# 4 Maintainability Analysis

At equipment and system level, *maintainability* has a great influence on *reliability* and *availability*. This holds in particular if *redundancy* has been implemented and redundant parts *can be repaired (restored) on line*, i. e., without interruption of operation at system level. Maintainability is thus an important parameter in the optimization of *reliability, availability,* and *life-cycle cost*. Achieving high maintainability in complex equipment and systems requires appropriate activities which must be started *early in design & development phase* and be coordinated by a *maintenance concept*. To this concept belong *failure detection* and *localization* (built-in tests), *partitioning* of equipment or system into (as far as possible) independent *line replaceable units*, and *logistic support*. A maintenance concept has to be *tailored* to the equipment or system considered. After some basic concepts, Section 4.2 deals with a *maintenance concept* for *complex equipment & systems*. Section 4.3 discusses maintainability aspects in *design reviews*. Section 4.4 gives methods and tools for *maintainability prediction*. *Spare parts provisioning & repair strategies* are carefully considered in Sections 4.5 & 4.6, respectively; *cost optimization* in Sections 4.5 - 4.7. *Design guidelines* for maintainability are given in Section 5.2. The influence of preventive maintenance, imperfect switching, and incomplete coverage on system's reliability & availability is investigated in Section 6.8. For simplicity, *delays* (administrative, logistic, technical) are neglected and *repair* is used for *restoration*.

## 4.1 Maintenance, Maintainability

*Maintenance* defines all those *activities* performed on the item to *retain* it in or to *restore* it to a specified state. Maintenance includes thus *preventive maintenance*, carried out at predetermined intervals, according to prescribed procedures to reduce the probability of failures or the degradation of the functionality of the item, and *corrective maintenance*, initiated after fault detection and intended to bring the item into a state in which it can again perform the required function (Fig. 4.1). The aim of preventive maintenance must also be to detect and repair *hidden failures*, i. e., undetected failures in redundant elements. Corrective maintenance is also known as *repair* (restoration) and can include any or all of following steps: *detection* (recognition), *localization* (isolation), *correction* (disassemble, remove, reassemble, adjust),

A. Birolini, *Reliability Engineering*, 6th ed., DOI 10.1007/978-3-642-14952-8_4,

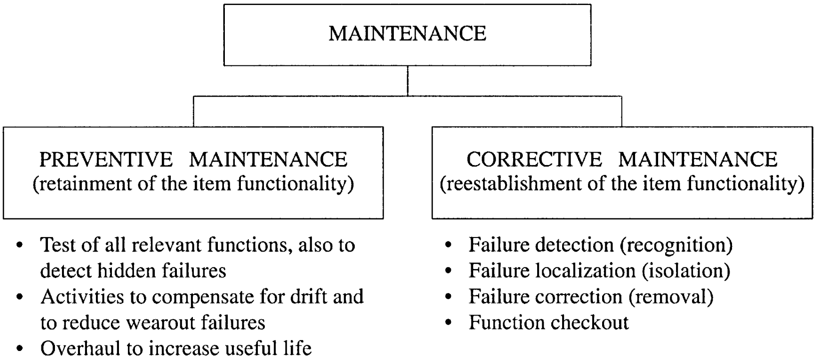



**Figure 4.1** Basic maintenance tasks, disregarded from administrative, logistic, and technical delays (*failure* can be replaced by *fault*, including failures and defects)

and *function checkout* (Fig. 4.1). The time elapsed from the failure occurrence until the start-up after failure correction, including all delays (administrative, logistic, technical), is often denoted as *restoration time* (see [A1.4 (2 nd Ed.)] for a comprehensive maintenance time diagram). For simplicity, in this book delays are neglected (ideal logistic support, apart in Example 6.7 (p. 201) and Fig. A7.12 (p. 504)); thus, *repair* will be used for *restoration*. The situation in which only a part of the item is repaired (minimal repair) is considered in Section 4.6.2.

*Maintainability* is a characteristic of the item, expressed by the *probability* that *preventive maintenance* (serviceability) or *repair* (repairability) of the item will be performed within a stated time interval by *given procedures and resources*. If $\tau'$ and $\tau''$ are the (random) times required to carry out a repair and a preventive maintenance, respectively, then

$$\text{Repairability} = \Pr\{\tau' \le x\} \qquad \text{and} \qquad \text{Serviceability} = \Pr\{\tau'' \le x\}. \tag{4.1}$$

Considering $\tau'$ and $\tau''$ like *interarrival times*, the variable $x$ is used instead of $t$ in Eq. (4.1). For a rough characterization, the means (expected values) of $\tau'$ and $\tau''$

$$\mathrm{E}[\tau'] = MTTR = \text{mean time to repair (restoration)}$$

$$\mathrm{E}[\tau''] = MTTPM = \text{mean time to preventive maintenance}$$

are often used. Assuming $x$ as a parameter, Eq. (4.1) gives the *distribution functions* of $\tau'$ and $\tau''$, respectively. These distribution functions characterize the *repairability* and the *serviceability* of the item considered. Experience shows that $\tau'$ and $\tau''$ often exhibit a *lognormal distribution* (Eq. (A6.110)). The typical shape of the corresponding density is shown in Fig. 4.2. A characteristic of the lognormal density is the sudden increase after a period of time in which its value is practically zero, and the relatively fast decrease after reaching the maximum (modal value $x_M$).

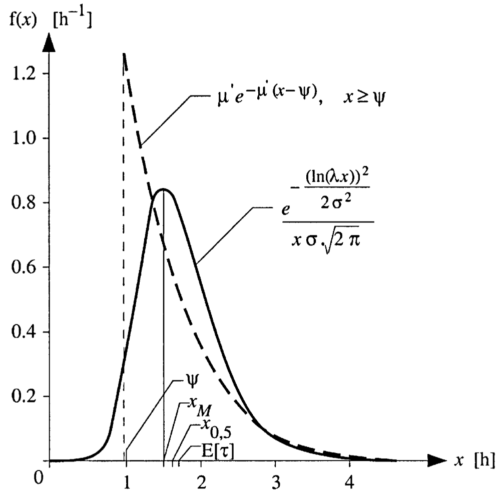



**Figure 4.2** Density of the lognormal distribution function for $\lambda = 0.6\,h^{-1}$ and $\sigma = 0.3$ (dashed is the approximation given by a shifted exponential distribution with same mean)

This shape can be accepted, taking into consideration the main terms of a repair time (Fig. 4.1). However, calculations using a lognormal distribution can become time-consuming. In practical applications it is therefore useful to distinguish between one of the following two situations:

1. *Investigation of maintenance times*, often under assumption of ideal logistic support: In this case, the actual distribution function must be considered, see Sections 7.3 and 7.5 for some examples with a lognormal distribution.
2. Investigation of the *reliability and availability of repairable systems*: The exact shape of the repair time distribution has in general less influence on the reliability and availability values at system level, as long as the *MTTR* is *unchanged* and *MTTR* << *MTTF* holds (Examples 6.8, 6.9, 6.10); in this case, the actual repair time distribution function can often be *approximated* by an exponential function *with same mean.*

A further possibility to Point 2 above, is to use e.g. a shifted exponential distribution function (Examples 6.9 and 6.10). Figure 4.2 shows (dashed) an example with

$$\psi = x_M - \sqrt{\mathrm{Var}[\tau']} = e^{-\sigma^2} / \lambda \; - \; \sqrt{e^{2\sigma^2} - e^{\sigma^2}} \; / \lambda .$$

The parameter $\mu'$ of the exponential d.f. follows from the equality of the mean values

$$MTTR = e^{\sigma^2/2} \,/\, \lambda = \psi + 1/\mu' \quad \rightarrow \quad \mu' = \lambda \,/\, (e^{\sigma^2/2} - \lambda\psi). \qquad (4.2)$$

For the numerical example given in Fig. 4.2 ($\lambda = 0.6\,\mathrm{h}^{-1}$, $\sigma = 0.3$; $MTTR = 1.75\,\mathrm{h}$, $\mathrm{Var} = 0.29\,\mathrm{h}^2$) one obtains $\psi = 0.99\,\mathrm{h}$ and $\mu' = 1.32\,\mathrm{h}^{-1}$. A shift which considers equal mean and variance leads to $\psi = 1.2\,\mathrm{h}$ & $\mu' = 1.9\,\mathrm{h}^{-1}$. For a deeper investigation, one can refer to Examples 6.8 - 6.10. In some cases, an Erlang distribution (Eq. (A6.102)) with $\beta \ge 3$ can be assumed for repair times, yielding simple results.

As in the case of the failure rate $\lambda(x)$, for a statistical evaluation of repair times ($\tau'$) it would be preferable to omit data attributable to *systematic failures*. For the remaining data, a *repair rate* $\mu(x)$ can be obtained from the distribution function $\mathrm{G}(x) = \Pr\{\tau' \le x\}$, with density $\mathrm{g}(x) = d\,\mathrm{G}(x)/dx$, as per Eq. (A6.25)

$$\mu(x) = \lim_{\delta x \downarrow 0} \frac{1}{\delta x} \Pr\{x < \tau' \le x + \delta x \mid \tau' > x\} = \frac{\mathrm{g}(x)}{1 - \mathrm{G}(x)}, \tag{4.3}$$

(considering that $\tau'$ starts anew at each repair (restoration), $x$ is used instead of $t$).

In evaluating the maintainability *achieved in the field*, the influence of the *logistic support* must be considered. *MTTR requirements* are discussed in Appendix A3.1. *MTTR estimation* and *demonstration* is considered in Section 7.3.

## 4.2 Maintenance Concept

Like for reliability, maintainability must be *built* into equipment and systems during the *design and development phase*. This in particular because maintainability cannot be easily predicted by analytical methods, and a maintainability improvement often requires important changes in layout or construction of the item (system) considered. For these reasons, attaining a prescribed maintainability in complex equipment and systems generally requires the *planning* and *realization* of a *maintenance concept*. Such a concept deals with the following aspects:

1. Fault *detection* and *localization*, including checkout after repair (localization can be subdivided in isolation and diagnosis, and fault is used to consider failures and defects).
2. Partitioning of the equipment or system into independent *line replaceable units* (LRUs), i.e., in spare parts at equipment or system level (*line repairable*, *last repairable*, or *last replaceable* is often used for *line replaceable*).
3. Preparation of the user documentation (operating & maintenance manuals).
4. Training of operating and maintenance personnel.
5. Logistic support for the user, including after-sales service.

This section introduces the above points for the case of *complex equipment and systems with high maintainability requirements*.

### 4.2.1 Fault Detection (Recognition) and Localization

For complex equipment and systems, *detection* of partial failures or of *hidden failures* (failure of redundant elements) can be difficult. For this reason, a *status test*, initiated by operating personnel, or an *operation monitoring*, running autonomously, must often be implemented. Properties, advantages, and disadvantages of both methods are summarized in Table 4.1. The choice between a *status test* or a (more complete) *operating monitoring* must consider cost, reliability, availability, and safety requirements at system level.

The goal of *fault localization* is to isolate faults (failures and defects) down to the *line replaceable units* (LRUs), i.e., to the part which is considered as a *spare part* at equipment or system level. LRUs are generally assemblies, e.g. populated printed circuit board, or units which for repair purposes are considered as an *entity* and replaced on a *plug-out/plug-in basis* to reduce repair times. Repair of LRUs is generally performed by specialized personnel and repaired LRUs are stored for *reuse*. Fault isolation should be performed using *built-in test* (BIT) facilities, if necessary supported by *built-in test equipment* (BITE). Use of external special tools should be avoided, however *check lists* and portable test equipment can be useful to limit the amount of built-in facilities.

Fault *detection* and *fault localization* are closely related and should be considered together using *common* hardware and/or software. A high degree of automation

**Table 4.1** Automatic and semiautomatic fault (failures and defects) detection (recognition )

| | Status Test | | Operation Monitoring |
|---|---|---|---|
| | Rough (quick test) | Complete (functional test) | |
| Properties | • Testing of all important functions, if necessary with help of external test equipment<br>• Initiated by the operating personnel, then runs automatically | • Periodic testing of all important functions<br>• Initiated by the operating personnel, then runs automatically or semi-autom. (possibly without external stimulation or test equipment) | • Monitoring of all important functions and automatic display of complete and partial faults<br>• Performed with built-in means (BIT / BITE) |
| Advantages | • Lower cost<br>• Allows fast checking of the functional conditions | • Gives a clear status of the functional conditions of the item considered<br>• Allows fault localization down to LRU level | • Runs automatically on-line, i. e. in background |
| Drawbacks | • Limited fault localization (isolation and diagnosis) capability | • Relatively expensive<br>• Runs generally off-line (i. e. not in background) | • Expensive |

LRU = line replaceable unit; BIT = built-in test; BITE = built-in test equipment should be striven for, and test results should be automatically recorded. A one-to-one correspondence between test messages and content of the *user documentation* (operating and maintenance manuals) must be assured.

*Built-in tests* (BIT) should be able to detect and localize also *hidden faults*, i.e., faults (defects or failures) of redundant elements and, as possible, *software defects* too. This ability is generally characterized by the following *testability* parameters:

- degree of fault *detection* (*coverage*, e.g. 99% of all relevant failures),
- degree of fault *localization* (e.g. down to LRUs),
- *correctness* of the fault localization (e.g. 95%),
- test *duration* (e.g. 1s).

The first two parameters can be expressed by a *probability*, and distinction between *failures* and *defects* is important. As a measure of the *correctness* of the fault isolation capability, one can use the ratio between the number of correctly isolated faults and the number of isolation tests performed. This figure, similar to that of *test coverage*, must often remain at an empirical level, because of the lack of exact information about the defects and failures really present or assumed in the item considered. For the test duration, it is generally sufficient to work with mean values. *Failure* (fault) *modes* analysis methods (FMEA /FMECA, FTA, cause-to-effect charts, etc.) are useful to check the effectiveness of built-in facilities (Section 2.6).

Built-in test facilities, in particular built-in test equipment (BITE), must be defined taking into consideration not only of price/performance aspects but also of their *impact* on the *reliability* and *availability* of the equipment or system in which they are used. Standard BITE can often be integrated into the equipment or system considered. However, project specific BITE is generally more efficient than standard solutions. For such a selection, the following aspects are important:

1. *Simplicity*: Test sequences, procedures, and documentation should be as easy as possible.
2. *Standardization*: The greatest possible standardization should be striven for, in hardware and software.
3. *Reliability*: Built-in facilities should have a failure rate of at least one order of magnitude lower than that of the equipment or system in which they are used; their failure should not influence the item's operation (FMEA/FMECA).
4. *Maintenance*: The maintenance of BIT/BITE must be simple and should not interfere with that of the equipment or system; the user should be connected to the *field data change service* of the manufacturer.

For some applications, it is important that fault localization (or at least part of the diagnosis) can be *remotely controlled*. Such a requirement can often be satisfied, if stated early in the design phase. *Remote diagnosis* must be investigated on a case-by-case basis, using results from a careful failure modes and effects analysis (FMEA).

A further step on above considerations leads to maintenance concepts which allow automatic or semiautomatic *reconfiguration* of the item after failure.

A new concept on design for fault tolerance, using time, structure, and information redundancy is presented in [4.26], see also [4.4] for diagnostic aspects.

D*esign guidelines* for maintainability are given in Section 5.2. Effects of imperfect switching and incomplete coverage are investigated in Section 6.8.

### 4.2.2 Equipment and System Partitioning

The consequent *partitioning* of complex equipment and systems into (as far as possible) independent *line replaceable units* (LRUs) is important for good maintainability. Partitioning must be performed *early in the design phase*, because of its impact on layout and construction of the equipment or system considered. LRUs should constitute *functional units* and have *clearly defined interfaces* with other LRUs. Ideally, LRUs should allow a *modular construction* of the equipment or system, i. e., constitute autonomous units which can be tested each one independently from every other, for hardware as well as for software.

Related to the above aspects are those of *accessibility, adjustment*, and *exchangeability*. Accessibility should be easy for LRUs with *limited useful life*, high failure rate, or wearout. The use of digital techniques largely reduces the need for *adjustment* (alignment). As a general rule, hardware adjustment in the field should be avoided. *Exchangeability* can be a problem for equipment and systems with long *useful life*. *Spare parts provisioning* and aspects of *obsolescence* can in such cases become mandatory (Section 4.5).

### 4.2.3 User Documentation

User (or product) documentation for complex equipment and systems can include all of the following Manuals or Handbooks

- General Description
- Operating Manual
- Preventive Maintenance (Service) Manual
- Corrective Maintenance (Repair) Manual
- Illustrated Spare Parts Catalog
- Logistic Support.

It is important for the content of the user documentation to be *consistent* with the hardware and software status of the item considered. Emphasis must be placed on a clear and concise presentation, with block diagrams, flow charts, check lists. The language should be easily understandable to non-specialized personnel. Procedures should be self sufficient and contain checkpoints to prevent the skipping of steps.

## 4.2.4 Training of Operating and Maintenance Personnel

Suitably equipped, well trained, and motivated maintenance personnel are an important prerequisite to achieve short maintenance times and to avoid *human errors.* Training must be comprehensive enough to cover present needs. However, for complex systems it should be periodically updated to cover technological changes introduced in the system and to further motivate operating and maintenance personnel.

## 4.2.5 User Logistic Support

For complex equipment and systems, customers (users) generally expect from the manufacturer a *logistic support* during the useful life of the item under consideration. This can range from support on an *on-call basis* up to a *maintenance contract* with manufacturer's personnel located at the user site. One important point in such a logistic support is the definition of *responsibilities.* For this reason, maintenance is often subdivided into different levels (four for military applications (Table 4.2) and three for industry, in general). The *first level* concerns simple maintenance work such as the status test, fault detection and fault localization down to the subsystem level. This task is generally performed by *operating personnel.* At the *second level,* fault localization is refined, the defective LRU is replaced by a good one, and the functional test is performed. For this task *first line maintenance personnel* is often required. At the *third level,* faulty LRUs are repaired by *maintenance personnel* and stored for reuse. The *fourth level* is generally relates to

**Table 4.2** Maintenance levels in the defense area

| | logistic level | Location | Carried out by | Tasks |
|---|---|---|---|---|
| Advanced maintenance service | Level 1 | Field | Operating personnel | • Simple maintenance work<br>• Status test<br>• Fault detection (recognition)<br>• Fault localization down to subsystem level |
| | Level 2 | Cover | First line maintenance personnel | • Preventive maintenance<br>• Fault localization down to LRU level<br>• First line repair (LRU replacement)<br>• Functional test |
| Back-up maintenance service | Level 3 | Depot | Maintenance personnel | • Difficult maintenance<br>• Repair of LRUs |
| | Level 4 | Arsenal or Industry | Specialists from arsenal or industry | • Reconditioning work<br>• Important changes or modifications |

LRU = line replaceable unit (spare part at system level); *fault* includes failures and defects *overhaul* or *revision* (essentially for large mechanical parts subjected to wear, erosion, scoring, etc.) and performed at the manufacturer's site by *specialized personnel.*

For large mechanical systems, maintenance can account for over 30% of the operating cost. A careful optimization of these cost may be necessary in many cases. The part contributed by preventive maintenance is more or less deterministic. For the corrective maintenance, cost equations weighted by probabilities of occurrence can be established from considerations similar as those given in Sections 1.2.9 and 8.4, see also Sections 4.5, 4.6, and 4.7.

**Table 4.3** Example of catalog of questions for the *preparation of project specific* checklists for the evaluation of maintainability aspects in preliminary design reviews (Appendices A3 and A4) of complex equipment and systems with high maintainability requirements

1. Has the equipment or system been conceived with modularity in mind? Are the modules functionally independent and separately testable?
2. Has a concept for fault (failure & defect) detection and localization been planned and realized? Is fault detection automatic? Which kind of faults are detected? How does fault localization work? Is localization down to *line replaceable units* (LRUs) possible? How large are values for fault detection and fault localization (coverage)? Is remote diagnostic possible?
3. Can redundant elements be repaired on-line?
4. Are enough test points provided? Do they have pull-up/pull-down resistors?
5. Have hardware adjustments (or alignments) been reduced to a minimum? Are the adjustable elements clearly marked and easily accessible? Is the adjustment uncritical?
6. Has the amount of external test equipment been kept to a minimum?
7. Has the standardization of components, materials, and maintenance tools been considered?
8. Are *line replaceable units* (LRUs) identical with spare parts? Can they be easily tested? Is a spare parts provisioning concept available?
9. Are all elements with limited useful life clearly marked and easily accessible?
10. Are access flaps (and doors) easy to open (without special tools) and self-latching? Have plug-in unit guide rails self-blocking devices? Can a standardized extender for PCBs be used?
11. Have indirect connectors been used? Is the plugging-out/plugging-in of PCBs (LRUs) easy? Are power supplies and ground distributed across different contacts?
12. Have wires and cables been conveniently placed? Also with regard to maintenance?
13. Are sensitive elements sufficiently protected against mishandling during maintenance?
14. Can preventive maintenance be performed on-line? Does preventive maintenance also allow the detection of hidden failures?
15. Which part of the item (system) can be considered as-good-as-new after a maintenance action?
16. Have man-machine aspects been sufficiently considered?
17. Have all safety aspects also for operating and maintenance personnel been considered? Also in the case of failure (FMEA / FMECA, FTA, etc.)?

## 4.3 Maintainability Aspects in Design Reviews

*Design reviews* are important to point out, discuss, and eliminate *design weaknesses*. Their objective is also to *decide about continuation or stopping* of the project on the basis of objective considerations (*feasibility checks* in Tables A3.3 & 5.3 and Fig. 1.6). The most important design reviews (PDR & CDR) are described in Table A3.3. To be effective, design reviews must be supported by *project specific checklists*. Table 4.3 gives an example of catalog of questions which can be used to generate project specific checklists for maintainability aspects in design reviews (see Table 2.8 for reliability and Appendix A4 for other aspects).

## 4.4 Predicted Maintainability

Knowing the reliability structure of a system and the reliability and maintainability of its elements, it is possible to calculate the maintainability of the system considered as a *one-item structure* (e. g. calculating the reliability function and the point availability at system level and extracting $g(t)$ as the density of the repair time at system level using Eqs. (6.14) and (6.18)). However, such a calculation soon becomes laborious for arbitrary systems (Chapter 6). For many practical applications it is often sufficient to know the *mean time to repair* at system level $MTTR_S$ (expected value of the repair (renewal) time at system level) as a function of the system reliability structure, and of the mean time to failure $MTTF_i$ and mean time to repair $MTTR_i$ of its elements. Such a calculation is discussed in Section 4.4.1. Section 4.4.2 deals then with the calculation of the *mean time to preventive maintenance* at system level $MTTPM_S$. The method used in Sections 4.4.1 and 4.4.2 is easy to understand and delivers mathematically exact results for $MTTR_S$ and $MTTPM_S$. Use of statistical methods to estimate or demonstrate a maintainability or a $MTTR$ are discussed in Sections 7.2.1, 7.3, 7.5, and 7.6.

### 4.4.1 Calculation of $MTTR_S$

Let us first consider a *system without redundancy*, with elements $E_1, \ldots, E_n$ in series as given in Fig. 6.4. $MTTF_i$ and $MTTR_i$ are the *mean time to failure* and the *mean time to repair* of element $E_i$, respectively ($i = 1, \ldots, n$). Assume now that each element works for the same *cumulative operating time* $T$ (the system is disconnected during repair, or repair times are neglected because of $MTTR_i \ll MTTF_i$) and let $T$ be *arbitrarily large*. In this case, the mean (expected value) of the number of failures of element $E_i$ during $T$ is given by (Eq. (A7.27))

$$\frac{T}{MTTF_i}.$$

The mean of the total repair time necessary to restore the $T / MTTF_i$ failures follows then from

$$MTTR_i \frac{T}{MTTF_i}.$$

For the whole system, there will be in *mean*

$$\sum_{i=1}^{n} \frac{T}{MTTF_i} \tag{4.4}$$

failures and a *mean* total repair time of

$$\sum_{i=1}^{n} MTTR_i \frac{T}{MTTF_i}. \tag{4.5}$$

From Eqs. (4.4) and (4.5) it follows then for the *mean time to repair* (restoration) at system level $MTTR_S$, the final value

$$MTTR_S = \frac{\sum_{i=1}^{n} MTTR_i / MTTF_i}{\sum_{i=1}^{n} 1 / MTTF_i}. \tag{4.6}$$

Equation (4.6) gives the *mathematically exact* value for the mean repair time at system level $MTTR_S$, under the assumption that at system down (during a repair) no further failures can occur and that switching is ideal (no influence on the reliability). From Eq. (4.6) one can easily verify that

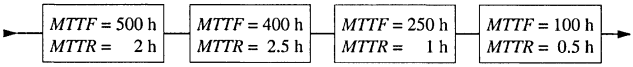
$$MTTR_S = MTTR, \quad \text{for} \quad MTTR_1 = \ldots = MTTR_n = MTTR,$$

and

$$MTTR_S = \frac{1}{n} \sum_{i=1}^{n} MTTR_i, \quad \text{for} \quad MTTF_1 = \ldots = MTTF_n.$$

**Example 4.1**

Give the mean time to repair at system level $MTTR_S$ for the following system.




How large is the mean of the *total system down time* during the interval $(0, t]$ for $t \to \infty$?

**Solution**

From Eq. (4.6) it follows that

$$MTTR_S = \frac{\dfrac{2\,\text{h}}{500\,\text{h}} + \dfrac{2.5\,\text{h}}{400\,\text{h}} + \dfrac{1\,\text{h}}{250\,\text{h}} + \dfrac{0.5\,\text{h}}{100\,\text{h}}}{\dfrac{1}{500\,\text{h}} + \dfrac{1}{400\,\text{h}} + \dfrac{1}{250\,\text{h}} + \dfrac{1}{100\,\text{h}}} = \frac{0.01925}{0.0185\,\text{h}^{-1}} \approx 1.04\,\text{h}.$$

The mean down time at system level is also $1.04\,\text{h}$, then for a *system without redundancy* it holds that down time = repair time. The *mean operating time* at system level in the interval $(0, t]$ can be obtained from the expression for the average availability $AA_S$ (Eqs. (6.23), (6.24), (6.48), and (6.49))

$$\lim_{t\to\infty} \text{E[total operating time in } (0, t]] = t \cdot AA_S = t \cdot MTTF_S \,/\, (MTTF_S + MTTR_S).$$

From this, the mean of the *total system down time* during $(0, t]$ for $t \to \infty$ follows from

$$\lim_{t\to\infty} \text{E[total system down time in } (0, t]] = t - t \cdot AA_S = t\, MTTR_S \,/\, (MTTF_S + MTTR_S).$$

Numerical computation then leads to

$$t\, MTTR_S \,/\, (MTTF_S + MTTR_S) \approx t\, MTTR_S \,/\, MTTF_S = t \cdot 1.04\text{h} \cdot 0.0185\text{h}^{-1} \approx 0.019t.$$

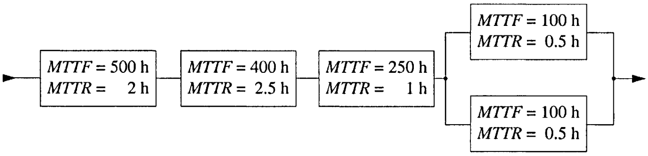
If every element exhibits a constant failure rate $\lambda_i$, then $MTTF_i = 1/\lambda_i$ and

$$MTTR_S = \frac{\sum_{i=1}^{n} \lambda_i\, MTTR_i}{\sum_{i=1}^{n} \lambda_i} = \sum_{i=1}^{n} \frac{\lambda_i}{\lambda_S} MTTR_i, \qquad \text{with } \lambda_S = \sum_{i=1}^{n} \lambda_i. \tag{4.7}$$

Equations (4.6) and (4.7) can also be used for systems with redundancy. However, in this case, a distinction at system level between *repair time* and *down time* is necessary. If the system contains only *active redundancy*, the *mean time to repair* at system level $MTTR_S$ is given by Eq. (4.6) or (4.7) by *summing over all elements* of the system, as if they were in series (a similar consideration holds for spare parts provisioning). By assuming that failures of redundant elements are repaired without interruption of operation at system level, Eq. (4.6) or (4.7) can be used to obtain an *approximate value* of the *mean down time* at system level, by *summing only over all elements without redundancy* (series elements), see Example 4.2.

**Example 4.2**

How does the $MTTR_S$ of the system in Example 4.1 change, if an active redundancy is introduced to the element with $MTTF = 100\,\text{h}$?




Under the assumption that the redundancy is repaired without interruption of operation at system level, is there a difference between the *mean time to repair* and the *mean down time* at system level?

**Solution**

Because of the assumed active redundancy, the operating elements and the reserve elements show the same mean number of failures. The mean system repair time follows from Eq. (4.6) by summing over all system elements, yielding

$$MTTR_S = \frac{\frac{2\,\text{h}}{500\,\text{h}} + \frac{2.5\,\text{h}}{400\,\text{h}} + \frac{1\,\text{h}}{250\,\text{h}} + \frac{0.5\,\text{h}}{100\,\text{h}} + \frac{0.5\,\text{h}}{100\,\text{h}}}{\frac{1}{500\,\text{h}} + \frac{1}{400\,\text{h}} + \frac{1}{250\,\text{h}} + \frac{1}{100\,\text{h}} + \frac{1}{100\,\text{h}}} = \frac{0.02425}{0.0285\,\text{h}^{-1}} \approx 0.85\,\text{h}\,.$$

However, the system down time differs now from the system repair time. Assuming for the redundancy an availability equal to one (for constant failure rate $\lambda = 1/MTTF$, constant repair rate $\mu = 1/MTTR$, and one repair crew, Table 6.6 (p. 200) gives for the 1-out-of-2 active redundancy $PA = AA = \mu(2\lambda + \mu) / (2\lambda(\lambda + \mu) + \mu^2)$ yielding $AA = 0.99995$ for this example), the system down time is defined by the elements in series on the reliability block diagram (see Point 9 in Section 6.8.9 (Eq. (6.295)) for precise considerations), thus

$$\text{mean down time at system level} \approx \frac{\frac{2\,\text{h}}{500\,\text{h}} + \frac{2.5\,\text{h}}{400\,\text{h}} + \frac{1\,\text{h}}{250\,\text{h}}}{\frac{1}{500\,\text{h}} + \frac{1}{400\,\text{h}} + \frac{1}{250\,\text{h}}} = \frac{0.01425}{0.0085\,\text{h}^{-1}} \approx 1.68\,\text{h}\,.$$

Similarly to Example 4.1, the mean of the system down time during the interval $(0, t]$ follows from

$$\lim_{t \to \infty} \text{E[total down time in } (0, t]] = t\,(1 - AA_S) \approx t\,\frac{MTTR_S}{MTTF_S} = t \cdot 1.68\,\text{h} \cdot 0.0085\,\text{h}^{-1} \approx 0.014\,t\,.$$

### 4.4.2 Calculation of $MTTPM_S$

Based on the results of Section 4.4.1, calculation of the *mean time to preventive maintenance* at system level $MTTPM_S$ can be performed for the following two cases:

1. Preventive maintenance is carried out at once for the *entire system*, one element after the other. If the system consists of elements $E_1, ..., E_n$ (arbitrarily grouped on the reliability block diagram) and the mean time to preventive maintenance of element $E_i$ is $MTTPM_i$, then

$$MTTPM_S = \sum_{i=1}^{n} MTTPM_i. \tag{4.8}$$

2. Every element $E_i$ of the system is serviced for preventive maintenance *independently* of all other elements and has a mean time to preventive maintenance $MTTPM_i$. In this case, Eq. (4.6) can be used with $MTBPM_i$ instead of $MTTF_i$ and $MTTPM_i$ instead of $MTTR_i$, where $MTBPM_i$ is the *mean time between preventive maintenance* for the element $E_i$.

Case 2 has a practical significance when preventive maintenance can be performed without interruption of the operation at system level.

## 4.5 Basic Models for Spare Parts Provisioning

*Spare parts provisioning* is important for systems with long *useful life* or when short repair times and/or independence from the manufacturer is required (spare part is used here e.g. for *line replaceable unit* (LRU)). Basically, a distinction is made between centralized and decentralized logistic support. Also important is to take into account whether spare parts are repairable or not. This section presents the basic models for the provision of nonrepairable and of repairable spare parts. For nonrepairable spare parts, the cases of centralized and decentralized logistic support are considered in order to quantify the advantage of a centralized logistic support with respect to a decentralized one. More general maintenance strategies are discussed in Section 4.6, cost specific aspects in Sections 4.5-4.7.

### 4.5.1 Centralized Logistic Support, Nonrepairable Spare Parts

In *centralized logistic support*, spare parts are stocked at *one place*. The basic problem can be formulated as follows:

*At time $t = 0$, the first part is put into operation, it fails at time $t = \tau_1$ and is replaced (in a negligible time) by a second part which fails at time $t = \tau_1 + \tau_2$ and so forth; asked is the number n of parts which must be stocked in order that the requirement for parts during the cumulative operating time T is met with a given* (fixed) *probability* $\gamma$.

To answer this question, the *smallest integer n* must be found for which

$$\Pr\{\tau_1 + \ldots + \tau_n > T\} \geq \gamma \tag{4.9}$$

holds. In general, $\tau_1, \ldots, \tau_n$ are assumed to be independent positive random variables with the same distribution function $F(x)$, density $f(x)$, and finite mean $E[\tau_i] = E[\tau] = MTTF$ & $\mathrm{Var}[\tau_i] = \mathrm{Var}[\tau]$. If the number of parts is calculated from

$$n = T \,/\, MTTF, \tag{4.10}$$

the requirement can only be covered (for $T$ large) with a probability of 0.5. Thus, more than $T / MTTF$ parts are necessary to meet the requirement with $\gamma > 0.5$.

According to Eq. (A7.12), the probability as per Eq. (4.9) can be expressed by the $(n-1)$th convolution of the distribution function $F(t)$ with itself, i.e.

$$\Pr\{\tau_1 + \ldots + \tau_n > T\} = 1 - F_n(T),$$

$$\text{with} \quad F_1(T) = F(T) \quad \text{and} \quad F_n(T) = \int_0^T F_{n-1}(T-x) f(x) dx, \quad n > 1. \tag{4.11}$$

Of the distribution functions $F(x)$ used in reliability theory, a closed, simple form for the function $F_n(x)$ exists only for the *exponential*, *gamma*, and *normal* distribution functions, yielding a Poisson, gamma, and normal distribution, respectively. In particular, the exponential distribution $F(x) = 1 - e^{-\lambda x}$ leads to (Eq. (A7.39))

$$\Pr\{\sum_{i=1}^{n} \tau_i > T\} = \sum_{i=0}^{n-1} \frac{(\lambda T)^i}{i!} e^{-\lambda T}. \tag{4.12}$$

The important case of the *Weibull distribution* $F(x) = 1 - e^{-(\lambda x)^{\beta}}$ must be solved numerically. Figure 4.3 shows the results with $\gamma$ and $\beta$ as parameters [4.3 (1974)].

For $n$ large, an *approximate solution* for a wide class of distribution functions $F(x)$ can be obtained using the *central limit theorem*. From Eq. (A6.148) if follows that (for $\mathrm{Var}[\tau] < \infty$)

$$\lim_{n \to \infty} \Pr\{\sum_{i=1}^{n} \frac{\tau_i - E[\tau]}{\sqrt{n\,\mathrm{Var}[\tau]}} > x\} = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-y^2/2} dy = 1 - \Phi(x) = \Phi(-x), \tag{4.13}$$

and thus, using $x\sqrt{n\,\mathrm{Var}[\tau]} + n\,E[\tau] = T$,

$$\lim_{n\to\infty} \Pr\{\sum_{i=1}^{n} \tau_i > T\} = \frac{1}{\sqrt{2\pi}} \int_{(T-n\,\mathrm{E}[\tau])/\sqrt{n\,\mathrm{Var}[\tau]}}^{\infty} e^{-y^2/2}\,dy = \gamma \tag{4.14}$$

Setting $(T - n\mathrm{E}[\tau])/\sqrt{n\mathrm{Var}[\tau]} = -d$ it follows that for $n \to \infty$

$$n = [\,\kappa d/2 + \sqrt{(\kappa d/2)^2 + T/\mathrm{E}[\tau]}\,]^2, \qquad \text{with } \kappa = \sqrt{\mathrm{Var}[\tau]}/\mathrm{E}[\tau]. \tag{4.15}$$

A similar approximation can also be obtained from Eq. (A7.34).

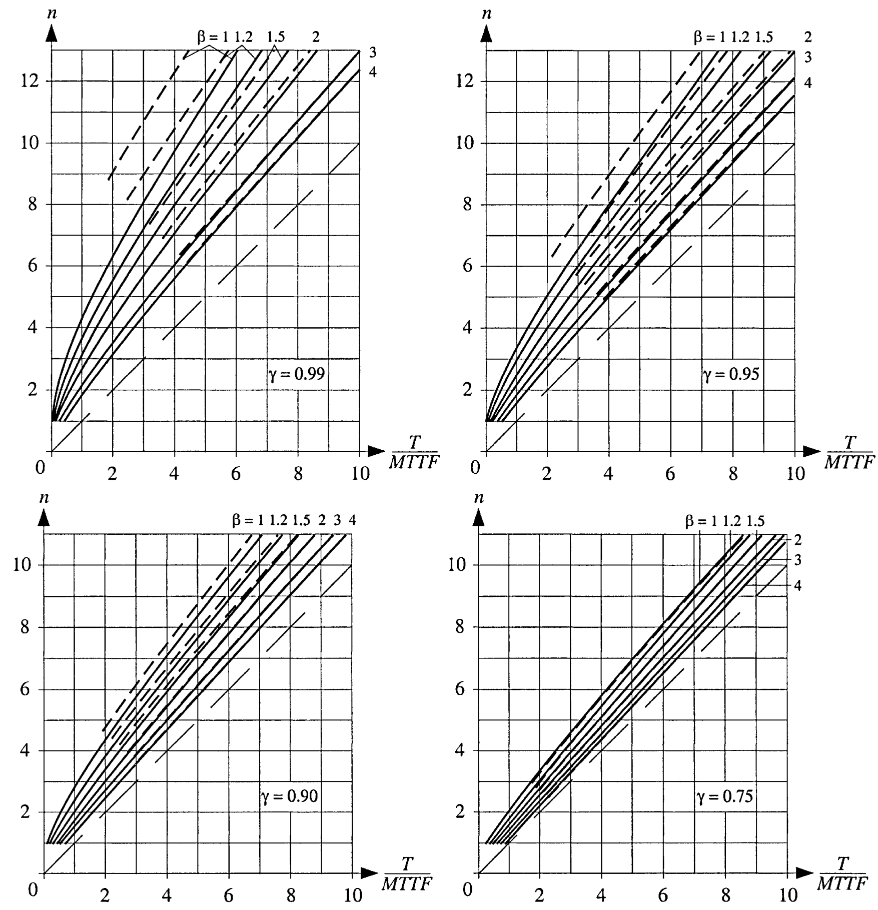



**Figure 4.3** Number of parts ($n$) which are necessary to cover a total cumulative operating time $T$ with a probability $\geq \gamma$, i.e. smallest $n$ for which $\Pr\{\tau_1 + \ldots + \tau_n > T\} \geq \gamma$ holds, with $\Pr\{\tau_i \leq x\} = 1 - e^{-(\lambda x)^\beta}$ and $MTTF = \Gamma(1 + 1/\beta)/\lambda$ (dashed are the results given by the central limit theorem as per Eq. (4.15), $\beta = 1$ yields the exponential distribution function)

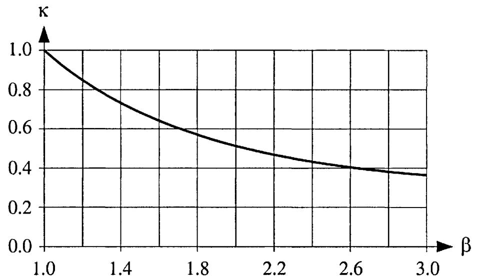



**Figure 4.4** Coefficient of variation for the Weibull distribution for $1 \leq \beta \leq 3$

From Eqs. (4.13) to (4.15) one recognizes that $d$ is the $\gamma$ quantile of the *standard normal distribution* ($\gamma = 1 - \Phi(-d) = \Phi(d)$), yielding e.g. (Table A9.1)

| | | | | | |
|---|---|---|---|---|---|
| $\gamma =$ | 0.99 | 0.95 | 0.90 | 0.75 | 0.5 |
| $d =$ | 2.33 | 1.64 | 1.28 | 0.67 | 0 |

Equation (4.15) gives for $\gamma \leq 0.95$ a good *approximation* of the number of parts $n$ down to low values of $n$ (see e.g. Fig. 4.3). $\kappa = \sqrt{\mathrm{Var}[\tau]}\,/\,\mathrm{E}[\tau]$ is the *coefficient of variation* ($\kappa = 1$ for the exponential distribution and $\kappa = \sqrt{\Gamma(1+2/\beta)/(\Gamma(1+1/\beta))^2 - 1}$ for the Weibull distribution (Fig. 4.4)).

For the case of a Weibull distribution with $\beta \geq 1$, approximate values for $n$ obtained using the central limit theorem (Eq. (4.15)) are shown *dashed* in Fig. 4.3. For $\beta = 1$, deviation from the exact value is $< 1.3$ for $\gamma \leq 0.95$ and $n \geq 5$; this deviation drops off rapidly for increasing values of $\beta$ ($\mathrm{F}_n(x)$ already approaches a normal distribution for small $n$). From Eqs. (4.13) - (4.15) one recognizes that for $\gamma = 0.5$, $d = 0$ and thus, for $n$ large, $n = T\,/\,\mathrm{E}[\tau]$ (Eq. (4.10)).

Let us now consider the case in which *the same part occurs k times* in the system. For $\mathrm{F}(x) = 1 - e^{-\lambda x}$, i.e. $\mathrm{E}[\tau] = 1/\lambda$ and $\kappa = 1$, Eqs. (4.12) - (4.15) hold with

$$\lambda' = k\,\lambda, \qquad k = 1, 2, \ldots, \tag{4.16}$$

instead of $\lambda$. This is because the *sum of independent Poisson processes* is a Poisson process (Eq. (7.27)) and $k$ parts must be operating for the required function. The same holds if $l$ systems use the same part, one or more per system with total $k$ parts of the same type, and *storage is centralized* (Example 4.3).

Considering that $k$ parts are available at $t = 0$ (operating at $t = 0$), it is reasonable to define as number of *spare parts* $n_{sp}$ the quantity

$$n_{sp} = n - k, \qquad k = 1, 2, \ldots, \tag{4.17}$$

where $n$ is the number of parts obtained from Eqs. (4.12) - (4.16), see Examples 4.3 and 4.4 for practical applications.

**Example 4.3**

A part with constant failure rate $\lambda = 10^{-3}\,h^{-1}$ is used three times in a system $(k = 3)$. Give the number of spare parts $n_{sp}$ which must be stored to cover a cumulative operating time $T = 10{,}000\,h$ with a probability $\gamma \geq 0.90$.

**Solution**

Considering $k\lambda T = 30$, the exact solution is given by the smallest integer $n_{sp} = n - 3$ for which

$$\sum_{i=0}^{n-1} \frac{30^i}{i!} e^{-30} \geq 0.9$$

holds (Eq. (4.12)). From Table A9.2 it follows, for $q = 1 - 0.9 = 0.1$ and $t_{\nu,q} = 2 \cdot 30 = 60$, the value $\nu = 75.2$ (lin. interpolation); thus, $\nu = 76$ and (Appendix A9.2 & Eq. (4.12)) $n = \nu / 2 = 38$ (same results from Fig. 7.3 for $m = 30$ & $\gamma = 0.9$, yielding $n = c + 1 = 38$, and with Eq. (4.15) for $\kappa = 1$ and $d = 1.28$, yielding $n = 38$ ($[0.64 + \sqrt{0.64^2 + 30}\,]^2 \approx 37.9$)). Thus, considering that 3 parts are operating at $t = 0$, it follows that (Eq. (4.17)) $n_{sp} = 38 - 3 = 35$.

## 4.5.2 Decentralized Logistic Support, Nonrepairable Spare Parts

For users who have the same system located at different places, spare parts are often stored *decentralized*, i.e., separately at each location (decentralized means that spare parts cannot be transferred from one location to another location). If there are $l$ systems, each with a given part, and the storage of spare parts is decentralized at each system (or location), a first approach could be to store with each system the same number of spare parts obtained using Eqs. (4.9) and (4.17). In this case, the total number of parts would be $n \cdot l$, i.e. $(n-k) \cdot l$ spare parts. This number $n$ of parts, which would be sufficient to meet, with a probability $> \gamma$ (often $\gg \gamma$) the needs of the $l$ systems with a *centralized storage* (Example 4.4), would now in general be too small to meet all the individual needs at each location. In fact, assuming that failures at each location are independent, and that with $n$ parts the probability of meeting the needs at any location individually is $\gamma$, then the probability of meeting the need at all locations is $\gamma^l$. Thus, to meet the need at the $l$ locations with a probability $\gamma$

$$n_{dec} = l \cdot n_l \tag{4.18}$$

parts are required, where $n_l$ is computed for each location individually with

$$\gamma_l = \sqrt[l]{\gamma}\,, \tag{4.19}$$

e.g. using Eq. (4.15) with $d_l$ instead of $d$ ($\Phi(d) = \gamma$, $\Phi(d_l) = \sqrt[l]{\gamma}$). To make a comparison between a centralized and a decentralized logistic support, let us assume that the part considered appears $k$ times in each of the $l$ locations, has *constant failure rate* $\lambda$, and $k\lambda T \gg d_l^2/4 > d^2/4$ holds. In this case, Eqs. (4.15) & (4.16) lead to

$$n \approx k\lambda T + d\sqrt{k\lambda T}, \qquad k\lambda T \gg d^2/4, \quad k = 1, 2, \ldots, \quad \text{probability } \gamma. \tag{4.20}$$

For *centralized logistic support*, Eq. (4.20) yields

$$n_{cen} \approx lk\lambda T + d\sqrt{lk\lambda T}, \qquad lk\lambda T >> d^2/4, \quad k,l = 1,2,..., \text{ probability } \gamma. \tag{4.21}$$

For *decentralized logistical support*, Eq. (4.20) yields

$$n_{dec} \approx l(k\lambda T + d_l\sqrt{k\lambda T}), \qquad k\lambda T >> d_l^2/4, \quad k,l = 1,2,..., \text{ probability } \gamma, \tag{4.22}$$

where $d_l$ is obtained as for $d$ in Eq. (4.15) with $\gamma_l = \sqrt[l]{\gamma}$ instead of $\gamma$ (for example, $d = 1.64$ for $\gamma = 0.95$ and $d_l = 2.57$ for $l = 10$ i.e. for $\gamma_l = 0.9949$, see Table A9.1). From the above considerations it follows that for $k\lambda T >> d_l^2/4 > d^2/4$

$$\frac{n_{dec}}{n_{cen}} \approx \frac{1+d_l/\sqrt{k\lambda T}}{1+d/\sqrt{lk\lambda T}} \quad \text{or} \quad \frac{n_{sp_{dec}}}{n_{sp_{cen}}} \approx \frac{1+(d_l/\sqrt{k\lambda T})-1/\lambda T}{1+(d/\sqrt{lk\lambda T})-1/\lambda T}, \tag{4.23}$$

with $\Phi(d) = \gamma$ and $\Phi(d_l) = \sqrt[l]{\gamma}$ (see Example 4.4). Setting $\lambda T = T/\mathrm{E}[\tau] = T/MTTF$, Eq. (4.23) can be used for arbitrary distribution of the spare parts failure-free time $\tau$.

### 4.5.3 Repairable Spare Parts

In Sections 4.5.1 and 4.5.2 it was assumed that the spare parts (LRUs) were *nonrepairable*, i.e., that a new spare part was necessary at each failure. In many cases, spare parts can be repaired and then stored for *reuse*. Calculation of the number of spare parts which should be stored can be performed in a way similar to the investigation of a *k-out-of-n standby redundancy*, where $k$ is the number of parts used in the system (as in Eq. (4.17)) and $n$ is the smallest integer *to be determined* such that the requirement is met with a given (fixed) probability $\gamma$. Following two cases have to be considered:

**Example 4.4**

Let $\lambda = 10^{-4}\,\mathrm{h}^{-1}$ be the constant failure rate of a part in a given system. The user has 6 locations ($l = 6$) and would like to achieve a cumulative operating time $T = 50{,}000\,\mathrm{h}$ at each location with a probability $\gamma \geq 0.95$. How many spare parts can be saved using a centralized logistic support?

**Solution**

From Fig. 4.3 ($T/MTTF = 5$, $\gamma = \sqrt[6]{0.95} \approx 0.99$), Fig 7.3 ($m = 5$, $\gamma = 0.99$, $c = n_l - 1$), or from a $\chi^2$-Table ($t_{\nu,q} = 10$, $q = 1-0.99 = 0.01$, $\nu = 2n_l$) each user would need $n_l = 12$ parts ($n_l = 14$ using Eq. (4.15) with $d = d_l = 2.33$ and $\lambda T = 5$); thus $n_{dec} = 6 \cdot 12 = 72$ parts and (Eq. (4.17)) $n_{sp_{dec}} = 72 - 6 = 66$ spare parts. Combining the storage ($l = 6$), it follows from Fig. 7.3 ($m = 30$, $\gamma = 0.95$, $c = n_{cen} - 1$) or from Table A9.2 ($t_{\nu,q} = 60$, $q = 0.05$, $\nu = 2n_{cen}$) that $n_{cen} = 40$ ($n_{cen} = 41$ using Eq. (4.15) with $d = 1.64$ and $\lambda T = 30$); thus, $n_{sp_{cen}} = 40 - 6 = 34$. A centralized storage would save $66 - 34$ (or $72 - 40$) $= 32$ spare parts (Eq. (4.23) gives 1.57 instead of 1.8 (left) and 1.67 instead of 1.94 (right), because $k\lambda T = 5$ is not $>> d_l^2/4 = 1.36$).

***Supplementary result:*** Provisioning independently for each location with $\gamma = 0.95$ yields $n_l = 10$ (Fig. 4.3 with $T/MTTF = 5$ & $\gamma = 0.95$) and thus $n = 6 \cdot 10 = 60$.

1. $\gamma$ is the probability that a request for a spare part *at a time point t* can be met without time delay; in this case, $\gamma$ can be considered as the *point availability* $PA_S$ (in steady-state to simplify investigations) and $n$ is the smallest integer such that $PA_S \geq \gamma$ for a given (fixed) $\gamma$.
2. $\gamma$ is the probability that any request for a spare part *during the time interval* $(0,t]$ will be met without time delay; in this case, $\gamma$ can be considered as the *reliability function* $R_{S0}(t)$ and $n$ is the smallest integer such that $R_{S0}(t) \geq \gamma$ for given (fixed) $\gamma$ and $t$.

If the *spare parts* have *constant failure rate* $\lambda = 1/MTTF$ and *constant repair rate* $\mu = 1/MTTR$, *birth-and-death processes* can be used (Section A7.5.5). To simplify investigations and to agree with results in Chapter 6, it is assumed that *only one spare part at a time can be repaired* (only 1 repair crew is available) and *no further failures are considered when a request for a spare part cannot be met* (corresponds to the assumption *no further failure at system down* (Fig. 6.13).

For Case 1 above, Eq. (6.138) with $\lambda_r \equiv 0$ and Eq. (6.140) yield

$$PA_S = \sum_{j=0}^{n-k} P_j = 1 - P_{n-k+1} \geq \gamma \tag{4.24}$$

with

$$P_j = \frac{\pi_j}{\sum_{i=0}^{n-k+1} \pi_i} \qquad \text{and} \qquad \pi_i = (k\lambda/\mu)^i, \quad i = 0, \ldots, n-k+1. \tag{4.25}$$

Sought is the smallest integer $n$ which satisfies Eq. (4.24) for given (fixed) $\gamma$, $k$, $\lambda$, and $\mu$. Often $n = k+1$ (*one spare part*) or $n = k+2$ (*two spare parts*) will be *sufficient*. In these cases, results of Table 6.8 yield

$$PA_{S_1} = \frac{1}{1 + k^2\lambda^2/(k\lambda\mu + \mu^2)} \approx 1 - (k\lambda/\mu)^2, \tag{4.26}$$

$n_{sp} = n - k = 1$ spare part, 1 repair crew, Case 1,

$$PA_{S_2} = \frac{1}{1 + k^3\lambda^3/(k^2\lambda^2\mu + k\lambda\mu^2 + \mu^3)} \approx 1 - (k\lambda/\mu)^3, \tag{4.27}$$

$n_{sp} = n - k = 2$ spare parts, 1 repair crew, Case 1.

If $PA_{S_2}$ is still $< \gamma$, more than 2 spare parts are necessary. A good approximation for the number $n_{sp}$ of spare parts can be obtained using the smallest integer $n_{sp} = n - k$ satisfying (Table 6.8)

$$PA_{S_{n_{sp}}} \approx 1 - (k\lambda/\mu)^{n_{sp}+1} \geq \gamma, \quad n_{sp} = n - k \text{ spare parts, 1 repair crew, Case 1.} \tag{4.28}$$

Using results of Appendix A7.5.5 (Eq. (A7.157)) and considering $k\lambda \ll \mu$, it can be shown that approximations given by Eqs. (4.26) - (4.28) hold also if the assumption "no further failures are considered when a request for a spare part cannot be met" is not made. The case in which $n_{sp}+1$ repair crews are available (instead of 1 repair crew) is considered by Eq. (4.32) for comparative investigations.

For Case 2 above, the reliability function can be approximated by an exponential function (Eq. (6.93)), yielding (Eqs. (6.144) & (6.145) with $\nu_i = k\lambda$)

$$R_{S0_1}(t) \approx e^{-t(k\lambda)^2/\mu}, \qquad n_{sp} = n - k = 1 \text{ spare part, 1 repair crew, Case 2}, \tag{4.29}$$

$$R_{S0_2}(t) \approx e^{-t(k\lambda)^3/\mu^2}, \qquad n_{sp} = n - k = 2 \text{ spare parts, 1 repair crew, Case 2}. \tag{4.30}$$

If $R_{S0_2}(t)$, with $t$ as mission time, is still $< \gamma$, more than 2 spare parts are necessary. A good approximation for the number $n_{sp}$ of spare parts can be obtained using the smallest integer $n_{sp} = n-k$ satisfying (Table 6.8)

$$R_{S0_{n_{sp}}}(t) \approx e^{-t\mu(k\lambda/\mu)^{n_{sp}+1}} \ge \gamma, \quad n_{sp} = n-k \text{ spareparts, 1 repair crew, Case2}. \tag{4.31}$$

For Eqs. (4.29) to (4.31) it holds necessarily that no further failures are considered when a request for a spare part cannot be met (system down states are made absorbing for reliability calculations). The case in which $n_{sp}$ repair crews are available is considered by Eq. (4.33) for comparative investigations. Example 4.5 gives a practical application.

Assuming *for comparative investigations* that each of the $n_{sp} = n-k$ spare parts can be repaired *independently* from each other ($n_{sp}+1$ repair crew, no further failures when a request for a spare part cannot be met), results of Section A7.5.5, with $\nu_i = k\lambda$, $i = 0, \ldots, n-k$, and $\theta_i = i\mu$, $i = 1, \ldots, n-k+1$, yield (see also Eq. (6.149))

**Example 4.5**

A system contains $k = 100$ identical parts (LRUs) with a constant failure rate $\lambda = 10^{-5}\,h^{-1}$ and which can be repaired with a constant repair rate $\mu = 10^{-1}\,h^{-1}$. (*i*) Give the number of spare parts which must be stored in order to meet without any time delay and with a probability $\gamma \ge 0.99$ a request for a spare part at a time point $t$ (consider the steady-state only, one repair crew, and no further failure when a request for a spare part cannot be met). (*ii*) If one spare part is stored ($n = k+1$), how large is the probability that any request for a spare part during the time interval $(0, 10^4\,h]$ will be met without any time delay?

**Solution**

(*i*) Taking $n = k+1$ (1 spare part), Eq. (4.26) yields

$$PA_{S1} = \frac{1}{1 + 10^4 \cdot 10^{-10} / (100 \cdot 10^{-5} \cdot 10^{-1} + 10^{-2})} \approx 1 - \left(\frac{100 \cdot 10^{-5}}{10^{-1}}\right)^2 \approx 0.9999.$$

Thus only one spare part ($n_{sp} = 1$) must be stored.

(*ii*) For $n = k+1$, Eq. (4.29) yields $R_{S0_1}(t) \approx e^{-0.00001\,t}$ and thus $R_{S0_1}(10^4\,h) \approx e^{-0.1} \approx 0.91$.

***Supplementary result:*** To reach $R_{S0}(10^4) \ge 0.99$ one needs $n_{sp}=2$ spare parts ($R_{S0_2}(10^4) = 0.999$).

$$PA_{S_{n_{sp}}} \approx 1 - (k\lambda / \mu)^{n_{sp}+1} / (n_{sp} + 1)!, \qquad \begin{array}{l} n_{sp} = n - k \text{ spare parts,} \\ n_{sp}+1 \text{ repair crews, Case 1,} \end{array} \tag{4.32}$$

and, with $\nu_i$ as before and $\theta_i = i\mu$, $i = 1, \ldots, n-k$,

$$R_{S0_{n_{sp}}}(t) \approx e^{-t\mu (k\lambda/\mu)^{n_{sp}+1} / (n_{sp})!}, \qquad \begin{array}{l} n_{sp} = n - k \text{ spare parts,} \\ n_{sp} \text{ repair crews, Case 2.} \end{array} \tag{4.33}$$

Using results of Appendix A7.5.5 (Eq. (A7.157)) and considering $k\lambda << \mu$, it can be shown that the approximation given by Eq. (4.32) holds also if the assumption "no further failures are considered when a request for a spare part cannot be met" is not made. For Eq. (4.33) it holds necessarily that no further failures are considered when a request for a spare part cannot be met (system down states are absorbing).

Generalization of the repair rate leads to semi-regenerative processes with $n-k+1$ regeneration and $n-k$ not regeneration states (Sections 6.4.2 & 6.5.2, Appendix A7.7). For instance, assuming for the repair time a density $g(t)$, a mean *MTTR*, and a variance Var $[\tau']$, Eq. (6.110) with $k\lambda$ instead of $\lambda$ and $\lambda_r \equiv 0$ (see supplementary results in Example A7.12) and $\tilde{g}(\lambda)$ per Eq. (6.113), lead to

$$PA_{S_1} = \frac{k\lambda}{(k\lambda)^2 MTTR + k\lambda\, \tilde{g}(k\lambda)} \approx \frac{1}{1 + (k\lambda)^2 (MTTR^2 + \text{Var}[\tau']) / 2}$$
$$\approx 1 - (k\lambda MTTR)^2 (1 + \text{Var}[\tau'] / MTTR^2) / 2 \gtrapprox 1 - (k\lambda MTTR)^2, \tag{4.34}$$
$$n_{sp} = n - k = 1 \text{ spare part, 1 repair crew, Case 1.}$$

Similarly, Eq. (6.108) with $k\lambda$ instead of $\lambda$ and $\lambda_r \equiv 0$ and Eq. (6.114) lead to

$$R_{S0_1}(t) \approx e^{-t k\lambda (1 - \tilde{g}(k\lambda))} \approx e^{-t (k\lambda)^2 MTTR}, \qquad n_{sp} = n - k = 1 \text{ spare part, 1 repair crew, Case2.} \tag{4.35}$$

The last approximation in Eq. (4.34) assumes for the coefficient of variation $\kappa$ that

$$\kappa^2 = \text{Var}[\tau'] / \text{E}^2[\tau'] \le 1, \tag{4.36}$$

which holds for distribution functions used for repair times (increasing repair rate). Assuming $MTTR = 1/\mu$, i. e., the same mean time to repair disregarding the distribution of the repair time, the last approximations in Eqs. (4.34) and (4.35) yield the same result as given by Eqs. (4.26) and (4.29), showing, once more, the *small influence of the repair time distribution on results at system level.* The last approximation in Eq. (4.35) is obtained by assuming $k\lambda\, MTTR << 1$, i. e. using $\tilde{g}(k\lambda) \approx 1 - k\lambda\, MTTR$ (Eq. (6.114). For the approximation in Eq. (4.34) it was necessary to use $\tilde{g}(k\lambda) \approx 1 - k\lambda MTTR + (k\lambda)^2 (MTTR^2 + \text{Var}[\tau'])/2$ (Eq. (6.113)).

Taking $R_S(t) = e^{-t/MTTF_S}$ in Eqs. (4.31), (4.33) & (4.35), and $PA_S$ as in Eqs. (4.28), (4.32) & (4.34), $PA_S$ can be expressed as (Eq. (A7.189))

$$PA_S \approx 1 - MTTR_S \,/\, MTTF_S \,, \tag{4.37}$$

with $MTTR_S = 1/\mu$, $MTTR_S = 1/(n-k+1)\mu$ & $MTTR_S = MTTR$, respectively.

The results of Sections 4.5.1 - 4.5.3, in particular those on *decentralized logistic support*, can be extended to cover the case of systems with *different spare parts.*

# 4.6 Maintenance Strategies

Maintenance strategies can be very different according to the objective to be reached (choice between maintenance policies, minimization of system down time or spare parts, availability maximization by given cost and/or logistic support, etc.). Among possible maintenance strategies [2.34, 4.1, 4.2, 4.6, 4.8, 4.14, 4.18, 4.29, 6.3, A7.4 (62)], this section unifies and extends basic repair/replacement policies. For a more pragmatic approach, developed for high safety systems, one can refer to [4.26]. Cost aspects are considered here and in Section 4.7.

In the following it is assumed that the item is new at $t = 0$, its failure-free time $\tau > 0$ has distribution function $F(x)$ and density $f(x)$ and, in Sections 4.6.1 & 4.6.2, repairs/replacements are performed in a negligible time. Section 4.6.1 considers the case in which the item is *as-good-as-new* after each maintenance action, planned (preventive maintenance) or at failure. In section 4.6.2, the item is *as-good-as-new* only after planned maintenance actions, but *as-bad-as-old* after repairs (*minimal repair* at failure). Further considerations are in Section 4.6.3. +)

## 4.6.1 Complete renewal at each maintenance action

In this section it is assumed that each maintenance action, planned or at failure, brings the item considered to *as-good-as-new* (see the remark on pp. 8 and 169 for complex items), yielding to a *renewal point* for the underlying point process.

Among possible strategies to avoid *wearout failures* or *effects of sudden failures*, replacements ++) can be performed basically

(*a*) at a given (fixed) operating time $T_{PM}$ or at failure if the operating time is shorter than $T_{PM}$ (*age replacement*, Fig. 4.5a),

(*b*) at given (fixed) time points $T_{PM}, 2T_{PM}, \ldots$ or at failure (*block replacement*, Fig. 4.5b),

(*fix*) only at given (fixed) time points $T_{PM}, 2T_{PM}, \ldots$ (*fix replacement*, Fig. 4.5c),

(*of*) *only at failure* (ordinary renewal process without truncation).

---

+) Considering remarks to Eqs. (A6.27) - (A6.30), preventive maintenance is useful only for items with *increasing failure rate*, tacitly assumed here. ++) As in the established literature, *replacement* is used instead of *renewal* for the case of an item which is as-good-as-new after repair.

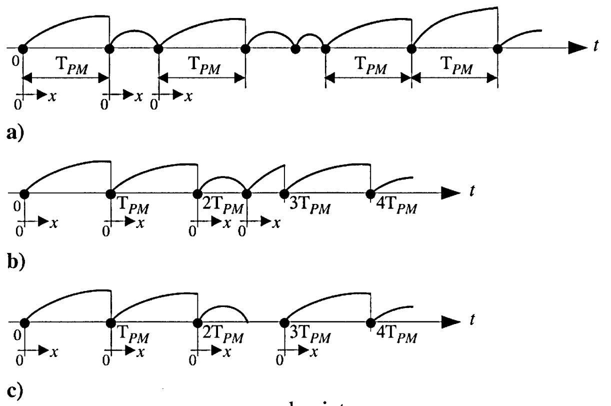



**Figure 4.5** Possible time schedules for a repairable item with preventive maintenance and replacements (renewals) of negligible length: **a)** After $T_{PM}$ operating hours or at failure (age replacement); **b)** At fixed times $T_{PM}, 2T_{PM}, \ldots$ or at failure (block repl.); **c)** At fixed times $T_{PM}, 2T_{PM}, \ldots$ (fix repl.) (item new at $t = 0$ and at each repair or preventive maintenance, $x$ starts by 0 at each renewal point)

Considering first the case of *age replacement* (Fig. 4.5a), results of Appendix A7.2 for renewal processes and Section 4.5 for spare parts provisioning can be used, taking for the failure-free time $\tau_{repl_a}$ the truncated distribution function $F_{repl_a}(x)$

$$F_{repl_a}(x) = \Pr\{\tau_{repl_a} \le x\} = \begin{cases} F(x) & \text{for } 0 < x < T_{PM}, \\ 1 & \text{for } x \ge T_{PM}, \end{cases} \qquad F(0) = F_{repl_a}(0) = 0. \quad (4.38)$$

Taking care of $\Pr\{\tau_{repl_a} = T_{PM}\} = 1 - F(T_{PM})$, the mean time to replacement follows as

$$E[\tau_{repl_a}] = \int_0^{T_{PM}} x\, f(x)\, dx + (1 - F(T_{PM}))\, T_{PM} = \int_0^{T_{PM}} (1 - F(x))\, dx < T_{PM}. \qquad (4.39)$$

Defining as $\nu_a(t)$ the number of renewals in $(0, t]$ on age replacement policy (replacements at failure & preventive maintenance), it follows from Eq. (A7.15) that

$$E[\nu_a(t)] = H_a(t), \qquad t > 0,\ \nu_a(0) = H_a(0) = 0, \qquad (4.40)$$

(with $F_1(x) = F(x) = F_{repl\,a}(x)$ in Eq. (A7.15)). Furthermore, Eq. (A7.27) yields

$$\lim_{t \to \infty} E[\nu_a(t)] = t \,/\, E[\tau_{repl_a}] = t \,/ \int_0^{T_{PM}} (1 - F(x))\, dx, \qquad (4.41)$$

in the proportion $F(T_{PM})$ for replacements at failure and $1 - F(T_{PM})$ for replacements at age. Thus, with $c_f$ and $c_{ar}$ as cost for replacement at failure and at age, the mean total cost per unit time (cost rate) is

$$\lim_{t \to \infty} E[c_a / t] = [c_f F(T_{PM}) + c_{ar}(1 - F(T_{PM}))] \,/ \int_0^{T_{PM}} (1 - F(x))\, dx. \qquad (4.42)$$

From Eq. (4.42) one recognizes that $E[c_a / t] \to \infty$ for $T_{PM} \to 0$ and $\to c_f / E[\tau]$ for $T_{PM} \to \infty$; with $E[\tau]$ as mean of the failure-free time $\tau$ of the item considered (Eq. (A6.38)). Optimization of $c_a / t$ is considered with Eq. (4.49). Reliability and availability is investigated in Section 6.8.2 (Eqs. (6.192) - (6.195)).

For the case of *block replacement* (Fig 4.5b), one or more failures can occur during an interval $(kT_{PM}, (k+1)T_{PM}]$ $(k = 0,1,...)$, with consequent repair. For the expected total number of renewals in $(0, nT_{PM}]$ on block replacement policy (replacements at failure and preventive maintenance) it follows that

$$E[\nu_b(nT_{PM})] = n\,H(T_{PM}) + n, \qquad n=1,2,...,\ T_{PM}>0,\ \nu_b(0)=H(0)=0, \quad (4.43)$$

where $H(T_{PM})$ is the renewal function at $T_{PM}$ (Eq. (A7.15) with $F(x)$ as distribution function of the failure-free time $\tau$ of the item considered). With $c_f$ & $c_{br}$ as cost for replacement at failure & at $T_{PM}, 2T_{PM}, ...$, the mean total cost per unit time is

$$E[c_b / nT_{PM}] = [c_f\,H(T_{PM}) + c_{br}] \,/\, T_{PM}, \qquad T_{PM}>0,\ H(0)=0. \quad (4.44)$$

From Eq. (4.44) one recognizes that $E[c_b / nT_{PM}] \to \infty$ for $T_{PM} \to 0$ and, using Eq. (A7.27), $E[c_b / nT_{PM}] \to c_f / E[\tau]$ for $T_{PM} \to \infty$; with $E[\tau]$ as mean of the failure free time $\tau$ of the item considered. Optimization of $c_b$ is considered with Eq. (4.52).

For *fix replacement* (Fig. 4.5c), i.e., replacement only at times $T_{PM}, 2T_{PM}, ...$ (taking in charge that for a failure in $(kT_{PM}, (k+1)T_{PM}]$ $(k = 0,1,...)$ the item is down from failure time to $(k+1)T_{PM}$), the expected number of renewals in $(0, nT_{PM}]$ is

$$E[\nu_{fix}(nT_{PM})] = n, \qquad n=1,2,...,\ T_{PM}>0,\ \nu_{fix}(0)=0. \quad (4.45)$$

With $c_{fix}$ as cost for replacement at $T_{PM}, 2T_{PM}, ...$, the mean total cost per unit time is

$$E[c_{fix} / nT_{PM}] = c_f \,/\, T_{PM}. \quad (4.46)$$

It can be noted that the number of failures in $(0, nT_{PM}]$ has a binomial distribution (Eq. (A6.120) with $p = F(T_{PM})$). Furthermore, setting $c_d$ = cost per unit down time and considering Eq. (A6.30) one obtains $E[c_{fix} / nT_{PM}] = [c_f + c_d \int_0^{T_{PM}} F(x)\,dx] / T_{PM}$.

The *replacement only at failure* leads to an ordinary renewal process (Appendix A7.2), yielding results of Section 4.5 on spare parts provisioning and in particular

$$\lim_{n\to\infty} E[\nu_{of}(nT_{PM})] = nT_{PM} \,/\, E[\tau], \qquad n=1,2,...,\ T_{PM}>0,\ \nu_{of}(0)=0, \quad (4.47)$$

with $E[\tau]$ as mean of the failure-free time $\tau$ of the item considered, and

$$\lim_{n\to\infty} E[c_{of} / nT_{PM}] = c_f \,/\, E[\tau], \qquad n=1,2,...,\ T_{PM}>0. \quad (4.48)$$

One recognizes that for large $nT_{PM}$, $E[\nu_{of}(nT_{PM})] \le E[\nu_a(nT_{PM})] \le E[\nu_b(nT_{PM})]$. This follows for $\nu_{of}$ versus $\nu_a$ by comparing Eqs. (4.41) and (4.47), and for $\nu_a$ versus $\nu_b$ heuristically from Fig. 4.5 (at least one failure-free time will be truncated for large $n$ and the probability for a truncation is greater for case b) than for case a)) or by considering $H(t) \ge t / (\int_0^t (1 - F(x))\,dx) - 1$ [2.34 (1965)].

For age and block replacement policy it is basically possible to *optimize* $T_{PM}$. Setting the derivative with respect to $T_{PM}$ equal to 0, Eq. (4.42) yields for $T_{PMa_{opt}}$

$$\lambda(T_{PMa_{opt}}) \int_0^{T_{PMa_{opt}}} (1-\mathrm{F}(x))dx - \mathrm{F}(T_{PMa_{opt}}) = \frac{c_{ar}}{c_f - c_{ar}}, \qquad c_f > c_{ar}, \tag{4.49}$$

with $\lambda(x)$ as failure rate of the item considered (Eq. (A6.25)), and thus (Eq. (4.42))

$$\lim_{t\to\infty} \mathrm{E}[c_{a_{opt}} / t] = (c_f - c_{ar})\,\lambda(T_{PMa_{opt}}), \tag{4.50}$$

if $T_{PMa_{opt}} < \infty$ exist. For *strictly increasing failure rate* $\lambda(x)$, $T_{PMa_{opt}} < \infty$ exist for

$$\lambda(\infty) > c_f / (\mathrm{E}[\tau](c_f - c_{ar})), \tag{4.51}$$

see Example 4.6. $\lambda(\infty) \le c_f / (\mathrm{E}[\tau](c_f - c_{ar}))$, $\lambda(x) = \lambda$, or $c_f \le c_{ar}$ leads to a replacement only at failure ($T_{PM} = \infty$). Similarly, Eq. (4.44) yields

$$T_{PMb_{opt}} \mathrm{h}(T_{PMb_{opt}}) - \mathrm{H}(T_{PMb_{opt}}) = c_{br} / c_f, \tag{4.52}$$

with $\mathrm{h}(x) = d\mathrm{H}(x)/dx$ as renewal density (Eq. (A7.18)), and thus (Eq. (4.44))

$$\lim_{t\to\infty} \mathrm{E}[c_{b_{opt}} / t] = c_f\, \mathrm{h}(T_{PMb_{opt}}), \tag{4.53}$$

if $T_{PMb_{opt}} < \infty$ exist. Equation (4.52) is a necessary condition (only). For *strictly increasing failure rate*, at least one $T_{PMb_{opt}} < \infty$ exist for

$$1 - \mathrm{Var}[\tau] / \mathrm{E}^2[\tau] > 2\, c_{br} / c_f, \qquad \text{implying also} \qquad c_f > 2\, c_{br}, \tag{4.54}$$

see Example 4.6. $1 - \mathrm{Var}[\tau]/\mathrm{E}^2[\tau] \le 2\, c_{br}/c_f$ or $\lambda(x) = \lambda$ leads to a replacement only at failure ($T_{PM} = \infty$).

**Example 4.6**

Investigate Eqs. (4.49) and (4.52).

**Solution**

(i) To Eq. (4.49), with $T_{PMa_{opt}}$ replaced by $T$ for simplicity, one can recognize that for *strictly increasing failure rate* $\lambda(x)$, $\lambda(T)\int_0^T (1-\mathrm{F}(x))dx - \mathrm{F}(T)$ is *strictly increasing in T*, from 0 to $\lambda(\infty)\mathrm{E}[\tau] - 1$. In fact, for $T_2 > T_1$ it holds that

$$\lambda(T_2)\int_0^{T_1}(1-\mathrm{F}(x))dx + \lambda(T_2)\int_{T_1}^{T_2}(1-\mathrm{F}(x))dx - \mathrm{F}(T_1) - \int_{T_1}^{T_2}\mathrm{f}(x))dx > \lambda(T_1)\int_0^{T_1}(1-\mathrm{F}(x))dx - \mathrm{F}(T_1),$$

considering $\lambda(T_2) > \lambda(T_1)$ and $\int_{T_1}^{T_2}\mathrm{f}(x))dx = \int_{T_1}^{T_2}\lambda(x)(1-\mathrm{F}(x))dx < \lambda(T_2)\int_{T_1}^{T_2}(1-\mathrm{F}(x))dx$. Thus, $T < \infty$ exist for $\lambda(\infty)\mathrm{E}[\tau] - 1 > c_{ar} / (c_f - c_{ar})$, i.e. for $\lambda(\infty) > c_f / (\mathrm{E}[\tau](c_f - c_{ar}))$. However, an analytical expression for $T_{PMa_{opt}}$ is rarely possible, see e.g. [4.8] for numerical solutions.

(ii) To Eq. (4.52) one can recognize that for *strictly increasing failure rate* $\lambda(x)$, $T\mathrm{h}(T) - \mathrm{H}(T) \to (1 - \mathrm{Var}[\tau]/\mathrm{E}^2[\tau])/2 > 0$ for $T \to \infty$ and thus, considering $\mathrm{H}(0) = 0$, at least one $T < \infty$ exist for $(1 - \mathrm{Var}[\tau]/\mathrm{E}^2[\tau])/2 > c_{br}/c_f$. This follows from Eqs. (A7.28) & (A7.31) by considering $\mathrm{Var}[\tau] < \mathrm{E}^2[\tau]$ for strictly increasing failure rate [2.34 (1965)], see e.g. Fig. 4.4.

Comparison of cost per unit time is straightforward for fix replacement versus replacement only at failure (Eqs. (4.46) & (4.48)), but can become laborious for age replacement versus block replacement and / or replacement only at failure (Eqs. (4.42), (4.44)), (4.48), and (4.49) - (4.54)). In general, it must be performed on a case by case basis, often taking care that $c_f > c_{ar} > c_{br}$ and of other aspects like e. g. the importance to avoid wearout or sudden failures. Besides remarks to Eqs. (4.51) and (4.54) for $\lambda(x) = \lambda$, the following general results can be given for large $t$ or $nT_{PM}$:

1. For strictly increasing failure rate $\lambda(x)$ and $\lambda(\infty) > c_f / (\mathrm{E}[\tau](c_f - c_{ar}))$ (Eq. (4.51)), $T_{PMa_{opt}} < \infty$ exist (see e. g. [4.8] for numerical solutions) and, for large $t$, optimal age replacement (Eq. (4.50)) is better (cheaper) than replacement only at failure ($\mathrm{E}[c_a / t]$ per Eq. (4.42) crosses from above $\mathrm{E}[c_{of} / t] = c_f / \mathrm{E}[\tau]$).
2. Considering Eq. (A7.28) for an ordinary renewal process ($MTTF_a = MTTF = \mathrm{E}[\tau]$), it follows that $\mathrm{H}(T_{PM}) \to T_{PM} / \mathrm{E}[\tau] + (\mathrm{Var}[\tau] / \mathrm{E}^2[\tau] - 1) / 2$ for $T_{PM} \to \infty$. Thus, considering Eqs. (4.53) and (4.48), for $c_{br} / c_f < (1 - \mathrm{Var}[\tau] / \mathrm{E}^2[\tau]) / 2$ optimal block replacement can be better (cheaper) than replacement only at failure; however, this implies $\mathrm{Var}[\tau] / \mathrm{E}^2[\tau] < 1$ (given by a strictly increasing failure rate) and $c_f > 2\, c_{br}$.
3. For $c_f > c_{br} \ge c_{ar}$ optimal age replacement is better (cheaper) than optimal block replacement [4.2]; however, often one has $c_{br} < c_{ar}$.
4. For $c_{ar} = c_{br} = c_f$, $\mathrm{E}[c_{of} / nT_{PM}] \le \mathrm{E}[c_a / nT_{PM}] \le \mathrm{E}[c_b / nT_{PM}]$ (follows from $\mathrm{E}[\nu_{of}(nT_{PM})] \le \mathrm{E}[\nu_a(nT_{PM})] \le \mathrm{E}[\nu_b(nT_{PM})]$, see remarks to Eq. (4.48)).

## 4.6.2 Block replacement with minimal repair at failure

Let now consider the situation in which the item is *as-good-as-new* after planned replacements, but *as-bad-as-old* after repairs, i. e., *minimal repair* is performed at failure and the *item's failure rate after repair is the same as just before failure* (only a small portion of the item has been repaired [2.34, 6.2, 6.3], see also pp. 419 & 511).

One can recognize that the case of *maintenance only at failure* leads to a *non-homogeneous Poisson* process with intensity $\mathrm{m}(t)$ *equal the failure rate* $\lambda(t)$ *of the item considered* and mean value function $\mathrm{M}(t) = \int_0^t \lambda(x)dx$, i. e. (considering $\mathrm{F}(0) = 0$)

$$\mathrm{m}(t) = \lambda(t) = \frac{\mathrm{f}(t)}{1 - \mathrm{F}(t)} \quad \text{and} \quad \mathrm{M}(t) = \int_0^t \mathrm{m}(x)dx = \int_0^t \frac{\mathrm{f}(x)}{1 - \mathrm{F}(x)}dx = -\ln(1 - \mathrm{F}(t)), \tag{4.55}$$

see Point 2 on p. 511. For this reason, minimal repair can not be considered for a *maintenance only at failure*, because for strictly increasing failure rate the item continue to degenerate and at a given time it will be necessary to reestablish the as-good-as-new situation.

Similar is for *age replacement.* In fact, because of the minimal repair, age replacement at the operating time $T_{PM}$ leads practically to a planned replacement at $T_{PM}, 2T_{PM}, \ldots$, i.e., to a block replacement with minimal repair.

For *block replacement with minimal repair,* change with respect to Section 4.6.1 is the fact that between consecutive replacements at $T_{PM}, 2T_{PM}, \ldots$ the involved point process is a nonhomogeneous Poisson process (Eq. (4.55), Appendix A7.8.2). Defining $c_{br}$ and $c_{f_{mr}}$ as cost for replacement at block and minimal repair, respectively, the total cost per unit time follows as (see also Eq. (4.44))

$$E[c_{b_{mr}} / nT_{PM}] = \frac{c_{fmr} M(T_{PM}) + c_{br}}{T_{PM}} = \frac{-c_{fmr} \ln(1 - F(T_{PM})) + c_{br}}{T_{PM}}. \tag{4.56}$$

From Eq. (4.56) one recognizes that $E[c_{b_{mr}} / nT_{PM}] \to \infty$ for $T_{PM} \to 0$ and $\to c_{f_{mr}} \lambda(\infty)$ for $T_{PM} \to \infty$. Optimization of $T_{PM}$ (using $\partial / \partial T_{PM} = 0$) yields for $T_{PMbmropt}$

$$T_{PMbmropt}\, \lambda(T_{PMbmropt}) - \int_0^{T_{PM\,bmropt}} \lambda(t)\,dt = c_{br} / c_{f_{mr}}. \tag{4.57}$$

and thus (Eq. (4.44))

$$E[c_{bmropt} / nT_{PMbmropt}] = c_{f_{mr}}\, \lambda(T_{PMbmropt}), \tag{4.58}$$

if $T_{PMbmropt} < \infty$ exist. For $\lambda(t)$ strictly increasing, with $\lambda(0) = 0$, $T\lambda(T) - \int_0^T \lambda(t)\,dt$ is strictly increasing in $T$ and can cross from below $c_{br} / c_{f_{mr}}$ at $T = T_{PMbmropt} < \infty$. This occurs for $\lambda(\infty) = \infty$; for $\lambda(\infty) = \lambda < \infty$, $T_{PMbmropt} < \infty$ exist for

$$\lim_{t \to \infty} [\lambda t - \int_0^t \lambda(x)\,dx] > c_{br} / c_{f_{mr}}, \qquad \lambda(\infty) = \lambda. \tag{4.59}$$

No solution exist for $\lambda(t)$ constant. Taking as an example a *Weibull* distribution (Eq. A.6.89), for which $\lambda(t) = \beta \lambda^{\beta} t^{\beta - 1}$, one obtains for $\beta > 1$

$$T_{PM\,bmropt} = \frac{\sqrt[\beta]{c_{br} / ((\beta - 1)\, c_{f_{mr}})}}{\lambda} \quad \text{and} \quad E[c_{bmropt} / nT_{PMbmropt}] = \frac{\beta\, c_{br}}{(\beta - 1)\, T_{PMbmropt}}. \tag{4.60}$$

Cost comparison with results of Section 4.6.1 has to be performed on a case by case basis. For the Weibull distribution, Eqs. (4.60) and (4.48) show, for instance, that for $c_{f_{mr}} > ((\beta - 1)/c_{br})^{\beta - 1} (c_f / \Gamma(1/\beta))^{\beta}$ replacement only at failure is better (cheaper) than block replacement with minimal repair (contrary by reversed inequality).

### 4.6.3 Further considerations on maintenance strategies

For the case of non negligible repair and preventive maintenance times, with mean *MTTR* and *MTTPM*, *asymptotic & steady-state overall availability* $OA_S$ (Eq. (6.196)) can be optimized with respect to preventive maintenance period $T_{PM}$.

In fact, considering Eq. (4.41), Eq. (6.196) leads to $OA_S = \mathrm{E}[\tau_{repl_a}] / [\mathrm{E}[\tau_{repl_a}] + \mathrm{F}(T_{PM})MTTR + (1-\mathrm{F}(T_{PM}))MTTPM]$ for *age replacement*, Eq. (4.43) to $OA_S = T_{PM} / [T_{PM} + \mathrm{H}(T_{PM})MTTR + MTTPM]$ for *block replacement*, and Eq. (4.56) to $OA_S = T_{PM} / [T_{PM} + MTTR \int_0^{T_M} \lambda(x)dx + MTTPM]$ for *block replacement with minimal repair*. Optimization follows using $\partial PA_S / \partial T_{PM} = 0$, and leads to Eqs. (4.49), (4.52), (4.57) with $c_{ar}$ & $c_{br}$ replaced by $MTTPM$, $c_f$ by $MTTR$, $c_{fmr}$ by $MTTMR$, respectively ($MTTMR$ = mean time to minimal repair).

Besides the previous replacement strategies, a further possibility is to assume that at times $T_{PM}, 2T_{PM}, \ldots$ the system is inspected, and replacement at $(k+1)T_{PM}$ is performed only if a failure is occurred between $kT_{PM}$ and $(k+1)T_{PM}$. If the failure-free time $\tau$ is $>0$ with $\mathrm{F}(x) = \Pr\{\tau \le x\}$, the replacement time $\tau_{rep}$ has distribution

$$\Pr\{\tau_{rep} = kT_{PM}\} = \mathrm{F}(kT_{PM}) - \mathrm{F}((k-1)T_{PM}), \qquad k = 1, 2, \ldots, \ \mathrm{F}(0) = 0. \tag{4.61}$$

This case has been investigated in [6.17] with cost considerations. If $c_i$ = inspection cost, $c_r$ = cost for replacement, and $c_d$ = cost for unit of time (h) in which the system is down waiting for replacement ($c_i$, $c_r$, $c_d > 0$), the total cost $C$ per unit time is for $t = n\,T_{PM} \to \infty$ given by

$$C = \frac{n c_i}{n T_{PM}} + \frac{c_r\, n T_{PM} / \mathrm{E}[\tau_{rep}]}{n T_{PM}} + \frac{c_d \mathrm{E}[\tau_{rep} - \tau]}{\mathrm{E}[\tau_{rep}]} = \frac{c_i}{T_{PM}} + \frac{c_r}{\mathrm{E}[\tau_{rep}]} - \frac{c_d\, MTTF}{\mathrm{E}[\tau_{rep}]} + c_d, \tag{4.62}$$

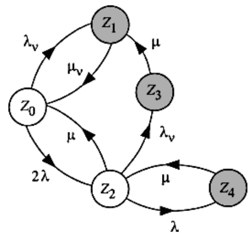
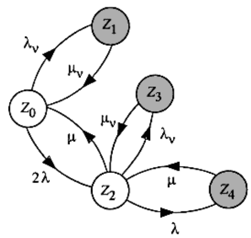
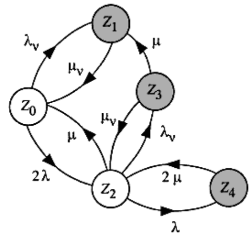
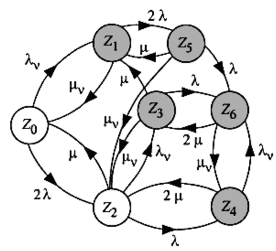
where $MTTF = \mathrm{E}[\tau]$. For $T_{PM} \to \infty$, $\mathrm{E}[\tau_{repl}] \to \infty$ and $C \to c_d$. Thus, inspection is useful for $C < c_d$. For given $\mathrm{F}(x)$ it is possible to find a $T_{PM}$ which minimizes $C$ [6.17].

For the *mission availability* and *work-mission availability*, as defined by Eqs. (6.28) and (6.31), it can be asked in some applications that the number of repairs (replacements) be limited to $N$ (e.g. because only $N$ spare parts are available). In this case, the summation in Eqs. (6.29) and (6.32) goes up to $n = N + 1$. If $k$ elements $E_1, \ldots, E_k$ with constant failure rates $\lambda_1, \ldots, \lambda_k$ and constant repair rates $\mu_1, \ldots, \mu_k$ are in series, a good approximation for the work-mission availability with *limited repairs* is obtained by multiplying the probability for total system down time $\le x$ for unlimited repairs (Eq. (7.22) with $\lambda = \lambda_S$ and $\mu = \mu_S$ from Table 6.10 (2 nd row)) with the $k$ probabilities that $N_i$ spare parts will be sufficient for element $E_i$ [6.11].

A strategy can also be based on the repair time $\tau'$ itself. Assuming for example that if the repair is not finished at time $\Delta$ the failed element is replaced at time $\Delta$ by a new one in a negligible time, the distribution function $\mathrm{G}(x)$ of the repair times $\tau'$ is truncated at $\Delta$ (Eq. (4.38)). For the case of const. repair rate $\mu$, the Laplace transform of $\mathrm{G}(x)$ to be used in reliability computations is given by (Appendix A9.7) $\tilde{\mathrm{G}}(s) = (\mu + s \cdot e^{-(s+\mu)\Delta}) / s(s+\mu)$, yielding $\mathrm{E}[\tau'] = (1 - e^{-\mu\Delta}) / \mu$ as per Eq. (4.39).

Further maintenance strategies are, for instance, in [2.34, 4.18, 4.30, A7.4 (62)]. A comparison between some different maintenance strategies with respect to reliability and availability is given in Table 4.4 for a basic reliability structure (Fig. 6.15). Expression for $MTTF_{S0}$ is the same for all cases in Table 4.4 and given by Eq. (6.158).

**Table 4.4** Basic series-parallel structure as per Figs. 6.15 & A7.5 for some relevant *repair strategies* (constant failure and repair rates $(\lambda, \lambda_v, \mu, \mu_v)$, active redundancy, ideal failure recognition & switch, Markov processes, no FF ≡ no further failures at system down, approximations valid for $\lambda_i << \mu_i$, $PA_S = AA_S$ = asymptotic & steady-state point and average availability; expressions for $MTTF_{S0}$ are here identical for all 4 cases and given by Eq. (6.158))

<table>
<tr><td><br><br>a) One repair crew, no repair priority, no FF</td><td>$PA_S = AA_S$, obtained by solving<br>$(2\lambda+\lambda_v)P_0 = \mu_v P_1 + \mu P_2$, $\mu_v P_1 = \lambda_v P_0 + \mu P_3$, $\mu P_3 = \lambda_v P_2$,<br>$(\lambda+\lambda_v+\mu)P_2 = 2\lambda P_0 + \mu P_4$, $P_0+P_1+P_2+P_3+P_4 = 1$,<br>is given by (Eq. (6.162))<br>$PA_S = AA_S = P_0 + P_2 = \frac{1}{1+\lambda_v/\mu_v + 2\lambda^2(1+\lambda_v/\lambda)/\mu^2(1+(2\lambda+\lambda_v)/\mu)}$<br>$\approx 1 - \lambda_v/\mu_v - 2(\lambda/\mu)^2 - 2\lambda\lambda_v/\mu^2 + 2\lambda(2\lambda^2+3\lambda\lambda_v+\lambda_v^2)/\mu^3$</td></tr>
<tr><td><br><br>b) One repair crew, repair priority on $E_v$, no FF</td><td>$PA_S = AA_S$, obtained by solving<br>$(2\lambda+\lambda_v)P_0 = \mu_v P_1 + \mu P_2$, $\mu_v P_1 = \lambda_v P_0$, $\mu_v P_3 = \lambda_v P_2$,<br>$(\lambda+\lambda_v+\mu)P_2 = 2\lambda P_0 + \mu_v P_3 + \mu P_4$, $P_0+P_1+P_2+P_3+P_4 = 1$,<br>is given by (Eq. (6.160))<br>$PA_S = AA_S = P_0 + P_2 = \frac{1}{1+\lambda_v/\mu_v + 2\lambda^2/\mu^2(1+2\lambda/\mu)}$<br>$\approx 1 - \lambda_v/\mu_v - 2(\lambda/\mu)^2 + 4(\lambda/\mu)^3$</td></tr>
<tr><td><br><br>c) 2 repair crews, no priority, no FF</td><td>$PA_S = AA_S$, obtained by solving<br>$(2\lambda+\lambda_v)P_0 = \mu_v P_1 + \mu P_2$, $\mu_v P_1 = \lambda_v P_0 + \mu P_3$, $(\mu+\mu_v)P_3 = \lambda_v P_2$,<br>$(\lambda+\lambda_v+\mu)P_2 = 2\lambda P_0 + \mu_v P_3 + 2\mu P_4$, $P_0+P_1+P_2+P_3+P_4 = 1$,<br>is given by<br>$PA_S = AA_S = P_0 + P_2 = \frac{1}{1+\lambda_v/\mu_v + \lambda^2/\mu^2(1+2\lambda/\mu+\lambda_v/(\mu+\mu_v))}$<br>$\approx 1 - \lambda_v/\mu_v - (\lambda/\mu)^2 + 2(\lambda/\mu)^3 + \lambda^2\lambda_v/\mu^2(\mu+\mu_v)$</td></tr>
<tr><td><br><br>d) 3 repair crews (same as completely independent elements)</td><td>$PA_S = AA_S$, obtained by solving<br>$(2\lambda+\lambda_v)P_0 = \mu_v P_1 + \mu P_2$, $(2\lambda+\mu_v)P_1 = \lambda_v P_0 + \mu P_3 + \mu P_5$,<br>$(\lambda+\lambda_v+\mu)P_2 = 2\lambda P_0 + \mu_v P_3 + 2\mu P_4 + \mu_v P_5$, $(\lambda+\mu+\mu_v)P_5 = 2\lambda P_1$,<br>$(\lambda+\mu+\mu_v)P_3 = \lambda_v P_2 + 2\mu P_6$, $(\lambda_v+2\mu)P_4 = \lambda P_2 + \mu_v P_6$, $P_0+\ldots+P_6 = 1$,<br>(or directly using Eq. (2.48) or Table 6.9), is given by<br>$PA_S = AA_S = P_0 + P_2 = \frac{1}{1+\lambda_v/\mu_v}\left(\frac{2}{1+\lambda/\mu} - \frac{1}{(1+\lambda/\mu)^2}\right)$<br>$\approx 1 - \lambda_v/\mu_v - (\lambda/\mu)^2 + 2(\lambda/\mu)^3 + \lambda^2\lambda_v/\mu^2\mu_v$</td></tr>
</table>

Approximations given up to $(\lambda/\mu)^3$; considering $(3\lambda+\lambda_v) < \mu$ it holds that $PA_{Sa)} \lessapprox PA_{Sb)} \lessapprox PA_{Sc)} \lessapprox PA_{Sd)}$

## 4.7 Basic Cost Considerations

Cost considerations are important in practical applications and apply in particular to spare parts provisioning (Section 4.5) and maintenance strategies (Section 4.6). In *addition* to the considerations in Sections 4.5 and 4.6, this section considers two basic models based on homogeneous Poisson processes (HPP) with fixed and random costs.

As a first example consider the case in which a constant cost $c_0$ is related to each repair of a given item. Assuming that repair duration is negligible and times between successive failures are independent and exponentially distributed with parameter $\lambda$, the failure flow is a homogeneous Poisson process and the probability for $n$ failures during the operating time $t$ is given by (Eq. (A7.41))

$$\Pr\{n \text{ failures in } (0,t] \mid \lambda\} = \Pr\{\nu(t)=n \mid \lambda\} = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \qquad \begin{matrix} n=0,1,2,\ldots \\ t>0,\ \nu(0)=0. \end{matrix} \tag{4.63}$$

Eq. (4.63) is also the probability that the cumulated repair cost over $t$ is $C = n c_0$. Mean and variance of $C$ are (Eqs. (A6.40) and (A6.46) with Eq. (A7.42))

$$\mathrm{E}[C] = c_0 \lambda t \qquad \text{and} \qquad \mathrm{Var}[C] = c_0^2 \lambda t. \tag{4.64}$$

For large $\lambda t$, $C$ is approximately normally distributed (Eqs. (A6.105)) with mean and variance as per Eq. (4.64), see e.g. [A8.8].

If repair cost is a random variable $\xi_i > 0$ distributed according to $\mathrm{F}(x) = \Pr\{\xi_i \le x\}$ ($\mathrm{F}(0) = 0$, $i = 1, 2, \ldots$), $\xi_1, \xi_2, \ldots$ are statistically independent and independent of the count function $\nu(t)$ giving the number of failures in the *operating time* interval $(0,t]$, and $\xi_t$ is the sum of $\xi_i$ over $(0,t]$, it holds that (Eq. (A7.218))

$$\xi_t = \sum_{i=1}^{\nu(t)} \xi_i, \qquad \nu(t) = 1, 2, \ldots, \ t>0,\ \nu(0)=0, \ \ \xi_t = 0 \text{ for } \nu(t) = 0. \tag{4.65}$$

$\xi_t$ is distributed as the (cumulative) repair time for failures occurred in a total operating time $t$ of a repairable item, and is thus given by the *work-mission availability* $\mathrm{WMA}_{S0}(T_0, x)$ (Eq. (6.32) with $T_0 = t$). Assuming that the failures flow is a homogeneous Poisson process (HPP) with parameter $\lambda$, all $\xi_i$ are statistically independent, independent of $\nu(t)$, and have the same exponential distribution with parameter $\mu$, Eq. (6.32) with constant failure and repair rates $\lambda(x) = \lambda$ and $\mu(x) = \mu$ and $T_0 = t$ yields (Eqs. (6.33), (A7.219))

$$\Pr\{\xi_t \le x\} = \mathrm{WMA}_{S0}(t,x) = e^{-\lambda t} + \sum_{n=1}^{\infty} \Big[\frac{(\lambda t)^n}{n!} e^{-\lambda t} \Big(1 - \sum_{k=0}^{n-1} \frac{(\mu x)^k}{k!} e^{-\mu x}\Big)\Big]$$

$$= 1 - e^{-(\lambda t + \mu x)} \sum_{n=1}^{\infty} \Big[\frac{(\lambda t)^n}{n!} \sum_{k=0}^{n-1} \frac{(\mu x)^k}{k!}\Big], \qquad t>0 \text{ given},\ x>0,\ \Pr\{\xi_t = 0\} = e^{-\lambda t}. \tag{4.66}$$

Mean and variance of $\xi_t$ follow as (Eq. (A7.220), see also Eqs. (4.66), (A6.38), (A6.45), (A6.41))

$$\mathrm{E}[\xi_t] = \lambda t / \mu \quad \text{and} \quad \mathrm{Var}[\xi_t] = 2\lambda t / \mu^2. \tag{4.67}$$

Furthermore, for $t \to \infty$ the distribution of $\xi_t$ approach a normal distribution with mean and variance as per Eq. (4.67). Moments of $\xi_t$ can also be obtained for arbitrary $\mathrm{F}(x)=\Pr\{\xi_i \le x\}$, with $\mathrm{F}(0)=0$ (Example A7.14, Eq. (A7.221))

$$\mathrm{E}[\xi_t] = \mathrm{E}[\nu(t)]\,\mathrm{E}[\xi_i] \quad \text{and} \quad \mathrm{Var}[\xi_t] = \mathrm{E}[\nu(t)]\,\mathrm{Var}[\xi_i] + \mathrm{Var}[\nu(t)]\,\mathrm{E}^2[\xi_i]. \tag{4.68}$$

Of interest in some practical applications can also be the distribution of the time $\tau_C$ at which the cumulative cost $\xi_t$ crosses a give (fixed) barrier $C$. For the case given by Eq. (4.66) (in particular for $\xi_i > 0$), the events

$$\{\tau_C > t\} \quad \text{and} \quad \{\xi_t \le C\} \tag{4.69}$$

are equivalent. Form Eq. (4.66) it follows then (Eq. (A7.223))

$$\Pr\{\tau_C > t\} = 1 - e^{-(\lambda t + \mu C)} \sum_{n=1}^{\infty} \left[ \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n-1} \frac{(\mu C)^k}{k!} \right], \qquad C > 0 \text{ given}, \; t > 0, \tag{4.70}$$

(in Eq. (4.70), $C$ has dimension of $\mu^{-1}$).

More general cost optimization strategies are often necessary in practical applications. For example, spare parts provisioning has to be considered as a parameter in the *optimization* between performance, reliability, availability, logistic support and *cost*, taking care of *obsolescence* aspects as well. In some cases, one parameter is given (e.g. cost) and the best logistic structure is sought to maximize system availability or system performance. Basic considerations, as discussed above and in Sections 1.2.9, 8.4, A6.10.7, A7.5.3.3, apply. However, even assuming constant failure and repair rates, numerical solutions can become necessary (e.g. [4.31]).