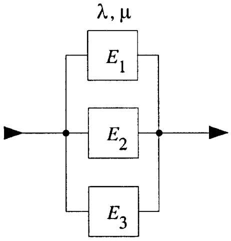
Method 2 of Section 6.7.1 yields, using Table 6.10,

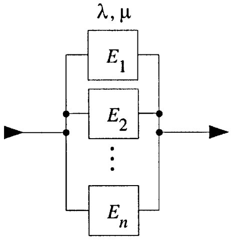


System

Table 6.9 Basic structures for the investigation of large series-parallel systems by assuming *totally independent elements* (each element operates and is repaired independently from every other element), constant failure & repair rates (λ, μ), active redundancy, ideal failure detection & switch, *n repair crews* (one for each element), Markov processes (for rows 1 to 5 see Eqs. (6.48), (2.48) & (6.60), (2.48) & (6.99), (2.48) & (6.171) with $\lambda_v = 0$, and (2.48) & (6.148), respectively; $\lambda_S = 1/MTTF_{S0}$ and $\mu_S = 1/MTTR_S \approx \lambda_S/(1-PA_S)$ are used to simplify the notation; approximations valid for $\lambda_i << \mu_i$; $PA_S = AA_S$ = asymptotic & steady-state point and average availability, often denoted by A)

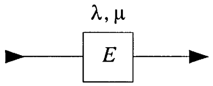
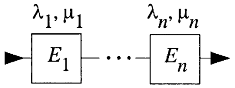
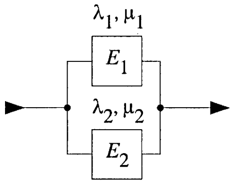
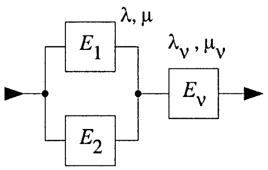
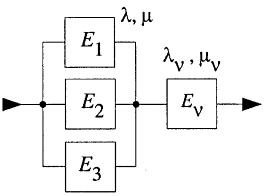
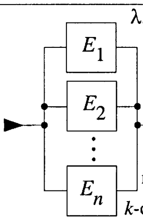
<table>
<tr>
<td>
</td>
<td>$\lambda_S = \lambda, \quad \mu_S = \mu, \quad PA_S = \frac{1}{1+\lambda_S/\mu_S} \approx 1 - \lambda_S/\mu_S$
$\Rightarrow \quad \mu_S = \frac{\lambda_S\, PA_S}{1-PA_S} \approx \frac{\lambda_S}{1-PA_S}$</td>
</tr>
<tr>
<td>
</td>
<td>$PA_S = PA_1 \dots PA_n = \frac{\mu_1}{\mu_1+\lambda_1} \cdots \frac{\mu_n}{\mu_n+\lambda_n} \approx 1 - (\frac{\lambda_1}{\mu_1} + \dots + \frac{\lambda_n}{\mu_n})$
$\lambda_S = \lambda_1 + \dots + \lambda_n \quad \Rightarrow \quad \mu_S \approx \frac{\lambda_S}{1-PA_S} \approx \frac{\lambda_1 + \dots + \lambda_n}{\lambda_1/\mu_1 + \dots + \lambda_n/\mu_n}$</td>
</tr>
<tr>
<td>

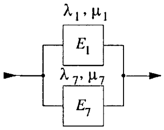
1-out-of-2 (active)</td>
<td>$PA_S = PA_1 + PA_2 - PA_1\,PA_2 = \frac{\mu_1\mu_2 + \mu_1\lambda_2 + \mu_2\lambda_1}{(\mu_1+\lambda_1)(\mu_2+\lambda_2)}^{+)} \approx 1 - \frac{\lambda_1\lambda_2}{\mu_1\mu_2}$
$\frac{1}{\lambda_S} \equiv MTTF_{S0} = \frac{(\lambda_1+\lambda_2+\mu_1)(\lambda_1+\lambda_2+\mu_2) - \lambda_1\lambda_2}{\lambda_1\lambda_2(\lambda_1+\lambda_2+\mu_1+\mu_2)}^{+)} \approx \frac{\mu_1\mu_2}{\lambda_1\lambda_2(\mu_1+\mu_2)}$
$\Rightarrow \quad \mu_S \approx \frac{\lambda_S}{1-PA_S} = \mu_1 + \mu_2$
+) Same result using Fig. A74b right hand</td>
</tr>
<tr>
<td>

2-out-of-3 active
($E_1 = E_2 = E_3 = E$)</td>
<td>$PA_S = 3PA^2 - 2PA^3 \approx 1 - \frac{3(\lambda/\mu)^2}{1+3\lambda/\mu} \approx 1 - 3(\frac{\lambda}{\mu})^2$
$1/\lambda_S \equiv MTTF_{S0} = \frac{5\lambda+\mu}{6\lambda^2} \approx \frac{\mu}{6\lambda^2}$
$\Rightarrow \quad \mu_S \approx \frac{\lambda_S}{1-PA_S} \approx 2\mu$</td>
</tr>
<tr>
<td>

k-out-of-n active
($E_1 = \dots = E_n = E$)</td>
<td>$PA_S \approx 1 - \frac{k}{n-k+1}\binom{n}{k}\left(\frac{\lambda}{\mu}\right)^{n-k+1}$
$1/\lambda_S \equiv MTTF_{S0} \approx \frac{1}{k\lambda\binom{n}{k}}\left(\frac{\mu}{\lambda}\right)^{n-k}$
$\Rightarrow \quad \mu_S \approx \frac{\lambda_S}{1-PA_S} \approx (n-k+1)\,\mu$</td>
</tr>
</table>

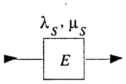
Table 6.10 Basic *macro-structures* for the investigation of *large* series-parallel systems by successive building of *macro-structures* bottom up to system level, const. failure & repair rates (λ, μ), active red., ideal failure detection & switch, one repair crew for each macro-structure, repair priority on E_ν, no further failure at system down, Markov proc. (for rows 1-6 see Eqs. (6.48), (6.65) & (6.60), (6.103) & (6.99), (6.160) & (6.158), (6.65) & (6.60) & Tab. 6.8, and as for row 5, resp.; $\lambda_S = 1/MTTF_{S0}$ and $\mu_S = 1/MTTR_S \approx \lambda_S/(1-PA_S)$ are used to simplify the notation; approximations valid for $\lambda_i << \mu_i$)

	$\lambda_S \equiv \lambda, \quad \mu_S \equiv \mu, \quad PA_S = 1/(1+\lambda_S/\mu_S) \approx 1 - \lambda_S/\mu_S$ $\Rightarrow \quad \mu_S = \frac{\lambda_S \, PA_S}{1 - PA_S} \approx \frac{\lambda_S}{1 - PA_S}$
	$PA_S \approx 1 - (\lambda_1/\mu_1 + \ldots + \lambda_n/\mu_n); \qquad \lambda_S = \lambda_1 + \ldots + \lambda_n$ $\Rightarrow \quad \mu_S \approx \frac{\lambda_S}{1 - PA_S} \approx \frac{\lambda_1 + \ldots + \lambda_n}{\lambda_1/\mu_1 + \ldots + \lambda_n/\mu_n}$ $\quad (\approx \mu \text{ for } \mu_1 = \ldots = \mu_n = \mu)$
 1-out-of-2 (active)	$PA_S \approx 1 - \frac{\lambda_1 \lambda_2}{\mu_1^2 \mu_2^2}(\mu_1^2 + \mu_2^2)$ $1/\lambda_S \equiv MTTF_{S0} \approx \mu_1 \mu_2 / (\lambda_1 \lambda_2 (\mu_1 + \mu_2))$ $\Rightarrow \quad \mu_S \approx \frac{\lambda_S}{1 - PA_S} \approx \mu_1 \mu_2 \frac{\mu_1 + \mu_2}{\mu_1^2 + \mu_2^2}$ $\quad (\approx \mu \text{ for } \mu_1 = \mu_2)$
 1-out-of-2 active ($E_1 = E_2 = E$) repair priority on E_ν	$PA_S \approx 1 - \frac{\lambda_\nu}{\mu_\nu} - \frac{2(\lambda/\mu)^2}{1 + 2\lambda/\mu}$ $1/\lambda_S \equiv MTTF_{S0} = 1/(\lambda_\nu + 2\lambda^2/(\mu + 3\lambda + \lambda_\nu)) \approx 1/(\lambda_\nu + 2\lambda^2/\mu)$ $\Rightarrow \quad \mu_S \approx \frac{\lambda_S}{1 - PA_S}$ $\quad (\approx \mu_\nu \text{ for } \mu_\nu = \mu)$
 2-out-of-3 active ($E_1 = E_2 = E_3 = E$) repair priority on E_ν	$PA_S \approx 1 - \frac{\lambda_\nu}{\mu_\nu} - \frac{6(\lambda/\mu)^2}{1 + 3\lambda/\mu}$ $1/\lambda_S \equiv MTTF_{S0} \approx 1/(\lambda_\nu + 6\lambda^2/\mu)$ $\Rightarrow \mu_S \approx \frac{\lambda_S}{1 - PA_S} \approx \mu_\nu \frac{\lambda_\nu + 6\lambda^2/\mu}{\lambda_\nu + \frac{6\lambda^2/\mu}{1 + 3\lambda/\mu} \cdot \frac{\mu_\nu}{\mu}}$ $\quad (\approx \mu_\nu \text{ for } \mu_\nu = \mu)$
 repair priority E_ν k-out-of-n active ($E_1 = \ldots = E_n = E$)	$PA_S \approx 1 - \frac{\lambda_\nu}{\mu_\nu} - \frac{n!}{(k-1)!}\left(\frac{\lambda}{\mu}\right)^{n-k+1}$ $1/\lambda_S \equiv MTTF_{S0} \approx 1/\left(\lambda_\nu + \lambda \frac{n!}{(k-1)!}\left(\frac{\lambda}{\mu}\right)^{n-k}\right)$ $\Rightarrow \quad \mu_S \approx \frac{\lambda_S}{1 - PA_S}$ $\quad (\approx \mu_\nu \text{ for } \mu_\nu = \mu)$



$$\lambda_7 \approx \lambda_3 + 2\lambda_2^2 / \mu_2,$$

$$\mu_7 \approx \frac{\mu_3 (2\lambda_2^2 + \mu_2 \lambda_3)(1 + 2\lambda_2 / \mu_2)}{\mu_2 \lambda_3 + 2\lambda_2 \lambda_3 + 2\lambda_2^2 \mu_3 / \mu_2}, \tag{6.180}$$



$$\lambda_S \approx \frac{\lambda_1 \lambda_7 (\mu_1 + \mu_7)}{\mu_1 \mu_7}, \quad \mu_S \approx \mu_1 \mu_7 \frac{\mu_1 + \mu_7}{\mu_1^2 + \mu_7^2}. \tag{6.181}$$

From Eqs. (6.180) and (6.181) it follows that

$$1 / MTTF_{S0} \equiv \lambda_S \approx \lambda_1 \left(\frac{2\lambda_2^2 + \mu_2 \lambda_3}{\mu_1 \mu_2} + \frac{\mu_2 \lambda_3 + 2\lambda_2 \lambda_3 + 2\mu_3 \lambda_2^2 / \mu_2}{\mu_2 \mu_3 (1 + 2\lambda_2 / \mu_2)} \right) \tag{6.182}$$

and

$$PA_S \approx 1 - \frac{\lambda_S}{\mu_S} \approx 1 - \frac{2\lambda_2^2 + \mu_2 \lambda_3}{\mu_2} \left(\frac{\lambda_1}{\mu_1^2} + \frac{\lambda_1 (\mu_2 \lambda_3 + 2\lambda_2 \lambda_3 + 2\mu_3 \lambda_2^2 / \mu_2)^2}{(2\lambda_2^2 + \mu_2 \lambda_3)^2 (1 + 2\lambda_2 / \mu_2)^2 \mu_3^2} \right)$$

$$= 1 - \frac{\lambda_1}{\mu_1} \left(\frac{\lambda_3}{\mu_3} + \frac{2\lambda_2^2}{\mu_2 \mu_3} \right) \left(\frac{\mu_3}{\mu_1} + \frac{(\mu_2 \lambda_3 + 2\lambda_2 \lambda_3 + 2\mu_3 \lambda_2^2 / \mu_2)^2 \mu_1 / \mu_3}{(2\lambda_2^2 + \mu_2 \lambda_3)^2 (1 + 2\lambda_2 / \mu_2)^2} \right). \tag{6.183}$$

Method 3 of Section 6.7.1 yields, using Table 6.2 and Fig. 6.20, the following system of algebraic equations for the *mean time to failure* ($M_i = MTTF_{Si}$)

$$\begin{aligned}
&\rho_0 M_0 = 1 + \lambda_1 M_1 + 2\lambda_2 M_2 + \lambda_3 M_3, && \rho_1 M_1 = 1 + \mu_1 M_0 + 2\lambda_2 M_7, \\
&\rho_2 M_2 = 1 + \mu_2 M_0 + \lambda_3 M_4 + \lambda_2 M_6 + \lambda_1 M_7, && \rho_3 M_3 = 1 + \mu_3 M_0 + 2\lambda_2 M_4, \\
&\rho_4 M_4 = 1 + \mu_3 M_2 + \lambda_2 M_5, && \rho_5 M_5 = 1 + \mu_3 M_6, \\
&\rho_6 M_6 = 1 + \mu_2 M_2 + \lambda_3 M_5, && \rho_7 M_7 = 1 + \mu_1 M_2,
\end{aligned} \tag{6.184}$$

where

$$\begin{aligned}
&\rho_0 = \lambda_1 + 2\lambda_2 + \lambda_3, && \rho_1 = \mu_1 + 2\lambda_2 + \lambda_3, && \rho_2 = \mu_2 + \lambda_1 + \lambda_2 + \lambda_3, \\
&\rho_3 = \mu_3 + 2\lambda_2 + \lambda_1, && \rho_4 = \mu_3 + \lambda_2 + \lambda_1, && \rho_5 = \mu_3 + \lambda_1, \\
&\rho_6 = \mu_2 + \lambda_3 + \lambda_1, && \rho_7 = \mu_1 + \lambda_3 + \lambda_2, && \rho_8 = \mu_1, \\
&\rho_9 = \mu_1, && \rho_{10} = \mu_1, && \rho_{11} = \mu_1.
\end{aligned} \tag{6.185}$$

From Eqs. (6.184) and (6.185) it follows that

$$1 / \lambda_S \equiv MTTF_{S0} = \frac{a_5 + a_6(a_8 + a_9 a_{10}) + a_7 a_{10}}{1 - a_6 a_{12} - a_{11}(a_7 + a_6 a_9)}, \tag{6.186}$$

with

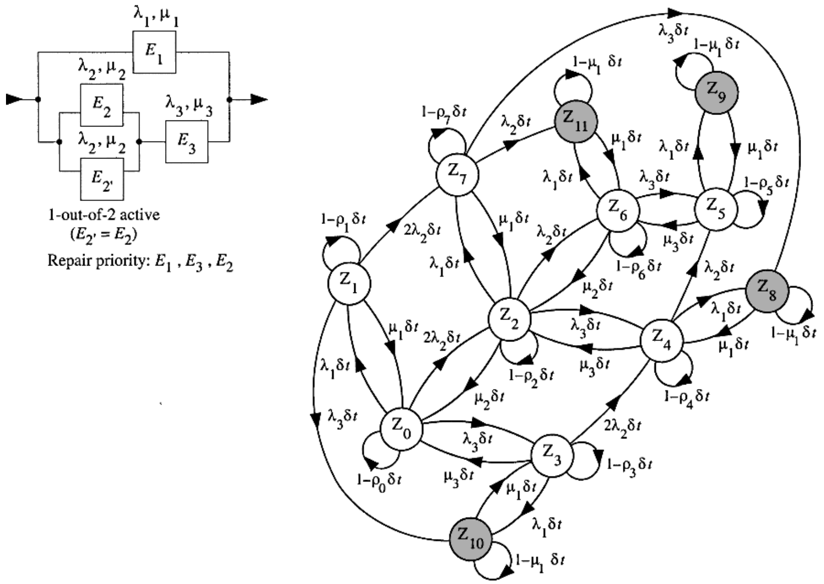
$$a_1 = \frac{1}{\rho_4} + \frac{\lambda_2}{\rho_4 \rho_5}(1 + \mu_3 \frac{\lambda_3 + \rho_5}{\rho_5 \rho_6 - \lambda_3 \mu_3}), \qquad a_2 = \frac{\lambda_2 \mu_2 \mu_3}{\rho_4 (\rho_5 \rho_6 - \lambda_3 \mu_3)} + \frac{\mu_3}{\rho_4},$$

$$a_3 = \frac{1}{\rho_3}(1 + 2\lambda_2 a_1), \qquad a_4 = \frac{2\lambda_2}{\rho_3} a_2, \qquad a_5 = \frac{1 + \lambda_3 a_3}{\rho_0 - \lambda_3 \mu_3 / \rho_3},$$

$$a_6 = \frac{\lambda_1}{\rho_0 - \lambda_3 \mu_3 / \rho_3}, \qquad a_7 = \frac{2\lambda_2 + \lambda_3 a_4}{\rho_0 - \lambda_3 \mu_3 / \rho_3}, \qquad a_8 = \frac{1 + 2\lambda_2 / \rho_7}{\rho_1},$$

$$a_9 = \frac{2\lambda_2 \mu_1}{\rho_1 \rho_7}, \qquad a_{10} = \frac{1 + \lambda_3 a_1 + (\lambda_2 \lambda_3 + \lambda_2 \rho_5)/(\rho_5 \rho_6 - \lambda_3 \mu_3) + \lambda_1 / \rho_7}{\rho_2 - \lambda_3 a_2 - \lambda_2 \mu_2 \rho_5 / (\rho_5 \rho_6 - \lambda_3 \mu_3) - \lambda_1 \mu_1 / \rho_7},$$

$$a_{11} = \frac{\mu_2}{\rho_2 - \lambda_3 a_2 - \lambda_2 \mu_2 \rho_5 / (\rho_5 \rho_6 - \lambda_3 \mu_3) - \lambda_1 \mu_1 / \rho_7}, \qquad a_{12} = \frac{\mu_1}{\rho_1}. \qquad (6.187)$$



$\rho_{01}=\lambda_1$; $\rho_{02}=2\lambda_2$; $\rho_{03}=\lambda_3$; $\rho_{10}=\mu_1$; $\rho_{17}=2\lambda_2$; $\rho_{1\,10}=\lambda_3$; $\rho_{20}=\mu_2$;
$\rho_{24}=\lambda_3$; $\rho_{26}=\lambda_2$; $\rho_{27}=\lambda_1$; $\rho_{30}=\mu_3$; $\rho_{34}=2\lambda_2$; $\rho_{3\,10}=\lambda_1$; $\rho_{42}=\mu_3$;
$\rho_{45}=\lambda_2$; $\rho_{48}=\lambda_1$; $\rho_{56}=\mu_3$; $\rho_{59}=\lambda_1$; $\rho_{62}=\mu_2$; $\rho_{65}=\lambda_3$; $\rho_{6\,11}=\lambda_1$;
$\rho_{72}=\mu_1$; $\rho_{78}=\lambda_3$; $\rho_{7\,11}=\lambda_2$; $\rho_{84}=\mu_1$; $\rho_{95}=\mu_1$; $\rho_{10\,3}=\mu_1$; $\rho_{11\,6}=\mu_1$

Figure 6.20 Reliability block diagram and diagram of transition probabilities in $(t, t+\delta t]$ for the system described by Fig. 6.19 (active redundancy, const. failure & repair rates (λ_i, μ_i), ideal failure detection & switch, one repair crew, repair priority in the sequence E_1, E_2, E_3, no further failures at system down, Z_8, Z_9, Z_{10}, Z_{11} down states, arbitrary t, $\delta t \downarrow 0$, Markov process, $\rho_i = \sum_j \rho_{ij}$)

Note: The diagram of transition probabilities would have 14 states for $E_2 \neq E_{2'}$, 16 states for totally independent elements, and 65 states for $E_2 \neq E_{2'}$, one repair crew & repair as per first-in first-out

Similarly, for the *asymptotic & steady-state* value of the point and average availability $PA_S = AA_S$ the following system of algebraic equations, can be obtained using Table 6.2 and Fig. 6.20

$$\begin{aligned}
&\rho_0 P_0 = \mu_1 P_1 + \mu_2 P_2 + \mu_3 P_3, && \rho_1 P_1 = \lambda_1 P_0,\\
&\rho_2 P_2 = 2\lambda_2 P_0 + \mu_3 P_4 + \mu_2 P_6 + \mu_1 P_7, && \rho_3 P_3 = \lambda_3 P_0 + \mu_1 P_{10},\\
&\rho_4 P_4 = \lambda_3 P_2 + 2\lambda_2 P_3 + \mu_1 P_8, && \rho_5 P_5 = \lambda_2 P_4 + \lambda_3 P_6 + \mu_1 P_9,\\
&\rho_6 P_6 = \lambda_2 P_2 + \mu_3 P_5 + \mu_1 P_{11}, && \rho_7 P_7 = 2\lambda_2 P_1 + \lambda_1 P_2,\\
&\rho_8 P_8 = \lambda_1 P_4 + \lambda_3 P_7, && \rho_9 P_9 = \lambda_1 P_5,\\
&\rho_{10} P_{10} = \lambda_3 P_1 + \lambda_1 P_3, && \rho_{11} P_{11} = \lambda_1 P_6 + \lambda_2 P_7.
\end{aligned} \tag{6.188}$$

with ρ_i as in Eq. (6.185). One (arbitrarily chosen) of the Eqs. (6.188) must be dropped and replaced by $P_0 + P_1 + \ldots + P_{11} = 1$. The solution yields P_0 to P_{11}, from which

$$PA_S = P_0(1 + b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7), \tag{6.189}$$

with

$$P_0 = 1 / (1 + \sum_{i=1}^{11} b_i) \tag{6.190}$$

and

$$\begin{aligned}
&b_1 = \frac{\lambda_1}{\rho_1}, \qquad b_2 = \frac{\rho_0 - \lambda_1 \mu_1 / \rho_1}{\mu_2} - \frac{\mu_3 \lambda_3 (1 + \lambda_1 / \rho_1)}{(\mu_3 + 2\lambda_2)\mu_2},\\
&b_3 = \frac{\lambda_3 (1 + \lambda_1 / \rho_1)}{\mu_3 + 2\lambda_2}, \qquad b_4 = \frac{\lambda_3 b_2 (1 + \lambda_1 / \rho_7) + 2\lambda_2 b_3 + 2\lambda_1 \lambda_2 \lambda_3 / \rho_7 \rho_1}{\rho_4 - \lambda_1},\\
&b_5 = \frac{\lambda_2 b_4 + \dfrac{\lambda_2 \lambda_3}{\rho_7 (\mu_2 + \lambda_3)} (b_2 (\rho_7 + \lambda_1) + 2\lambda_1 \lambda_2 / \rho_1)}{\rho_5 - \lambda_1 - \mu_3 \lambda_3 / (\mu_2 + \lambda_3)}, \qquad b_7 = \frac{2\lambda_1 \lambda_2}{\rho_1 \rho_7} + \frac{\lambda_1}{\rho_7} b_2,\\
&b_6 = \frac{\lambda_2}{\mu_2 + \lambda_3} (b_2 + \frac{2\lambda_1 \lambda_2}{\rho_1 \rho_7} + \frac{\lambda_1}{\rho_7} b_2 + \frac{\mu_3}{\lambda_2} b_5), \qquad b_8 = \frac{\lambda_1}{\mu_1} b_4 + \frac{\lambda_3}{\mu_1} b_7,\\
&b_9 = \frac{\lambda_1}{\mu_1} b_5, \qquad b_{10} = \frac{\lambda_3 \lambda_1}{\mu_1 \rho_1} + \frac{\lambda_1}{\mu_1} b_3, \qquad b_{11} = \frac{\lambda_1}{\mu_1} b_6 + \frac{\lambda_2}{\mu_1} b_7.
\end{aligned} \tag{6.191}$$

An analytical comparison of Eqs. (6.186) with Eqs. (6.178) and (6.182) or of Eq. (6.189) with Eqs. (6.179) and (6.183) is time consuming. Numerical evaluation yields (λ and μ in h^{-1}, *MTTF* in h)

λ_1	1/100	1/100	1/1,000	1/1,000
λ_2	1/1,000	1/1,000	1/10,000	1/10,000
λ_3	1/10,000	1/10,000	1/100,000	1/100,000
μ_1	1	1/5	1	1/5
μ_2	1/5	1/5	1/5	1/5
μ_3	1/5	1/5	1/5	1/5
$MTTF_{S0}$ (Eq. (6.178), totally IE)	$1.575 \cdot 10^{+5}$	$9.302 \cdot 10^{+4}$	$1.657 \cdot 10^{+7}$	$9.926 \cdot 10^{+6}$
$MTTF_{S0}$ (Eq. (6.182), MS)	$1.528 \cdot 10^{+5}$	$9.136 \cdot 10^{+4}$	$1.652 \cdot 10^{+7}$	$9.906 \cdot 10^{+6}$
$MTTF_{S0}$ (Eq. (6.186), no FF)	$1.589 \cdot 10^{+5}$	$9.332 \cdot 10^{+4}$	$1.658 \cdot 10^{+7}$	$9.927 \cdot 10^{+6}$
$MTTF_{S0}$ (Method 4, Cutting)	$1.487 \cdot 10^{+5}$	$9.294 \cdot 10^{+4}$	$1.645 \cdot 10^{+7}$	$9.917 \cdot 10^{+6}$
$MTTF_{S0}$ (only one repair crew)	$1.596 \cdot 10^{+5}$	$9.327 \cdot 10^{+4}$	$1.657 \cdot 10^{+7}$	$9.922 \cdot 10^{+6}$
$1 - PA_S$ (Eq. (6.179), totally IE)	$5.250 \cdot 10^{-6}$	$2.625 \cdot 10^{-5}$	$5.025 \cdot 10^{-8}$	$2.513 \cdot 10^{-7}$
$1 - PA_S$ (Eq. (6.183), MS)	$2.806 \cdot 10^{-5}$	$5.446 \cdot 10^{-5}$	$2.621 \cdot 10^{-7}$	$5.045 \cdot 10^{-7}$
$1 - PA_S$ (Eq. (6.189), no FF)	$6.574 \cdot 10^{-6}$	$5.598 \cdot 10^{-5}$	$6.060 \cdot 10^{-8}$	$5.062 \cdot 10^{-7}$
$1 - PA_S$ (Method 4, Cutting)	$2.995 \cdot 10^{-5}$	$5.556 \cdot 10^{-5}$	$2.647 \cdot 10^{-7}$	$5.059 \cdot 10^{-7}$
$1 - PA_S$ (only one repair crew)	$6.574 \cdot 10^{-6}$	$5.627 \cdot 10^{-5}$	$6.061 \cdot 10^{-8}$	$5.062 \cdot 10^{-7}$

Also given in the above numerical comparison are the results obtained by method 4 of Section 6.7.1 (for a given precision of 10^{-8} on the unavailability $1 - PA_S$) and by dropping the assumption of no further failures at system down in method 3. These results confirm that for $\lambda_i << \mu_i$ good approximate expressions for practical applications can be obtained from all the methods presented in Section 6.7.1. The influence of λ_i / μ_i appears when comparing column 1 with column 2 and column 3 with column 4. The results obtained with method 1 of Section 6.7.1 (Eqs. (6.178) and (6.179)) give higher values for $MTTF_{S0}$ and PA_S than those obtained with method 2 (Eqs. (6.182) and (6.183)), because of the assumption that each element has its own repair crew (totally independent elements). Comparing the results form Eqs. (6.186) and (6.189) with those for the case in which the assumption of no further failures at system down is dropped (only one repair crew), shows (for this example) the small influence of this assumption on final results.

For indicative purpose and to support the validity of *approximate expressions*, the following are the state probabilities for the numerical example according to the first column above, obtained by solving (Eq. (6.188), i.e., with the assumption of *one repair crew and no further failure at system down* as per Fig. 6.20 [6.21]:

$P_0 = 0.98$, $P_1 = 0.98 \cdot 10^{-2}$, $P_2 = 0.99 \cdot 10^{-2}$, $P_3 = 0.49 \cdot 10^{-3}$, $P_4 = 0.98 \cdot 10^{-5}$, $P_5 = 0.74 \cdot 10^{-7}$,
$P_6 = 0.50 \cdot 10^{-4}$, $P_7 = 0.12 \cdot 10^{-3}$, $P_8 = 0.11 \cdot 10^{-6}$, $P_9 = 0.74 \cdot 10^{-9}$, $P_{10} = 0.59 \cdot 10^{-5}$, $P_{11} = 0.62 \cdot 10^{-6}$

(more exactly: $P_0 = 0.976499684018$, $P_0 + \ldots + P_7 = 0.9999933933087$, $P_8 + \ldots + P_{11} = 0.0000066066913$).

6.8 Systems with Complex Structure

Structures and models investigated in the previous sections of this chapter were based on the existence of a reliability block diagram and on some simplifying assumptions ((6.1) - (6.7)). In particular, elements with only two states (good/failed) and ideal fault coverage & switching. This was, so far, good to understand basic investigation methods and tools, see e.g. Figs. 6.9 & 6.10, Examples 6.8 & 6.9, Section 6.7.2, Table 6.2. However, in practical applications more complex situations can arise. This section uses tools developed in Appendix A7 (summarized in Table 6.2 for Markov & semi-Markov processes) to investigate *complex fault tolerant repairable systems* for cases in which a reliability block diagram does not exist or can not easily be found. Constant failure and, in general, also constant repair rates are assumed. On the basis of practical examples it is shown that working with the *diagram of transition probabilities* or a *time schedule,* problems occurring in practical applications can be solved on a *case-by-case basis*. To improve readability, the *diagram of transition probabilities in* $(t, t+\delta t]$ will be replaced in Sections 6.8 & 6.9 by the *diagram of transition rates, which considers* transition rates ρ_{ij} only, by omitting δt and $1-\rho_i \delta t$. Of course, new systems can provide a starting point for *new models*, and a large number of papers is known on this subject too.

After some general considerations, Section 6.8.2 deals with aspects of *preventive maintenance*. Sections 6.8.3 & 6.8.4 consider *imperfect switching* & *incomplete coverage*. Elements with more than two states or one failure mode are discussed in Section 6.8.5. Section 6.8.6 investigates fault tolerant reconfigurable systems by considering reconfiguration because of mission profile (phased-mission systems) or failure. For this last case, *reward and frequency / duration aspects* are involved in the analysis. Section 6.8 7 considers systems with *common cause failures*. Section 6.8.8 presents some basic considerations on network reliability, and Section 6.8.9 summarizes the procedure for modeling systems with complex structure. Alternative investigation methods (dynamic FTA, BDD, ETA, Petri nets, computer-aided analysis) are introduced in Section 6.9 and a Monte Carlo procedure, useful for *rare events* is given. As a general rule, modeling complex systems is a task which must be solved in close cooperation between project and reliability engineers.

6.8.1 General Considerations

In the context of this book, a structure is *complex* when the reliability block diagram either does *not exist* or cannot be reduced to a *series-parallel structure* with *independent elements* (p. 52). If the reliability block diagram exists, but not as series-parallel structure, reliability and availability analysis can be performed using *one or more* of the following assumptions (as in previous sections, *failure-free time* is used as a synonym for *failure-free operating time*, *repair* as a synonym for *restoration*):

1. For each element in the reliability block diagram, failure-free times and repair times are statistically *independent.*
2. Failure and repair rates of each element are *constant* (time independent).
3. Each element in the reliability block diagram has *constant failure rate.*
4. The flow of failures is a *Poisson process* (homogeneous or nonhomogeneous).
5. *No further failures* are considered (can occur) at *system down* (no FF).
6. Redundant elements are repaired on-line (no interruptions at system level).
7. After each repair, the repaired *element* is as-good-as-new.
8. After each repair, the *entire system* is as-good-as-new.
9. *Only one repair crew* is available, repair is started as soon as the repair crew is free (*first-in first-out*) or according to a given *repair priority.*
10. Totally independent elements, i.e., each element operates and is repaired *independently of every other element* (n repair crews for n elements).
11. Ideal failure detection (in particular no *hidden failures* or false alarms).
12. Failure-free & repair times are > 0 and continuous with *finite* mean & variance.
13. For each element, the mean time to repair is *much lower* than the mean time to failure ($MTTR_i << MTTF_i$).
14. Switches and *switching operations* are 100% reliable and have no influence on the reliability of the system.
15. *Preventive maintenance* is not considered.

A clear formulation of the assumptions stated is important to *fix the validity of the results* obtained. Often it is tacitly assumed that each element has only 2 states (good / failed), one *failure mode* (e.g. shorts or opens), and a time invariant required function (e.g. continuous operation of all elements). Elements with more than two states or one failure mode are discussed in Section 6.8.5 (see also Section 2.3.6 for the nonrepairable case). A time dependent operation and/or required function can be investigated when constant failure rate is assumed (Section 6.8.6.2).

The following is a brief discussion of the above assumptions. With assumptions 1 and 2, the time behavior of the system can be described by a (time-homogeneous) *Markov process* with a finite number of states. Equations can be established using the *diagram* of *transition probabilities in* $(t, t+\delta t]$ and Table 6.2. Difficulties can arise because of the *large number of states involved.* In such cases, a first possibility is to limit investigation to the calculation of the mean time to failure $MTTF_{S0}$ and the asymptotic & steady-state value of the point and average availability $PA_S = AA_S$, i.e., to the solution of *algebraic equations.* A second possibility is to use *approximate expressions* (Section 6.7) or special software tools (Section 6.9.6). Assumption 3 assures existence of a regenerative process. Assumption 4 often applies to systems with a large number of elements. As shown in Sections 6.3 - 6.6, assumption 5 simplifies calculation of the point availability and interval reliability. It has *no influence on the reliability function* and $MTTF_{S0}$, and can be used for approximate expressions when assumption 13 applies (see Section 6.7.2 for an example).

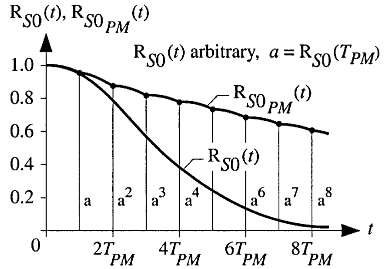
Assumption 6 must be met during the *system design*. If not satisfied, improvements given by redundancy are *questionable*; in such cases, at least *fault detection* and *localization* should be required and implemented (see Section 6.8.4). Assumptions 7 and 8 are satisfied if either assumption 2 or 3 holds. Assumption 7 is frequently used, its validity must be checked. Assumption 8 is rarely used (only with 2 or 3). Assumption 9 simplifies calculation and is useful for deriving *approximate expressions* (especially if assumption 13 holds; together with assumption 3, the system behavior can be described by a *semi-regenerative process* (embedded semi-Markov proc.). Assumption 3 alone assures that the involved process is *regenerative*. With assumption 10, point availability can be computed using the reliability equation for the nonrepairable case (Eqs. (2.47) & (2.48)). This assumption rarely applies in practical applications. However, it allows a simple calculation of an *upper bound* for the point availability. Assumption 13 is generally met. It leads to *approximate expressions*, as illustrated e.g. in Section 6.7 or by using asymptotic expansions, see e.g. [6.19, A7.26]. As shown in Examples 6.8 - 6.10, the shape of the distribution function of the repair time has small influence on the results at system level ($MTTF_{S0}, PA_S, \mathrm{IR}_S(\theta)$), if assumption 13 holds. Assumptions 14 and 15 simplify investigations, they are valid for *all models* discussed in Sections 6.2 - 6.7.

Investigation of *large series - parallel structures* or of *complex structures* is in general time-consuming and can become mathematically intractable. As a first step it is useful to operate with *Markov models*, refinements can then be considered on a *case-by-case basis* (see Section 6.8.9, pp. 273 - 275).

If the *reliability block diagram does not exists*, stochastic processes and tools introduced in Appendix A7 can be used to investigate reliability and availability of *fault tolerant systems*, on the basis of the *diagram of transition rates* or a *time schedule*, see Sections 6.8.3 - 6.8.7 for some examples on systems with imperfect switching, incomplete coverage, more than two states or one failure mode, reconfigurable structure, and common cause failures. A general procedure for investigating complex fault tolerant systems is given in Section 6.8.9. Alternative investigation methods (dynamic FTA, BDD, ETA, Petri nets, computer-aided analysis) are introduced in Section 6.9 and a Monte Carlo procedure, useful for *rare events* is given.

6.8.2 Preventive Maintenance

Preventive maintenance is necessary to avoid *wearout failures* and to identify and repair *hidden (undetected, latent) failures*, i.e., failures of redundant elements which cannot be detected during normal operation. This section investigates a *one-item repairable structure* with preventive maintenance at $T_{PM}, 2T_{PM}, \ldots$. Results are basic for the investigation of more complex structures and will be useful in the following sections to investigate fault tolerant repairable systems (Section 6.8.6). Further models / strategies for preventive maintenance are possible (see Section 4.6).



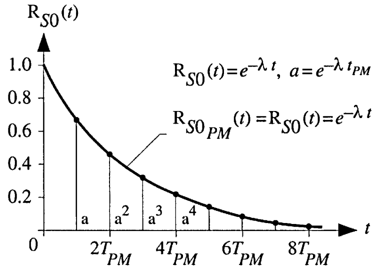


Figure 6.21 Reliability functions for a one-item structure with preventive maintenance (of negligible duration) at times T_{PM}, $2T_{PM}$, ... for two distribution functions $F(t) = 1 - R_{S0}(t)$ of the failure-free times (item new at $t = 0$, T_{PM}, $2T_{PM}$,...; left *strictly increasing*, right *constant* failure rate)

The item considered is new at $t = 0$ and has failure-free & repair times distributed according to $F(x)$ & $G(x)$ with densities $f(x)$ & $g(x)$ $(F(0)=G(0)=0)$. Preventive maintenance is of negligible time duration (specialized personnel, no logistic delays) and restores the item to *as-good-as-new*. If a preventive maintenance is due at a time in which the item is under repair, one of the following cases will apply:

1. Preventive maintenance will not be performed (included in the running repair, considering that after each repair the item is as-good-as-new).
2. Preventive maintenance is performed, i.e., a running repair is terminated with the preventive maintenance in a negligible time span (this maintenance strategy is known as *block replacement policy* (Section 4.6)).

Both situations can occur in practical applications. In case 2, times $0, T_{PM}, 2T_{PM}, ...$ are *renewal points*. Case 2 will be considered in the following.

The *reliability function* $R_{S0_{PM}}(t)$ for case 2 above can be calculated from

$$R_{S0_{PM}}(t) = R_{S0}(t) = 1 - F(t), \qquad \text{for } 0 < t \le T_{PM}, \; R_{S0_{PM}}(0) = R_{S0}(0) = 0,$$

$$R_{S0_{PM}}(t) = R^n_{S0}(T_{PM}) R_{S0}(t - n\,T_{PM}), \qquad \text{for } nT_{PM} < t \le (n+1)T_{PM}, \; n \ge 1, \quad (6.192)$$

with $R_{S0}(x) = 1 - F(x)$ (Eq. (6.14)). Figure 6.21 shows the shape of $R_{S0}(t)$ and $R_{S0_{PM}}(t)$ for an item with strictly increasing (left) and constant (right) failure rate. Because of the *memoryless property* of the exponential distribution function,

$$R_{S0_{PM}}(t) = R_{S0}(t) = e^{-\lambda t} \quad \text{holds for} \quad F(x) = 1 - e^{-\lambda x}. \qquad (6.193)$$

From Eq. (6.192), the mean time to failure with preventive maintenance $MTTF_{S0_{PM}}$ is

$$MTTF_{S0_{PM}} = \int_0^\infty R_{S0_{PM}}(t)dt = [1 + \sum_{n=1}^\infty R^n_{S0}(T_{PM})] \int_0^{T_{PM}} R_{S0}(t)dt$$
$$= \int_0^{T_{PM}} R_{S0}(t)dt \,/\, [1 - R_{S0}(T_{PM}) = \int_0^{T_{PM}} (1 - F(x))dx \,/\, F_{PM}(T). \qquad (6.194)$$

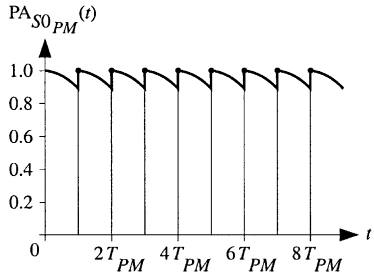


Figure 6.22 Point availability for a repairable one-item structure with preventive maintenance (of negligible duration) at times T_{PM}, $2T_{PM}$, ... (item new at $t = 0$, T_{PM}, $2T_{PM}$, ... and after each repair)

For $F(x) = 1 - e^{-\lambda x}$, Eq. (6.194) yields $MTTF_{S0_{PM}} = 1/\lambda = E[\tau]$. For a strictly increasing failure rate $\lambda(x)$ it holds that $MTTF_{S0_{PM}} > E[\tau]$; the contrary is for strictly decreasing $\lambda(x)$. To see this, consider that

$$\int_0^{T_{PM}} R_{S0}(t)\,dt = \int_0^{\infty} R_{S0}(t)\,dt - \int_{T_{PM}}^{\infty} R_{S0}(t)\,dt = E[\tau] - R_{S0}(T_{PM})E[\tau - T_{PM} \mid \tau > T_{PM}],$$

with τ as failure-free time of the item considered and $E[\tau - T_{PM} \mid \tau > T_{PM}]$ as per Eq. (A6.28); the rest of the proof follows from remark 2 to Eq. (A6.28). Optimization of *preventive maintenance period* must consider Eq. (6.194) as well as cost, logistic support, and other relevant aspects ($MTTF_{S0_{PM}} \to \infty$ for $T_{PM} \to 0$ and $f(+0) = 0$).

Calculation of the *point availability* $PA_{S0_{PM}}(t)$ for case 2 above leads to

$$PA_{S0_{PM}}(t) = PA_{S0}(t), \qquad \text{for } 0 \le t < T_{PM},$$

$$PA_{S0_{PM}}(t) = PA_{S0}(t - n\,T_{PM}), \qquad \text{for } nT_{PM} \le t < (n+1)T_{PM},\ n \ge 1, \tag{6.195}$$

with $PA_{S0}(t)$ from Eq. (6.17). Figure 6.22 shows a typical shape of $PA_{S0_{PM}}(t)$.

If the time duration for the preventive maintenance is not negligible, it is useful to define, in addition to the availability introduced in Section 6.2.1, the *overall* (or *operational*) *availability* OA_S, defined for $t \to \infty$ as the ratio of the total up time in $(0, t]$ to the sum of total up and down time in $(0, t]$. Defining $MTTF$ = mean time to failure and MDT = mean down time (with $MTTR$ = mean time to repair (restore), $MTTPM$ = mean time to carry out preventive maintenance, MLD = *mean logistic delay*, and T_{PM} = preventive maintenance period) it follows that (see e.g. p. 122)

$$OA_S = \frac{MTTF}{MTTF + MDT} = \frac{MTTF}{MTTF + MTTR + MLD + MTTPM(MTTF / T_{PM})}. \tag{6.196}$$

For $MLD = 0$, the *overall availability* is often called *technical availability*. Other availability measures are possible, e.g. as in [6.12] for railway applications.

Further maintenance strategies are investigated in Section 4.6. Distribution and mean of the *undetected (latent) fault time* τ_{UFT} is considered by Eq. (A6.30).

Example 6.14

Assume a nonrepairable (up to system failure) 1-out-of-2 active redundancy with two identical elements with constant failure rate λ. Give the mean time to failure $MTTF_{S0_{PM}}$ by assuming a preventive maintenance with period $T_{PM} << 1/\lambda$. The preventive maintenance is performed in a negligible time span and restores the 1-out-of-2 active redundancy to as-good-as-new.

Solution

For a nonrepairable (up to system failure) 1-out-of-2 active redundancy with two identical elements with constant failure rate λ, the reliability function is given by Eq. (2.21)

$$R_{S0}(t) = 2e^{-\lambda t} - e^{-2\lambda t}$$

The mean time to failure with preventive maintenance follows from Eq. (6.194) as

$$MTTF_{S0_{PM}} = \frac{\int_0^{T_{PM}} R_{S0}(t)dt}{1 - R_{S0}(T_{PM})} = \frac{\int_0^{T_{PM}} (2e^{-\lambda t} - e^{-2\lambda t})dt}{1 - 2e^{-\lambda T_{PM}} + e^{-2\lambda T_{PM}}} = \frac{\frac{2}{\lambda}(1 - e^{-\lambda T_{PM}}) - \frac{1}{2\lambda}(1 - e^{-2\lambda T_{PM}})}{1 - 2e^{-\lambda T_{PM}} + e^{-2\lambda T_{PM}}}.$$

Using $e^{-x} \approx 1 - x + x^2/2$ it follows that

$$MTTF_{S0_{PM}} \approx \frac{2T_{PM} - T_{PM}}{\lambda^2 T_{PM}^2} = \frac{1}{\lambda^2 T_{PM}} \qquad (= MTBF \cdot MTBF / T_{PM} \text{ for } MTBF = 1/\lambda). \qquad (6.197)$$

Without preventive maintenance, Eq. (2.21) yields $MTTF_{S0} = 3/2\lambda$. Equation (6.197) clearly shows the gain given by the preventive maintenance.

6.8.3 Imperfect Switching

In practical applications, *switching* is necessary for powering down failed elements and powering up repaired elements. In some cases it is sufficient to locate the *switching element* in series with the redundancy on the reliability block diagram, yielding series - parallel structures as investigated in Section 6.6. However, such an approach is often too simple to cover real situations. This section shows this on the basis of practical examples. Further considerations are given in Section 6.8.4 dealing with incomplete coverage.

As a *first example*, Fig. 6.23 shows a situation in which measurement points M_1 and M_2, switches S_1 and S_2, as well as a control unit C must be considered. To simplify, let us consider only the reliability function in the nonrepairable case (up to system failure). From a reliability point of view, switch S_i, element E_i, and measurement point M_i in Fig. 6.23 are in series $(i = 1, 2)$. Let τ_{b1} and τ_{b2} be the corresponding failure-free times with distribution function $F_b(x)$ and density $f_b(x)$. τ_c is the failure-free time of the control device with distribution function $F_c(x)$ and density $f_c(x)$. Consider first the case of *standby redundancy* and assume that at $t = 0$

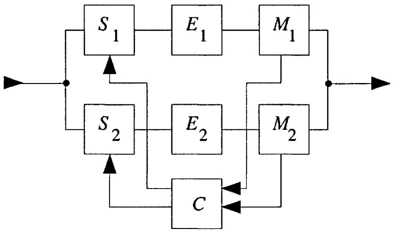


Figure 6.23 Functional block diagram for a 1-out-of-2 redundancy with switches S_1 and S_2, measurement points M_1 and M_2, and control device C

element E_1 is switched on. A system failure in the interval $(0, t]$ occurs with one of the following mutually exclusive events

$$\{\tau_c > \tau_{b1} \cap (\tau_{b1} + \tau_{b2}) \le t\} \qquad \text{or} \qquad \{\tau_c < \tau_{b1} \le t\}.$$

It is implicitly assumed here that a failure of the control device has no influence on the operating element, and does not lead to a commutation to E_2. A verification of these conditions by an *FMEA* (Section 2.6) is necessary. With these assumptions, the *reliability function* $R_{S0}(t)$ of the system described by Fig. 6.23 is given by (nonrepairable case, system new at $t = 0$)

$$R_{S0}(t) = 1 - [\int_0^t f_b(x)(1 - F_c(x))\, F_b(t - x)\, dx + \int_0^t f_b(x)\, F_c(x)\, dx]. \tag{6.198}$$

Assuming further $f_b(x) = \lambda_b\, e^{-\lambda_b x}$ and $f_c(x) = \lambda_c\, e^{-\lambda_c x}$, Eq. (6.198) yields

$$R_{S0}(t) = e^{-\lambda_b t} + (1 - e^{-\lambda_c t}) \frac{\lambda_b}{\lambda_c} e^{-\lambda_b t} \tag{6.199}$$

and

$$MTTF_{S0} = \frac{2\lambda_b + \lambda_c}{\lambda_b(\lambda_b + \lambda_c)}. \tag{6.200}$$

$\lambda_c \equiv 0$ leads to the results of Section 2.3.5 for the 1-out-of-2 standby redundancy (Eqs. (2.63), (2.64)). Assuming now an *active redundancy* (at $t = 0$, E_1 is put into operation and E_2 into the reserve state), a system failure occurs in the interval $(0, t]$ with one of the following *mutually exclusive events*

$$\{\tau_{b1} \le t \cap \tau_c > \tau_{b1} \cap \tau_{b2} \le t\} \qquad \text{or} \qquad \{\tau_c < \tau_{b1} \le t\}.$$

The r*eliability function* is then given by (nonrepairable case, system new at $t = 0$)

$$R_{S0}(t) = 1 - [F_b(t) \int_0^t f_b(x)(1 - F_c(x))\, dx + \int_0^t f_b(x)\, F_c(x)\, dx]. \tag{6.201}$$

From Eq.(6.201) and assuming $f_b(x) = \lambda_b e^{-\lambda_b x}$ and $f_c(x) = \lambda_c e^{-\lambda_c x}$ it follows that

$$R_{S0}(t) = \frac{2\lambda_b + \lambda_c}{\lambda_b + \lambda_c} e^{-\lambda_b t} - \frac{\lambda_b}{\lambda_b + \lambda_c} e^{-(2\lambda_b + \lambda_c)t} \tag{6.202}$$

and

$$MTTF_{S0} = \frac{2\lambda_b + \lambda_c}{\lambda_b(\lambda_b + \lambda_c)} - \frac{\lambda_b}{(\lambda_b + \lambda_c)(2\lambda_b + \lambda_c)} . \tag{6.203}$$

$\lambda_c \equiv 0$ leads to the results of Section 2.2.6.3 for the 1-out-of-2 active redundancy (Eq. (2.22)). From Eqs. (6.200) and (6.203) one recognizes that for $\lambda_c >> \lambda_b$

$$MTTF_{S0} \approx 1/\lambda_b, \qquad \text{for } \lambda_c >> \lambda_b , \tag{6.204}$$

for both standby and active redundancy, i.e., to a situation as *where no redundancy.*

As a *second example* consider a *1-out-of-2 warm redundancy* with constant failure rate λ, λ_r and repair rate μ. The switching element can fail with constant failure rate λ_σ and failure mode *stuck at the state occupied just before failure*. At first, let us consider the case in which the failure of the switch can be immediately detected and repaired with constant repair rate μ_σ. Furthermore, assume only one repair crew, *repair priority on the switch*, and no further failure at system down. Asked are the mean time to system failure $MTTF_{S0}$ for system new (state Z_0) at $t = 0$ and the asymptotic & steady-state (stationary) point and average availability $PA_S = AA_S$. The involved process is a (time-homogeneous) Markov process. Figure 6.24 give the *diagrams of transition rates* for reliability and availability calculation, respectively (down states $Z_2, Z_{2'}, Z_{2''}$). From Fig. 6.24a & Table 6.2 or Eq. (A7.126) it follows that $MTTF_{S0}$ is given as solution of the following system ($M_i \equiv MTTF_{Si}$)

$$\begin{aligned} \rho_0 M_0 &= 1 + \lambda_\sigma M_{0'} + (\lambda + \lambda_r) M_1 , & \rho_{0'} M_{0'} &= 1 + \lambda_r M_{1'} + \mu_\sigma M_0 , \\ \rho_1 M_1 &= 1 + \lambda_\sigma M_{1'} + \mu M_0 , & \rho_{1'} M_{1'} &= 1 + \mu_\sigma M_1 , \end{aligned} \tag{6.205}$$

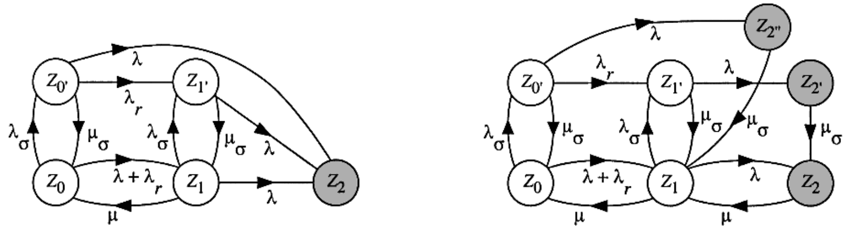
yielding

$$\begin{aligned} MTTF_{S0} &= \frac{(\rho_1\rho_{1'} - \lambda_\sigma\mu_\sigma)[\rho_{0'}\rho_{1'} + \lambda_\sigma(\rho_{1'} + \lambda_r)] + (\rho_{1'} + \lambda_\sigma)[\lambda_r\lambda_\sigma\mu_\sigma + (\lambda + \lambda_r)\rho_{0'}\rho_{1'}]}{(\rho_1\rho_{1'} - \lambda_\sigma\mu_\sigma)(\rho_0\rho_{0'}\rho_{1'} - \rho_{1'}\lambda_\sigma\mu_\sigma) - \mu\rho_{1'}[\lambda_r\lambda_\sigma\mu_\sigma + \rho_{0'}\rho_{1'}(\lambda + \lambda_r)]} \\ &\approx \frac{1 + (3\lambda + \lambda_r + \lambda_\sigma)/\mu_\sigma + (2\lambda + \lambda_r)/\mu}{\lambda\lambda_\sigma/\mu_\sigma + \lambda(\lambda + \lambda_r)/\mu} \approx \frac{\mu}{\lambda(\lambda + \lambda_r + \lambda_\sigma\mu/\mu_\sigma)} . \end{aligned} \tag{6.206}$$

The approximation assumes $\lambda, \lambda_\sigma << \mu, \mu_\sigma$. From this approximate expression it follows that the *effect of imperfect switching with failure mode stuck at the state occupied just before failure,* immediately detected and repaired, is minor and becomes negligible for (see Eqs. (6.212) and (6.239) for more severe conditions)

$$\lambda_\sigma \mu / \mu_\sigma << \lambda + \lambda_r, \qquad \text{i.e. } \lambda_\sigma << \lambda + \lambda_r \text{ for } \mu = \mu_\sigma >> \lambda, \lambda_r, \lambda_\sigma . \tag{6.207}$$

The case $\lambda_\sigma = 0$ implies $\mu_\sigma = 0$ and must be investigated using the exact expression for $MTTF_{S0}$, yielding $MTTF_{S0} = (\mu + 2\lambda + \lambda_r)/\lambda(\lambda + \lambda_r)$ as per Table 6.6.



a) For reliability

$\rho_{00'}=\rho_{11'}=\lambda_\sigma;\ \rho_{01}=\lambda+\lambda_r;\ \rho_{0'0}=\rho_{1'1}=\mu_\sigma;$
$\rho_{10}=\mu;\ \rho_{0'1'}=\lambda_r;\ \rho_{0'2}=\rho_{12}=\rho_{1'2}=\lambda$

$\rho_0=\lambda+\lambda_r+\lambda_\sigma;\ \rho_{0'}=\lambda+\lambda_r+\mu_\sigma$
$\rho_1=\lambda+\lambda_\sigma+\mu;\ \rho_{1'}=\lambda+\mu_\sigma;\ \rho_2=0$

b) For availability

$\rho_{00'}=\rho_{11'}=\lambda_\sigma;\ \rho_{01}=\lambda+\lambda_r;\ \rho_{10}=\rho_{21}=\mu;\ \rho_{0'1'}=\lambda_r;$
$\rho_{0'0}=\rho_{1'1}=\rho_{2'2}=\rho_{2''1}=\mu_\sigma;\ \rho_{0'2''}=\rho_{12}=\rho_{1'2'}=\lambda$

$\rho_0=\lambda+\lambda_r+\lambda_\sigma;\ \rho_{0'}=\lambda+\lambda_r+\mu_\sigma;\ \rho_1=\lambda+\lambda_\sigma+\mu;$
$\rho_{1'}=\lambda+\mu_\sigma;\ \rho_2=\mu;\ \rho_{2'}=\rho_{2''}=\mu_\sigma$

Figure 6.24 Diagram of transition rates for a repairable 1-out-of-2 warm redundancy with constant failure & repair rates $(\lambda, \lambda_r, \mu)$, imperfect switching (failure rate λ_σ, repair rate μ_σ, failure mode stuck at the state occupied), ideal failure detection, one repair crew, switch repaired with *repair priority*, no further failure at system down, (Z_2, $Z_{2'}$, $Z_{2''}$ down states, Markov process)

From Fig. 6.24b and Table 6.2 or Eq. (A7.127) it follows that $PA_S = AA_S$ is given as solution of the following system of algebraic equations

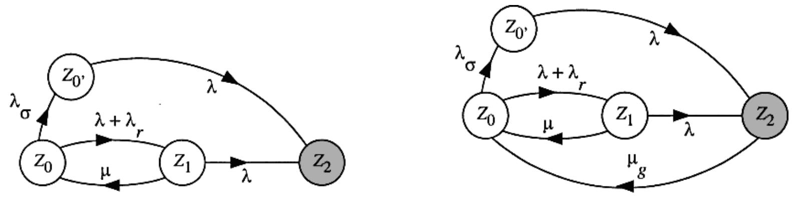
$$\rho_0 P_0=\mu_\sigma P_{0'}+\mu P_1,\quad \rho_{0'} P_{0'}=\lambda_\sigma P_0,\quad \rho_1 P_1=(\lambda+\lambda_r)P_0+\mu_\sigma P_{1'}+\mu P_2+\mu_\sigma P_{2''},$$
$$\rho_{1'} P_{1'}=\lambda_r P_{0'}+\lambda_\sigma P_1,\quad \rho_2 P_2=\lambda P_1+\mu_\sigma P_{2'},\quad \rho_{2'} P_{2'}=\lambda P_{1'},\quad \rho_{2''} P_{2''}=\lambda P_{0'}. \qquad (6.208)$$

One of the Eq. (6.208), arbitrarily chosen, must be replaced by $\sum P_i = 1$. The *asymptotic* & *steady-state* point and average availability follows then from

$$PA_S = AA_S = P_0 + P_{0'} + P_1 + P_{1'} =$$
$$\frac{1}{1+\dfrac{\rho_{1'}\lambda(\lambda+\lambda_r)(\rho_0+\mu_\sigma)/\mu+(\lambda/\mu+\lambda/\mu_\sigma)(\mu\lambda_r\lambda_\sigma+\rho_0\rho_{0'}\lambda_\sigma-\lambda_\sigma^2\mu_\sigma)+\rho_{1'}\lambda\lambda_\sigma\mu/\mu_\sigma}{\mu\rho_{0'}\rho_{1'}+\mu\lambda_\sigma(\rho_{1'}+\lambda_r)+(\rho_0\rho_{0'}-\lambda_\sigma\mu_\sigma)(\rho_{1'}+\lambda_\sigma)}}$$
$$\approx 1-\frac{\lambda(\lambda+\lambda_r+\lambda_\sigma)}{\mu(\mu+\lambda+\lambda_r+\lambda_\sigma)}, \quad \text{for } \mu=\mu_\sigma \gg \lambda, \lambda_r, \lambda_\sigma. \qquad (6.209)$$

The approximation assumes $\mu_\sigma=\mu$ & $\lambda, \lambda_r, \lambda_\sigma \ll \mu$ and Eq. (6.209) allows same conclusions as for Eq. (6.206). $\lambda_\sigma=0$ implies $\mu_\sigma=0$ and yields results for ideal switch.

Further models for imperfect switching are conceivable. For instance, by assuming that for the model of Fig. 6.24 failure of the switch (with *stuck at the state occupied just before failure* and failure rate λ_σ) can only be detected and repaired at system down together with failed elements (one or both) at a repair rate μ_g. This situation occurs e.g. in power systems (*refuse to start*). Figure 6.25 gives the corresponding *diagrams of transition rates* for reliability and availability calculation, respectively (down state Z_2). Results are given in Example 6.15. A further possibility



a) For reliability

$\rho_{00'}=\lambda_\sigma;\ \rho_{01}=\lambda+\lambda_r;\ \rho_{10}=\mu;\ \rho_{12}=\rho_{0'2}=\lambda$
$\rho_0=\lambda+\lambda_r+\lambda_\sigma;\ \rho_{0'}=\lambda;\ \rho_1=\lambda+\mu;\ \rho_2=0$

b) For availability

$\rho_{00'}=\lambda_\sigma;\ \rho_{01}=\lambda+\lambda_r;\ \rho_{10}=\mu;\ \rho_{12}=\rho_{0'2}=\lambda;\ \rho_{20}=\mu_g$
$\rho_0=\lambda+\lambda_r+\lambda_\sigma;\ \rho_{0'}=\lambda;\ \rho_1=\lambda+\mu;\ \rho_2=\mu_g$

Figure 6.25 Diagram of transition rates for a repairable 1-out-of-2 warm redundancy with const. failure & repair rates $(\lambda, \lambda_r, \mu)$, imperfect switching (failure rate λ_σ, failure mode stuck at the state occupied), ideal failure detection, failure of the switch repaired only at system down with failed elements at a repair rate μ_g, no further failure at system down (Z_2 down state, Markov process)

is to assume *no connection* as failure mode (Fig. 6.31) or a constant probability c that the switch will perform correctly when called to operate (Figs. 6.27, 6.28).

Example 6.15

Compute the mean time to system failure $MTTF_{S0}$ for system new (in Z_0) at $t=0$ and the *steady-state* point and average availability $PA_S=AA_S$ of the 1-out-of-2 warm redundancy as per Fig. 6.25.

Solution

From Fig. 6.25a and Table 6.2, $MTTF_{S0}$ is given as solution of (with $M_i \equiv MTTF_{Si}$)

$$\rho_0\, M_0 = 1+\lambda_\sigma M_{0'} + (\lambda+\lambda_r) M_1\,, \qquad \rho_1\, M_1 = 1+\mu M_0\,, \qquad \rho_{0'}\, M_{0'} = 1\,, \qquad (6.210)$$

yielding

$$MTTF_{S0} = \frac{\rho_{0'}\rho_1 + \lambda_\sigma \rho_1 + (\lambda+\lambda_r)\rho_{0'}}{\rho_0 \rho_{0'}\rho_1 - (\lambda+\lambda_r)\mu\rho_{0'}} = \frac{\lambda(2\lambda+\lambda_r+\mu)+\lambda_\sigma(\lambda+\mu)}{\lambda^2(\lambda+\lambda_r)+\lambda\lambda_\sigma(\lambda+\mu)}\,. \qquad (6.211)$$

Because of the not detected failure of the switch, the condition on λ_σ to yield results for ideal switching (Table 6.6) is more severe as Eq. (6.207) and is given by (see also Eqs. (6.207), (6.239))

$$\lambda_\sigma << \lambda(\lambda+\lambda_r)/\mu, \qquad (\text{for } \mu, \mu_g >> \lambda, \lambda_r, \lambda_\sigma). \qquad (6.212)$$

From Fig. 6.25b and Table 6.2 or Eq. (A7.127) it follows that $PA_S=AA_S$ is given as solution of

$$\rho_0 P_0 = \mu P_1 + \mu_g P_2\,, \quad \rho_{0'} P_{0'} = \lambda_\sigma P_0\,, \quad \rho_1 P_1 = (\lambda+\lambda_r) P_0\,, \quad \rho_2 P_2 = \lambda P_1 + \lambda P_{0'}\,. \qquad (6.213)$$

One of the Eq. (6.213), arbitrarily chosen, must be replaced by $P_0+P_{0'}+P_1+P_2=1$. The *asymptotic & steady-state* point and average availability follows then from

$$PA_S = AA_S = P_0 + P_{0'} + P_1 = \frac{1}{1+\dfrac{\lambda^2(\lambda+\lambda_r+\lambda_\sigma)+\mu\lambda\lambda_\sigma}{\mu_g(\lambda+\mu)(\lambda+\lambda_\sigma)+\mu_g\lambda(\lambda+\lambda_r)}} \approx 1-\frac{\lambda\lambda_\sigma}{\mu_g(\lambda+\lambda_\sigma)}\cdot \qquad (6.214)$$

Equation (6.214) allows (before the approximation) same conclusions for λ_σ as for Eq. (6.211). If Eq. (6.212) is not satisfied, and in particular for $\mu\lambda_\sigma >> \lambda(\lambda+\lambda_r)$, Eq. (6.211) yields $MTTF_{S0} \approx 1/\lambda_\sigma + 1/\lambda$ (*nonrepairable 1-out-of-2 standby redundancy* with λ_σ & λ); and, for $\lambda_\sigma >> \lambda$, $MTTF_{S0} \approx 1/\lambda$ and $PA_S = AA_S \approx 1-\lambda/\mu_g$ (*repairable one-item*).

6.8.4 Incomplete Coverage

Incomplete fault (failure) coverage occurs because of lack or failure in the diagnosis. *Fault coverage* is defined as the *proportion of faults of an item that can be detected under given conditions.* A fault coverage greater as 0.9 is often required for complex equipment and systems (see e. g. [A2.5 (61508)]). Lacks in the diagnosis lead to *hidden (undetected, latent) failures,* i.e., failures which are not covered by diagnosis and can be detected only during a repair or a preventive maintenance. Hidden or latent failures can cause *serious reduction of the advantage offered by a redundancy* (see e.g. Eqs. (6.221) & (6.223)). Failure modes of a diagnosis have to be investigated on a *case-by-case basis,* starting often from following two modes

- false alarm,
- no alarm emitted (alarm defection).

Basically, *incomplete coverage* acts on the switching operation and is often investigated as part of *imperfect switching* (Section 6.8.3). Following an illustrative example, this section discusses some possibilities to investigate incomplete coverage. Because of practical difficulties in implementing some models, the use of a *majority redundancy* (e.g. a 2-out-of-3 instead of a 1-out-of-2 redundancy) remains often the best way to compensate incomplete coverage. In a 2-out-of-3 red., the first failure is captured on line, irrespective of coverage, and no switch is necessary (Example 2.5).

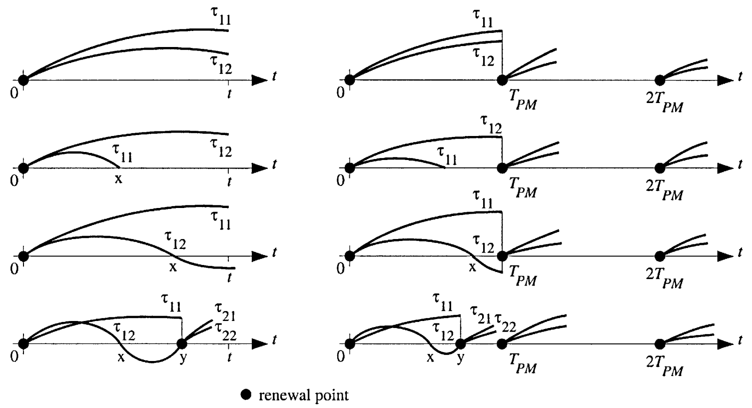
Consider first a 1-out-of-2 active redundancy with 2 different elements E_1 & E_2, and assume that failures of E_1 can be *detected* only at the end of a repair of E_2 or at a preventive maintenance (*hidden failures* in E_1). Elements E_1 and E_2 have constant failure rates (λ_1, λ_2), the repair time of E_2 is distributed according to $G(x)$ ($G(0)=0$, density $g(x)$), and preventive maintenance as well as repair of E_1 takes a *negligible time* (see Example 6.17 for constant repair rate). [+)] If no *preventive maintenance* is performed, Fig. 6.26a shows a possible time schedule of the system (new at $t=0$), yielding for the *reliability function*

$$R_{S0}(t) = e^{-(\lambda_1+\lambda_2)t} + \int_0^t \lambda_1 e^{-\lambda_1 x} e^{-\lambda_2 t} dx + \int_0^t \lambda_2 e^{-\lambda_2 x} e^{-\lambda_1 t} (1 - G(t-x))\, dx$$
$$+ \int_0^t \int_0^y \lambda_2 e^{-\lambda_2 x} e^{-\lambda_1 y} g(y-x) R_{S0}(t-y)\, dx\, dy. \qquad (6.215)$$

The Laplace transform of $R_{S0}(t)$ follows as

$$\tilde{R}_{S0}(s) = \frac{(s+\lambda_1)(s+\lambda_1+\lambda_2) + \lambda_2 (s+\lambda_2)(1-\tilde{g}(s+\lambda_1))}{(s+\lambda_1)(s+\lambda_2)(s+\lambda_1+\lambda_2) - (s+\lambda_1)(s+\lambda_2)\lambda_2\, \tilde{g}(s+\lambda_1)}, \qquad (6.216)$$

[+)] This situation arises, for instance, when for the repair of E_2 a travel time is involved (see e.g. pp. 202, 504). Also, it is tacitly assumed that at each renewal point ($t=0$, end of a repair of E_2 or of a preventive maintenance (Fig. 6.26b)), E_2 is put in operation and E_1 in reserve state; furthermore, failure detection in E_2 and switch to E_1 are ideal (Fig. 6.26 and graph in Example 6.17).



a) Without preventive maintenance **b)** With preventive maintenance (period T_{PM})

Figure 6.26 Possible time schedules for a repairable 1-out-of-2 parallel redundancy with hidden (latent) failures in element E_1 (reliability investigation, new at $t = 0$, repair times greatly exaggerated)

and the *mean time to failure* becomes

$$MTTF_{S0} = \frac{\lambda_1(\lambda_1+\lambda_2)+\lambda_2^2(1-\tilde{g}(\lambda_1))}{\lambda_1\lambda_2(\lambda_1+\lambda_2)-\lambda_1\lambda_2^2\,\tilde{g}(\lambda_1)}. \tag{6.217}$$

Example 6.16 gives a discussion of Eq. (6.217). The *point availability* $PA_{S0}(t)$ is investigated in Example 6.17 for the case of constant repair rate μ ($g(x) = \mu e^{-\mu x}$). If *preventive maintenance* is performed at times $0, T_{PM}, 2T_{PM}, \ldots$ (independently of the state of element E_2) and after each preventive maintenance (assumed of negligible duration, also considering a possible repair of E_2 and/or E_1) the system is *as-good-as-new*, then the times $0, T_{PM}, 2T_{PM}, \ldots$ are *renewal points* for the system. The *reliability function* $R_{S0_{PM}}(t)$ is given by Eq. (6.192) with $R_{S0}(t)$ as per Eq. (6.215); similarly for $MTTF_{S0_{PM}}$ (Eq. (6.194)). For $T_{PM} >> MTTR$, the approximation $PA_{S0_{PM}}(t) \approx PA_{S0}(t) \approx PA_S = AA_S$ can often be used (Example 6.17). Optimization of T_{PM} must consider cost and logistic aspects too ($MTTF_{S0_{PM}} \to \infty$ for $T_{PM} \to 0$).

Example 6.16

Give *approximate expressions* for the mean time to failure $MTTF_{S0}$ given by Eq. (6.217).

Solution

For $\tilde{g}(\lambda_1) \to 1$, it follows from Eq. (6.217) that

$$MTTF_{S0} \approx (\lambda_1 + \lambda_2)/\lambda_1\lambda_2 = 1/\lambda_1 + 1/\lambda_2. \tag{6.218}$$

A better approximation using $\tilde{g}(\lambda_1) = 1 - \lambda_1\, MTTR$ yields (with $MTTR$ as per Eq. (6.111))

$$MTTF_{S0} \approx (\lambda_1 + \lambda_2 + \lambda_2^2\, MTTR)/(\lambda_1\lambda_2(1+\lambda_2\, MTTR)). \tag{6.219}$$

Equation (6.218) shows that a repairable 1-out-of-2 active redundancy with hidden failures in one element behaves like a nonrepairable 1-out-of-2 standby redundancy; this result bears out, how important it is in the presence of redundancy to investigate failure detection and failure modes.

Example 6.17

Investigate $R_{S0}(t)$ per Eq. (6.216), $MTTF_{S0}$ per Eq. (6.217), $MTTF_{S0_{PM}}$ per Eq. (6.194), and the asymptotic & steady-state point and average availability $PA_S = AA_S$ and $PA_{S_{PM}} = AA_{S_{PM}}$ for the case of constant repair rate μ (i.e. for $g(x) = \mu e^{-\mu x}$).

Solution

With $\tilde{g}(s+\lambda_1) = \mu/(s+\lambda_1+\mu)$ it follows from Eq. (6.216) that

$$\tilde{R}_{S0}(s) = \frac{(s+\lambda_1+\lambda_2)(s+\lambda_1+\mu)+\lambda_2(s+\lambda_2)}{(s+\lambda_1)(s+\lambda_2)(s+\lambda_1+\lambda_2+\mu)} \tag{6.220}$$

and thus

$$R_{S0}(t) = A e^{-\lambda_1 t} + B e^{-\lambda_2 t} + C e^{-(\lambda_1+\lambda_2+\mu)t}$$

with

$$A = \frac{\lambda_2(\lambda_2-\lambda_1+\mu)}{(\lambda_2-\lambda_1)(\lambda_2+\mu)} \approx \frac{\lambda_2}{(\lambda_2-\lambda_1)}, \quad B = \frac{-\lambda_1(\lambda_1-\lambda_2+\mu)}{(\lambda_2-\lambda_1)(\lambda_1+\mu)} \approx \frac{-\lambda_1}{(\lambda_2-\lambda_1)}, \quad C = \frac{-\lambda_1\lambda_2}{(\lambda_1+\mu)(\lambda_2+\mu)} \approx 0.$$

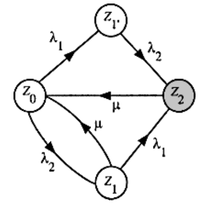
The mean time to failure $MTTF_{S0}$ follows from Eq. (6.220) as

$$MTTF_{S0} = \tilde{R}_{S0}(0) = \frac{(\lambda_1+\lambda_2)(\lambda_1+\mu)+\lambda_2^2}{\lambda_1\lambda_2(\lambda_1+\lambda_2+\mu)} \approx \frac{\lambda_1+\lambda_2}{\lambda_1\lambda_2} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}. \tag{6.221}$$

One recognizes, that $\lambda_1+\lambda_2 << \mu$ yields directly to

$$R_{S0}(t) \approx (\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t})/(\lambda_2-\lambda_1) \quad \text{and} \quad MTTF_{S0} \approx 1/\lambda_1 + 1/\lambda_2. \tag{6.222}$$

$R_{S0}(t)$ as well as the point availability $PA_{S0}(t)$ can be obtained using a 4 states Markov process with up states $Z_0, Z_1, Z_{1'}$ and down state Z_2 (Z_2 absorbing for $R_{S0}(t)$), see graph and the model discussion on p. 246). The asymptotic & steady-state point and average availability $PA_S = AA_S$ is obtained by solving (Tab. 6.2) $(\lambda_1+\lambda_2)P_0 = \mu P_1 + \mu P_2$, $\lambda_2 P_{1'} = \lambda_1 P_0$, $(\lambda_1+\mu)P_1 = \lambda_2 P_0$, $P_0+P_1+P_{1'}+P_2 = 1$, yielding



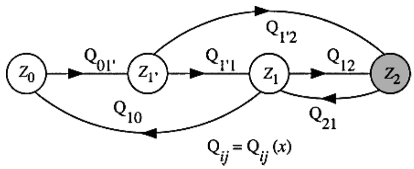
$$PA_S = AA_S = P_0 + P_1 + P_{1'} = \frac{1}{1 + \dfrac{\lambda_1\lambda_2(\lambda_1+\lambda_2+\mu)}{\mu[(\lambda_1+\mu)(\lambda_1+\lambda_2)+\lambda_2^2]}} \approx 1 - \frac{\lambda_1\lambda_2}{\mu(\lambda_1+\lambda_2)}. \tag{6.223}$$

Investigation of $PA_{S0}(t)$ (Table 6.2 and above graph) shows that $PA_{S0}(t)$ converges rapidly to $PA_S = AA_S$ given by Eq. (6.223) and for $\lambda_1, \lambda_2 << \mu$ the approximation given by Eq. (6.88), with PA_S and μ per Eq. (6.223), can often be used.

In the case of *preventive maintenance* at $T_{PM}, 2T_{PM}, \ldots$ (renewal points at $t = 0, T_{PM}, 2T_{PM}, \ldots$), Eq. (6.194) with $R_{S0}(t)$ as per Eq. (6.222) yields

$$MTTF_{S0_{PM}} \approx \frac{\lambda_2(1-e^{-\lambda_1 T_{PM}})/\lambda_1 - \lambda_1(1-e^{-\lambda_2 T_{PM}})/\lambda_2}{\lambda_2(1-e^{-\lambda_1 T_{PM}}) - \lambda_1(1-e^{-\lambda_2 T_{PM}})} \approx \frac{2}{\lambda_1\lambda_2 T_{PM}}. \tag{6.224}$$

The last part of Eq. (6.224) follows with $e^{-\lambda x} \approx 1 - \lambda x + (\lambda x)^2/2$. For $T_{PM} >> 1/\mu = MTTR$ $PA_{S0_{PM}}(t) \approx PA_{S0}(t) \approx PA_S = AA_S$ with $PA_S = AA_S$ per Eq. (6.223) can often be used.



$Q_{01'}(x) = 1 - e^{-2\lambda x}$; $Q_{1'2}(x) = (1 - c)u(x)$;

$Q_{1'1}(x) = c\,u(x)$; $Q_{10}(x) = \frac{\mu}{\lambda + \mu}(1 - e^{-(\lambda+\mu)x})$;

$Q_{12}(x) = \frac{\lambda}{\mu} Q_{10}(x)$; $Q_{21}(x) = 1 - e^{-\mu x}$;

$u(x) = 0$ for $x < 0$ and 1 for $x > 0$; $Q_{ij}(\infty) = P_{ij}$

Figure 6.27 State transition diagram for a repairable 1-out-of-2 active redundancy with const. failure & repair rates (λ, μ), incomplete coverage (detection of the failed element with probability c), one repair crew (Z_2 down state (absorbing for rel. calculation), semi-Markov process; see also Fig. 6.28)

A basic possibility to consider incomplete coverage is to assume that *a failure will be detected* (internal BIT) *only with* a *probability* c .This will be considered in the following for the case of identical elements in a 1-out-of-2 active redundancy. At a failure, outputs of both elements differ and with probability $1-c$ the failed element *can not be detected and disconnected*, yielding a system failure. This case is similar to that of imperfect switching mentioned at the end of Section 6.8.3 and is known in the literature [6.47 (2001)]. Figure 6.27 gives the *state transition diagram* of the involved semi-Markov process. The transition from state $Z_{1'}$ occurs instantaneously to Z_1 with probability $P_{1'1} = c$ or to Z_2 with $P_{1'2} = 1-c$. Assuming constant failure and repair rates, the model of Fig. 6.27 can be investigated using a Markov process with the diagram of transition rates given in Fig 6.28 (known in power systems as redundancy with *no start at call*, see e. g. [6.34]). Examples 6.18 and 6.19 investigate the models of Figs. 6.27 and 6.28, showing their equivalence.

Example 6.18

Give the mean time to system failure $MTTF_{S0}$ (system new, enters Z_0, at $t=0$) and the asymptotic & steady-state point and average availability $PA_S = AA_S$ of the 1-out-of-2 warm red. as per Fig. 6.27.

Solution

From Fig. 6.27 and Table 6.2 or Eq. (A7.173), $MTTF_{S0}$ is given as solution of $M_{1'} = T_{1'} + c\,M_1$, $M_0 = T_0 + M_{1'}$, $M_1 = T_1 + (\mu / (\lambda + \mu)) M_0$, with $M_i \equiv MTTF_{Si}$, $T_i = \int_0^\infty (1 - Q_i(x))dx$, $Q_i(x) = \sum_j Q_{ij}(x)$ (Eqs.(A7.166) and (A7.165)). Considering Fig. 6.27 it follows that $T_0 = 1/2\lambda$, $T_{1'} = 0$, $T_1 = 1/(\lambda + \mu)$, and $T_2 = 1/\mu$, yielding

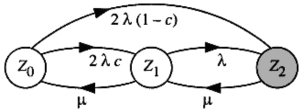
$$MTTF_{S0} = \frac{T_0 + T_{1'} + cT_1}{1 - c\mu/(\lambda+\mu)} = \frac{\lambda + \mu + 2\lambda c}{2\lambda(\lambda + \mu - \mu c)} \approx \frac{\mu}{2\lambda^2 + 2\lambda\mu(1-c)} . \qquad (6.225)$$

From Fig. 6.27 and Table 6.2 or Eq. (A7.178), $PA_S = AA_S = P_0 + P_{1'} + P_1$ is given as

$$PA_S = AA_S = (P_0 T_0 + P_{1'} T_{1'} + P_1 T_1) / (P_0 T_0 + P_{1'} T_{1'} + P_1 T_1 + P_2 T_2). \qquad (6.226)$$

Thereby, P_j are the state probabilities of the *embedded Markov chain*, obtained as solution of $P_0 = P_1 \mu / (\lambda + \mu)$, $P_{1'} = P_0$, $P_1 = P_{1'} c + P_2$, $P_2 = P_{1'}(1-c) + P_1 \lambda / (\lambda + \mu)$ (Table 6.2), yielding (considering $P_0 + P_1 + P_{1'} + P_2 = 1$) $P_1 = (\lambda + \mu) / (2(\lambda + 2\mu) - \mu c)$, $P_2 = (\lambda + \mu - \mu c) / (2(\lambda + 2\mu) - \mu c)$ $P_0 = P_{1'} = \mu / (2(\lambda + 2\mu) - \mu c)$. From Eq. (6.226) it follows then

$$PA_S = AA_S = (\mu^2 + 2\lambda\mu) / (\mu^2 + 2\lambda\mu + 2\lambda(\lambda + \mu - \mu c)) \approx 1 - 2\lambda(\lambda + \mu - \mu c) / (\mu^2 + 2\lambda\mu). \qquad (6.227)$$



$\rho_{01} = 2\lambda c$; $\rho_{02} = 2\lambda(1-c)$; $\rho_{10} = \mu$;
$\rho_{12} = \lambda$; $\rho_{21} = \mu$ ($\rho_{21} = 0$ for reliability)
$\rho_0 = 2\lambda$; $\rho_1 = \lambda + \mu$; $\rho_2 = \mu$ ($\rho_2 = 0$ for reliability)

Figure 6.28 State transition diagram for a repairable 1-out-of-2 active redundancy with constant failure & repair rates (λ, μ), incomplete coverage (detection of failed element with probability c, i. e., with probability $1-c$ the system goes down because the outputs of both elements differ), one repair crew (Z_2 down state (absorbing for reliability calculation), Markov process; see also Fig. 6.27)

Example 6.19

Give the mean time to system failure $MTTF_{S0}$ (system new, in Z_0, at $t=0$) and the asymptotic & steady-state point and average availability $PA_S=AA_S$ of the 1-out-of-2 warm red. as per Fig. 6.28.

Solution

From Fig. 6.28 & Table 6.2 or Eq. (A7.126), $MTTF_{S0}$ is given as solution of (with $M_i \equiv MTTF_{Si}$) $2\lambda M_0 = 1 + 2\lambda c M_1$ and $(\lambda+\mu)M_1 = 1 + \mu M_0$, yielding

$$MTTF_{S0} = \frac{\lambda + \mu + 2\lambda c}{2\lambda(\lambda + \mu - \mu c)} \approx \frac{\mu}{2\lambda^2 + 2\lambda\mu(1-c)} . \qquad (6.228)$$

From Fig. 6.28 and Table 6.2 or Eq. (A7.127), $PA_S = AA_S$ is given as solution of $2\lambda P_0 = \mu P_1$, $(\lambda+\mu) P_1 = 2\lambda c P_0 + \mu P_2$ and $P_0 + P_1 + P_2 = 1$, yielding

$$PA_S = AA_S = P_0 + P_1 = \frac{\mu^2 + 2\lambda\mu}{\mu^2 + 2\lambda\mu + 2\lambda(\lambda + \mu - \mu c)} \approx 1 - \frac{2\lambda^2 + 2\lambda\mu(1-c)}{\mu^2 + 2\lambda\mu} . \qquad (6.229)$$

Comparison of Eqs. (6.225) with (6.228) and (6.227) with (6.229) shows the equivalence of the models given by Figs. 6.27 and 6.28 (for constant failure and repair rates). For $c=1$, Eqs. (6.228) & (6.229) yield results of Table 6.6 for a 1-out-of-2 active redundancy. For $c=0$, Eqs. (6.228) and (6.229) yield results for a one-item with failure rate 2λ and repair rate μ ($\mu >> 2\lambda$ for $PA_S = AA_S$); most unfavorable case, because at the first failure it is not possible to identify the failed element, yielding to a system down. Comparison of Eqs. (6.92) with (6.228) and (6.87) with (6.229) shows that the *effect of incomplete coverage is negligible for*

$$2\lambda\mu(1-c) << 2\lambda^2, \qquad \text{e. g.} \quad \mu(1-c) < 0.1\lambda \rightarrow c > 1 - 0.1\lambda/\mu . \qquad (6.230)$$

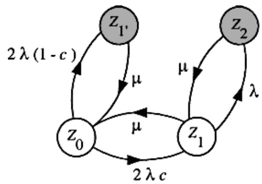
Condition (6.230) applies to $MTTF_{S0}$ (Eq. (6.228)) and to $1 - PA_S$ (Eq. (6.229)). It can be hard to realize for complex systems and remains practically the same even if in the model of Fig. 6.28, repair of a hidden (latent) failure brings the system to state Z_0 instead of Z_1, (Example 6.20). A further possibility is investigated in Example 6.21 by assuming that at the occurrence of a hidden failure, one of the two elements is selected with probability p to continue operation. However, *majority redundancy* should be preferred for critical applications.

Example 6.20

Investigate $MTTF_{S0}$ and $PA_S = AA_S$ for the model given by Fig. 6.28 by assuming that a repair for a hidden (latent) failure (transition $Z_0 \to Z_2$) brings the system to state Z_0 and not to Z_1.

Solution

$MTTF_{S0}$ is given by Eq. (6. 228). The point availability $PA_{S0}(t)$ can be obtained using a 4 states Markov process with up states Z_0 and Z_1 and down states $Z_{1'}$ and Z_2, see graph. The asymptotic & steady-state point and average availability $PA_S = AA_S$ is obtained by solving (Table 6.2) $2\lambda P_0 = \mu P_{1'} + \mu P_1$, $(\lambda + \mu) P_1 = 2\lambda c P_0 + \mu P_2$, $\mu P_{1'} = 2\lambda(1-c) P_0$, and $P_0 + P_1 + P_{1'} + P_2 = 1$, yielding



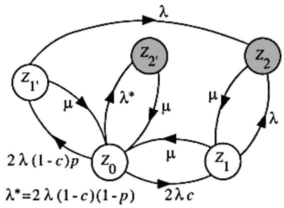
$$PA_S = AA_S = P_0 + P_1 = \frac{1}{1 + \dfrac{2\lambda[\lambda c + \mu(1-c)]}{\mu^2 + 2\lambda\mu c}} \approx 1 - \frac{2\lambda[\lambda c + \mu(1-c)]}{\mu^2}. \qquad (6.231)$$

Example 6.21

Investigate $MTTF_{S0}$ and $PA_S = AA_S$ for the model considered in Example 6.20 by assuming that at the occurrence of a hidden (latent) failure (outputs of both elements differ and failed element can not be detected), one of the two elements is instantaneously selected to continue operation and the selected element is with probability p failure-free (safety is not relevant).

Solution

$R_{S0}(t)$ and $PA_{S0}(t)$ can be obtained using a 5 states Markov process with up states $Z_0, Z_1, Z_{1'}$ and down states $Z_2, Z_{2'}$ ($Z_2, Z_{2'}$ absorbing for $R_{S0}(t)$), see graph. $MTTF_{S0}$ is given as solution of (with $M_i \equiv MTTF_{Si}$), $2\lambda M_0 = 1 + 2\lambda c M_1 + 2\lambda(1-c) p M_{1'}$, $(\lambda + \mu) M_1 = 1 + \mu M_0$, $(\lambda + \mu) M_{1'} = 1 + \mu M_0$, yielding



$$MTTF_{S0} = \frac{\lambda + \mu + 2\lambda(c + (1-c)p)}{2\lambda^2 + 2\lambda\mu(1-c)(1-p)} \approx \frac{\mu}{2\lambda^2 + 2\lambda\mu(1-c)(1-p)}. \qquad (6.232)$$

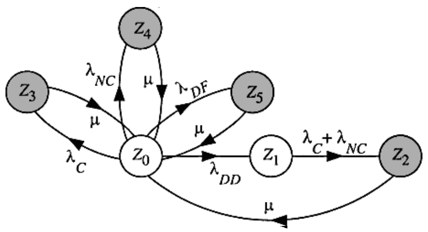
The asymptotic & steady-state point and average availability $PA_S = AA_S$ is obtained by solving (Table 6.2) $2\lambda P_0 = \mu(P_1 + P_{1'} + P_{2'})$, $(\lambda + \mu) P_1 = 2\lambda c P_0 + \mu P_2$, $(\lambda + \mu) P_{1'} = 2\lambda(1-c) p P_0$, $\mu P_2 = \lambda(P_1 + P_{1'})$, $P_0 + P_1 + P_{1'} + P_2 + P_{2'} = 1$, yielding

$$PA_S = AA_S = P_0 + P_1 + P_{1'} = \frac{1}{1 + \dfrac{2\lambda[\lambda + (\mu - \lambda)(1-c)(1-p)]}{\mu^2 + 2\lambda\mu[p(1-c) + c]}} \approx 1 - \frac{2\lambda[\lambda + \mu(1-c)(1-p)]}{\mu^2}. \qquad (6.233)$$

Comparison of Eq. (6.232) with (6.228) and Eq. (6.233) with (6.231) shows that

$$\frac{MTTF_{S0\,p=0.5}}{MTTF_{S0\,p=0}} \approx \frac{2\lambda^2 + 2\lambda\mu(1-c)}{2\lambda^2 + \lambda\mu(1-c)} \lesssim 2 \quad \text{and} \quad \frac{(1 - PA_S)_{p=0}}{(1 - PA_S)_{p=0.5}} \approx \frac{\lambda c + \mu(1-c)}{\lambda + \mu(1-c)/2} \lesssim 2. \qquad (6.234)$$

Both ratios are 1 for a coverage probability $c = 1$ and ≈ 2 for $c = 0$. One recognizes also that results of Example 6.20 are those of Example 6.21 for $p = 0$ and that for $p = 1$, Eq. (6.233) yields Eq. (6.87) and Eq. (6.232) yields Eq. (6.92), as for $c = 1$.



$$\rho_{01}=\lambda_{DD}\,;\ \rho_{03}=\lambda_C\,;\ \rho_{04}=\lambda_{NC}\,;\ \rho_{05}=\lambda_{DF}\,;$$
$$\rho_{12}=\lambda_C+\lambda_{NC}=\lambda\,;\ \rho_{20}=\rho_{30}=\rho_{40}=\rho_{50}=\mu$$
$$\rho_0=(\lambda_C+\lambda_{NC})+(\lambda_{DF}+\lambda_{DD})=\lambda+\lambda_D\,;$$
$$\rho_1=\lambda_C+\lambda_{NC}=\lambda;\ \rho_2=\rho_3=\rho_4=\rho_5=\mu$$

C = covered, NC = not covered

DF = false alarm, DD = alarm defection

Figure 6.29 Diagram of transition rates for a one-item structure with incomplete coverage and 2 failures modes for the diagnosis (constant failure and repair rates $\lambda_C, \lambda_{NC}, \lambda_{DF}, \lambda_{DD}, \mu$, ideal failure detection, Z_2, Z_3, Z_4, Z_5 down states (absorbing for rel. calculation), Markov process)

Influence of preventive maintenance at $T_{PM}, 2T_{PM}, \ldots$ (renewal points at $t=0$, $T_{PM}, 2T_{PM}, \ldots$) can be investigated as discussed Section 6.8.2, on p. 247, and in Example 6.17, often using

$$\mathrm{R}_{S0}(t) \approx e^{-t/MTTF_{S0}} \quad \text{and} \quad \left|\mathrm{PA}_{S0}(t)-\mathrm{PA}_S\right| \approx (1-\mathrm{PA}_S)\, e^{-\mu t}, \qquad (6.235)$$

or $\mathrm{PA}_{S0}(t) \approx PA_S = AA_S$, see Eqs. (6.94) & (6.88) for a deeper investigation (the two sided bound can be necessary if $\mathrm{PA}_{S0}(t)$ oscillate, as often for systems with many states).

Other possibilities to consider for incomplete coverage are conceivable. Assuming, for instance, that in a 1-out-of-2 active redundancy at a failure of one element (outputs of both elements differ and failed element can not be detected), one element is instantaneously selected to continue operation at system level and the selected element is failure-free with probability p, leads to the model considered in Example 6.20 with $c=p$. $p=0$ yields results for a one-item structure with failure rate 2λ and repair rate μ.

A more elaborated model which considers 2 failure modes for the diagnosis, *false alarm* with failure rate λ_{DF} and *alarm defection* with failure rate λ_{DD} has been proposed in [6.43]. Figure 6.29 shows this model by considering a repair rate μ for all failure modes. Investigation of this model using Fig. 6.29 and Table 6.2 leads to

$$MTTF_{S0}=\frac{\lambda+\lambda_{DD}}{\lambda(\lambda+\lambda_D)} \quad \text{and} \quad PA_S=AA_S=\frac{\mu(\lambda+\lambda_{DD})}{\mu(\lambda+\lambda_{DD})+\lambda(\lambda+\lambda_D)} \approx 1-\frac{\lambda(\lambda+\lambda_D)}{\mu(\lambda+\lambda_{DD})}. \qquad (6.236)$$

$\lambda_{DD}=\lambda_{DF}=\lambda_D=0$ yields results for a one-item structure with failure rate λ and repair rate μ. A possible diagram of transition rates for a 1-out-of-2 active redundancy with 2 repair crews on the basis of Fig. 6.29 is Fig. 3 of [6.43]. A further example for a duplex system is Fig. 1 of [1.13].

6.8.5 Elements with more than two States or one Failure Mode

Elements with more than two states (good / failed for instance) or one failure mode (e.g. open or short) often arise in practical applications. Some considerations have been given in Sections 2.3.6 and 6.8.4. This section shows, on the basis of practical examples, that items with more than two states or one failure mode can often be investigated using the diagram of transition rates, see also pp. 262-265.

As a *first example* consider an item with the three states *good, waiting for repair, repair* [6.14]. Figure 6.30 shows this model. From Fig. 6.30 & Table 6.2 it holds that

$$MTTF_{S0} = \frac{1}{\lambda} \qquad \text{and} \qquad PA_S = AA_S = \frac{\mu\mu'}{\mu\,\mu'+\lambda\,(\mu+\mu')} \approx 1-\lambda\,(\frac{1}{\mu}+\frac{1}{\mu'}). \qquad (6.237)$$

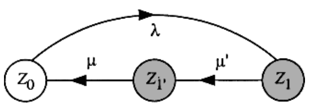
The item in Fig. 6.30 behaves like a one-item structure with failure rate λ and repair time Erlang distributed (n=2, Eq. (A6.102)) with mean $MTTR_{tot} = 1/\mu + 1/\mu'$. More complex structures can also be investigated, see e.g. [6.14].

As a *second example* consider a *1-out-of-2 warm redundancy* with constant failure rate λ, λ_r and repair rate μ. The switching element can fail with constant failure rate λ_σ for failure mode *stuck at the state occupied just before failure* or λ_0 for failure mode *no connection*. Failure of the switch can be immediately detected and repaired with constant repair rate μ_σ or μ_0. Furthermore, assume only one repair crew, *repair priority* on the switch, and no further failure at system down (also for the switch, no further failure is possible after a failure with one of the two possible failure modes). Asked is the mean time to system failure $MTTF_{S0}$ for system new (in state Z_0) at $t = 0$. The involved process is a (time-homogeneous) Markov process. Figure 6.31 gives the *diagrams of transition rates* for reliability calculation (see Example 6.22 for availability). Comparing Fig. 6.31 with Fig. 6.24a, one recognizes that $MTTF_{S0}$ is given by Eq. (6.206) with $\rho_0 = \lambda+\lambda_r+\lambda_\sigma+\lambda_0$ and $\rho_1 = \lambda+\lambda_0+\lambda_\sigma+\mu$ (i.e. adding λ_0 to ρ_0 and ρ_1). From this,

$$MTTF_{S0} \approx \frac{1+(3\lambda+\lambda_r+\lambda_\sigma)/\mu_\sigma+(2\lambda+\lambda_r+\lambda_0)/\mu}{\lambda_0\,[1+(3\lambda+\lambda_r)/\mu_\sigma+(\lambda+\lambda_r+\lambda_0)/\mu+\lambda\lambda_\sigma/\mu_\sigma\lambda_0+\lambda\,(\lambda+\lambda_r+\lambda_0)/\mu\lambda_0]}. \qquad (6.238)$$

The approximation assumes $\lambda, \lambda_\sigma, \lambda_0 << \mu, \mu_\sigma$. The failure mode *no connection* (λ_0) acts similarly as the failure mode *stuck at the state occupied just before failure* (λ_σ) in Example 6.15 (Eq (6.212)), and the *effect of imperfect switching* is *negligible* for

$$\lambda_0 << \lambda(\lambda+\lambda_r)/\mu \quad \text{and} \quad \lambda_\sigma << \lambda+\lambda_r, \qquad (\mu=\mu_\sigma>>\lambda,\lambda_r,\lambda_\sigma,\lambda_0). \qquad (6.239)$$

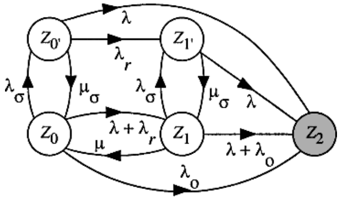


$\rho_{01} = \rho_0 = \lambda$; $\rho_{11'} = \rho_1 = \mu'$; $\rho_{1'0} = \rho_{1'} = \mu$

λ = failure rate, μ = repair rate,

μ' = failure detection rate (including possible travel time)

Figure 6.30 Diagram of transition rates for a one-item with 3 states *good, waiting for repair, repair* (constant failure, failure detection & repair rates (λ, μ'&μ), Z_1, $Z_{1'}$ down states, Markov process)



$$\rho_{00'} = \lambda_\sigma ; \quad \rho_{01} = \lambda + \lambda_r ; \quad \rho_{0'0} = \mu_\sigma ;$$
$$\rho_{02} = \lambda_o ; \quad \rho_{0'1'} = \lambda_r ; \quad \rho_{0'2} = \lambda ; \quad \rho_{10} = \mu ;$$
$$\rho_{11'} = \lambda_\sigma ; \quad \rho_{12} = \lambda + \lambda_o ; \quad \rho_{1'1} = \mu_\sigma ; \quad \rho_{1'2} = \lambda$$

$$\rho_0 = \lambda + \lambda_r + \lambda_\sigma + \lambda_o ; \quad \rho_{0'} = \lambda + \lambda_r + \mu_\sigma ;$$
$$\rho_1 = \lambda + \lambda_o + \lambda_\sigma + \mu ; \quad \rho_{1'} = \lambda + \mu_\sigma ; \quad \rho_2 = 0$$

Figure 6.31 Diagram of transition rates for reliability calculation of a repairable 1-out-of-2 warm redundancy with const. failure & repair rates λ, λ_r, μ, switch with failure modes *stuck at the state occupied* and *no connection* with constant failure & repair rates λ_σ, μ_σ and λ_o, μ_o, respectively (ideal failure detection, 1 repair crew, repair priority on switch, Z_2 down state, Markov process)

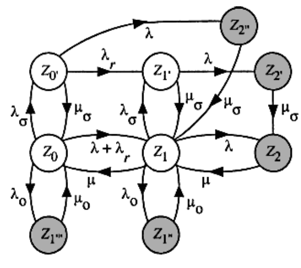
Condition given by Eq. (6.239) is for λ_o similar to that given by Eq. (6.212) for λ_σ. Example 6.22 investigates the *asymptotic & steady-state* point and average availability $PA_S = AA_S$ for the system described by Fig. 6.31 by assuming a repair rate μ_o for failure mode *no connection* and μ_σ for failure mode *stuck at the state occupied just before failure*, one repair crew, and repair priority for switch failures (for the switch only a failure mode is possible at a time). From Eq. (6.240) one recognizes that imperfect switching acts for $PA_S = AA_S$ in a similar way as for $MTTF_{S0}$ (Eq. (6.239)).

A more complex system is considered in Section 6.8.6.3 (pp. 262 - 65). Further models for systems with more than two states or one failure mode are conceivable.

Example 6.22

Investigates the asymptotic & steady-state point and average availability $PA_S = AA_S$ for the model considered in Fig. 6.31 by assuming no further failures at system down.

Solution



$PA_S = AA_S$ (as well as $PA_{S0}(t)$) can be obtained using a 9 states Markov process with up states $Z_0, Z_{0'}, Z_1, Z_{1'}$ and down states $Z_{1''}, Z_{1'''}, Z_2, Z_{2'}, Z_{2''}$ (absorbing for reliability calculation), see graph, by solving (Table 6.2) $(\lambda+\lambda_r+\lambda_\sigma+\lambda_o)P_0 = \mu_\sigma P_{0'} + \mu P_1 + \mu_o P_{1'''}$, $\mu_o P_{1'''} = \lambda_o P_0$, $(\lambda+\lambda_r+\mu_\sigma)P_{0'} = \lambda_\sigma P_0$, $(\lambda+\mu_\sigma)P_{1'} = \lambda_r P_{0'} + \lambda_\sigma P_1$, $\mu_o P_{1''} = \lambda_o P_1$, $\mu P_2 = \lambda P_1 + \mu_\sigma P_{2'}$, $\mu_\sigma P_{2'} = \lambda P_{1'}$, $\mu_\sigma P_{2''} = \lambda P_{0'}$, and $P_0 + P_{0'} + P_1 + P_{1'} + P_{1''} + P_{1'''} + P_2 + P_{2'} + P_{2''} = 1$, yielding

$$PA_S = AA_S = P_0 + P_{0'} + P_1 + P_{1'} = \frac{1}{1 + \dfrac{a(\lambda+\lambda_r)(a+\lambda_r+\lambda_\sigma)[\mu_\sigma(\mu_o\lambda+\mu\lambda_o)+\mu_o\lambda\lambda_\sigma]+a\mu[\mu\mu_\sigma\lambda_o(a+\lambda_r)+\mu_o\lambda\lambda_\sigma(\lambda_r+\mu)]}{\mu\mu_o\mu_\sigma(\lambda+\lambda_\sigma+\mu_\sigma)[(\lambda+\lambda_r+\mu_\sigma)(\lambda+\lambda_r+\mu)+\lambda_\sigma(\lambda+\lambda_r)]}}$$

$$\approx 1 - \frac{\mu^2\mu_\sigma^2\lambda_o + \mu^2\mu_o\lambda\lambda_\sigma + \mu_\sigma^2(\lambda+\lambda_r)(\mu\lambda_o + \mu_o\lambda)}{\mu^2\mu_\sigma^2\mu_o} \lessapprox 1 - \frac{\lambda_o}{\mu_\sigma}, \qquad a = (\lambda + \mu_\sigma). \quad (6.240)$$

Investigation of Eq. (6.240) leads to a condition similar (same for $\mu_o = \mu_\sigma = \mu$) to Eq. (6.239).

6.8.6 Fault Tolerant reconfigurable Systems

Fault tolerant structures are able to detect and localize faults (failures & defects) and reconfigure themselves to continue operation with minimum loss of performance and / or safety (*graceful degradation*). Such a characteristic must be built in during design & development. Typical examples of fault tolerant systems are safety circuits as well as power and telecommunication networks. Following a short discussion on ideal reconfiguration, this section deals with reconfiguration occurring at given fixed times or at failure by considering also non ideal conditions, for instance imperfect switching in Section 6.8.6.3. Investigation is based on tools introduced in Appendix A7 and summarized in Table 6.2. Constant failure and repair rates are assumed, yielding to (time-homogeneous) Markov processes. Procedures are illustrated on a *case-by-case basis using diagrams of transition rates.*

6.8.6.1 Ideal case

Each redundant structure belongs to a fault tolerant reconfigurable structure and must be validated for this purpose during design & development, for instance with an FMEA (Section 2.6). For the redundant structures investigated in Sections 2.2, 2.3.1 - 2.3.5, 6.4 - 6.7 and Appendix A7, independent elements (p. 52), ideal fault coverage, ideal switching, and no reduction of system performance at failure of a redundant element was assumed. Because of these assumptions, investigations often lead to series - parallel structures (Sections 6.6 & 6.7). Imperfect switching, incomplete coverage, and items (systems) with more than two states or failure modes are considered in Sections 2.3.6, 6.8.3 - 6.8.5, 6.8.6.3. Sections 6.8.6.2 and 6.8.6.3 investigate *time and failure censored reconfiguration*, and Section 6.8.6.4 considers *reward & frequency/duration* aspects. In addition, Sections 6.8.7 - 6.8.9 deal with common cause failures, basic considerations on reliability networks and a general procedure for complex repairable systems, and Section 6.9 introduces alternative investigation methods for complex systems.

6.8.6.2 Time Censored Reconfiguration (Phased-Mission Systems)

In some practical applications, systems are used for different required functions. If each required function can be considered separately from one another, investigation is performed by considering a reliability block diagram (if it exist) for each required function (p. 29). Otherwise, if mission phases follow each other, investigation must consider the *system reconfiguration* at the end of each phase and one define this as a *phased-mission system.* Investigation of phased-mission systems can be more time consuming as stated e. g. in [2.7, 2.18, 6.24, 6.33, 6.41], dealing with binary state assignment (basically limited to totally independent elements (p. 52)), considering time dependent failure or repair rates (breaking the Markov property), using semi-Markov processes (of limited validity), or missing Assumption 4 below (important when transferring state probabilities at the end of phase k to initial probabilities for phase $k+1$). A *lower bound* R_{S0_l} for the mission reliability R_{S0} is obtained by connecting the reliability block diagrams for each phase in series for the whole mission duration (Example 2.5). An *upper bound* for R_{S0} is given by the smaller of the reliability for each phase taken separately by assuming that all elements involved are as-good-as-new at begin of the phase considered; thus,

$$R_{S0_l} \le R_{S0} \le \min(R_{k,S0}) \qquad k=1,\dots,n \text{ (for } n \text{ phases)}. \tag{6.241}$$

Examples 6.23 - 6.25 illustrate some general considerations and Example 6.26 gives a numerical application of Eq. (6.241). For availability, Eq. (6.246) applies.

The following practice oriented procedure (Point (ii) below) for reliability and availability analysis of repairable phased-mission systems allows, in particular, consideration of standby redundancy and arbitrary repair strategy.

(i) *General assumptions*:

1. Failure and repair rates (λ_i and μ_i) of all elements are constant during the sojourn time in any state within each phase, but can change (stepwise) at a state (or phase) change because of change in configuration, component use, stress, repair strategy or other; for all elements it holds that $\lambda_i << \mu_i$.
2. At the begin of the mission (phased-mission) all elements are as-good-as-new.
3. Phase duration $T_1,\dots,T_n$ are given (fixed) values, each of them so large that *asymptotic & steady-state values* for availability can be assumed for every phase ($T_1,\dots,T_n >> 1/\mu_i$ for all elements, see Section 6.2.5 and Table 6.6).
4. For availability investigation, not used elements in a phase are either as-good-as-new and put in standby (failure rate $\lambda \equiv 0$) at begin of the phase or repaired (Assumption 3) and then put in standby (repair priority on elements used); for reliability investigation, down states at system level are absorbing states and the above rule holds for elements which have not caused system down.
5. The system has only one repair crew and no further failures can occur at system down; system down is an absorbing state for reliability; for availability, the system is restored to an operating state according to a given repair strategy.
6. Fault coverage, switch, and logistic support are ideal.
7. For each phase, a reliability block diagram exists.

Example 6.23

A one-item is used in a mission with phase 1 (duration T_1, const. failure rate λ_1), followed by phase 2 (duration T_2, const. failure rate λ_2). Compute the reliability function for item new at $t=0$.

Solution

For the reliability function of the whole mission it holds that (T_1, T_2 given (fixed))

$$R_{S0} = \Pr\{\text{phase 1 failure free} \cap \text{phase 2 failure free}\} = \Pr\{\text{phase 1 failure free}\} \cdot \Pr\{\text{phase 2 failure free} \mid \text{phase 1 failure free}\} = e^{-\lambda_1 T_1} \cdot e^{-\lambda_2 T_2} = e^{-(\lambda_1 T_1 + \lambda_2 T_2)}. \tag{6.242}$$

The product rule in Eq. (6.242) holds only because of *constant failure rates* (see also Eq. (6.27)).

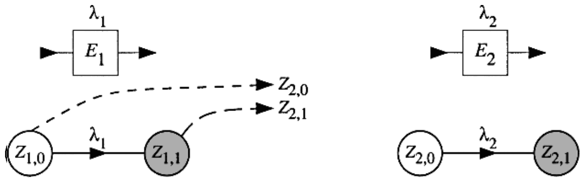


Figure 6.32 Diagrams of transition rates for a one-item used in a mission with phase 1 of duration T_1 and constant failure rate λ_1, followed by phase 2 of duration T_2 and constant failure rate λ_2 ($Z_{1,1}$, $Z_{2,1}$ down states, Markov process)

Example 6.24

Show that Eq. (6.242) can be obtained using a Markov approach, i.e., working with two separate transition rate diagrams for phase 1 and for phase 2, and setting final state probabilities from phase 1 as initial-state probabilities for phase 2.

Solution

Figure 6.32 gives the diagrams of transition rates for phase 1 and 2 (separately). For phase 1, the state probability $P'_{1,0}(t)$ follows from $\dot{P}'_{1,0}(t) = -\lambda_1 P'_{1,0}(t)$ (Table 6.2, Eq. (A7.115), yielding $P'_{1,0}(t) = e^{-\lambda_1 t}$, for $P'_{1,0}(0) = 1$. Thus,

$$R_{S0}(T_1) = P'_{1,0}(T_1) = e^{-\lambda_1 T_1} \quad \text{and} \quad P'_{1,1}(T_1) = 1 - e^{-\lambda_1 T_1}.$$

$P'_{1,1}(t)$ follows from $P'_{1,0}(t) + P'_{1,1}(t) = 1$ or by solving $\dot{P}'_{1,1}(t) = \lambda_1 P'_{1,0}(t)$ with $P'_{1,1}(0) = 0$. Similarly, for phase 2 with t starting at $t = T_1$,

$$\dot{P}'_{2,0}(t - T_1) = -\lambda_2 P'_{2,0}(t - T_1), \quad \text{with} \quad P'_{2,0}(T_1) = P'_{1,0}(T_1) = e^{-\lambda_1 T_1};$$

yielding

$$P'_{2,0}(t - T_1) = e^{-\lambda_1 T_1} e^{-\lambda_2 (t - T_1)}, \qquad T_1 \le t < T_1 + T_2,$$

and thus,

$$R_{S0}(T_1 + T_2) = P'_{2,0}(T_1 + T_2) = e^{-\lambda_1 T_1} e^{-\lambda_2 T_2} = e^{-(\lambda_1 T_1 + \lambda_2 T_2)} = R_{S0}. \tag{6.243}$$

Example 6.25

A one-item system with reliability function $R_{S0}(t)$ is used for a mission of random duration $\tau_W > 0$ distributed according to $F_W(t) = \Pr\{\tau_W \le t\}$ with $F_W(0) = 0$ and density $f_W(t)$. Give the reliability, first for the general case and then by assuming constant failure rate λ and exponentially distributed mission duration ($f_W(t) = \delta e^{-\delta t}$).

Solution

As mission duration can take any time between $(0, \infty)$, reliability takes a constant value given by

$$R_{S0} = \int_0^{\infty} f_W(t) R_{S0}(t)\, dt, \tag{6.244}$$

(see also Eq. (2.76)). For $f_W(t) = \delta e^{-\delta t}$ and constant failure rate λ, Eq. (6.244) yields

$$R_{S0} = \delta / (\delta + \lambda) \approx 1 \quad \text{for } \delta >> \lambda \qquad \text{or} \qquad \approx \delta / \lambda \quad \text{for } \delta << \lambda. \tag{6.245}$$

Supplementary results: In practical application, mission duration is limited to T_W and $\tau_W > 0$ is a truncated random variable with $\Pr\{\tau_W = T_W\} = 1 - F_W(T_W - 0)$; for this case, Eq. (6.245) becomes $R_{S0} = \delta / (\delta + \lambda) + e^{-(\delta + \lambda) T_W} \lambda / (\delta + \lambda)$.

(ii) *Procedure for reliability & availability computation of repairable phased-mission systems with fixed phase duration $T_1, \ldots, T_n$, satisfying the general assumptions* (i):

1. Group *series elements used in all phases* (power supply, cooling, etc.) in one element to be considered in final results (Table 6.10, 2nd row, Eqs. (6.257), (6.258)).
2. Draw the *diagram of transition rates for reliability* evaluation, separately for each phase $(1, \ldots, n)$, beginning by phase 1 with $Z_{1,0}$ (1 referring to phase 1 and 0 being the state in which all elements are as-good-as-new); down states at system level are absorbing states; use the same state numbering for the same state appearing in successive phases; however, state $Z_{k,i}$ corresponding to a state $Z_{c,i}$ in a phase c preceding phase k can also contain as-good-as-new elements appearing in phase k but not in a pervious phase, or standby elements (not used in phase k) with failure rate $\lambda \equiv 0$; for $k > 1$, state $Z_{k,0}$ contains all as-good-as-new elements used in phase k and (as necessary) elements not used in phase k which are standby with failure rate $\lambda \equiv 0$ (*as-good-as-new* is same as *operating or ready to operate*, because of λ_i const.).
3. For *availability* investigation, use results of Table 6.10 (or extend diagrams of transition rates, allowing a return to an operating state after system down according to a given repair strategy) to compute the asymptotic & steady-state availability for *each phase separately* ($PA_{k,S} = AA_{k,S}$ for phase k), taking care of elements which are not used in the phase considered and can act as standby redundancy ($\lambda \equiv 0$) for working elements; for the whole *mission* it holds then

$$PA_S = AA_S \geq \min(PA_{k,S} = AA_{k,S}), \qquad k = 1, \ldots, n \text{ (for } n \text{ phases)}. \tag{6.246}$$

4. For *reliability* investigation, compute the reliability function $R_{1,S0}(T_1)$ at the end of phase 1 starting in state $Z_{1,0}$ at $t = 0$ in the same way as for a one mission system (Table 6.2), as well as states probabilities $P'_{1,j}(T_1)$ for all up states $Z_{1,j}$; if $Z_{1,j}$ (possibly with further as-good-as-new elements used in phase 2) is an up state in phase 2, $P'_{1,j}(T_1)$ becomes the probability $P'_{2,j}(0)$ to start phase 2 in $Z_{2,j}$; if $Z_{1,j}$ is a down state in phase 2, $P'_{1,j}(T_1)$ adds to the initial probability of starting phase 2 in the down state; if $Z_{1,j}$ does not appear in phase 2, $P'_{1,j}(T_1)$ adds to the initial probability in state $Z_{2,0}$ to give $P'_{2,0}(0)$ (from rule 2 above and verifying that for each phase the sum of all states probabilities is 1); reliability calculation must take care of elements which are not used in the phase considered and can act as standby redundancy ($\lambda \equiv 0$) for working elements; continuing in this way, following equation can be found for the *mission reliability* R_{S0} starting phase 1 in $Z_{1,0}$

$$R_{S0} = \sum_{Z_j \in U_n} P'_{n,j}(T_n), \qquad U_n = \text{set of up states in phase } n. \tag{6.247}$$

To simplify the notation used in Example 6.24, the variable x starting by $x = 0$ at the begin of each phase is used in Rule 4 instead of t (starting by $t = 0$ with phase 1).

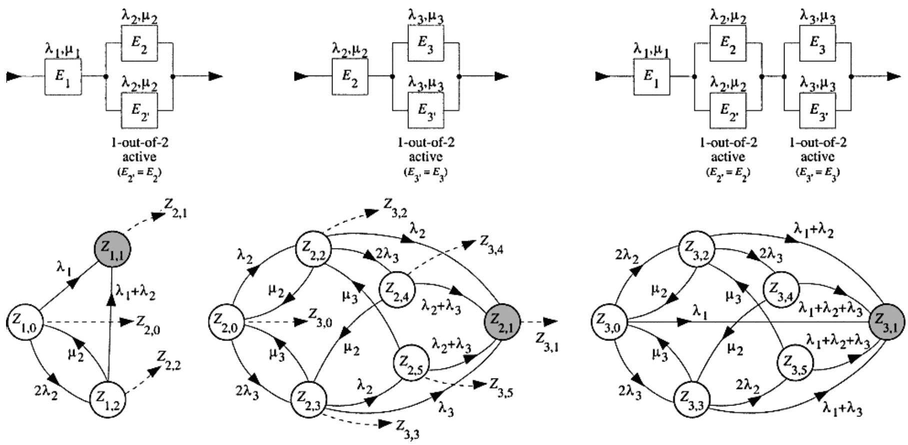


Figure 6.33 Reliability block diagrams and diagram of transition rates for reliability calculation of a phased-mission system with 3 phases (the diagram of transition rates for phase 2 takes care that one element E_2 is put in standby with $\lambda_2 \equiv 0$ as soon as available from phase 1); dashed are indicated to which states the final state probabilities of phase 1 and phase 2 are transferred as initial probabilities for phase 2 and phase 3, respectively (constant failure and repair rates (λ_i, μ_i), ideal failure detection & switch, one repair crew, repair as per first-in first-out, $Z_{1,1}$, $Z_{2,1}$, $Z_{3,1}$ down states, Markov proc.)

As an example let us consider the phased-mission system with 3 phases of given (fixed) duration T_1, T_2 and T_3, described by the 3 reliability block diagrams and the corresponding diagrams of transition rates for reliability investigation given in Fig. 6.33. The diagram of transition rates for phase 2 considers that in phase 2 only one element E_2 is used and assumes that the second element E_2 is put in standby redundancy with failure rate $\lambda_2 \equiv 0$ (either from state $Z_{1,0}$ or as soon as repaired if from state $Z_{1,2}$). Dashed is given to which states the final state probabilities at time T_1 for phase 1 and T_2 ($T_1 + T_2$ with respect to time t) for phase 2 are transferred as initial probabilities for the successive phase. Let us first consider the *asymptotic & steady-state mission availability* $PA_S = AA_S$. From Tables 6.10 and 6.6, it follows for the 3 phases (taken separately) that

$$PA_{1,S} = AA_{1,S} \approx 1 - (\lambda_1 / \mu_1) - 2(\lambda_2 / \mu_2)^2 \approx 1 - (\lambda_1 / \mu_1),$$
$$PA_{2,S} = AA_{2,S} \approx 1 - (\lambda_2 / \mu_2)^2 - 2(\lambda_3 / \mu_3)^2,$$
$$PA_{3,S} = AA_{3,S} \approx 1 - (\lambda_1 / \mu_1) - 2(\lambda_2 / \mu_2)^2 - 2(\lambda_3 / \mu_3)^2 \approx 1 - (\lambda_1 / \mu_1). \qquad (6.248)$$

The 2nd equation considers that in phase 2 one of the elements E_2 acts as standby redundancy with failure rate $\lambda_2 \equiv 0$, combining thus results from Table 6.6 $(1 - (\lambda_2 / \mu_2)^2)$ and Table 6.10 (2nd row). Equation (6.246) yields then

$$PA_S = AA_S \geq \min(PA_{k,S} = AA_{k,S}) \approx 1 - \lambda_1 / \mu_1, \qquad k = 1,2,3. \qquad (6.249)$$

For the *mission reliability* R_{S0}, starting in state $Z_{1,0}$ (all elements are as-good-as-new) at $t=0$, the diagrams of transition rates of Fig. 6.33 yield for phases 1, 2, 3 to following coupled system of differential equations for state probabilities (Table 6.2)

$$\dot{P}'_{1,0}=-(\lambda_1+2\lambda_2)P'_{1,0}+\mu_2 P'_{1,2}, \quad \dot{P}'_{1,2}=-(\lambda_1+\lambda_2+\mu_2)P'_{1,2}+2\lambda_2 P'_{1,0},$$
$$\dot{P}'_{1,1}=\lambda_1 P'_{1,0}+(\lambda_1+\lambda_2)P'_{1,2}, \quad \text{with } P'_{1,0}(0)=1,\ P'_{1,1}(0)=P'_{1,2}(0)=0;$$

$$\dot{P}'_{2,0}=-(\lambda_2+2\lambda_3)P'_{2,0}+\mu_2 P'_{2,2}+\mu_3 P'_{2,3}, \quad \dot{P}'_{2,4}=-(\lambda_2+\lambda_3+\mu_2)P'_{2,4}+2\lambda_3 P'_{2,2},$$
$$\dot{P}'_{2,2}=-(\lambda_2+2\lambda_3+\mu_2)P'_{2,2}+\lambda_2 P'_{2,0}+\mu_3 P'_{2,5}, \quad \dot{P}'_{2,5}=-(\lambda_2+\lambda_3+\mu_3)P'_{2,5}+\lambda_2 P'_{2,3},$$
$$\dot{P}'_{2,3}=-(\lambda_2+\lambda_3+\mu_3)P'_{2,3}+2\lambda_3 P'_{2,0}+\mu_2 P'_{2,4}, \quad \dot{P}'_{2,1}=\lambda_2 P'_{2,2}+\lambda_3 P'_{2,3}+(\lambda_2+\lambda_3)(P'_{2,4}+P'_{2,5}),$$
$$\text{with } P'_{2,0}(0)=P'_{1,0}(T_1),\ P'_{2,2}(0)=P'_{1,2}(T_1),\ P'_{2,1}(0)=P'_{1,1}(T_1),\ P'_{2,3}(0)=P'_{2,4}(0)=P'_{2,5}(0)=0;$$

$$\dot{P}'_{3,0}=-(\lambda_1+2\lambda_2+2\lambda_3)P'_{3,0}+\mu_2 P'_{3,2}+\mu_3 P'_{3,3}, \quad \dot{P}'_{3,3}=-(\lambda_1+2\lambda_2+\lambda_3+\mu_3)P'_{3,3}+2\lambda_3 P'_{3,0}+\mu_2 P'_{3,4},$$
$$\dot{P}'_{3,2}=-(\lambda_1+\lambda_2+2\lambda_3+\mu_2)P'_{3,2}+2\lambda_2 P'_{3,0}+\mu_3 P'_{3,5}, \quad \dot{P}'_{3,5}=-(\lambda_1+\lambda_2+\lambda_3+\mu_3)P'_{3,5}+2\lambda_2 P'_{3,3},$$
$$\dot{P}'_{3,4}=-(\lambda_1+\lambda_2+\lambda_3+\mu_2)P'_{3,4}+2\lambda_3 P'_{3,2}, \quad \dot{P}'_{3,1}=\lambda_1 P'_{3,0}+(\lambda_1+\lambda_2)P'_{3,2}+(\lambda_1+\lambda_3)P'_{3,3}+(\lambda_1+\lambda_2+\lambda_3)(P'_{3,4}+P'_{3,5}),$$
$$\text{with } P'_{3,0}(0)=P'_{2,0}(T_2),\ P'_{3,2}(0)=P'_{2,2}(T_2),\ P'_{3,3}(0)=P'_{2,3}(T_2),\ P'_{3,4}(0)=P'_{2,4}(T_2),$$
$$P'_{3,5}(0)=P'_{2,5}(T_2),\ P'_{3,1}(0)=P'_{2,1}(T_2). \tag{6.250}$$

In Eq. (6.250), $P'_{i,j}$ is used instead of $P'_{i,j}(x)$. From Eq. (6.247) it follows then

$$R_{S0}=P'_{3,0}(T_3)+P'_{3,2}(T_3)+P'_{3,3}(T_3)+P'_{3,4}(T_3)+P'_{3,5}(T_3). \tag{6.251}$$

Analytical solution of the system given by Eq. (6.250) is possible, but time consuming. Numerical solution can be quickly obtained (Example 6.26). A lower bound R_{S0_l} for the mission reliability R_{S0} is obtained by connecting the reliability block diagrams for each phase in series. For Fig. 6.33, this corresponds (practically) to consider phase 3 for a time span $T_1+T_2+T_3$ (in phase 2, for element E_2 a second element E_2 is available in standby redundancy). A good approximation for R_{S0_l} is

Example 6.26

Give the numerical solution of Eqs. (6.250) and (6.251) for $\lambda_1=10^{-4}\,h^{-1}$, $\lambda_2=10^{-2}\,h^{-1}$ $\lambda_3=10^{-3}\,h^{-1}$, $\mu_1=\mu_2=\mu_3=0.5\,h^{-1}$, $T_1=168\,h$, $T_2=336\,h$, and $T_3=672\,h$.

Solution

Numerical solution of the 3 coupled systems of differential equations given by Eq. (6.250) yields

$$P'_{3,0}(T_3)=0.598655,\ P'_{3,2}(T_3)=0.023493,\ P'_{3,3}(T_3)=0.002388,$$
$$P'_{3,4}(T_3)=0.000092,\ P'_{3,5}(T_3)=0.000094,\ P'_{3,1}(T_3)=0.375278 \tag{6.252}$$

(with 6 digits because of $P'_{3,4}(T_3)$ and $P'_{3,5}(T_3)$). R_{S0} follows then from Eq. (6.251)

$$R_{S0}=1-P'_{3,1}(T_3)=0.625. \tag{6.253}$$

Supplementary results: Computing lower and upper bound for R_{S0} as per Eqs. (6.241) and (6.254), yields for the above numerical example $0.55 \le R_{S0} \le 0.71$.

quickly obtained by computing $MTTF_{S0}$ using Table 6.10 and setting this in $R_{S0_l} \approx e^{-(T_1+T_2+T_3)/MTTF_{S0}}$; from this, $MTTF_{S0} \approx 1/(\lambda_1 + 2\lambda_2^2/\mu_2 + 2\lambda_3^2/\mu_3)$ and

$$R_{S0} > R_{S0_l} \approx e^{-(T_1+T_2+T_3)(\lambda_1+2\lambda_2^2/\mu_2+2\lambda_3^2/\mu_3)}. \tag{6.254}$$

Eq. (6.241) allows computation of an upper bound for R_{S0} (Example 6.26).

If the second element E_2 were not available in phase 2 as standby redundancy, $PA_{2,S} = AA_{2,S} \approx 1-\lambda_2/\mu_2$ and, from Eq. (6.249), $PA_S = AA_S \approx 1-\lambda_2/\mu_2$, since $\lambda_1/\mu_1 < \lambda_2/\mu_2$ can be assumed when considering the reliability block diagram for phase 1. Assuming furthermore that the second element E_2 would be repaired before the end of phase 2, if in a failed state at the end of phase 1 $(Z_{1,2})$, the diagram of transition rates for phase 2 would be equal to that for phase 1, with $\lambda_1 \to \lambda_2$, $\lambda_2 \to \lambda_3$, $\mu_2 \to \mu_3$, and $Z_{1,0} \to Z_{2,0}$, $Z_{1,1} \to Z_{2,1}$, $Z_{1,2} \to Z_{2,3}$ with

$$P'_{2,0}(0) = P'_{1,0}(T_1) + P'_{1,2}(T_1), \quad P'_{2,1}(0) = P'_{1,1}(T_1), \quad P'_{2,3}(0) = 0. \tag{6.255}$$

The corresponding initial probabilities for phase 3 would be

$$P'_{3,0}(0) = P'_{2,0}(T_2), \quad P'_{3,1}(0) = P'_{2,1}(T_2), \quad P'_{3,3}(0) = P'_{2,3}(T_2),$$
$$P'_{3,2}(0) = P'_{3,4}(0) = P'_{3,5}(0) = 0. \tag{6.256}$$

If an element E_{ser} where common to all 3 phases in Fig. 6.33 (i.e. in series with all 3 reliability block diagrams), Table 6.10 (2nd row) can be used to find

$$PA_{S_{tot}} = AA_{S_{tot}} \approx 1 - \lambda_{ser}/\mu_{ser} - \lambda_1/\mu_1 \tag{6.257}$$

(considering Eq. (6.249)) and, with R_{S0} from Eq. 6.251,

$$R_{S0_{tot}} \approx R_{S0} \cdot e^{-\lambda_{ser}(T_1+T_2+T_3)}. \tag{6.258}$$

The above procedure can be extended to consider more than one repair crew at system level or any kind of repair (restore) strategy. Other procedures (models) are conceivable. For instance, for *nonrepairable* systems (up to system failure) of complex structure, and with independent elements (parallel redundancy), it can be useful to number the states using binary considerations.

For randomly distributed phase duration, Eq. (6.246) can be used for availability. Reliability can be obtained by expanding results in Examples 6.23 - 6.25.

An alternative approach for phased-mission systems is to assume that at the begin of each mission phase, the system is *as-good-as-new* with respect to the elements used in the mission phase considered (required elements are repaired in a negligible time at the begin of the mission phase, if they are in a failed state, and not required elements can be repaired during a phase in which they are not used). This assumption can be reasonable for some repairable systems and highly simplifies investigation. For this case, results developed in Section 6.8.2 for preventive maintenance lead to (for phases 1,2,...)

$$\begin{aligned} R_S(t) &= R_{S1}(t), && \text{for} \quad 0 \le t < T_1^* \\ &= R_{S1}(T_1^*) R_{S2}(t - T_1^*), && \text{for} \quad T_1^* \le t < T_2^* \\ &= R_{S1}(T_1^*) R_{S2}(T_2^* - T_1^*) R_{S3}(t - T_2^*), && \text{for} \quad T_2^* \le t < T_3^* \\ &\vdots \end{aligned} \tag{6.259}$$

for the reliability function, and

$$\begin{aligned} PA_S(t) &= PA_{S1}(t), && \text{for} \quad 0 \le t < T_1^* \\ &= PA_{S2}(t - T_1^*), && \text{for} \quad T_1^* \le t < T_2^* \\ &= PA_{S3}(t - T_2^*), && \text{for} \quad T_2^* \le t < T_3^* \\ &\vdots \end{aligned} \tag{6.260}$$

for the point availability. S_i is the state from which the ith mission phase starts; $0, T_1^*, T_2^*, \ldots$ are the time points on the time axis at which the mission phase 1, 2, 3, ... begin (the mission duration of phase i being here $T_i^* - T_{i-1}^*$ with $T_0^* = 0$).

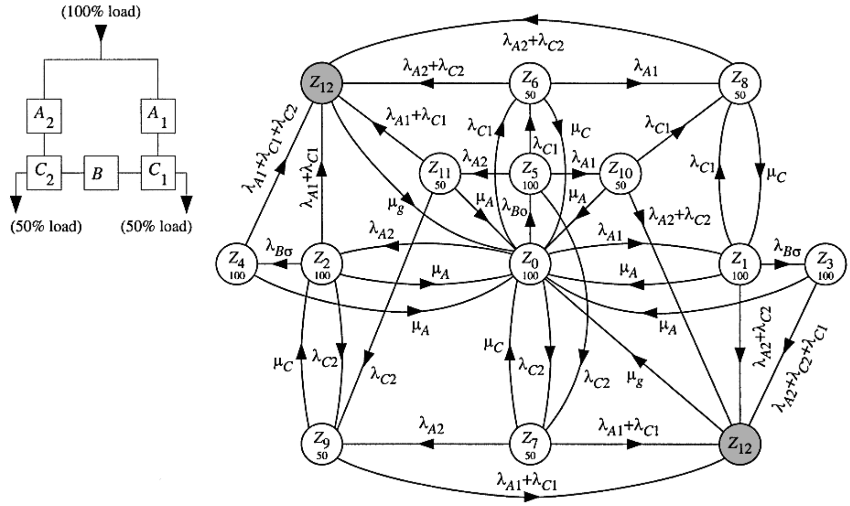
6.8.6.3 Failure Censored Reconfiguration

In most applications, reconfiguration occurs at the failure of a redundant element. Besides cases with ideal fault coverage, ideal switching, and no system performance reduction at failure (Sections 2.2, 2.3, and 6.4-6.7), more complex structures often arise in practical applications (see Sections 6.8.3 - 6.8.5 for some examples). Such structures must be investigated on a case-by-case basis, and an FMEA / FMECA (Section 2.6) is mandatory to validate investigations. Often it is necessary to consider that after a reconfiguration, the system performance is reduced, i.e., *reward and frequency / duration* aspects have to be involved in the analysis.

A reasonably simple and comprehensive example is a power system substation. Figure 6.34 gives the functional block diagram and the *diagram of transition rates* for availability calculation, $\mu_g \equiv 0$ for reliability investigation. Z_{12} is the down state. The substation is powered by a reliable network and consists of:

- Two branch *designated by* A_1 & A_2 *and capable of performing 100% load,* each with HV switch, HV circuit breaker and control elements, transformer, measurement & control elements, and LV switch.
- Two busbars *designated by* C_1 & C_2 *and capable of performing 100% load* (failure rate basically given by double contingency of faults on control elements).
- A coupler between the busbars, *designated by* B *and capable of performing 100% load*; failure modes *stuck at the state occupied just before failure* (does not open), failure rate $\lambda_{B\sigma}$, and *no connection* (does not close), failure rate λ_{B0}.

Load is distributed between C_1 and C_2 at 50% rate each. The *diagram of transition rates* is based on an extensive FMEA / FMECA [6.20 (2002)] showing in particular the key position of the coupler B in the reconfiguration strategy. Coupler B is normally open. A failure of B is recognized only at a failure of A or C. From state Z_0, B can fail only with failure mode *no connection,* from Z_1 or Z_2 only with failure mode



$\rho_{01}=\lambda_{A1}$; $\rho_{02}=\lambda_{A2}$; $\rho_{05}=\lambda_{Bo}$; $\rho_{06}=\lambda_{C1}$; $\rho_{07}=\lambda_{C2}$; $\rho_{10}=\mu_A$; $\rho_{13}=\lambda_{B\sigma}$; $\rho_{18}=\lambda_{C1}$; $\rho_{1\,12}=\lambda_{A2}+\lambda_{C2}$;
$\rho_{20}=\mu_A$; $\rho_{24}=\lambda_{B\sigma}$; $\rho_{29}=\lambda_{C2}$; $\rho_{2\,12}=\lambda_{A1}+\lambda_{C1}$; $\rho_{30}=\mu_A$; $\rho_{3\,12}=\lambda_{A2}+\lambda_{C2}+\lambda_{C1}$; $\rho_{4\,12}=\lambda_{A1}+\lambda_{C1}+\lambda_{C2}$;
$\rho_{40}=\mu_A$; $\rho_{56}=\lambda_{C1}$; $\rho_{57}=\lambda_{C2}$; $\rho_{5\,10}=\lambda_{A1}$; $\rho_{5\,11}=\lambda_{A2}$; $\rho_{60}=\mu_C$; $\rho_{68}=\lambda_{A1}$; $\rho_{6\,12}=\lambda_{A2}+\lambda_{C2}$; $\rho_{70}=\mu_C$;
$\rho_{79}=\lambda_{A2}$; $\rho_{7\,12}=\lambda_{A1}+\lambda_{C1}$; $\rho_{81}=\mu_C$; $\rho_{8\,12}=\lambda_{A2}+\lambda_{C2}$; $\rho_{92}=\mu_C$; $\rho_{9\,12}=\lambda_{A1}+\lambda_{C1}$; $\rho_{10\,0}=\mu_A$; $\rho_{10\,8}=\lambda_{C1}$;
$\rho_{10\,12}=\lambda_{A2}+\lambda_{C2}$; $\rho_{11\,0}=\mu_A$; $\rho_{11\,9}=\lambda_{C2}$; $\rho_{11\,12}=\lambda_{A1}+\lambda_{C1}$; $\rho_{12\,0}=\mu_g$; $\rho_i=\sum_j \rho_{ij}$

Figure 6.34 Functional block diagram and diagram of transition rates for availability calculation of a power system substation (active redundancy, constant failure and repair rates $\lambda_{A1}, \lambda_{A2}, \lambda_{B\sigma}, \lambda_{Bo}, \lambda_{C1}, \lambda_{C2}, \mu_A, \mu_C, \mu_g$, imperfect switching of B with failure modes *does not open* ($\lambda_{B\sigma}$, from Z_1 and Z_2) or *no connection* (λ_{Bo}, from Z_0), failure of B recognized only at failure of A or C, ideal failure detection for A and C, one repair crew, repair priority on C, no further failure at system down, Z_{12} down state, Markov process, $\mu_g \equiv 0$ for reliability calculation)

stuck at the state occupied just before failure. Constant failure rates $\lambda_{A1}, \lambda_{A2}, \lambda_{B\sigma}, \lambda_{Bo}, \lambda_{C1}, \lambda_{C2}$ and constant repair rates μ_A, μ_C, μ_g are assumed. μ_A and μ_C remain the same also if a repair of B is necessary; μ_g is larger than μ_A and μ_C. From the down state (Z_{12}) the system returns to state Z_0. Furthermore, only one repair crew, *repair priority* on C (followed by $C+B, A, A+B$), and no further failure at system down (50% load is an up state with reduced performance) are assumed. Sought are mean time to system failure $MTTF_{S0}$ for system new (in state Z_0) at $t=0$ and *asymptotic & steady-state* point and average availability $PA_S = AA_S$. The involved process is a (time-homogeneous) Markov process. If results are required for 100% load, $Z_6 - Z_{11}$ are down states (see Section 6.8.6.4 for reward considerations). To simplify investigation, $\lambda_{A1}=\lambda_{A2}=\lambda_A$ and $\lambda_{C1}=\lambda_{C2}=\lambda_C$ are assumed. To increase readability, the number of states in Fig. 6.34 has been reduced as per Point 2 on p. 273.

From Fig. 6.34 and Table 6.2 or Eq. (A7.126) it follows that $MTTF_{S0}$ is given as solution of the following system of algebraic equations (with $M_i = MTTF_{Si}$)

$$\rho_0 M_0 = 1 + \lambda_{B0} M_5 + \lambda_A (M_1 + M_2) + \lambda_C (M_6 + M_7), \quad \rho_5 M_5 = 1 + \lambda_A (M_{10} + M_{11}) + \lambda_C (M_6 + M_7),$$
$$\rho_1 M_1 = 1 + \mu_A M_0 + \lambda_C M_8 + \lambda_{B\sigma} M_3, \quad \rho_2 M_2 = 1 + \mu_A M_0 + \lambda_C M_9 + \lambda_{B\sigma} M_4, \quad \rho_3 M_3 = 1 + \mu_A M_0,$$
$$\rho_4 M_4 = 1 + \mu_A M_0, \quad \rho_6 M_6 = 1 + \lambda_A M_8 + \mu_C M_0, \quad \rho_7 M_7 = 1 + \lambda_A M_9 + \mu_C M_0, \quad \rho_8 M_8 = 1 + \mu_C M_1,$$
$$\rho_9 M_9 = 1 + \mu_C M_2, \quad \rho_{10} M_{10} = 1 + \mu_A M_0 + \lambda_C M_8, \quad \rho_{11} M_{11} = 1 + \mu_A M_0 + \lambda_C M_9. \qquad (6.261)$$

Because of $\lambda_{A1} = \lambda_{A2} = \lambda_A$, $\lambda_{C1} = \lambda_{C2} = \lambda_C$ and the symmetry in Fig. 6.34 it follows that $\rho_2 = \rho_1$, $\rho_4 = \rho_3$, $\rho_7 = \rho_6$, $\rho_9 = \rho_8$, $\rho_{11} = \rho_{10}$ and $M_2 = M_1$, $M_4 = M_3$, $M_7 = M_6$, $M_9 = M_8$, $M_{11} = M_{10}$. This has been considered in solving the system of algebraic equations (6.261). From Eq. (6.261) it follows that

$$MTTF_{S0} = \frac{a_1 a_2 + 2 a_1 a_3 \lambda_A + 2 a_2 \lambda_C \rho_5 \rho_{10} (\rho_8 + \lambda_A) + 2 a_3 \lambda_A \lambda_C \mu_C \rho_5 \rho_{10} + \lambda_{B0} (a_2 a_4 + a_3 a_6)}{a_1 a_2 \rho_0 - 2 \lambda_A \mu_A \rho_8 (\rho_3 + \lambda_{B\sigma})(a_1 + \lambda_C \mu_C \rho_5 \rho_{10}) - 2 a_2 \lambda_C \mu_C \rho_5 \rho_8 \rho_{10} - \lambda_{B0} (a_2 a_5 + a_6 \mu_A \rho_8 (\rho_3 + \lambda_{B\sigma}))},$$

with (6.262)

$$a_1 = \rho_5 \rho_6 \rho_8 \rho_{10}, \quad a_2 = \rho_1 \rho_3 \rho_8 - \rho_3 \lambda_C \mu_C, \quad a_4 = \rho_6 \rho_8 \rho_{10} + 2 \lambda_A \rho_6 (\rho_8 + \lambda_C) + 2 \lambda_C \rho_{10} (\rho_8 + \lambda_A),$$
$$a_3 = \rho_8 (\rho_3 + \lambda_{B\sigma}) + \rho_3 \lambda_C, \quad a_5 = 2 \lambda_A \mu_A \rho_6 \rho_8 + 2 \lambda_C \mu_C \rho_8 \rho_{10}, \quad a_6 = 2 \lambda_A \lambda_C \mu_C (\rho_6 + \rho_{10}), \qquad (6.263)$$

and

$$\rho_0 = 2\lambda_A + 2\lambda_C + \lambda_{B0}, \quad \rho_1 = \rho_2 = \lambda_A + 2\lambda_C + \lambda_{B\sigma} + \mu_A, \quad \rho_3 = \rho_4 = \lambda_A + 2\lambda_C + \mu_A, \quad \rho_5 = 2(\lambda_A + \lambda_C),$$
$$\rho_6 = \rho_7 = 2\lambda_A + \lambda_C + \mu_C, \quad \rho_8 = \rho_9 = \lambda_A + \lambda_C + \mu_C, \quad \rho_{10} = \rho_{11} = \lambda_A + 2\lambda_C + \mu_A, \quad \rho_{12} = \mu_g. \qquad (6.264)$$

$MTTF_{S0}$ per Eq. (6.262) can be approximated by

$$MTTF_{S0} \approx \frac{\mu_A + 5(\lambda_A + \lambda_C) + (4\lambda_A + 5\lambda_C)\mu_A / \mu_C + \lambda_{B\sigma} + \lambda_{B0}(\mu_A + \lambda_{B\sigma}) / 2(\lambda_A + \lambda_C)}{(2\lambda_A + 2\lambda_C + \lambda_{B0})(\lambda_A + \lambda_C \mu_A / \mu_C)},$$

(6.265)

yielding $MTTF_{S0} \approx \mu / 2(\lambda_A + \lambda_C)^2$ for $\lambda_{B\sigma} = \lambda_{B0} = 0$ and $\mu_A = \mu_C = \mu$ (1-out-of-2 active redundancy with A and C in series, as per Table 6.10, 2nd & 3rd row).

From Fig. 6.34 and Table 6.2 or Eq. (A7.127) it follows that the *asymptotic & steady-state* point and average availability $PA_S = AA_S$ is given as solution of

$$\rho_0 P_0 = \mu_A (P_1 + P_2 + P_3 + P_4 + P_{10} + P_{11}) + \mu_C (P_6 + P_7) + \mu_g P_{12}, \quad \rho_1 P_1 = \lambda_A P_0 + \mu_C P_8,$$
$$\rho_2 P_2 = \lambda_A P_0 + \mu_C P_9, \quad \rho_3 P_3 = \lambda_{B\sigma} P_1, \quad \rho_4 P_4 = \lambda_{B\sigma} P_2, \quad \rho_5 P_5 = \lambda_{B0} P_0,$$
$$\rho_6 P_6 = \lambda_C (P_0 + P_5), \quad \rho_7 P_7 = \lambda_C (P_0 + P_5), \quad \rho_8 P_8 = \lambda_A P_6 + \lambda_C (P_1 + P_{10}),$$
$$\rho_9 P_9 = \lambda_A P_7 + \lambda_C (P_2 + P_{11}), \quad \rho_{10} P_{10} = \lambda_A P_5, \quad \rho_{11} P_{11} = \lambda_A P_5,$$
$$\rho_{12} P_{12} = (\lambda_A + \lambda_C)(P_1 + P_2 + P_6 + P_7 + P_8 + P_9 + P_{10} + P_{11}) + (\lambda_A + 2\lambda_C)(P_3 + P_4). \qquad (6.266)$$

One of the Eq. (6.266) must be dropped and replaced by $\sum P_i = 1$. The solution yields

$$P_1 = P_2 = b_1 P_0, \quad P_3 = P_4 = b_1 P_0 \lambda_{B\sigma} / \rho_3, \quad P_5 = P_0 \lambda_{B0} / \rho_5, \quad P_8 = P_9 = b_2 P_0,$$
$$P_6 = P_7 = P_0 \lambda_C (\rho_5 + \lambda_{B0}) / \rho_5 \rho_6, \quad P_{10} = P_{11} = P_0 \lambda_A \lambda_{B0} / \rho_5 \rho_{10}, \quad P_{12} = b_3 P_0, \qquad (6.267)$$

with

$$P_0 = \frac{1}{1+2b_2+2b_1(1+\lambda_{B\sigma}/\rho_3)+\lambda_{Bo}/\rho_5+2\lambda_C(\rho_5+\lambda_{Bo})/\rho_5\rho_6+2\lambda_A\lambda_{Bo}/\rho_5\rho_{10}+b_3}$$

and (6.268)

$$b_1 = \frac{\rho_5\rho_6\rho_8\rho_{10}\lambda_A+\rho_{10}\lambda_A\lambda_C\mu_C(\rho_5+\lambda_{Bo})+\rho_6\lambda_A\lambda_C\lambda_{Bo}\mu_C}{\rho_1\rho_5\rho_6\rho_8\rho_{10}-\rho_5\rho_6\rho_{10}\lambda_C\mu_C},$$

$$b_2 = b_1\frac{\lambda_C}{\rho_8}+\frac{\lambda_A\lambda_C(\rho_5+\lambda_{Bo})}{\rho_6\rho_8\rho_5}+\frac{\lambda_A\lambda_C\lambda_{Bo}}{\rho_5\rho_8\rho_{10}},$$

$$b_3 = \frac{2(\lambda_A+\lambda_C)}{\rho_{12}}\left[b_1+b_2+\frac{\lambda_C(\rho_5+\lambda_{Bo})}{\rho_5\rho_6}+\frac{\lambda_A\lambda_{Bo}}{\rho_5\rho_{10}}\right]+2(\lambda_A+2\lambda_C)\frac{\lambda_{B\sigma}}{\rho_3\rho_{12}}. \qquad (6.269)$$

From Eqs. (6.267) - (6.269) it follows that

$$PA_S = AA_S = \sum_{i=0}^{11} P_i = 1-P_{12} = 1-b_3P_0$$

$$= \frac{1}{1+\dfrac{b_3}{1+2b_2+2b_1(1+\lambda_{B\sigma}/\rho_3)+\lambda_{Bo}/\rho_5+2\lambda_C(\rho_5+\lambda_{Bo})/\rho_5\rho_6+2\lambda_A\lambda_{Bo}/\rho_5\rho_{10}}}.$$

6.270)

$PA_S = AA_S$ per Eq. (6.270) can be approximated by

$$PA_S = AA_S \approx 1-\frac{2(\lambda_A+\lambda_C)(\lambda_C\mu_A+\mu_C(\lambda_A+\lambda_{B\sigma}))+\lambda_{Bo}(\lambda_A\mu_C+\lambda_C(\mu_A+\mu_C\lambda_{B\sigma}/\lambda_{Bo}))}{\mu_g[\mu_A\mu_C+2(\lambda_A\mu_C+\lambda_C\mu_A)(1+\lambda_{Bo}/(\lambda_A+\lambda_C))+2\lambda_A\lambda_{B\sigma}\mu_C/\mu_A]}. \qquad (6.271)$$

yielding $PA_S = AA_S \approx 1-2((\lambda_A+\lambda_C)/\mu)^2$ for $\lambda_{B\sigma} = \lambda_{Bo} = 0$ and $\mu_A = \mu_C = \mu_g = \mu$ (1-out-of-2 active redundancy with A and C in series, as per Table 6.10). Equations (6.265) and (6.271) show the small influence of the coupler B. A numerical evaluation with

$\lambda_{A1} = \lambda_{A2} = \lambda_A = 4\cdot 10^{-6}\ h^{-1}$ ($\approx$ 0.035 expected failures per year)
$\lambda_{C1} = \lambda_{C2} = \lambda_C = 0.12\cdot 10^{-6}\ h^{-1}$ ($\approx$ 0.001 expected failures per year)
$\lambda_{B\sigma} = 0.08\cdot 10^{-6}\ h^{-1}$ ($\approx$ 0.0007 expected failures per year)
$\lambda_{Bo} = 0.6\cdot 10^{-6}\ h^{-1}$ ($\approx$ 0.005 expected failures per year)
$\mu_A = \mu_C = 1/4h$, $\mu_g = 1/12h$

yields

$$MTTF_{S0} \approx 7.36\ 10^9 h \quad \text{and} \quad PA_S = AA_S \approx 1-1.63\cdot 10^{-9}$$

from Eqs.(6.262) & (6.270), as well as $MTTF_{S0} \approx 7.3\cdot 10^9 h$ and $PA_S = AA_S \approx 1-0.9\cdot 10^{-9}$ from Eqs. (6.265) & (6.271), respectively; moreover,

$P_0 = 0.932096$, $P_1 = P_2 = 1.491\cdot 10^{-5}$, $P_3 = P_4 = 4.77\cdot 10^{-12}$, $P_5 = 0.067871$,
$P_6 = P_7 = 4.80\cdot 10^{-7}$, $P_8 = P_9 = 1.53\cdot 10^{-11}$, $P_{10} = P_{11} = 1.09\cdot 10^{-6}$, $P_{12} = 1.63\cdot 10^{-9}$.

Considering the substation as a *macro-structure* (first row in Table 6.10), it holds that $PA_S = AA_S \approx 1-\lambda_S/\mu_S$ and $R_S(t) \approx e^{-\lambda_S t}$, with $\mu_S = \mu_g$ and $\lambda_S = 1/MTTF_{S0}$.

6.8.6.4 Reward and Frequency / Duration Aspects

For some applications, e. g. in power and communication systems, it is of importance to consider system performance also in the presence of failures. *Reward* and *frequency/duration* aspects are of interest to evaluate *system performability.* For constant failure and repair rates (Markov processes), asymptotic & steady-state *system failure frequency* f_{udS} and *system mean down time* MDT_S (mean repair (restoration) duration at system level) are given as (Eqs. (A7.143) & (A7.144))

$$f_{udS} = \sum_{Z_j \in U, Z_i \in \bar{U}} P_j \, \rho_{ji} = \sum_{Z_j \in U} P_j \, (\sum_{Z_i \in \bar{U}} \rho_{ji}) \tag{6.272}$$

and

$$MDT_S = (1 - \sum_{Z_j \in U} P_j) \, / \, f_{udS} = (1 - PA_S) \, / \, f_{udS}, \tag{6.273}$$

respectively (Eq. (6.273) can be heuristically explained, considering that for $T \to \infty$,. $(1 - PA_S)T$ is the *mean down time* in $(0, T]$ and $T \, / \, f_{udS}$ the *mean number of repairs* (and failures) in $(0, T]$. Similar results hold for semi-Markov processes. U is the set of states considered as *up states* for f_{udS} and MDT_S calculation, $\bar{U}$ is the complement to the totality of states considered. P_j is the asymptotic & steady-state probability of state Z_j and ρ_{ji} the transition rate from Z_j to Z_i. In Eq. (6.272), all transition rates ρ_{ji} leaving state $Z_j \in U$ toward $Z_i \in \bar{U}$ are considered (cumulated states). Example 6.27 gives an application to the substation investigated in Fig. 6.34. Considering $f_{udS} = f_{duS}$ (Eq. (A7.145)), f_{udS} can be replaced by f_{duS}.

Example 6.27

Give the *failure frequency* f_{udS} and the *mean failure duration* MDT_S in steady-state for the substation of Fig. 6.34 for failures referred to a load loss of 100% and 50%, respectively.

Solution

For loss of 100% load, Fig. 6.34 with $U = \{Z_0, ..., Z_{11}\}$, $\bar{U} = \{Z_{12}\}$ yields (P_i as per Eq. (6.267))

$$f_{udS \text{ loss } 100\%} = 2(P_1 + P_6 + P_8 + P_{10})(\lambda_A + \lambda_C) + 2P_3(\lambda_A + 2\lambda_C).$$

For loss of 50% load, Fig. 6.34 with $U = \{Z_0 - Z_5\}$ and $\bar{U} = \{Z_6 - Z_{12}\}$ yields

$$f_{udS \text{ loss } 50\%} = P_0 2\lambda_C + 2P_1(\lambda_A + 2\lambda_C) + 2P_3(\lambda_A + 2\lambda_C) + P_5 2(\lambda_A + \lambda_C).$$

From Eq. (6.273) it follows that

$$MDT_{S \text{ loss } 100\%} = P_{12} \, / \, f_{udS \text{ loss } 100\%},$$

and

$$MDT_{S \text{ loss } 50\%} = 1 - (P_0 + 2P_1 + 2P_3 + P_5) \, / \, f_{udS \text{ loss } 50\%}.$$

The numerical example on p. 265 yields $f_{udS \text{ loss } 100\%} \approx 136 \cdot 10^{-12} \text{h}^{-1}$ ($\approx 10^{-6}$ expected failures per year), $f_{udS \text{ loss } 50\%} \approx 783 \cdot 10^{-9} \text{h}^{-1}$ ($\approx 8 \cdot 10^{-3}$ expected failures per year), $MDT_{S \text{ loss } 100\%} \approx 12\,\text{h}$, and $MDT_{S \text{ loss } 50\%} \approx 4\,\text{h}$.

Example 6.28
Give the expected *instantaneous reward rate* in steady-state for the substation of Fig. 6.34.
Solution
Considering Fig. 6.34 and the numerical example on p. 265 it follows that

$$MIR_S = 1 \cdot (P_0 + 2P_1 + 2P_3 + P_5) + 0.5 \cdot (2P_6 + 2P_8 + 2P_{10}) \approx 0.9999984.$$

The *reward rate* r_i takes care of the performance reduction in the state considered, ($r_i = 0$ for down states, $0 < r_i < 1$ for partially down states, and $r_i = 1$ for up states with 100% performance). From this, the expected *instantaneous reward rate* in steady-state or for $t \to \infty$, MIR_S, is given as (Eq. (A7.147))

$$MIR_S = \sum_{i=0}^{m} r_i P_i, \tag{6.274}$$

The *expected accumulated reward* in steady-state (or for $t \to \infty$) follows as $MAR_S(t) = MIR_S \cdot t$, see Example 6.28 for an application. P_i in Eq. (6.274) is the asymptotic & steady-state probability of state Z_i, giving also the expected percentage of time the system stays at the performance level specified by Z_i (Eq. (A7.132)).

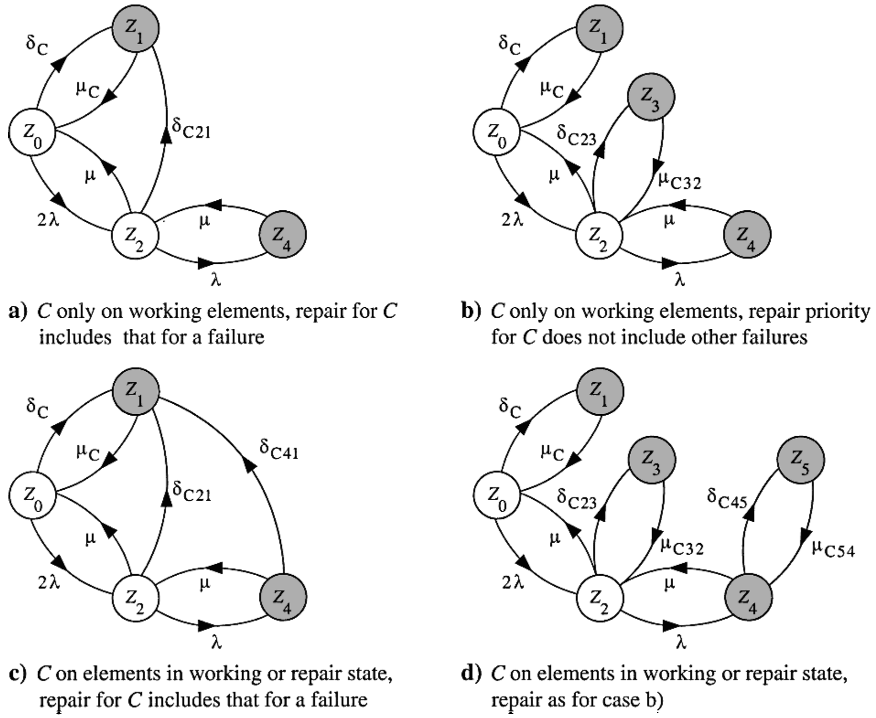
6.8.7 Systems with Common Cause Failures

In some practical applications it is necessary to consider that common cause failures can occur. *Common cause failures* (C) are multiple failures resulting from a single cause. They must be distinguished from *common mode failures*, which are multiple failures showing the same symptom. Common cause failures can occur in hardware as well as in software. Their causes can be quite different. Some possible causes for common cause failures in hardware are:

- overload (electrical, thermal, mechanical),
- technological weakness (material, design, production),
- misuse (caused e.g. by operating or maintenance personnel),
- external event.

Similar causes can be found for software.

In the following, a 1-out-of-2 active redundancy is used as a basic example for investigating effects of common cause failures. Results (Eqs. (6.276) & (6.280)) show that common cause failure acts (in general) as *a series element* in the system's reliability structure, with failure rate equal the occurrence rate δ_C of the common cause failure and repair (restoration) rate equal the remove rate μ_C of the common cause failure. Graphs given by Figs. 2.8 & 6.17 and rules (2.28) & (6.174) can be used to limit effects of common cause failures.



a) *C* only on working elements, repair for *C* includes that for a failure

b) *C* only on working elements, repair priority for *C* does not include other failures

c) *C* on elements in working or repair state, repair for *C* includes that for a failure

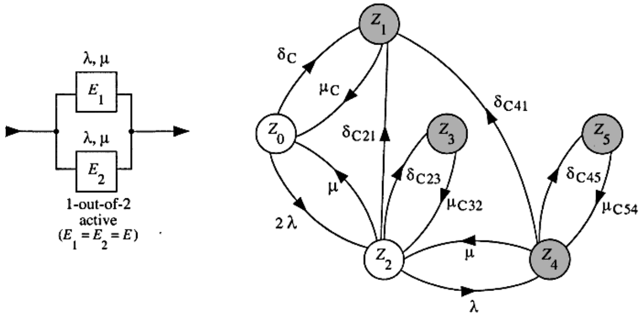
d) *C* on elements in working or repair state, repair as for case b)

Figure 6.35 Diagram of transition rates of the 1-out-of-2 active redundancy of Fig. 6.36 with *common cause failures* (C) for 4 different basic possibilities (constant failure and repair rates ($\lambda, \mu, \mu_C, \mu_{Ci}$), constant occurrence rates for C (δ_C, δ_{Ci} often with $\delta_{Ci} = \delta_C$), ideal failure detection and switch, one repair crew, repair priority on C, no further failures at system down (except for $\delta_{C41}, \delta_{C45}$), Z_1, Z_3, Z_4, Z_5 down states (absorbing for reliability calculation), Markov processes)

Figure 6.35 gives the diagrams of transition rates for the repairable 1-out-of-2 active redundancy of Fig. 6.36 with common cause failures for 4 different basic possibilities (C refers to *common cause failures*, repair priority for C, one repair crew, no further failures at system down except for $\delta_{C41}, \delta_{C45}$). To clarify results, occurrence rates δ_{Ci} and repair rates μ_{Ci} for common cause failures are assumed to be each other different when moving from one state to the other (for simplicity in the final Eqs. (6.276) & (6.280), $\delta_{C01} = \delta_C$ and $\mu_{C10} = \mu_C$). The 4 possibilities of Fig. 6.35 are resumed in Fig. 6.36 for investigation. From Fig. 6.36 and Table 6.2, $MTTF_{S0}$ is given as solution of the following system of algebraic equations (all down states (Z_1, Z_3, Z_4, Z_5) are absorbing for reliability investigation)

$$(2\lambda + \delta_C)\, MTTF_{S0} = 1 + 2\lambda\, MTTF_{S2}, \qquad (\lambda + \delta_{C21} + \delta_{C23} + \mu)\, MTTF_{S2} = 1 + \mu\, MTTF_{S0}. \tag{6.275}$$

From Eq. (6.275), $MTTF_{S0}$ follows as (for $\delta_C = \delta_{C01} \le \lambda$),



$\rho_{01}=\delta_C$; $\rho_{02}=2\lambda$ ($\lambda+\lambda_r$ for warm redundancy); $\rho_{24}=\lambda$; $\rho_{21}=\delta_{C21}$;
$\rho_{23}=\delta_{C23}$; $\rho_{41}=\delta_{C41}$; $\rho_{45}=\delta_{C45}$; $\rho_{10}=\mu_C$; $\rho_{32}=\mu_{C32}$; $\rho_{20}=\rho_{42}=\mu$;
$\rho_{54}=\mu_{C54}$; $\rho_i=\Sigma\rho_{ij}$; for $R_S(t)$ set $\rho_{10}=\rho_{32}=\rho_{41}=\rho_{42}=\rho_{45}=0$

Figure 6.36 Reliability block diagram and diagram of transition rates for availability calculation of a 1-out-of-2 active redundancy with *common cause failures* (*C*) for different possibilities as per Fig.6.35

$$MTTF_{S0} = \frac{3\lambda+\delta_{C21}+\delta_{C23}+\mu}{(2\lambda+\delta_C)(\lambda+\delta_{C21}+\delta_{C23})+\mu\delta_C} = \frac{1}{\delta_C+2\lambda(\lambda+\delta_{C21}+\delta_{C23}-\delta_C)\,/\,(3\lambda+\delta_{C21}+\delta_{C23}+\mu)} \le \frac{1}{\delta_C}. \tag{6.276}$$

Furthermore, from Fig. 6.36 and Table 6.2, the *asymptotic & steady-state* point and average availability $PA_S = AA_S$ is given as solution of the following system of algebraic equations

$$\rho_0 P_0 = \mu_C P_1 + \mu P_2; \quad \rho_1 P_1 = \delta_C P_0 + \delta_{C21} P_2 + \delta_{C41} P_4; \quad \rho_5 P_5 = \delta_{C45} P_4;$$
$$\rho_3 P_3 = \delta_{C23} P_2; \quad \rho_4 P_4 = \lambda P_2 + \mu_{C54} P_5; \quad \rho_2 P_2 = 2\lambda P_0 + \mu_{C32} P_3 + \mu P_4. \tag{6.277}$$

One of the Eq. (6.277) must be dropped and replaced by $P_0+...+P_5=1$ (the first equation because of the particular cases investigated below). The solution yields

$$PA_S = AA_S = P_0 + P_2 = \frac{1+a_2}{a_3+\lambda_C/\rho_1+a_1a_2\lambda_{41}/\rho_1+a_2\lambda_{21}/\rho_1}. \tag{6.278}$$

with

$$a_1 = \lambda\rho_5\,/\,(\rho_4\rho_5-\delta_{C45}\mu_{C54}), \quad a_2 = 2\lambda\rho_3\,/\,(\rho_2\rho_3-\delta_{C23}\mu_{C32}-a_1\mu\rho_3),$$
$$a_3 = 1+a_2(1+\delta_{C23}/\rho_3)+a_1a_2(1+\delta_{C45}/\rho_5), \tag{6.279}$$

and $\rho_0=2\lambda+\delta_C$, $\rho_1=\mu_C$, $\rho_2=\lambda+\delta_{C21}+\lambda_{C23}+\mu$, $\rho_3=\mu_{C32}$, $\rho_4=\delta_{C41}+\delta_{C41}+\mu$, $\rho_5=\mu_{C54}$ (Fig. 6.36, Eq. (A7.103)). Considering $\lambda \ll \mu$, $\delta_C \ll \mu_C$, $\delta_{Ci} \ll \mu_{Ci}$ it follows that

$$PA_S = AA_S \lessapprox \frac{\mu_C}{\delta_C+\mu_C} \approx 1-\delta_C/\mu_C. \tag{6.280}$$

Equations (6.276) & (6.278) can be used to investigate Fig. 6.35, yielding (for $\delta_C \le \lambda$)

$$MTTF_{S0} = \frac{1}{\delta_C + \dfrac{2\lambda(\lambda+\delta_{C21}-\delta_C)}{3\lambda+\delta_{C21}+\mu}} \le \frac{1}{\delta_C}$$

$$PA_S \approx 1 - \frac{\delta_C}{\mu_C} - \frac{2\lambda}{2\lambda+\delta_{C21}+\mu}\left(\frac{\lambda}{\mu} + \frac{\delta_{C21}}{\mu_C} - \frac{\delta_C}{\mu_C}\right)$$

a) Common cause failures (C) only on working elements, repair for C includes that for failure

$$MTTF_{S0} = \frac{1}{\delta_C + \dfrac{2\lambda(\lambda+\delta_{C23}-\delta_C)}{3\lambda+\delta_{C23}+\mu}} \le \frac{1}{\delta_C}$$

$$PA_S \approx 1 - \frac{\delta_C}{\mu_C} - \frac{2\lambda}{2\lambda+\mu}\left(\frac{\lambda}{\mu} + \frac{\delta_{C23}}{\mu_{C32}} - \frac{\delta_C}{\mu_C}\right)$$

b) C only on working elements, repair priority for C does not include other failures

$$MTTF_{S0} = \frac{1}{\delta_C + \dfrac{2\lambda(\lambda+\delta_{C21}-\delta_C)}{3\lambda+\delta_{C21}+\mu}} \le \frac{1}{\delta_C}$$

$$PA_S \approx 1 - \frac{\delta_C}{\mu_C} - \frac{2\lambda}{2\lambda+\delta_{C21}+\mu}\left(\frac{\lambda}{\mu} + \frac{\delta_{C21}}{\mu_C} - \frac{\delta_C}{\mu_C} + \frac{\lambda\,\delta_{C41}}{\mu\,\mu_C}\right)$$

c) C on elements in working or repair state, repair for C includes that for failure ($\delta_{C41} << \mu$)

$$MTTF_{S0} = \frac{1}{\delta_C + \dfrac{2\lambda(\lambda+\delta_{C23}-\delta_C)}{3\lambda+\delta_{C23}+\mu}} \le \frac{1}{\delta_C}$$

$$PA_S \approx 1 - \frac{\delta_C}{\mu_C} - \frac{2\lambda}{2\lambda+\mu}\left(\frac{\lambda}{\mu} + \frac{\delta_{C23}}{\mu_{C32}} - \frac{\delta_C}{\mu_C} + \frac{\lambda\,\delta_{C45}}{\mu\,\mu_{C54}}\right)$$

d) C on elements in working or repair state, repair as for case b)

Often $\delta_{C21} = \delta_{C23} = \delta_{C41} = \delta_{C45} = \delta_C$ and/or $\mu_{C32} = \mu_{C54} = \mu_C$ can be assumed. Case b) corresponds to a 1-out-of-2 active redundancy in series with a switch (Eqs. (6.158), (6.160)). Further approximations are possible, e.g. using $1 - PA_S = \overline{PA}_S = P_1 + P_3 + P_4 + P_5$.

Equations (6.276) & (6.280) clearly show the effect (consequence) of a common cause failure on a 1-out-of-2 active redundancy:

The common cause failure acts as a series element with failure rate equal the occurrence rate δ_C of the common cause failure and repair (restoration) rate μ_C equal the remove rate of the common cause failure; results given by Figs. 6.17 & 6.18 and rule (6.174)) apply.

The above rule holds quite general if the common cause failure acts at the same time on all redundant elements of a redundant structure. From this:

Good protection against common cause failures can only be given if each element of a redundant structure is realized with different technology (materials & tools), electrically, mechanically and thermally separated, and not designed by the same designer (true also for software).

Concrete protection against common cause failures must be worked out on a case-by-case basis, see Example 2.3 for a simple practical situation. In verifying such a protection, an FMEA/FMECA (Section 2.6) is mandatory for hardware and software. In some applications, common cause failures can occur with a time delay on elements of a redundant structure (e.g. because of the drop of a cooling ventilator); in this cases, automatic fault detection can avoid multiple failures. Some practical considerations on failure rates for common cause failures in electronic equipment are in [A2.5 (61508-6)], giving $\delta_C / \lambda \approx 0.005$ as achievable value (see rule (6.174)).

6.8.8 Basic Considerations on Network Reliability

A network (telecommunication, power, neuronal, or other) can often be regarded, for modeling purposes, as a graph with N nodes and up to $\binom{N}{2}$ edges (or links). Edges can be directed or bi-directional. Nodes and/or edges can fail, have 2 or more states, and for reliability investigations distinction is made between 2-terminal and k-terminal $(2 < k \leq N)$ connections. Networks can thus have very complex reliability structures, some of which have been investigated since the 1950s, with increasing interest in the last years, see e. g. [2.37, 6.61 - 6.80].

For the case of only two states for nodes and edges, small networks can be investigated with methods introduced in Sections 2.3.1 - 2.3.3 (nonrepairabe) or 6.2 - 6.8.7 (repairable). For large networks, solutions using minimal path or cut sets (i. e. based on Boolean functions, Section 2.3.4) are possible, manually (for instance using binary decision diagrams, Section 6.9.3) or with help of dedicated computer programs, see e. g. [6.63 (2007, 2009), 6.66, 6.68, 6.69, 6.74]. Multi-states for nodes and/or edges have to be considered when dealing with capacity problems, and some results for 2-terminal networks are known, see e. g. [6.63 (2009), 6.66 - 6.70, 6.74].

In the following, two basic network structures are investigated using the *key item method* given in Section 2.3.1 (see also Points 7 & 8 of Table 2.1 for further examples).

Figure 6.37a shows a network with 3 nodes N_1, N_2, N_3 and 3 bi-directional edges E_{12}, E_{13}, E_{23}. The reliability block diagram (RBD) for connection N_1, N_2 is given in Fig. 6.37b if only edges can fail and in Fig. 6.37c if nodes and edges can fail. The reliability function (nonrepairable) related to Fig. 6.37c follows as for Eq. (2.26)

$$R_{S0_{N_1,N_2}} = R_{N_1} R_{N_2} [R_{E_{12}} + R_{E_{13}} R_{E_{23}} R_{N_3} - R_{E_{12}} R_{E_{13}} R_{E_{23}} R_{N_3}], \qquad (6.281)$$

with $R_{S0_{N_1,N_2}} = R_{S0_{N_1,N_2}}(t)$, $R_i = R_i(t)$, $R_i(0) = 1$. Figure 6.37d gives the RBD for all-terminal. For this case, all nodes appear in series and the connection N_2, N_3 is included in the connections N_1, N_2 and $(\cap)$ N_1, N_3. The reliability function (nonrepairable) can be computed using the *key item* method (Eq. (2.29), on E_{12}), yielding

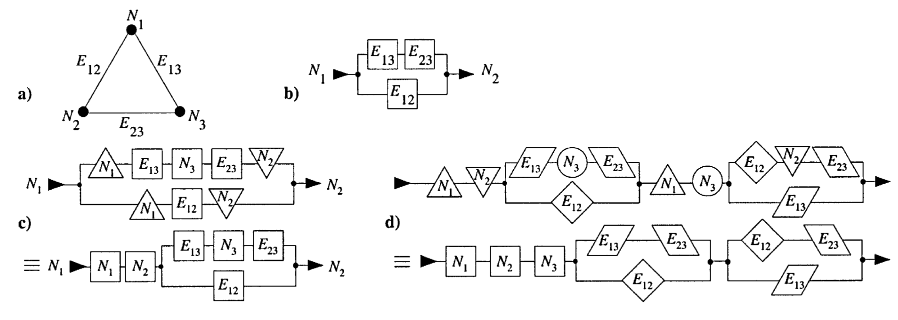


Figure 6.37 a) Network with 3 nodes & bi-directional connection from each node to each other node; **b)** 2-terminal RBD for nodes N_1 & N_2, 100% reliable nodes; **c)** 2-terminal RBD for nodes N_1 & N_2, edges and nodes can fail; **d)** RBD for all-terminal reliability, edges and nodes can fail

$$R_{S0_{all}} = R_{N_1} R_{N_2} R_{N_3} [R_{E_{12}} (R_{E_{13}} + R_{E_{23}} - R_{E_{13}} R_{E_{23}}) + \bar{R}_{E_{12}} R_{E_{13}} R_{E_{23}}], \qquad (6.282)$$

with $\bar{R}_i = 1 - R_i$, $R_{S0_{all}} = R_{S0_{all}}(t)$, $R_i = R_i(t)$, $R_i(0) = 1$. Substituting in Eqs. (6.281) & (6.282) $R_i(t)$ with $PA_i(t)$ one obtains the point availability $PA_{S0}(t)$ for the case of *totally independent elements* $N_1, N_2, N_3, E_{12}, E_{13}, E_{23}$ (p.52). To compute the reliability for the repairable case or the point availability for non totally independent elements, the states space method introduced above in this chapter can be used.

Figure 6.38a shows a network with 4 nodes N_1, N_2, N_3, N_4 and 6 bi-directional edges $E_{12}, E_{13}, E_{14}, E_{23}, E_{24}, E_{34}$. Assuming that nodes and edges can fail, the reliability block diagram is given in Fig. 6.37b for connection N_1, N_2 and Fig. 6.38c for all-terminal. Successively use of the *key item method* (on E_{12}, E_{34}, N_3, N_4) yields

$$R_{S0_{N_1,N_2}} = R_{N_1} R_{N_2} [R_{E_{12}} + \bar{R}_{E_{12}} \{ R_{E_{34}} [R_{N_3} \{ R_{N_4} (R_{E_{13}} + R_{E_{14}} - R_{E_{13}} R_{E_{14}})(R_{E_{23}} + R_{E_{24}} - R_{E_{23}} R_{E_{24}})$$
$$+ \bar{R}_{N_4} R_{E_{13}} R_{E_{23}} \} + \bar{R}_{N_3} R_{N_4} R_{E_{14}} R_{E_{24}}] + \bar{R}_{E_{34}} (R_{N_3} R_{E_{13}} R_{E_{23}} + R_{N_4} R_{E_{14}} R_{E_{24}} - R_{N_3} R_{E_{13}} R_{E_{23}} R_{N_4} R_{E_{14}} R_{E_{24}}) \}], \qquad (6.283)$$

and $R_{S0_{N1,N2}} = R_N^2 \, [R + 2R^2 R_N - 2R^3 R_N (1 - R_N) - R_N^2 (7R^4 - 7R^5 + 2R^6)]$ for $R_{N_i} = R_N$, $R_{E_{ij}} = R$. Similarly, Fig. 6.38c leads to (*key item method* on $E_{12}, E_{13}, E_{14}, E_{24}$)

$$R_{S0_{all}} = R_{N_1} R_{N_2} R_{N_3} R_{N_4} [R_{12} \{ R_{13} [1 - (1 - R_{14})(1 - R_{24})(1 - R_{34})] + \bar{R}_{13} [R_{14} (R_{23} + R_{34} - R_{23} R_{34})$$
$$+ \bar{R}_{14} R_b] \} + \bar{R}_{12} \{ R_{13} [R_{14} (R_{23} + R_{24} - R_{23} R_{24}) + \bar{R}_{14} R_b] + \bar{R}_{13} R_{14} R_b \}], \qquad (6.284)$$

with $R_b = R_{24}(R_{23} + R_{34} - R_{23} R_{34}) + \bar{R}_{24} R_{23} R_{34}$; from this, $R_{S0_{all}} = R_N^4 \, [16R^3 - 33R^4 + 24R^5 - 6R^6]$ for $R_{N_i} = R_N$, $R_{E_{ij}} = R$ (see also remarks to Eq. (6.282)).

Besides deterministic networks, some kinds of stochastic and evolving networks have been investigated, for instance by assuming that for bi-directional edges, every pair of nodes has a probability p to be connected (Erdős-Renyi) or there is a probability $p(k)$ that a randomly selected node has k edges ($p(k)$ can be a Poisson distribution (Erdős-Renyi) or a given power law), see e. g. [6.61-6.65] for greater details. However, because of their complexity, investigation of networks is still in progress.

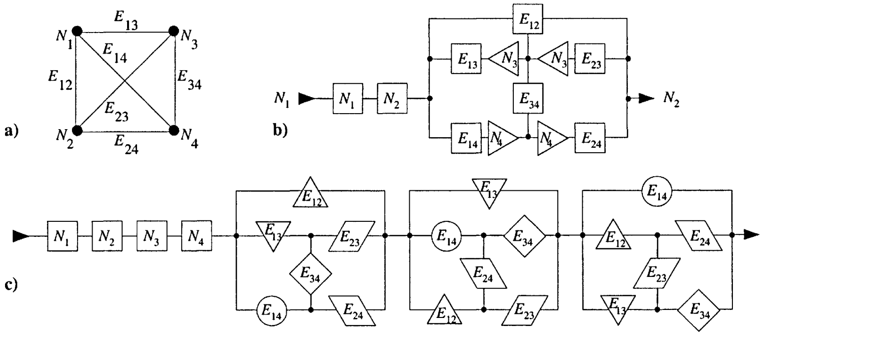


Figure 6.38 a) Network with 4 nodes & bi-directional connection from each node to each other node; **b)** 2-terminal RBD for nodes N_1 & N_2, edges and nodes can fail; **c)** RBD for all-terminal reliability, edges and nodes can fail (RBD = reliability block diagram)

6.8.9 General Procedure for Modeling Complex Systems

On the basis of the tools introduced in Appendix A7 and results in Sections 6.8.1 - 6.8.8, following procedure can be given for reliability and availability investigation of complex systems, both when a reliability block diagram exists or not (for series-parallel structures, Section 6.7 applies, in particular Table 6.10, p. 231).

1. As a first step operate with (time-homogeneous) *Markov processes*, i. e., assume that *failure and repair rates of all elements are constant* during the stay time in every state, and can change (stepwise) only at state changes, e. g. because of change in configuration, component use, stress, repair strategy or other (dropping this assumption leads to *non markovian* processes, as shown e. g. in Section 6.4.2, pp. 202 - 205). In a further step, refinements can be considered on a *case-by-case basis* using semi-regenerative processes.
2. Group series elements and assign to each macro-structure $E_1, ..., E_n$ a failure rate $\lambda_S = \lambda_1 + ... + \lambda_n$ and repair (restoration) rate $\mu_S = \lambda_S / (\lambda_1/\mu_1 + ... + \lambda_n/\mu_n)$ (Table 6.10). A further *reduction of a diagram of transition rates* is possible in some cases (see e.g. [6.32, 6.40], p. 227, Figs. 6.27 & 6.28, 6.30, 6.39).
3. Perform an FMEA (Section 2.6) to fix all relevant *failure modes* and to verify actual system capability for *detection, localization, reconfiguration, graceful degradation* at failure, and protection against *common cause/mode failures.*
4. Draw the *diagram of transition rates* and verify its correctness (see Fig. 6.20, p. 233 & Fig. 6.34, p. 263 for two comprehensive examples); important is the identification of up states which have a direct transition to a down state at system level (e.g. $Z_1, Z_3 - Z_7$ in Fig. 6.20), i. e. of *critical operating states.*
5. Identify the transition rates between each state (combination of failure and repair rates), by considering assumed repair (restoration) priorities, retained failure modes, and particularities specific to the system considered (dependence between elements, sequence of failure or failure modes, etc.).
6. For reliability calculation, the *mean time to system failure* $MTTF_{Si}$ for system entering state Z_i at $t = 0$ is obtained by solving (Eq. (A7.126))

$$\rho_i MTTF_{Si} = 1 + \sum_{Z_j \in U, j \neq i} \rho_{ij} \, MTTF_{Sj}, \qquad Z_i \in U, \quad \rho_i = \sum_{j=0, j\neq i}^{m} \rho_{ij} \, . \qquad (6.285)$$

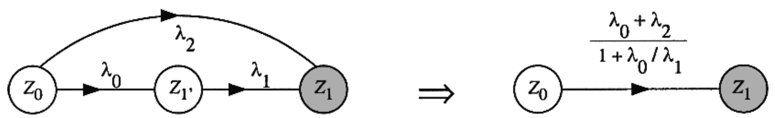


Figure 6.39 Example for a reduction of a diagram of transition rates for $MTTF_{S0}$ calculation (note that $(\lambda_0 + \lambda_2)/(1 + \lambda_0/\lambda_1) = (1 + \lambda_2/\lambda_0)/(1/\lambda_0 + 1/\lambda_1)$)

Thereby, U is the set of *up states*, $\bar{U}$ the set of down states ($U \cup \bar{U} = \{Z_0, \ldots, Z_m\}$), ρ_{ij} the transition rate from state $Z_i \in U$ to state $Z_j \in U$, and ρ_i the *sum of all transition rates leaving state* Z_i (Table 6.2). The system of algebraic equations (6.285) delivers all $MTTF_{Si}$ for any $Z_i \in U$ entered at $t=0$ (note that for Markov processes, the condition "Z_i is entered at $t=0$" can be replaced by "system in Z_i at $t=0$"). At system level,

$$R_{S0}(t) \approx e^{-t/MTTF_{S0}} \tag{6.286}$$

can often be used (in Z_0 all elements are operating or ready to operate, i. e., as-good-as-new because of the *memoryless* Markov property).

7. The *asymptotic* ($t \to \infty$) *& steady-state (stationary) point and average availability* $PA_S = AA_S$ is given as (Eq. (A7.134))

$$PA_S = AA_S = \sum_{Z_j \in U} P_j \tag{6.287}$$

with P_j as solution of (Eq. (A7.127, for irreducible embedded Markov chain)

$$\rho_j P_j = \sum_{i=0,\, i \neq j}^{m} P_i \rho_{ij}, \qquad \text{with } P_j > 0, \ \sum_{j=0}^{m} P_j = 1, \ \rho_j = \sum_{i=0,\, i \neq j}^{m} \rho_{ji}, \quad j = 0, \ldots, m. \tag{6.288}$$

One equation for P_j, arbitrarily chosen, must be replaced by $\sum P_j = 1$. Equation (6.288) states that *in steady-state, the probability to live* Z_j *is equal to the probability to come to* Z_j. For further availability figures see pp. 178-180.

8. Considering the constant failure rate for all elements, the asymptotic & steady-state *interval reliability* follows as (Eq. (6.27))

$$IR_S(t, t+\theta) \approx PA_S \, e^{-\theta/MTTF_{S0}} = \Big(\sum_{Z_j \in U} P_j\Big) \, e^{-\theta/MTTF_{S0}}. \tag{6.289}$$

9. The asymptotic & steady-state *system failure frequency* f_{udS} and *system mean up time* MUT_S are given as (Eqs.(A7.141) & (A7.142))

$$f_{udS} = \sum_{Z_j \in U,\, Z_i \in \bar{U}} P_j \rho_{ji} = \sum_{Z_j \in U} P_j \Big(\sum_{Z_i \in \bar{U}} \rho_{ji}\Big) \tag{6.290}$$

and

$$MUT_S = PA_S / f_{udS}, \tag{6.291}$$

respectively. U is the set of states considered as *up states for* f_{udS} *and* MUT_S *calculation*, $\bar{U}$ the complement to the totality of states considered. The same is for the *system repair (restoration) frequency* f_{duS} and the *system mean down time* MDT_S, given as (Eqs.(A7.143) & (A7.144))

$$f_{duS} = \sum_{Z_i \in \bar{U},\, Z_j \in U} P_i \rho_{ij} = \sum_{Z_i \in \bar{U}} P_i \Big(\sum_{Z_j \in U} \rho_{ij}\Big) \tag{6.292}$$

and

$$MDT_S = (1 - PA_S) / f_{duS}, \tag{6.293}$$

respectively. MUT_S is the mean of the time in which the system is moving in the set of up states $Z_j \in U$ ($Z_0 - Z_7$ in Fig. 6.20) before a transition in the set of down states $Z_i \in \bar{U}$ (($Z_8 - Z_{11}$ in Fig. 6.20) occurs, in steady-state or for $t \to \infty$. MDT_S is the mean repair (restoration) duration at system level. f_{udS} is the *system failure intensity* $z_S(t) = z_S$, as defined by Eq. (A7.230), in steady-state or for $t \to \infty$. It is not difficult to recognize that one has

$$f_{udS} = f_{duS} = z_S = 1/(MUT_S + MDT_S), \tag{6.294}$$

see example 6.29 for a practical application. Equations (6.291), (6.2.93), (6.294) lead to the following important relation

$$MDT_S = MUT_S\,(1 - PA_S)\,/\,PA_S \quad \text{i.e.} \quad PA_S = MUT_S\,/\,(MUT_S + MDT_S). \tag{6.295}$$

Considering that the asymptotic & steady-state probability P_0 is much greater than all other P_j, the approximation $MUT_S \approx MTTF_{S0}$ can often be used ($\sum_{Z_j \in U} P_j \; MTTF_{Sj}$ for MUT_S is basically not allowed, see example 6.29).

10. The asymptotic & steady-state expected *instantaneous reward rate* MIR_S is given by (Eq. (A7.147))

$$MIR_S = \sum_{i=0}^{m} r_i P_i\,. \tag{6.296}$$

Thereby, $r_i = 0$ for down states, $0 < r_i < 1$ for partially down states, and $r_i = 1$ for up states with 100% performance. The asymptotic & steady-state *expected accumulated reward* MAR_S follows as (Eq. (A7.148))

$$MAR_S(t) = MIR_S \cdot t\,. \tag{6.297}$$

In some cases it can be useful to operate with a time schedule (e.g. Fig. A7.11). Alternative investigation methods are introduced in Section 6.9. *Failure-free time* means *failure-free operating* time and *repair* is used as a synonym for *restoration*.

Example 6.29

Investigate MUT_S, MDT_S, f_{udS}, and f_{duS} for the 1-out-of-2 redundancy of Fig. 6.8a.

Solution

The solution of Eq. (6.84) with $\dot{P}_i(t) = 0$ yields (Eq. (6.87))

$$P_0 = \mu^2 / [(\lambda + \lambda_r)(\lambda + \mu) + \mu^2] \quad \text{and} \quad P_1 = \mu(\lambda + \lambda_r) / [(\lambda + \lambda_r)(\lambda + \mu) + \mu^2].$$

From Fig. 6.8a and Eqs. (6.290)-(6.294) it follows that

$$MUT_S = \frac{\lambda + \lambda_r + \mu}{\lambda(\lambda + \lambda_r)}, \quad MDT_S = 1/\mu, \quad \text{and} \quad f_{udS} = f_{duS} = z_S = \frac{\mu\lambda(\lambda + \lambda_r)}{(\lambda + \lambda_r)(\lambda + \mu) + \mu^2}.$$

For this example it holds that $MUT_S = MTTF_{S1}$ (with $MTTF_{S1}$ as solution of Eq. (6.89) with $P_1'(0) = 1$ or Eq. (6.285), see also Example A7.9), this because the system enters state Z_1 after each system failure; furthermore, $MDT_S = 1/\mu$ because only one repair crew is available.

6.9 Alternative Investigation Methods

The methods given in sections 6.1 to 6.8 are based on Markov, semi-Markov and semi-regenerative processes, according to the involved distributions for failure-free and repair times. They have the advantage of great flexibility (arbitrary redundancy and repair strategy, incomplete coverage or switch, common cause failures, etc.) and transparency. Further tools are known to model repairable systems, e. g. based on dynamic fault trees or Petri nets. For very large or complex systems, numerical solution or Monte Carlo simulation can become necessary. Many of these tools are similar in performance and versatility (Petri nets are equivalent to Markov models), other have limitations (fault tree analyses are basically limited to totally independent elements and Monte Carlo simulations delivers only numerical solutions), so that choice of the tool is often related to the personal experience of the analyst (see e. g. [A2.5 (61165, 60300-3-1), 6.30, 6.39 (2005)] for comparisons). However, modeling large complex systems requires a close cooperation between project and reliability engineers. After a recall for systems with totally independent elements, Sections 6.9.2 to 6.9.5 introduce dynamic fault trees, BDD, event trees, and Petri nets. Section 6.9.6 considers then basic aspects of numerical solutions and Section 6.9.7 reviews some considerations to approximate expressions for large and complex systems.

6.9.1 Systems with Totally Independent Elements

Totally independent elements means (pp. 52, 224) that each element operates and, if repairable, is repaired independently of any other element in the system considered. Elements are boxes in a reliability block diagram (Example 2.1) and, for repairable elements, total independence implies that each element *has its own repair crew and continues operation during the repair of a failed element.* This does not imply that the (physically) same element cannot appear more times in a reliability block diagram (Example 2.3). The reliability function $R_{S0}(t)$ of nonrepairable (up to system failure) systems with totally independent elements has been carefully investigated in Chapter 2. As stated with Eq. (2.48), *equations for* $R_{S0}(t)$ *are also valid for the point availability* $PA_{S0}(t)$ *of repairable systems, substituting* $PA_i(t)$ *to* $R_i(t)$. This rule can be used to get an upper bound of $PA_{S0}(t)$ for the case in which each element does not have its own repair crew. However the reliability function for repairable systems can not be computed using Boolean methods.

6.9.2 Static and Dynamic Fault Trees

A fault tree (FT) is a graphical representation of the conditions or other factors causing or contributing to the occurrence of a defined undesirable event, referred as *top event* [A2.5 (IEC 61025)]. In its original form, as introduced in Section 2.6 (p. 76), a fault tree contains only static gates (essentially AND and OR for coherent systems) and is thus termed *static fault tree*. Such a fault tree can handle combinatorial events, qualitatively (similar as for an FMEA, Section 2.6) or quantitatively (as with Boolean functions, Section 2.3.4). However, as the top event is in general a failure at system level, "0" is used for operating and "1" for failure. This is opposite to the notation used in Sections 2.2 and 2.3 for reliability investigations based on the reliability block diagram. With this notation OR gates represent in fault trees a *series structure* and AND gates a *parallel structure with active redundancy* (Figs. 2.14, 6.40-6.42). In setting up a fault tree, a reliability block diagram can be useful. However, fault trees can also easily consider external events. Figure 6.40 gives two examples of reliability structures with corresponding static fault trees (see Table 2.1 and Example 6.30 for computations based on the reliability block diagram and Section 6.9.3 for computations based on binary decision diagrams).

Static fault trees can be used to compute reliability and availability for the case of *totally independent elements* (active redundancy and each element has its own repair crew), see e.g. [A2.5 (IEC 61025)]. Reliability computation for the non-repairable case (up to system failure) using fault tree analysis (FTA) leads to

$$1-\mathrm{R}_{S0}(t)=1-\prod_{i=1}^{n}\mathrm{R}_i(t) \quad \text{or} \quad \overline{\mathrm{R}}_{S0}(t)=1-\prod_{i=1}^{n}(1-\overline{\mathrm{R}}_i(t)), \tag{6.298}$$

for the *series structure with independent elements*, and to

$$1-\mathrm{R}_{S0}(t)=1-\sum_{i=k}^{n}\binom{n}{i}\mathrm{R}^i(t)(1-\mathrm{R}(t))^{n-i} \quad \text{or} \quad \overline{\mathrm{R}}_{S0}(t)=1-\sum_{i=k}^{n}\binom{n}{i}(1-\overline{\mathrm{R}}(t))^i\,\overline{\mathrm{R}}(t)^{n-i}, \tag{6.299}$$

for the *k-out-of-n active redundancy with identical and independent elements* (Eqs. (2.17) and (2.23), $\overline{\mathrm{R}}_i(t)=1-\mathrm{R}_i(t)=$ failure probability). For complex structures, computation uses binary decision diagrams (based on the Shannon decomposition of the fault tree structure function, see Section 6.9.3) or minimal path or cut sets (Eqs. (2.42), (2.44)), often supported by computer programs.

However, because of their basic structure, static fault trees can not handle *states or time dependencies* (in particular standby redundancy or repair strategy). For these cases, it is necessary to extend static fault trees, adding so called *dynamic gates* to obtain *dynamic fault trees*. Important dynamic gates are [2.85, 6.38, A2.5 (IEC 61025)]:

- Priority AND gate (PAND), the output event (failure) occurs only if all input events occur and in sequence from left to right.
- Sequence enforcing gate (SEQ), the output event occurs only if input events occur in sequence from left to right and there are more than two input events.
- Spare gate (SPARE), the output event occurs if the number of spares is less than required.

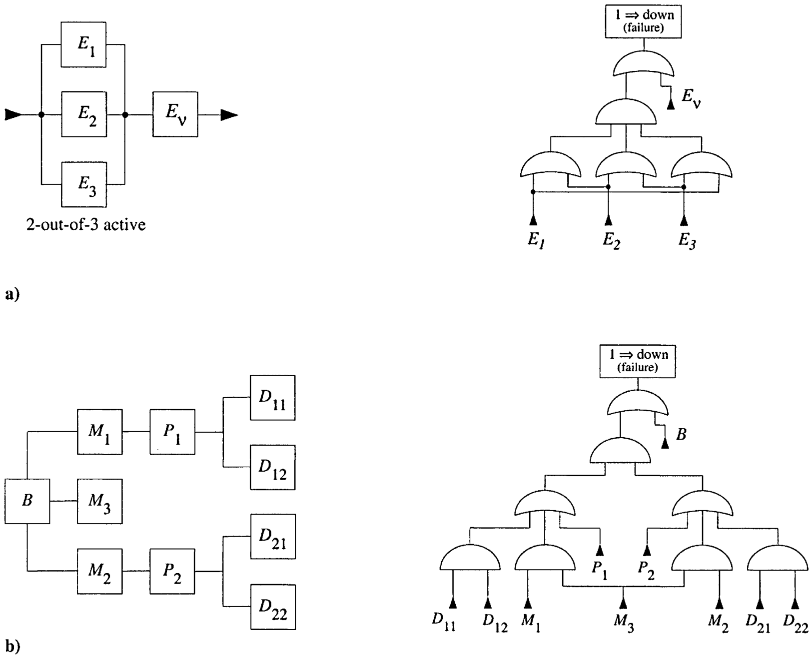


Figure 6.40 **a)** Reliability block diagram and corresponding static fault tree for a 2-out-of-3 active redundancy with switch element; **b)** Functional block diagram and corresponding static fault tree for a redundant computer system [6.30]; ***Note:*** "0" holds for operating (up) and "1" for failure (down)

Further gates (choice gate, redundancy gate, warm spare gate) have been suggested, e. g. in [6.38]. All above dynamic gates requires a Markov analysis, i. e., states probabilities must be computed by a Markov approach (constant failures & repair rates), yielding results used as occurrence probability for the basic event replacing the corresponding dynamic gate. Use of dynamic gates in dynamic fault tree analysis, with corresponding computer programs, has been carefully investigated, e. g. in [2.85, 6.36, 6.38].

Fault tree analysis (FTA) is an established methodology for reliability and availability analysis (emerging in the nineteen-sixties with investigations on nuclear power plants). However, the necessity to use Markov approaches to solve dynamic gates can limit its use in practical applications. Moreover, FTA has the same limits as those of methods based on binary considerations (fault trees, reliability block diagrams (RBD), binary decision diagrams (BDD), etc.). However, reliability block diagrams and fault trees are valid support in generating transition rates diagrams for Markov analysis. So once more, combination of investigation tools is often a good way to solve difficult problems.

6.9.3 Binary Decision Diagrams

A *binary decision diagram* (BDD) is a directed acyclic graph obtained by successive *Shannon decomposition* (Eq. (2.38)) of a *Boolean function*. It applies in particular to the *structure functions* developed in Section 2.3.4 for coherent systems, using *minimal path or cut sets*. This allows for easy computation of the reliability function $R_{S0}(t)$ for the *nonrepairable case* (Eqs. (2.45), (2.47)) or point availability $PA_{S0}(t)$ for *repairable totally independent elements* (Eqs. (2.45), (2.48)). Frequently, BDDs are used to compute $R_{S0}(t)$ or $PA_{S0}(t)$ for systems completely described by a *fault tree* with corresponding *fault tree structure function* $\phi_{FT}(\zeta_1,\ldots,\zeta_n)$. $\phi_{FT}(\zeta_1,\ldots,\zeta_n)$ follows from a fault tree, see e.g. Figs. 6.41 & 6.42, or from the corresponding reliability block diagram, considering "0" for operating (up) and "1" for failure (down).

In relation to fault trees, a BDD is constructed starting from the top event, i.e. from $\phi_{FT}(\zeta_1,\ldots,\zeta_n)$, down to the sink boxes using the Shannon decomposition (Eq. (2.38)) of the fault tree structure function at the node considered. Each node refers to a variable of $\phi_{FT}(\zeta_1,\ldots,\zeta_n)$ and has 2 outgoing edges, 0-edge for operating and 1-edge for failure. Input to a node can be one or more outgoing edges from other nodes. The BDD terminates in 2 sink boxes labeled 0 for operating (up), 1 for failure (down). Indication 0 or 1 and an arrow help to identify the outgoing edge. Figure 6.41 gives two basic reliability block diagrams with corresponding fault trees, ϕ_{FT}, and BDDs. Also given are the reliability functions for the *nonrepairable case* $R_{S0}(t)$ and $\overline{R}_{S0}(t)$:

> *To obtain* $R_{S0}(t)$*, one moves from the top of the BDD following all possible paths down to the sink box "0", taking in a multiplicative way* $R_i(t)$ *or* $\overline{R}_i(t) = 1 - R_i(t)$ *according to the value 0 or 1 assumed by the variable* ζ_i *considered* (similarly for $\overline{R}_{S0}(t)$, for $PA_{S0}(t)$ consider Eq. (2.48) or (2.45)).

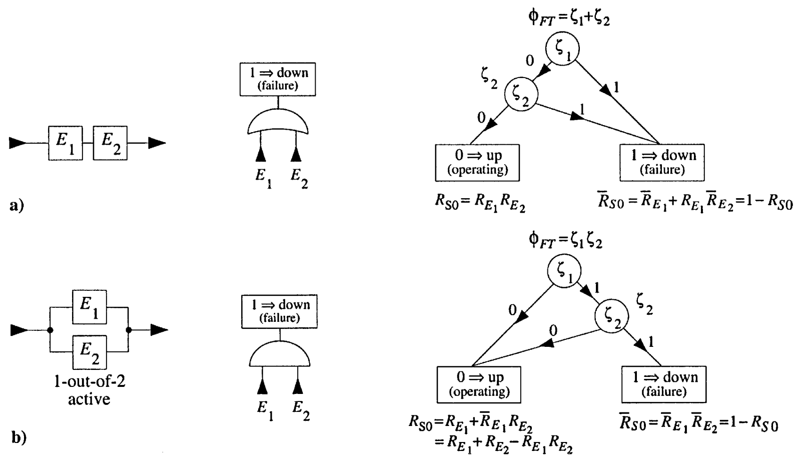


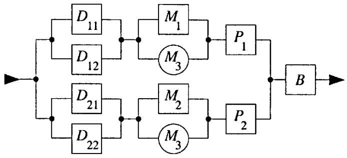
Figure 6.41 Basic reliability block diagrams with corresponding fault trees, ϕ_{FT}, and binary digital diagrams (ζ_i refers to E_i; "0" for operating, "1" for failure; $R_i = R_i(t)$, $R_i(0)=1$)

Example 6.30

Give the reliability function $R_{S0}(t)$ and the point availability $PA_{S0}(t)$ for the system of Fig. 6.40b, by assuming totally independent elements and using the reliability block diagram's method, for simplicity with $R_{D_{11}} = R_{D_{12}} = R_{D_{21}} = R_{D_{22}} = R_D$, $R_{M_1} = R_{M_2} = R_M$, $R_{P_1} = R_{P_2} = R_P$, $R_i(t) = R_i$.

Solution

The reliability block diagram follows from the functional block diagram of Fig. 6.40b (Section 2.2.2), or from the corresponding fault tree (Fig. 6.40b, considering "0" for operating (up) and "1" for failure (down)). As element M_3 appears twice in the reliability block diagram, computation of the reliability function make use of the key item method given in Section 2.3.1, yielding



$$R_{S0} = R_{M3}\{2[(2R_D - R_D^2)R_P] - [(2R_D - R_D^2)R_P]^2\}R_B + (1 - R_{M3})\{2[(2R_D - R_D^2)R_M R_P] - [(2R_D - R_D^2)R_M R_P]^2\}R_B, \tag{6.300}$$

with $R_i = R_i(t)$, and $R_i(0) = 1$. Following the assumed totally independence of the elements (each of the 10 elements has its own repair crew), the point availability $PA_{S0}(t)$ is also given by Eq. (6.300) substituting R_i with $PA_i(t)$ ($PA_i = AA_i$ for steady-state or $t \to \infty$).

Figure 6.42 considers the basic structures given in Fig. 6.40. The reliability function $R_{S0}(t)$ for the *nonrepairable case* follows, for the structure of Fig. 6.42b, from

$$\begin{aligned} R_{S0} = R_B\{ & R_{M_3}[R_{P_1}R_{D_{11}} + R_{P_1}\overline{R}_{D_{11}}R_{D_{12}} + R_{P_1}\overline{R}_{D_{11}}\overline{R}_{D_{12}}R_{P_2}(R_{D_{21}} + \overline{R}_{D_{21}}R_{D_{22}}) \\ & + \overline{R}_{P_1}R_{P_2}(R_{D_{21}} + \overline{R}_{D_{21}}R_{D_{22}})] \\ & + \overline{R}_{M_3}[R_{M_1}R_{M_2}R_{P_1}(R_{D_{11}} + \overline{R}_{D_{11}}R_{D_{12}}) + R_{M_1}R_{M_2}R_{P_1}\overline{R}_{D_{11}}\overline{R}_{D_{12}}R_{P_2}(R_{D_{21}} + \overline{R}_{D_{21}}R_{D_{22}}) \\ & + R_{M_1}R_{M_2}\overline{R}_{P_1}R_{P_2}(R_{D_{21}} + \overline{R}_{D_{21}}R_{D_{22}}) + R_{M_1}\overline{R}_{M_2}R_{P_1}(R_{D_{11}} + \overline{R}_{D_{11}}R_{D_{12}}) \\ & + \overline{R}_{M_1}R_{M_2}R_{P_2}(R_{D_{21}} + \overline{R}_{D_{21}}R_{D_{22}})]\}, \end{aligned} \tag{6.301}$$

with $R_{S0} = R_{S0}(t)$, $R_i = R_i(t)$, $\overline{R}_i = 1 - R_i(t)$. Setting $R_{D_{11}} = R_{D_{12}} = R_{D_{21}} = R_{D_{22}} = R_D$, $R_{M_1} = R_{M_2} = R_M$, $R_{P_1} = R_{P_2} = R_P$, one obtains Eq. (6.300). Similarly,

$$\begin{aligned} \overline{R}_{S0} = \overline{R}_B + R_B\{ & R_{M_3}[(R_{P_1}\overline{R}_{D_{11}}\overline{R}_{D_{12}} + \overline{R}_{P_1})(\overline{R}_{P_2} + R_{P_2}\overline{R}_{D_{21}}\overline{R}_{D_{22}})] \\ & + \overline{R}_{M_3}[\overline{R}_{M_1}\overline{R}_{M_2} + R_{M_1}R_{M_2}(\overline{R}_{P_1} + R_{P_1}\overline{R}_{D_{11}}\overline{R}_{D_{12}})(\overline{R}_{P_2} + R_{P_2}\overline{R}_{D_{21}}\overline{R}_{D_{22}}) \\ & + R_{M_1}\overline{R}_{M_2}(\overline{R}_{P_1} + R_{P_1}\overline{R}_{D_{11}}\overline{R}_{D_{12}}) + \overline{R}_{M_1}R_{M_2}(\overline{R}_{P_2} + R_{P_2}\overline{R}_{D_{21}}\overline{R}_{D_{22}})]\}, \end{aligned} \tag{6.302}$$

which verify $\overline{R}_{S0} = 1 - R_{S0}$. Assuming totally independent elements (Section 6.9.1), Eq. (6.301) delivers $PA_{S0}(t)$ by substituting R_i with $PA_i(t)$.

Evaluation of binary decision diagrams (and fault trees) is generally supported by dedicated computer programs, see e.g. [2.32, 2.36, 2.37, 6.63 (2009), 6.66]. For hand evaluation (e.g. for a great transparency), it is often more favorable to work directly with the *key item method* introduced in Section 2.3.1 (as in Example 6.30).

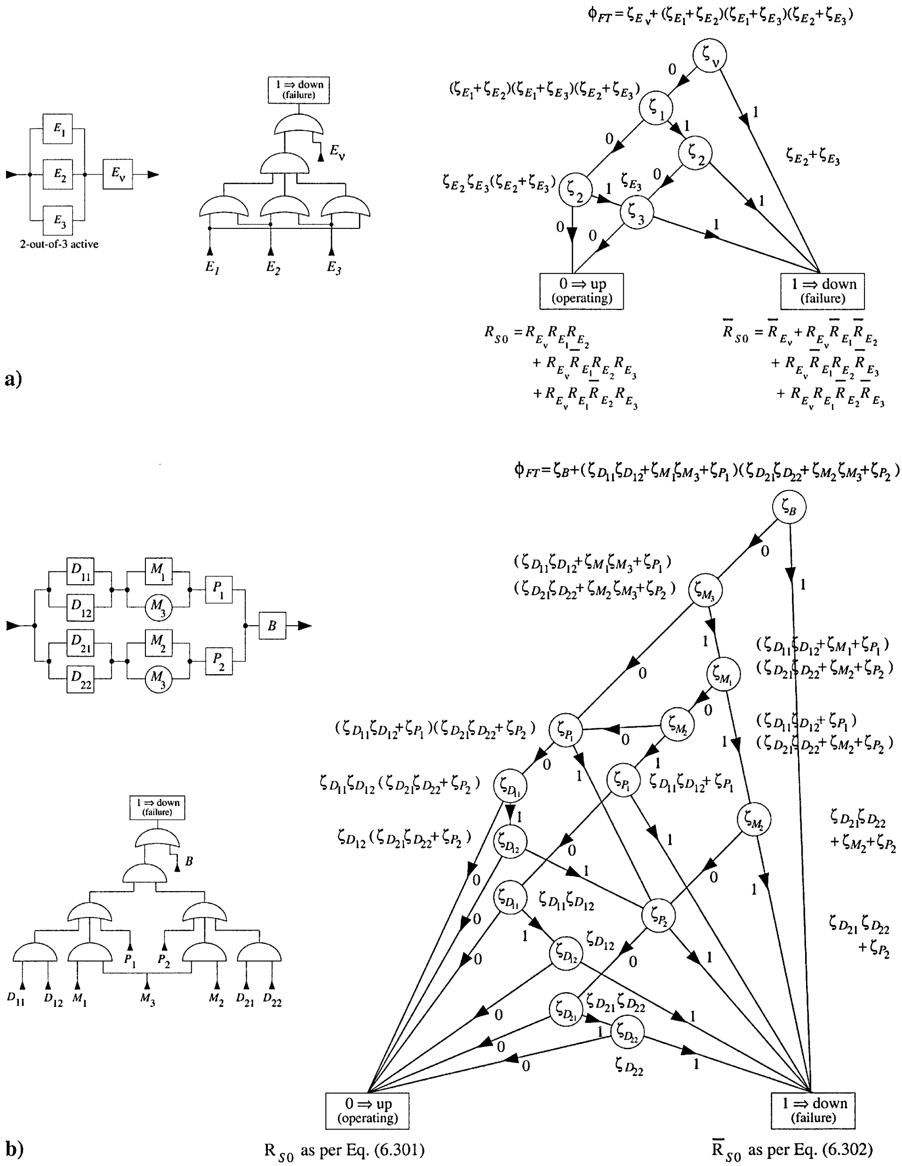


Figure 6.42 Reliability block diagrams with corresponding fault trees, ϕ_{FT}, and binary digital diagrams (BDDs) for the 2 structures of Fig. 6.40 (ζ_i refers to E_i; "0" holds for operating and "1" for failure; $R_{S0}=R_{S0}(t)$, $R_{S0}(0)=1$, $R_i=R_i(t)$, $R_i(0)=1$)

To consider "1" for operating (up) and "0" for failure (down), as in Sections 2.2 and 2.3, it is sufficient to change AND with OR and R_i with $\bar{R}_i$.

6.9.4 Event Trees

Event trees can be used to support and extend effectiveness of failure modes and effects analyses introduced in Section 2.6. *Event tree analysis* (ETA) is a bottom-up (inductive) logic procedure combining advantages of FMEA / FMECA and FTA. It applies, in particular, for risk analysis of large complex systems of any type with interacting internal and external factors (technical, environmental, human). The basic idea is to give an answer to the question *what happens if a given initiating event occurs*? The answer is given by investigating *propagation of initiating events*, in particular efficacy of mitigations (barriers) introduced to limit effects of the initiating event considered (column 8 in Table 2.6). An initiating event can be a fault or an external event (e. g. loss of power, fire, sabotage). A comprehensive list of initiating events must be prepared at the begin of the analysis.

Figure 6.43 shows the basic structure of an event tree for the case of two coupled systems (A and B), each with two mitigating factors (barriers) δ_i for the initiating event α considered. Each mitigation is successful with $\Pr\{\delta_i\}$ and unsuccessful (failure) with $\Pr\{\bar{\delta}_i\} = 1 - \Pr\{\delta_i\}$. The probability for the outcome ω in Fig. 6.43 is computed following the path leading to ω and is given by (Eq. (A6.12))

$$\Pr\{\omega\} = \Pr\{\alpha \cap \delta_{A1} \cap \delta_{A2} \cap \bar{\delta}_{B1} \cap \delta_{B2}\} = \Pr\{\alpha\}\Pr\{\delta_{A1} \mid \alpha\}\Pr\{\delta_{A2} \mid (\alpha \cap \delta_{A1})\}$$
$$\Pr\{\bar{\delta}_{B1} \mid (\alpha \cap \delta_{A1} \cap \delta_{A2})\}\Pr\{\delta_{B2} \mid (\alpha \cap \delta_{A1} \cap \delta_{A2} \cap \bar{\delta}_{B1})\}. \quad (6.303)$$

Computation of conditional probabilities can be laborious. Substituting λ_α to $\Pr\{\alpha\}$, Eq. (6.303) delivers the failure rate (occurrence frequency) of the outcoming event ω.

As for FMEA/FMECA & FTA, time evolution can not be easily considered in ETA. An extension like for dynamic FT (Section 6.9.2) is possible. In particular, $\Pr\{\delta_i\}$ can be issued from the top event of an FT, allowing handling of *common cause events*. A standard on event trees analysis is in preparation as IEC 62502 [A2.5].

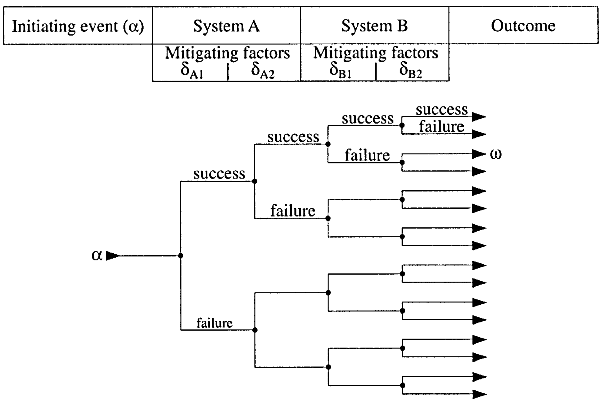


Figure 6.43 Basic structure of an event tree

6.9.5 Petri Nets

Petri nets (PN) were introduced 1962 [6.35, 6.6] to investigate in particular synchronization, sequentiality, concurrency, and conflict in parallel working digital systems. Several extensions have been at the origin of a large literature [6.1, 6.6, 6.8, 6.30, 6.39 (1999), 2.37]. Important for reliability investigations was the possibility to create algorithmically the diagram of transition rates belonging to a given Petri net. With this, investigation of time behavior on the basis of (time-homogeneous) Markov processes was open (stochastic Petri nets). Extension to semi-Markov process is straightforward [6.8], but less useful for reliability investigations (Sections 6.3 & 6.4). This section gives an introduction to Petri nets from a reliability analysis point of view. A Petri net (PN) is a directed graph involving 3 kind of elements:

- *Places* $P_1,...,P_n$ (drawn as circles): A place P_i is an *input* to a transition T_j if an arc exist from P_i to T_j and is an *output* of a transition T_k and input to a place P_l if an arc exist from T_k to P_l; places may contain *token* (black spots) and a PN with token is a *marked* PN.
- *Transitions* $T_1,...,T_m$ (drawn as empty rectangles for timed transitions or bars for immediate transitions): A transition *can fire*, taking one token from each input place and putting one token in each output place.
- *Directed arcs*: An arc connects a place with a transition or vice versa and has an arrowhead to indicate the direction; multiple arcs are possible and indicate that by firing of the involved transition a corresponding number of tokens is taken from the involved input place (for input multiple arc) or put in the involved output place (for output multiple arc); inhibitor arcs with a circle instead of the arrowhead are also possible and indicate that for firing condition no token must be contained in the corresponding place.

Firing rules for a transition are:

1. A transition is enabled (can fire) only if all places with an input arc to the given transition contain at least one token (no token for inhibitor arcs).
2. Only one transition can fire at a given time; the selection occurs according to the embedded Markov chain describing the stochastic behavior of the PN.
3. Firing of a transition can be immediate or occurs after a time interval $\tau_{ij} > 0$ (timed PN); $\tau_{ij} > 0$ is in general a random variable (stochastic PN) with distribution function $F_{ij}(x)$ when firing occurs from transition T_i to place P_j (yielding a Markov process for $F_{ij}(x) = 1 - e^{-\lambda_{ij}x}$, i.e. with transition rate λ_{ij}, or a semi-Markov process for $F_{ij}(x)$ arbitrary, with $F_{ij}(0) = 0$).

From rule 3, practically only Markov processes (i.e. constant failure and repair rates) will occur in Petri nets for reliability applications (Section 6.4.2). Two further concepts useful when dealing with Petri nets are those of *marking* and *reachability*:

A *marking* $M = \{m_1, ..., m_n\}$ gives the number m_i of token in the place P_i at a given time point and defines thus the *state* of the PN; M_j is immediately *reachable* from M_i if M_j can be obtained by firing a transition enabled by M_i.

With M_0 as marking at time $t=0$, $M_1, ..., M_k$ are all the (different) marking reachable from M_0; they define the PN states and give the *reachability tree*, from which, the diagram of transition rates of the corresponding Markov model follows. Figure 6.44 gives some examples of reliability structures with corresponding PN.

6.9.6 Numerical Reliability and Availability Computation

Investigation of large series - parallel structures or of *complex systems* (for which a reliability block diagram does not exist) is in general time-consuming and can become mathematically intractable. A large number of computer programs for numerical solution of reliability and availability equations as well as for Monte Carlo simulation have been developed. Such a numerical computation can be in some cases the only way to get results. Section 6.9.6.1 discusses requirements for a versatile program for the numerical solution of reliability and availability equations. Section 6.9.6.2 gives basic considerations on Monte Carlo simulation and introduces an approach useful for rare events. Although appealing, numerical solutions can deliver only case-by-case solutions and can causes problems (instabilities in the presence of sparse matrices, prohibitive run times for Monte Carlo simulation of rare events or if confidence limits are required). As a general rule, analytical solutions (Sections 6.2 - 6-6, 6.8) or approximate expressions (Sections 6.7, 6.9.7) should be preferred whenever possible.

6.9.6.1 Numerical Computation of System's Reliability and Availability

Analytical solution of algebraic or differential / integral equations for reliability and availability computation of large or complex systems can become time-consuming. Software tools exist to solve this kind of problems. From such a software package one generally expects high *completeness*, *usability*, *robustness*, *integrity*, and *portability* (Table 5.4). The following is a comprehensive list of requirements:

General requirements:

1. Support interface with CAD/CAE and *configuration management* packages.
2. Provide a large component data bank with the possibility for manufacturer and company-specific labeling, and storage of non application-specific data.
3. Support different failure rate models [2.21 - 2.30].
4. Have flexible output (regarding medium, sorting capability, weighting), graphic interface, single & multi-user capability, high usability & integrity.
5. Be portable to different platforms.

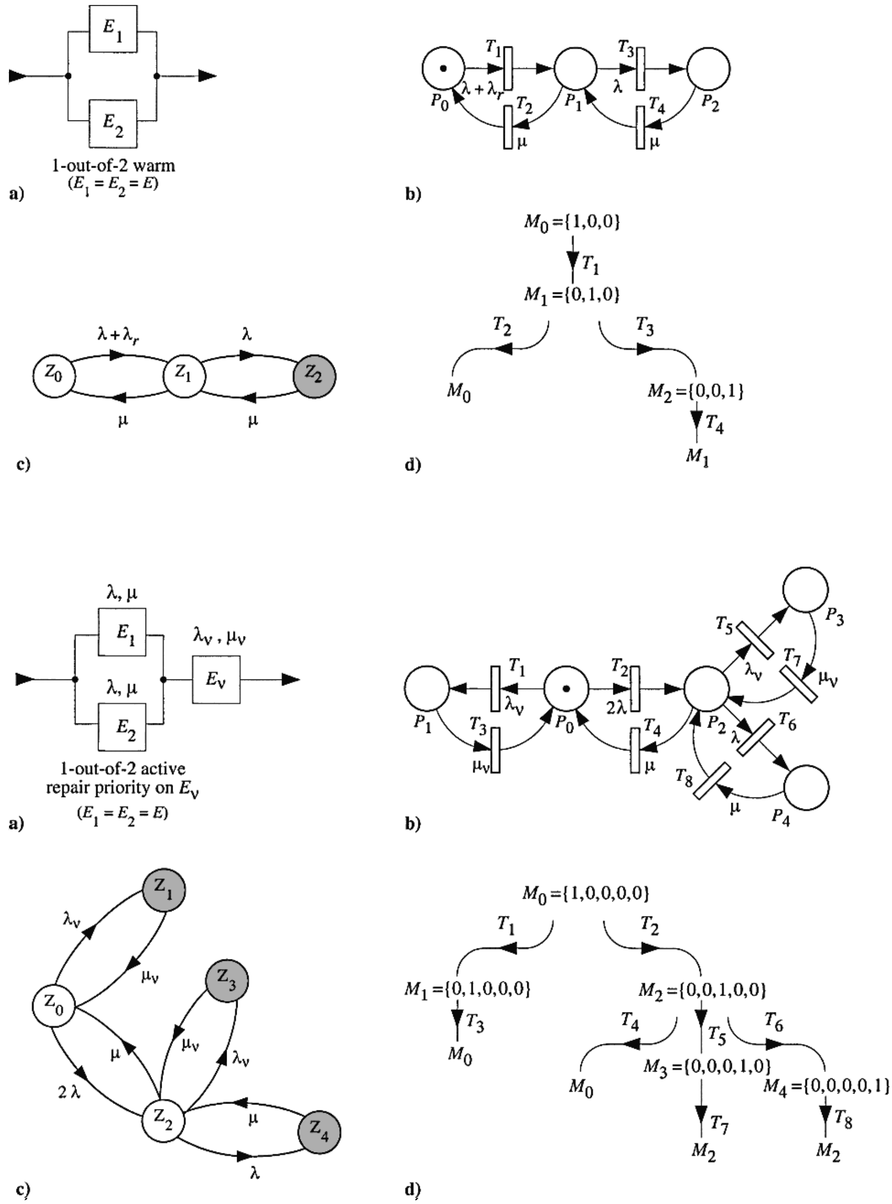


Figure 6.44 Top: Reliability block diagram (a), diagram of transition rates (c), Petri net (PN) (b), and reachability tree (d) for a repairable 1-out-of-2 warm redundancy (two identical elements, const. failure (λ, λ_r) and repair (μ) rates, one repair (restoration) crew, Z_2 down state, Markov process) **Bottom:** Reliability block diagram (a), diagram of transition rates (c), Petri net (b), and reachability tree (d) for a repairable 1-out-of-2 active redundancy with two identical elements and switch in series (constant failure (λ, λ_v) and repair (μ, μ_v) rates, one repair (restoration) crew, repair priority on switch, no further failures at system down, Z_1, Z_3, Z_4 down states, Markov process)

Specific for nonrepairable (up to system failure) systems:

1. Consider reliability block diagrams (RBD) of *arbitrary complexity* and with a large number of elements ($\geq 1{,}000$) and levels (≥ 10); possibility for any element to appear more than once in the RBD; automatic editing of series and parallel models; powerful algorithms to handle complex structures; constant or time dependent failure rate for each element; possibility to handle as element macro-structures or items with more than one failure mode.
2. Easy editing of *application-specific* data, with user features such as:
   - automatic computation of the ambient temperature at component level with freely selectable temperature difference between elements,
   - freely selectable duty cycle from the system level downwards,
   - global change of environmental and quality factors, manual selection of stress factors for tradeoff studies or risk assessment, manual introduction of field data and of *default values* for component families or assemblies.
3. Allow reuse of elements with arbitrary complexity in a RBD (libraries).

Specific for repairable systems:

1. Consider elements with *constant failure rate* and constant or *arbitrary repair rate*, i.e., handle Markov and (as far as possible) semi-regenerative processes.
2. Have *automatic* generation of the transition rates ρ_{ij} for Markov model and of the involved semi Markov transition probabilities $Q_{ij}(x)$ for systems with constant failure rates, one repair crew, and arbitrary repair rate (starting e.g. from a given set of *successful paths)*; automatic generation and solution of the equations describing the system's behavior.
3. Allow *different repair strategies* (first-in first-out, one repair crew or other).
4. Use sophisticated algorithms for quick inversion of *sparse matrices*.
5. Consider at least 20,000 states for the *exact solution* of the *asymptotic & steady-state* availability $PA_S = AA_S$ and mean time to system failure $MTTF_{Si}$.
6. Support investigations yielding approximate expressions (macro-structures, totally independent elements, cutting states or other, see Section 6.7.1).

A scientific software package satisfying many of the above requirements has been developed at the Reliability Lab. of the ETH [2.50]. Refinement of the requirements is possible. For basic reliability computation, commercial programs are available [2.51-2.60]. Specialized programs are e.g. in [2.6, 2.18, 2.59, 2.85, 6.23, 6.24, 6.43]; considerations on numerical methods for reliability evaluation are e.g. in [2.56].

6.9.6.2 Monte Carlo Simulations

The Monte Carlo technique is a numerical method based on a probabilistic interpretation of quantities obtained from algorithmically generated random variables. It was introduced 1949 by N. Metropolis and S. Ulman [6.31]. Since this time, a large amount of literature has been published, see e. g. [6.4, 6.13, 6.31, A7.18]. This section deals with some basic considerations on Monte Carlo simulation useful for reliability analysis and gives an approach for the simulation of rare events which avoids the difficulty of time truncation because of amplitude quantization of the digital number used.

For reliability purposes, a Monte Carlo simulation can basically be used to estimate a value (e. g. an unknown probability) or simulate (reproduce) the stochastic process describing the behavior of a complex system. In this sense, a Monte Carlo simulation is useful to achieve results, numerically verify an analytical solution, get an idea of the possible time behavior of a complex system or determine interaction among variables. Two main problems related to Monte Carlo simulation are the generation of uniformly distributed *random numbers* in the interval (0,1) and the transformation of these numbers in random variables with prescribed distribution function. A congruential relation

$$\varsigma_{n+1} = (a\varsigma_n + b) \mod m, \tag{6.304}$$

where *mod* is used for *modulo*, is frequently used to generate *pseudo-random numbers* (for simplicity, *pseudo* will be omitted in the following). Transformation to an arbitrary distribution function $F(x)$ is often performed with help of the inverse function $F^{-1}(x)$, see Example A6.18. The method of the inverse function is simple but not necessarily good enough for critical applications.

A further question arising with Monte Carlo simulation is that of how many repetitions n must be run to have an estimate of the unknown quantity within a given interval $\pm\varepsilon$ at a given confidence level γ. For the case of an event with probability p and assuming n sufficiently large as well as p or $(1-p)$ not very small, Eq. (A6.152) yields for p known

$$n = \left(\frac{t_{(1+\gamma)/2}}{\varepsilon}\right)^2 p(1-p) \qquad \text{i.e} \qquad n_{\max} = \left(\frac{t_{(1+\gamma)/2}}{2\varepsilon}\right)^2 \text{ for } p = 0.5, \tag{6.305}$$

where $t_{(1+\gamma)/2}$ is the $(1+\gamma)/2$ quantile of the standard normal distribution; for instance, $t_{(1+\gamma)/2} = 1.645$ for $\gamma = 0.9$ and 1.96 for $\gamma = 0.95$ (Appendix A9.1). For p totally unknown, the value $p = 0.5$ has to be taken. Knowing the number of realizations k in n trials, Eq. (A8.43) can be used to find confidence limits for p.

To simulate (reproduce) a (time-homogeneous) Markov process, following procedure is useful, starting by a transition in state Z_i at the arbitrary time $t = 0$:

1. Select the next state Z_j to be visited by generating an event with probability

$$\mathcal{P}_{ij} = \frac{\rho_{ij}}{\rho_i}, \qquad j \neq i \quad (\mathcal{P}_{ii} \equiv 0), \quad \rho_i = \sum_{j=0,\, j\neq i}^{m} \rho_{ij}, \quad \sum_{j=0,\, j\neq i}^{m} \mathcal{P}_{ij} = 1, \tag{6.306}$$

according to the *embedded Markov chain* (for uniformly distributed random numbers ξ in (0,1) it holds that $\Pr\{\xi \leq x\} = x$).

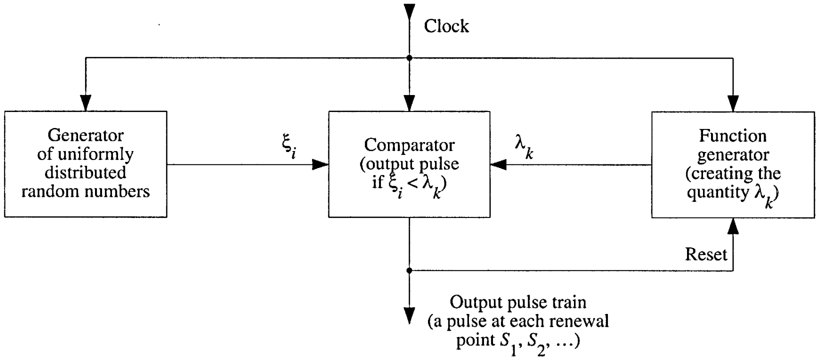


Figure 6.45 Block diagram of the programmable generator for renewal processes

2. Find the stay time (sojourn time) in state Z_i up to jump to the next state Z_j by generating a random variable with distribution function (Example A6.18)

$$F_{ij}(x) = 1 - e^{-\rho_i x}. \qquad (6.307)$$

3. Jump to state Z_j.

Extension to semi-Markov processes is easy [A7.2 (1974 & 1977)]. For semi regenerative processes, states visited during a cycle must be considered (e.g. Fig. A7.11). The advantage of this procedure is that transition sequence and stay (sojourn) times are generated with only a few random numbers. A disadvantage is that the stay times are *truncated* because of the amplitude quantization of $F_{ij}(x)$.

To avoid truncation problems, in particular when dealing with *rare events* distributed on the time axis, an alternative approach implemented as hardware generator for semi-Markov processes in [A7.2 (1974 & 1977)] can be used. To illustrate the basic idea, Fig. 6.45 shows the structure of the generator for renewal processes. The generator is driven by a clock $\Delta t = \Delta x$ and consists of three main elements:

- a generator for (pseudo-) random numbers ξ_i uniformly distributed in (0,1);
- a comparator, comparing at each clock the actual random number ξ_i with λ_k and giving an output pulse, *marking a renewal point*, for $\xi_i < \lambda_k$;
- a function generator creating λ_k and starting with λ_1 at each renewal point.

It can be shown ($\lambda_k = w_k$ in [A7.2 (1974 & 1977)]) that for

$$\lambda_k = (F(k\Delta x) - F((k-1)\Delta x)) / (1 - F((k-1)\Delta x)), \qquad k = 1, 2, \ldots, \qquad (6.308)$$

the sequence of output pulses is a realization of an *ordinary renewal process* with distribution function $F(k\Delta x)$ for times between successive renewal points. λ_k is the *failure rate* of the arithmetic random variable with distribution function $F(k\Delta x)$.

Generated random times are *not truncated*, since the last part of $F(k\Delta x)$ can be approximated by a *geometric distribution* (λ_k const., Eq. (A6.132)). A software realization of the generator of Fig 6.45 is easy, and hardware limitations can be avoided.

The homogeneous Poisson process (HPP), is a particular renewal process (Appendix A7.2.5) and can thus be generated (reproduced) with the generator given by Fig. 6.45; λ_k is constant, and the generated random time interval have a geometric distribution. For a nonhomogeneous Poisson process (NHPP) with mean value function $M(t) = E[\nu(t)]$, generation can be based on the considerations given on pp. 509 - 510 (for fixed $t = T$, generate k according to a Poisson distribution with parameter $M(t)$ (Eq. (A7.190)) and then k random variables with density $m(t) / M(T)$; the ordered values are the k occurrence times of the NHPP on $(0, T)$).

6.9.7 Approximate Expressions for Large Complex Systems: Basic Considerations

Approximate expressions for the reliability and availability of large series-parallel structures, which elements $E_1, E_2, ..., E_n$ have constant failure and repair rates λ_i, μ_i, $i = 1, ..., n$, have been developed in Section 6.7, in particular using *macro-structures* (Table 6.10) or *totally independent elements* (Table 6.9). Thereby, based on the results obtained for the repairable 1-out-of-2 redundancy (Eqs. (6.88) & (6.94) with $\lambda_r = \lambda$), a series, parallel, or simple series - parallel structure is considered as a one-item structure with constant failure and repair rates λ_S, μ_S for calculations, and integrated into further macro-structures bottom up to system level.

Expressions for small complex systems, for which a reliability block diagram either does not exist or cannot be reduced to a series-parallel structure with independent elements, have been carefully investigated in Sections 6.8.2 - 6.8.7, assuming no further failures at system down and taking care of imperfect switching, incomplete coverage, more than one failure mode, reconfiguration strategy (time censored (phased-mission) or failure censored), and common cause failures.

Investigation methods and tools for large complex systems are still in progress. Clustering of states (p. 227) is often possible by conserving exact results. Cutting states with more than one failure (p. 227) is applicable, simplify investigations and delivers approximate expressions for reliability and availability often sufficiently good in practical applications (see, for instance, the numerical evaluations on pp. 235, 265). State merging in Markov transition diagrams is conceivable, but basically limited to the case in which transitions from a block of merged states to an unmerged state have the same transition rates [6.40]. Also of limited applicability is the exploitation of symmetries in Markov transition diagrams [6.32].

A general procedure delivering often useful upper bounds for the mean time to failure $MTTF_{S0}$ and the asymptotic & steady-state availability $PA_S = AA_S$ at system level can be (for coherent systems):

1. Assume *totally independent elements* (Section 6.9.1) $E_1, \ldots, E_n$ with constant failure rates λ_i and repair rates $\mu_i = \mu$, $i = 1, \ldots, n$.
2. Compute $PA_S = AA_S$ as per Eq. (2.48), i.e., substituting in the structure function $\phi(\zeta_1, \ldots, \zeta_n)$ given by Eqs. (2.42) or (2.44),

$$PA_{Si} = \mu \,/\, (\lambda_i + \mu), \qquad i = 1, \ldots, n, \tag{6.309}$$

   for ζ_i (p. 59 bottom).
3. Compute $MTTF_{S0}$ from $PA_S = MTTF_S \,/\, (MTTF_S + MTTR_S)$ (Eq. (A7.189))

$$MTTF_{S0} \approx \frac{PA_S}{\mu(1 - PA_S)}, \tag{6.310}$$

   i.e. by assuming

$$MTTR_S \approx 1/\mu. \tag{6.311}$$

On the basis of the results obtained for the 1-out-of-2 redundancy (Eqs. (6.88) and (6.94) with $\lambda_r = \lambda$),

$$\mathrm{R}_{S0}(t) \approx e^{-t/MTTF_{S0}} \qquad \text{and} \qquad \mathrm{PA}_{S0}(t) \approx PA_S \tag{6.312}$$

can often be assumed at system level. To give a touch for the above approximations, consider a k-out-of-n active redundancy. Comparison of results in Table 6.9 (or Eq. (6.148)) for totally independent elements *(IE)* and in Table 6.10 for macro-structures *(MS)* with one repair crew and no further failures at system down, yields

$$MTTF_{S0_{IE}} \,/\, MTTF_{S0_{MS}} \approx (n-k)! \tag{6.313}$$

and

$$(1 - PA_{S0_{IE}}) / (1 - PA_{S0_{MS}}) = \overline{PA}_{S0_{IE}} \,/\, \overline{PA}_{S0_{MS}} \approx 1 \,/\, (n-k+1)!. \tag{6.314}$$

Thus, for weak redundancy levels (small values of $n-k$), the assumption of totally independent elements can yield acceptable upper bounds for the mean time to failure $MTTF_{S0}$ and the asymptotic & steady-state availability $PA_S = AA_S$ at system level. However, exact evaluation of the validity of Eqs. (6.310) - (6.312) can be performed only on a case-by-case basis, and for very complex systems a dedicated computer program or a Monte Carlo simulation remains often the only practicable way to get results.