3

Valuation portfolio in life insurance

In this chapter we define the valuation portfolio for a life insurance liability portfolio. The construction is done with the help of an explicit example. We proceed in two steps: First, we assume that the cash flows have deterministic insurance technical risk, i.e. we have a deterministic mortality table, and only the value of the financial instruments describe a stochastic process. Then, we map the cash flows onto these financial instruments. In the second step, we introduce stochastic mortality rates yielding stochastic insurance technical risk. In that case we follow the construction in step 1, but we add loadings for the insurance technical risks coming from the stochastic mortality table. This construction gives us a replicating portfolio (protected against insurance technical risks) in terms of financial instruments.

3.1 Deterministic life insurance model

To define the valuation portfolio VaPo we start with a deterministic life insurance model where no insurance technical risk is involved (see also Baumgartner et al. [BBK04]). We assume that we have a deterministic mortality table (second order life table) giving the mortalities *without loadings*. Let l_x denote the number of insured lives aged x and d_x the number of insured lives aged x who die before reaching age $x+1$.

$$\begin{array}{lcl} l_x & & \\ \downarrow & \longrightarrow & d_x = l_x - l_{x+1} \\ l_{x+1} & & \\ \downarrow & \longrightarrow & d_{x+1} = l_{x+1} - l_{x+2} \\ l_{x+2} & & \\ \downarrow & \longrightarrow & d_{x+2} = l_{x+2} - l_{x+3} \\ \vdots & & \vdots \end{array}$$

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Example 3.1 (Endowment insurance policy).

We assume that the initial sum insured (death benefit) is CHF 1, the age at policy inception is $x = 50$ and the contract term is $n = 5$. Moreover, we assume that:

- The annual premium $\Pi_t = \Pi$, $t = 50, \ldots, 54$, is due in non-indexed CHF at the beginning of each year.
- The benefits are indexed by a well-defined index I_t, $t = 50, 51, \ldots, 55$, with initial value $I_{50} = 1$.
  - Death benefit is the indexed maximum of I_t and $(1+i)^{t-50}$ for some fixed minimal guaranteed interest rate i.
  - Survival benefit is I_{55}, i.e. no minimal guarantee in the case of survival.

The benefits are always paid at the end of each period $(t-1, t]$.

This means that the survival benefit is given by a financial instrument $\mathbf{I}$ whose price is a stochastic process $(I_t)_{t \geq 50}$ with initial value $I_{50} = 1$. This index can be any financial instrument like a stock, a fund, etc. Hence, to hedge the survival benefit we need to buy one unit of index $\mathbf{I}$ at the price $I_{50} = 1$ and it generates the (random) survival benefit I_{55} at time $t = 55$.

Thus, the endowment contract gives the following cash flow diagram for $\mathbf{X} = (X_{50}, \ldots, X_{55}) \in L^2_{n+1}(P, \mathcal{G})$: for initially l_{50} persons alive we have (if we only consider 1 person we divide by l_{50})

time	cash flow	premium	death benefit	survival benefit
50	X_{50}	$-l_{50}\ \Pi$		
51	X_{51}	$-l_{51}\ \Pi$	$d_{50}\ \left(I_{51} \vee (1+i)^1\right)$	
52	X_{52}	$-l_{52}\ \Pi$	$d_{51}\ \left(I_{52} \vee (1+i)^2\right)$	
53	X_{53}	$-l_{53}\ \Pi$	$d_{52}\ \left(I_{53} \vee (1+i)^3\right)$	
54	X_{54}	$-l_{54}\ \Pi$	$d_{53}\ \left(I_{54} \vee (1+i)^4\right)$	
55	X_{55}		$d_{54}\ \left(I_{55} \vee (1+i)^5\right)$	$l_{55}\ I_{55}$

Cash inflows (premium) have a negative sign, cash outflows have a positive sign, and $x \vee y = \max\{x, y\}$.

□

Task: Value this endowment policy at the beginning of the contract and at every successive year!

3.2 Valuation portfolio for the deterministic life insurance model

For the life insurance portfolio considered in Example 3.1 (with deterministic mortality rates) we now want to construct the valuation portfolio. Roughly speaking the valuation portfolio (VaPo) is a portfolio of financial instruments that replicates the future cash flows arising from the insurance contracts. The procedure is the following: to replicate the insurance cash flow $\mathbf{X} \in L^2_{n+1}(P, \mathcal{G})$ we specify in a first step the set of financial instruments that will be used for the replication purposes. Second, for each financial instrument the appropriate number of units must be determined, this gives the VaPo for $\mathbf{X}$. Thirdly, we define the market-consistent value of the cash flow $\mathbf{X}$ to be equal to the value of the VaPo. This convention is consistent with the well-known "law of one price"-principle which says that in an arbitrage-free economy two instruments with the same cash flows must have the same price.

Step 1. Define units, choose a financial basis.

- The premium Π is due at time $t = 50, \ldots, 54$ in non-indexed CHF. Hence, as units we choose the zero coupon bonds $Z^{(50)}, \ldots, Z^{(54)}$ (the units are denoted by $Z^{(t)}$, whereas the cash flow of the zero coupon bond $Z^{(t)}$ is denoted by $\mathbf{Z}^{(t)}$, see (2.28) and (3.4)).
- Survival benefit: Unit is the indexed fund $\mathbf{I}$ with price process $(I_t)_{t=50,\ldots,55}$.
- Death benefit $I_t \vee (1+i)^{t-50}$ can be measured in an indexed fund $\mathbf{I}$ plus a put option on $\mathbf{I}$ with strike time t and strike $(1+i)^{t-50}$. We denote this put option by $\mathrm{Put}^{(t)} = \mathrm{Put}^{(t)}(\mathbf{I}, (1+i)^{t-50})$.

Hence we have the following units (financial instruments)

$$\begin{aligned}(\mathcal{U}_1, \ldots, \mathcal{U}_{11}) \\ = \left(Z^{(50)}, \ldots, Z^{(54)}, \mathbf{I}, \mathrm{Put}^{(51)}\left(\mathbf{I}, (1+i)^1\right), \ldots, \mathrm{Put}^{(55)}\left(\mathbf{I}, (1+i)^5\right)\right),\end{aligned} \tag{3.1}$$

i.e. we have that the total number of different units equals 11. These units play the role of the basis (financial instruments) in which we measure the insurance liabilities.

Step 2. Determine the number/amount of each unit needed.

At the beginning of the policy we need:

<u>*Valuation Scheme A*</u> (for l_{50} persons)

time	premium	death benefit	survival benefit
50	$-l_{50}\ \Pi\ Z^{(50)}$		
51	$-l_{51}\ \Pi\ Z^{(51)}$	$d_{50}\left(\mathbf{I} + \mathrm{Put}^{(51)}\left(\mathbf{I}, (1+i)^1\right)\right)$	
52	$-l_{52}\ \Pi\ Z^{(52)}$	$d_{51}\left(\mathbf{I} + \mathrm{Put}^{(52)}\left(\mathbf{I}, (1+i)^2\right)\right)$	
53	$-l_{53}\ \Pi\ Z^{(53)}$	$d_{52}\left(\mathbf{I} + \mathrm{Put}^{(53)}\left(\mathbf{I}, (1+i)^3\right)\right)$	
54	$-l_{54}\ \Pi\ Z^{(54)}$	$d_{53}\left(\mathbf{I} + \mathrm{Put}^{(54)}\left(\mathbf{I}, (1+i)^4\right)\right)$	
55		$d_{54}\left(\mathbf{I} + \mathrm{Put}^{(55)}\left(\mathbf{I}, (1+i)^5\right)\right)$	$l_{55}\ \mathbf{I}$

This immediately leads to the summary of units:

Valuation Scheme B (for l_{50} persons)

unit $\mathcal{U}_i$	number of units
$Z^{(50)}$	$-l_{50}\ \Pi$
$Z^{(51)}$	$-l_{51}\ \Pi$
$Z^{(52)}$	$-l_{52}\ \Pi$
$Z^{(53)}$	$-l_{53}\ \Pi$
$Z^{(54)}$	$-l_{54}\ \Pi$
$\mathbf{I}$	$d_{50} + d_{51} + d_{52} + d_{53} + d_{54} + l_{55} = l_{50}$
$\text{Put}^{(51)}\left(\mathbf{I}, (1+i)^1\right)$	d_{50}
$\text{Put}^{(52)}\left(\mathbf{I}, (1+i)^2\right)$	d_{51}
$\text{Put}^{(53)}\left(\mathbf{I}, (1+i)^3\right)$	d_{52}
$\text{Put}^{(54)}\left(\mathbf{I}, (1+i)^4\right)$	d_{53}
$\text{Put}^{(55)}\left(\mathbf{I}, (1+i)^5\right)$	d_{54}

Observe that the number of units of $\mathbf{I}$ needed is exactly l_{50} because every person insured receives one index $\mathbf{I}$, no matter whether he dies during the term of the contract or not.

Our valuation portfolio VaPo($\mathbf{X}$) is a point in an 11-dimensional vector space (see also (3.2) in Section 3.3 below) where we have specified a basis of financial instruments $\mathcal{U}_i$ (dimension of vector space) and the number of instruments we need to hold to replicate the insurance liabilities.

Step 3. To obtain the (monetary) value for our cash flow we need to apply an accounting principle to this VaPo($\mathbf{X}$), see Section 3.3 below.

□

Conclusion. In a first and second step, we decompose the liability cash flow $\mathbf{X} = (X_{50}, \ldots, X_{55})$ into a 11-dimensional vector VaPo($\mathbf{X}$), whose basis consists of financial instruments $\mathcal{U}_1, \ldots, \mathcal{U}_{11}$. Only in a third step, we calculate the monetary value of the cash flow $\mathbf{X}$ by applying an accounting principle to the units $\mathcal{U}_i$, and thus to VaPo($\mathbf{X}$).

Hence we have found the following general valuation procedure:

3.3 General valuation procedure for deterministic insurance technical risks

1. For every policy with cash flow $\mathbf{X} \in L^2_{n+1}(P, \mathcal{G})$ with deterministic insurance technical risk we construct the VaPo($\mathbf{X}$) as follows: Define units $\mathcal{U}_i$ (basis of a multidimensional vector space) and determine the (deterministic) number $\lambda_i(\mathbf{X}) \in \mathbb{R}$ of each unit $\mathcal{U}_i$:

$$\mathbf{X} \;\mapsto\; \mathrm{VaPo}(\mathbf{X}) = \sum_i \lambda_i(\mathbf{X})\, \mathcal{U}_i. \tag{3.2}$$

From a theoretical point of view the VaPo mapping needs to be a multi-dimensional positive continuous linear function that maps the insurance liabilities $\mathbf{X}$ onto a valuation portfolio $\mathrm{VaPo}(\mathbf{X})$ which replicates the insurance liabilities in terms of financial instruments.

2. Apply then an accounting principle $\mathcal{A}_t$ to the valuation portfolio to obtain a monetary value at time $t \geq 0$

$$\mathrm{VaPo}(\mathbf{X}) \mapsto \mathcal{A}_t\left(\mathrm{VaPo}(\mathbf{X})\right) = Q_t\left[\mathbf{X}\right] \in \mathbb{R}. \tag{3.3}$$

This mapping must be a positive, continuous, linear functional. Moreover, the sequence of accounting principles $(\mathcal{A}_t)_{t=0,\ldots,n}$ must satisfy certain consistency properties in order to have an arbitrage-free pricing system. In fact, we require a martingale property (2.57) for deflated price processes. This is further discussed below.

For the zero coupon bond with maturity m we have at time 0 ($\mathcal{U}_1 = Z^{(m)}$)

$$Q_0[\mathbf{Z}^{(m)}] = \mathcal{A}_0\left(\mathrm{VaPo}(\mathbf{Z}^{(m)})\right) = \mathcal{A}_0\left(\lambda_1(\mathbf{Z}^{(m)})\; Z^{(m)}\right) = \mathcal{A}_0\left(Z^{(m)}\right). \tag{3.4}$$

The construction of the VaPo adds enormously to the understanding and communication between actuaries and asset managers and investors, respectively. In a first step the actuary decomposes the insurance portfolio into financial instruments, in a second step the asset manager evaluates the financial instruments. Indeed, it is the key step to a successful *asset and liability management* (ALM) technique, and it clearly highlights the sources of uncertainties involved in the process. It also allocates the responsibilities for the uncertainties to the different parties involved.

Remark 1. For a cash flow $\mathbf{X} \in L^2_{n+1}(P, \mathcal{G})$ with no insurance technical risk involved we obtain for the value at time 0

$$Q_0\left[\mathbf{X}\right] = \mathcal{A}_0\left(\mathrm{VaPo}(\mathbf{X})\right) = \sum_i \lambda_i(\mathbf{X})\; \mathcal{A}_0\left(\mathcal{U}_i\right) \quad \in \mathbb{R}, \tag{3.5}$$

which should be a positive, continuous, linear functional on $L^2_{n+1}(P, \mathcal{G})$. One has to be a little bit careful with the positivity: In order to obtain a positive linear functional, we must have that $U_t^{(i)} = \mathcal{A}_t\left(\mathcal{U}_i\right) > 0$ for all i as long as a policy is in force, which must be kept in mind whenever the units are selected.

Remark 2. By linearity the individual policies can be added up to a portfolio, i.e. individual cash flows $\mathbf{X}^{(k)} \in L^2_{n+1}(P, \mathcal{G})$ easily merge to $\sum_k \mathbf{X}^{(k)}$ which has value

$$Q_t\left[\sum_k \mathbf{X}^{(k)}\right] = \sum_k \mathcal{A}_t\left(\mathrm{VaPo}(\mathbf{X}^{(k)})\right). \tag{3.6}$$

This means that we can value portfolios of a single contract as well as of the whole insurance company. Note that this aggregation needs to be done very carefully as soon as also insurance technical risks are involved.

Examples of accounting principles $\mathcal{A}_t$. An accounting principle $\mathcal{A}_t$ attaches a value to the financial instruments. There are different ways to choose an appropriate accounting principle. In fact, choosing an appropriate accounting principle very much depends on the problem under consideration. We give two examples.

- Classical actuarial discounting. In many situation, for example in (traditional) communication with regulators, the value of the financial instruments are determined by a mathematical model (such as amortized costs, etc.). If we choose the model where we discount with a fixed constant interest rate we denote the accounting principle by $\mathcal{D}_t$.
- In modern actuarial valuation, the financial instruments are often valued at an economic value, market value or value according to the IASB accounting rule. In general, this means that the value of the asset is essentially the price at which it can be exchanged at the financial market. If we use such an economic accounting principle we use the symbol $\mathcal{E}_t$.

Both principles $\mathcal{D}_t$ and $\mathcal{E}_t$ need to fulfill some time consistency properties in order to have an arbitrage-free pricing system. That is, assume we choose the economic accounting principles $\mathcal{E}_t$, $t = 0, \ldots, n$. Then, for cash flows $\mathbf{X} \in L^2_{n+1}(P, \mathcal{G})$ (with deterministic insurance technical risk) we have the following value at time 0

$$Q_0[\mathbf{X}] = \mathcal{E}_0(\mathrm{VaPo}(\mathbf{X})) = \sum_i \lambda_i(\mathbf{X})\ \mathcal{E}_0(\mathcal{U}_i). \tag{3.7}$$

Using Riesz' representation theorem (Theorem 2.5) we find the state price deflator $\boldsymbol{\varphi} \in L^2_{n+1}(P, \mathbb{F})$ with

$$< \mathbf{X}, \boldsymbol{\varphi} > \ = \ Q_0[\mathbf{X}] \ = \ \mathcal{E}_0(\mathrm{VaPo}(\mathbf{X})). \tag{3.8}$$

Using price definition (2.38), Lemma 2.8 then implies that we need to have

$$\left(\varphi_t \mathcal{E}_t(\mathrm{VaPo}(\mathbf{X}))\right)_{t=0,\ldots,n} \qquad \text{forms an } \mathbb{F}\text{-martingale under } P. \tag{3.9}$$

3.4 Self-financing property of the VaPo (deterministic insurance technical risk)

In (2.49) we have defined $\mathbf{X}_{(k)} \in L^2_{n+1}(P, \mathcal{G})$ as the remaining cash flow after time $k-1$. Moreover, define the cash flow

$$\mathbf{X}_k = X_k\ \mathbf{Z}^{(k)} = (0, \ldots, 0, X_k, 0 \ldots, 0) \in L^2_{n+1}(P, \mathcal{G}). \tag{3.10}$$

Hence, note

$$\mathbf{X}_{(k)} = \mathbf{X}_{(k+1)} + \mathbf{X}_k, \tag{3.11}$$

and using the linearity of the valuation portfolio (3.2) we have the following lemma.

Lemma 3.1 (Self-financing property as portfolio) *For* $\mathbf{X} \in L^2_{n+1}(P, \mathcal{G})$

$$\mathrm{VaPo}\left(\mathbf{X}_{(k)}\right) = \mathrm{VaPo}\left(\mathbf{X}_{(k+1)}\right) + \mathrm{VaPo}\left(\mathbf{X}_k\right). \tag{3.12}$$

Of course, in this lemma we assume that the vector space is spanned by the financial instruments determined by $\mathbf{X}_{(k)}$.

Remark. At time k, the last term in (3.12) is simply cash value, which we abbreviate

$$\mathrm{VaPo}\left(\mathbf{X}_k\right) = X_k \qquad \text{at time } k, \tag{3.13}$$

i.e. we omit in this case to write the unit because it is just 1 at time k.

Studying now the values given by the accounting principle $\mathcal{A}_t$, we have by the linearity of $\mathcal{A}_t$ the following lemma:

Lemma 3.2 (Self-financing property in value) *For* $\mathbf{X} \in L^2_{n+1}(P, \mathcal{G})$ *and* $t \leq k$

$$\mathcal{A}_t\left(\mathrm{VaPo}\left(\mathbf{X}_{(k)}\right)\right) = \mathcal{A}_t\left(\mathrm{VaPo}\left(\mathbf{X}_{(k+1)}\right)\right) + \mathcal{A}_t\left(\mathrm{VaPo}\left(\mathbf{X}_k\right)\right). \tag{3.14}$$

In particular, if the valuation portfolio of $\mathbf{X}_k$ is evaluated at time k then

$$X_k = \frac{1}{\varphi_k}\ E\left[\varphi_k\ X_k \middle| \mathcal{F}_k\right] = Q_k\left[\mathbf{X}_k\right] = \mathcal{A}_k\left(\mathrm{VaPo}(\mathbf{X}_k)\right), \tag{3.15}$$

hence

$$\mathcal{A}_k\left(\mathrm{VaPo}\left(\mathbf{X}_{(k)}\right)\right) = \mathcal{A}_k\left(\mathrm{VaPo}\left(\mathbf{X}_{(k+1)}\right)\right) + X_k, \tag{3.16}$$

which tells again that the VaPo for $\mathbf{X}_k$ at time k is simply X_k. This observation is fundamental and should hold independently of the value assigned to the VaPo by the accounting principle $\mathcal{A}_t$.

For a more detailed analysis of the self-financing property in monetary value over time we refer to Subsection 6.2.

3.5 VaPo protected against insurance technical risks

So far we have considered an ideal situation which is an important point of reference to measure deviations.

<u>ideal</u>	<u>realistic</u>	<u>deviation</u>
deterministic mortality	stochastic mortality	technical risk
VaPo	real investment portfolio $\mathcal{S}$	financial risk

The ideal situation is often called base scenario and one then studies deviations from this base scenario.

In this section we want to consider insurance technical risks. They come from the fact that the insurance liabilities are not deterministic, i.e. $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$. This means in our example that we have stochastic mortality rates.

For the deviations from the deterministic model (which are expectations, best-estimates for the liabilities) we add a protection. Such a protection can be obtained e.g. via reinsurance products, risk loadings or risk bearing capital. The VaPo with this additional protection will be called **VaPo protected against insurance technical risks**.

3.5.1 Construction of the VaPo protected against insurance technical risks

Let us return to our Example 3.1. The stochastic mortality table reads as:

$$
\begin{array}{lcl}
l_x & & \\
\downarrow & \longrightarrow & D_x = l_x - L_{x+1} \\
L_{x+1} & & \\
\downarrow & \longrightarrow & D_{x+1} = L_{x+1} - L_{x+2} \\
L_{x+2} & & \\
\downarrow & \longrightarrow & D_{x+2} = L_{x+2} - L_{x+3} \\
\vdots & & \vdots
\end{array}
$$

where now L_{x+k} and D_{x+k-1} are random variables for $k \geq 1$. From

$$D_{50} = l_{50} - L_{51}, \tag{3.17}$$

$$d_{50} = l_{50} - l_{51}, \tag{3.18}$$

we obtain

$$D_{50} - d_{50} = l_{51} - L_{51}, \tag{3.19}$$

which describes the deviations of D_{50} and L_{51} from their expected values d_{50} and l_{51}, respectively. In fact, in a first step we use the expected value d_{50} as a predictor for the random variable D_{50}, and in a second step we need to study the prediction uncertainty or the deviation of the random variable D_{50} around its predictor d_{50}.

The Valuation Scheme A then reads as follows for the stochastic mortality table:

time	premium	death benefit	survival benefit
50	$-l_{50}\ \Pi\ Z^{(50)}$		
51	$-L_{51}\ \Pi\ Z^{(51)}$	$D_{50}\ \left(\mathbf{I} + \mathrm{Put}^{(51)}\left(\mathbf{I}, (1+i)^1\right)\right)$	
52	$-L_{52}\ \Pi\ Z^{(52)}$	$D_{51}\ \left(\mathbf{I} + \mathrm{Put}^{(52)}\left(\mathbf{I}, (1+i)^2\right)\right)$	
53	$-L_{53}\ \Pi\ Z^{(53)}$	$D_{52}\ \left(\mathbf{I} + \mathrm{Put}^{(53)}\left(\mathbf{I}, (1+i)^3\right)\right)$	
54	$-L_{54}\ \Pi\ Z^{(54)}$	$D_{53}\ \left(\mathbf{I} + \mathrm{Put}^{(54)}\left(\mathbf{I}, (1+i)^4\right)\right)$	
55		$D_{54}\ \left(\mathbf{I} + \mathrm{Put}^{(55)}\left(\mathbf{I}, (1+i)^5\right)\right)$	$L_{55}\ \mathbf{I}$

Let us define the expected survival probabilities and the expected death probabilities (second order life table) for $t \geq x$:

$$p_t = \frac{l_{t+1}}{l_t} \qquad \text{and} \qquad q_t = 1 - p_t = \frac{d_t}{l_t}. \tag{3.20}$$

Denote by $\mathrm{VaPo}(\mathbf{X}_{(t+1)})$ the valuation portfolio for the cash flows after time t with deterministic insurance technical risks (deterministic mortality table as defined in Section 3.2). I.e. $\mathrm{VaPo}(\mathbf{X}_{(t+1)})$ denotes the valuation portfolio with the expected cash flows (L_t is replaced by its mean l_t).

If we allow for a stochastic survival in period $(50, 51]$ we have the following deviations from the expected VaPo (deterministic insurance technical risks): For $t = 51$ we obtain the following deviations form the expected payments

$$(D_{50} - d_{50})\ \left(\mathbf{I} + \mathrm{Put}^{(51)}\left(\mathbf{I}, (1+i)^1\right)\right), \tag{3.21}$$

$$(l_{51} - L_{51})\ \Pi\ Z^{(51)}, \tag{3.22}$$

$$(L_{51} - l_{51})\ \frac{\mathrm{VaPo}(\mathbf{X}_{(52)})}{l_{51}}, \tag{3.23}$$

if $\mathrm{VaPo}(\mathbf{X}_{(52)})$ denotes the deterministic cash flows of our endowment policy after time $t = 51$ (according to Section 3.2). This means that we have deviations in the payments at time $t = 51$ due to the stochastic mortality, and then at $t = 51$, we start with a new basis of L_{51} insured lives (instead of l_{51}), which gives a new expected VaPo after time $t = 51$ of (use the linearity of the VaPo)

$$L_{51}\ \frac{\mathrm{VaPo}(\mathbf{X}_{(52)})}{l_{51}}. \tag{3.24}$$

Using (3.19) and equations (3.21)-(3.23) we see that we need additional reserves of

$$(D_{50} - d_{50})\ \left(\mathbf{I} + \mathrm{Put}^{(51)}\left(\mathbf{I}, (1+i)^1\right) + \Pi\ Z^{(51)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(52)})}{l_{51}}\right) \tag{3.25}$$

for the deviations from the expected mortality table within $(50, 51]$. Note that this deviation is stochastic seen from time $t = 50$. Hence the portfolio at risk is

$$\mathbf{I} + \mathrm{Put}^{(51)}\left(\mathbf{I}, (1+i)^1\right) + \Pi\ Z^{(51)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(52)})}{l_{51}}. \tag{3.26}$$

We can now iterate this procedure:
For $t = 52$ we have the following deviation from the expected VaPo. The expected VaPo starts now after $t = 51$ with the new basis of L_{51} insured lives (we have to build the additional VaPo reserves for the new basis in (3.25)). Note that conditionally, given L_{51}, we expect $q_{51}L_{51}$ persons to die within the time interval $(51, 52]$ and we observe D_{51} at time $t = 52$. This gives the following deviations

$$(D_{51} - q_{51}\ L_{51})\ \left(\mathbf{I} + \mathrm{Put}^{(52)}\left(\mathbf{I}, (1+i)^2\right)\right), \tag{3.27}$$

$$(p_{51}\ L_{51} - L_{52})\ \Pi\ Z^{(52)}, \tag{3.28}$$

$$(L_{52} - p_{51}\ L_{51})\ \frac{\mathrm{VaPo}(\mathbf{X}_{(53)})}{p_{51}\ L_{51}}\ \frac{L_{51}}{l_{51}}, \tag{3.29}$$

where the last term can be simplified to

$$\frac{\mathrm{VaPo}(\mathbf{X}_{(53)})}{p_{51}\ L_{51}}\ \frac{L_{51}}{l_{51}} = \frac{\mathrm{VaPo}(\mathbf{X}_{(53)})}{l_{52}}. \tag{3.30}$$

Hence we need for the deviation in $(51, 52]$ additional reserves of

$$(D_{51} - q_{51}\ L_{51})\ \left(\mathbf{I} + \mathrm{Put}^{(52)}\left(\mathbf{I}, (1+i)^2\right) + \Pi\ Z^{(52)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(53)})}{l_{52}}\right). \tag{3.31}$$

And analogously for $t = 53, 54, 55$ we obtain the deviations

$$\begin{aligned}
&(D_{52} - q_{52}\ L_{52})\ \left(\mathbf{I} + \mathrm{Put}^{(53)}\left(\mathbf{I}, (1+i)^3\right) + \Pi\ Z^{(53)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(54)})}{l_{53}}\right), \\
&(D_{53} - q_{53}\ L_{53})\ \left(\mathbf{I} + \mathrm{Put}^{(54)}\left(\mathbf{I}, (1+i)^4\right) + \Pi\ Z^{(54)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(55)})}{l_{54}}\right), \\
&(D_{54} - q_{54}\ L_{54})\ \left(\mathbf{I} + \mathrm{Put}^{(55)}\left(\mathbf{I}, (1+i)^5\right) - \mathbf{I}\right).
\end{aligned} \tag{3.32}$$

Remark. One can see that when adding up the terms inside in (3.25) and (3.31)-(3.32) the unit $\mathbf{I}$ cancels since $\mathrm{VaPo}(\mathbf{X}_{(t+1)})$ contains exactly l_t units of $\mathbf{I}$ for $t = 50$. This is immediately clear because the number of units $\mathbf{I}$ we need to buy at the beginning of the policy does not depend on the mortality table (see Valuation Scheme B on page 46), i.e. no matter whether a person dies or stays alive it receives $\mathbf{I}$.

Hence we find the following portfolios at risk:

$$
\begin{aligned}
t = 51: \ & \mathbf{I} + \mathrm{Put}^{(51)} \left(\mathbf{I}, (1+i)^1\right) + \Pi \ Z^{(51)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(52)})}{l_{51}}, \\
t = 52: \ & \mathbf{I} + \mathrm{Put}^{(52)} \left(\mathbf{I}, (1+i)^2\right) + \Pi \ Z^{(52)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(53)})}{l_{52}}, \\
t = 53: \ & \mathbf{I} + \mathrm{Put}^{(53)} \left(\mathbf{I}, (1+i)^3\right) + \Pi \ Z^{(53)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(54)})}{l_{53}}, \\
t = 54: \ & \mathbf{I} + \mathrm{Put}^{(54)} \left(\mathbf{I}, (1+i)^4\right) + \Pi \ Z^{(54)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(55)})}{l_{54}}, \\
t = 55: \ & \mathbf{I} + \mathrm{Put}^{(55)} \left(\mathbf{I}, (1+i)^5\right) - \mathbf{I}.
\end{aligned}
\tag{3.33}
$$

The interpretation of (3.33) is the following. Consider for example the period $(52, 53]$, if more people die than expected $(D_{52} > q_{52} \ L_{52})$ we have to pay an additional death benefit of

$$
(D_{52} - q_{52} \ L_{52}) \ \left(\mathbf{I} + \mathrm{Put}^{(53)} \left(\mathbf{I}, (1+i)^3\right)\right). \tag{3.34}
$$

On the other hand for all these people the contracts are terminated which means that our liabilities are reduced by

$$
(D_{52} - q_{52} \ L_{52}) \ \left(-\Pi \ Z^{(53)} + \frac{\mathrm{VaPo}(\mathbf{X}_{(54)})}{l_{53}}\right). \tag{3.35}
$$

These insurance technical risks are now protected against adverse developments by adding a security loading. This gives us the following **reinsurance premium loadings as a portfolio**:

$$
\begin{aligned}
\mathrm{RPP}_{50} = l_{50} \ & (q^*_{50} - q_{50}) \\
& \left(\mathbf{I} + \mathrm{Put}^{(51)} \left(\mathbf{I}, (1+i)^1\right) + \Pi \ Z^{(51)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(52)})}{l_{51}}\right), \\
\mathrm{RPP}_{51} = l_{51} \ & (q^*_{51} - q_{51}) \\
& \left(\mathbf{I} + \mathrm{Put}^{(52)} \left(\mathbf{I}, (1+i)^2\right) + \Pi \ Z^{(52)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(53)})}{l_{52}}\right), \\
\mathrm{RPP}_{52} = l_{52} \ & (q^*_{52} - q_{52}) \\
& \left(\mathbf{I} + \mathrm{Put}^{(53)} \left(\mathbf{I}, (1+i)^3\right) + \Pi \ Z^{(53)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(54)})}{l_{53}}\right), \\
\mathrm{RPP}_{53} = l_{53} \ & (q^*_{53} - q_{53}) \\
& \left(\mathbf{I} + \mathrm{Put}^{(54)} \left(\mathbf{I}, (1+i)^4\right) + \Pi \ Z^{(54)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(55)})}{l_{54}}\right), \\
\mathrm{RPP}_{54} = l_{54} \ & (q^*_{54} - q_{54}) \ \left(\mathbf{I} + \mathrm{Put}^{(55)} \left(\mathbf{I}, (1+i)^5\right) - \mathbf{I}\right),
\end{aligned}
\tag{3.36}
$$

where $q^*_t - q_t$ denote the loadings charged by the reinsurer against insurance technical risks, and l_t is the number of units we need to buy. Here, q^*_t can be interpreted as the yearly renewable term (YRT) rates charged by the reinsurer.

Valuation Portfolio protected against insurance technical risks is now defined as

$$\mathrm{VaPo}^{prot}(\mathbf{X}) = \mathrm{VaPo}(\mathbf{X}) + \sum_{t=50}^{54} \mathrm{RPP}_t. \tag{3.37}$$

Remarks.

- For a monetary reinsurance premium we need to apply an accounting principle $\mathcal{A}_t$ to the reinsurance premium portfolio (yearly renewable term):

$$\begin{aligned} \Pi_t^R &= \mathcal{A}_t\left(\mathrm{RPP}_t\right) \\ &= l_t\ \left(q_t^* - q_t\right) \\ &\quad \times \mathcal{A}_t\left(\mathbf{I} + \mathrm{Put}^{(t+1)}\left(\mathbf{I}, (1+i)^{t-50+1}\right) + \Pi Z^{(t+1)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(t+2)})}{l_{t+1}}\right). \end{aligned} \tag{3.38}$$

- The last term in (3.38) highlights that the choice of the loadings $q_t^* - q_t$ needs some care. The monetary value of the portfolio at risk (3.33) may have both signs. Therefore the sign of the loading may depend on the monetary value of the portfolio at risk. For example, for death benefits we decrease the survival probabilities p_t, whereas for annuities we increase the survival probabilities.
- There are different possibilities to determine the premium: We could choose an actuarial accounting principle $\mathcal{D}_t$ or an economic accounting principle $\mathcal{E}_t$ (which gives an economic yearly renewable term, see also page 48). This idea opens interesting *new reinsurance products*: Offer a reinsurance cover against insurance technical risks in terms of a valuation portfolio.
- A static hedging strategy is to invest the reinsurance premium into the valuation portfolios of the reinsurer.

3.5.2 Probability distortion of life tables

The choice of the death probabilities q_t^* may look artificial at the first sight. They often come from a first order life table. A first order life table refers to survival or death probabilities that are chosen prudent (i.e. with some safety margin), whereas the second order life table refers to best-estimate survival and death probabilities. However, the choice of a first order life table fits perfectly into our modelling framework. Indeed, the first order life tables can be explained by probability distortions: In (2.105) we have considered the term $\Lambda_{t,k} = \frac{1}{\varphi_t^{(\mathcal{T})}} E\left[\varphi_k^{(\mathcal{T})} \Lambda_k \middle| \mathcal{T}_t\right]$, $k > t$, referring to the price of the insurance cover in units.

To explain this term, we revisit our Example 3.1 with a stochastic mortality table: for illustrative purposes we choose $t = 52$. The σ-field $\mathcal{T}_{52}$ tells us that there are L_{52} persons alive at time $t = 52$, i.e. L_{52} is $\mathcal{T}_{52}$-measurable. Moreover, we choose $k = 53$ and we assume that Λ_{53} models the death benefit. Thus we study (set $\varphi_{52}^{(\mathcal{T})} = 1$)

$$E\left[\varphi_{53}^{(\mathcal{T})}\Lambda_{53}\middle|\mathcal{T}_{52}\right] = E\left[\varphi_{53}^{(\mathcal{T})}D_{52}\middle|\mathcal{T}_{52}\right], \tag{3.39}$$

which describes for how many financial units we build insurance technical reserves.

In a first step we choose $\varphi_{53}^{(\mathcal{T})} \equiv 1$, then we obtain

$$E\left[\varphi_{53}^{(\mathcal{T})}\Lambda_{53}\middle|\mathcal{T}_{52}\right] = E\left[D_{52}\middle|\mathcal{T}_{52}\right] = q_{52}\ L_{52}, \tag{3.40}$$

i.e. q_{52} describes the single death probability within $(52, 53]$ and (3.40) leads to the VaPo that covers expected liabilities.

We now model the probability distortion (insurance technical deflator) $\varphi_{53}^{(\mathcal{T})}$ so that we obtain the first order life table q_{52}^*. Note that

$$E\left[\varphi_{53}^{(\mathcal{T})}\Lambda_{53}\middle|\mathcal{T}_{52}\right] = E\left[\varphi_{53}^{(\mathcal{T})}D_{52}\middle|\mathcal{T}_{52}\right] = \sum_{i=1}^{L_{52}} E\left[\varphi_{53}^{(\mathcal{T})}I_i\middle|\mathcal{T}_{52}\right], \tag{3.41}$$

where I_i is the indicator whether person i dies within $(52, 53]$.

We assume that single life times (of persons all of the same age) are i.i.d. Then we assume that the probability distortion $\varphi_{53}^{(\mathcal{T})}$ is of the form

$$\varphi_{53}^{(\mathcal{T})} = \prod_{i=1}^{L_{52}} \varphi_{53}^{(\mathcal{T})}(I_i), \tag{3.42}$$

such that each factor of this product has expectation 1. Henceforth, we write

$$\sum_{i=1}^{L_{52}} E\left[\varphi_{53}^{(\mathcal{T})}I_i\middle|\mathcal{T}_{52}\right] = \sum_{i=1}^{L_{52}} E\left[\varphi_{53}^{(\mathcal{T})}(I_i)\ I_i\middle|\mathcal{T}_{52}\right]. \tag{3.43}$$

The factors of the probability distortions are now chosen as follows: Take $q_{52}^* \in (0, 1)$ and define

$$\varphi_{53}^{(\mathcal{T})}(1) = \frac{q_{52}^*}{q_{52}}, \tag{3.44}$$

$$\varphi_{53}^{(\mathcal{T})}(0) = \frac{1 - q_{52}^*}{1 - q_{52}}. \tag{3.45}$$

We then obtain the required normalization

$$E\left[\varphi_{53}^{(\mathcal{T})}(I_i)\middle|\mathcal{T}_{52}\right] = q_{52}\ \frac{q_{52}^*}{q_{52}} + p_{52}\ \frac{1 - q_{52}^*}{1 - q_{52}} = 1, \tag{3.46}$$

and the first order life table

$$E\left[\varphi_{53}^{(\mathcal{T})}(I_i)I_i \middle| \mathcal{T}_{52}\right] = q_{52}\ \frac{q_{52}^*}{q_{52}} = q_{52}^*, \tag{3.47}$$

i.e., note that we have set $\varphi_{52}^{(\mathcal{T})} = 1$, and

$$E\left[\varphi_{53}^{(\mathcal{T})}\Lambda_{53} \middle| \mathcal{T}_{52}\right] = q_{52}^*\ L_{52}. \tag{3.48}$$

In other words the transition from the second order life table p_t to the first order life table p_t^* exactly refers to a probability distortion $\varphi_{t+1}^{(\mathcal{T})}$.

Exercise 3.2 (Life-Time Annuity).

Consider a life-time annuity for a man aged x at time 0. We assume that the life-time annuity contract is paid by a single premium installment π_0 at the beginning of the insurance period (initial lump sum) and that the insured receives an annual payment of M until he dies.

- Determine the valuation portfolio VaPo based on the second order life table p_t, $t \geq x$.
- Calculate the portfolios at risk and the VaPo protected against insurance technical risks.
- Determine the sign of the loadings $p_t^* - p_t$.
- Express the second order life table p_t^* with the help of probability distortions $\varphi_{t+1}^{(\mathcal{T})}$.

□

3.6 Back to the basic model

In Chapter 2 we have chosen a deflator

$$\boldsymbol{\varphi} = (\varphi_0, \ldots, \varphi_n) \in L_{n+1}^2(P, \mathbb{F}) \tag{3.49}$$

to value cash flows $\mathbf{X} = (X_0, \ldots, X_n) \in L_{n+1}^2(P, \mathbb{F})$. The basic assumption was that $\boldsymbol{\varphi}$ and $\mathbf{X}$ are $\mathbb{F}$-adapted. Moreover, we have assumed that on our filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F})$ we can decompose $\mathbb{F}$ into independent filtrations $\mathcal{T}$ and $\mathcal{G}$ such that

$$X_k = \Lambda_k\ U_k^{(k)}, \tag{3.50}$$

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}^{(\mathcal{T})}\ \boldsymbol{\varphi}^{(\mathcal{G})}, \tag{3.51}$$

where $\boldsymbol{\Lambda}, \boldsymbol{\varphi}^{(\mathcal{T})} \in L_{n+1}^2(P, \mathcal{T})$ and $\boldsymbol{\varphi}^{(\mathcal{G})}, (U_t^{(k)})_{t=0,\ldots,n} \in L_{n+1}^2(P_{\mathcal{G}}, \mathcal{G})$ for all $k = 0, \ldots, n$, see Assumption 2.15. This means that we can split the problem into two independent problems, one measuring insurance technical risks $\mathcal{T}$ and one describing (financial) price processes on $\mathcal{G}$.

To avoid ambiguity we have assumed that the expectation of the probability distortion is 1, (see also (2.100))

$$E\left[\varphi_t^{(\mathcal{T})}\right] = 1 \tag{3.52}$$

for all $t = 0, \dots, n$, and moreover, we have assumed that $(\varphi_t^{(\mathcal{T})})_{t=0,\dots,n}$ is a $\mathcal{T}$-martingale under P, see (2.101).

The VaPo construction in this chapter has now led to a multidimensional approach, i.e. the cash flow $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$ is decomposed into a vector consisting of different financial instruments/units (see also (3.2))

$$\mathbf{X} \;\mapsto\; \sum_{i=1}^{p} \Lambda_i(\mathbf{X})\, \mathcal{U}_i, \tag{3.53}$$

if $\mathcal{U}_1, \dots, \mathcal{U}_p$ represent the p financial instruments by which $\mathbf{X}$ can be described, and Λ_i the (random) number of units $\mathcal{U}_i$ needed. The value/price process of $\mathcal{U}_i$ is denoted by $(U_t^{(i)})_{t=0,\dots,n}$ and is independent of $\mathcal{T}$. If we now use vector notation, (3.53) can be rewritten as (we have linear mappings)

$$\mathbf{X} = \begin{pmatrix} X_0 \\ X_1 \\ \vdots \\ X_n \end{pmatrix} \mapsto \sum_{i=1}^{p} \begin{pmatrix} \Lambda_i(\mathbf{X}_0) \\ \Lambda_i(\mathbf{X}_1) \\ \vdots \\ \Lambda_i(\mathbf{X}_n) \end{pmatrix}^T \begin{pmatrix} \mathcal{U}_i \\ \mathcal{U}_i \\ \vdots \\ \mathcal{U}_i \end{pmatrix}, \tag{3.54}$$

where $\mathbf{X}_t = X_t\, \mathbf{Z}^{(t)} = (0, \dots, 0, X_t, 0, \dots, 0)$.

For the **VaPo construction seen from time 0** we have then replaced the random $\Lambda_i(\mathbf{X}_k)$ by deterministic numbers (expected values):

$$\Lambda_i(\mathbf{X}_k) \;\mapsto\; l_{i,k} = l_{i,k}^{(0)} = E\left[\Lambda_i(\mathbf{X}_k)\middle|\, \mathcal{T}_0\right]. \tag{3.55}$$

If $\Lambda_i(\mathbf{X}_k)$ is deterministic as in Section 3.1, then we have $\Lambda_i(\mathbf{X}_k) = \lambda_i(\mathbf{X}_k) = l_{i,k}$ (see (3.2)).

For the **VaPo protected against insurance technical risks** (seen from time 0) we replace $\Lambda_i(\mathbf{X}_k)$ by the following deterministic numbers (distorted expected values):

$$\Lambda_i(\mathbf{X}_k) \;\mapsto\; l_{i,k}^* = l_{i,k}^{*,0} = E\left[\varphi_k^{(\mathcal{T})}\, \Lambda_i(\mathbf{X}_k)\middle|\, \mathcal{T}_0\right], \tag{3.56}$$

which adds a loading to $l_{i,k}$ for insurance technical risks. If $\Lambda_i(\mathbf{X}_k)$ is deterministic as in Section 3.1, i.e. $\mathbf{X} \in L^2_{n+1}(P, \mathcal{G})$, then we have $\Lambda_i(\mathbf{X}_k) = \lambda_i(\mathbf{X}_k) = l_{i,k} = l_{i,k}^*$ due to (2.100), i.e. we do not need a loading for insurance technical risks. The loading in $l^*_{i,k}$ could also have been chosen directly, not via the definition of a probability distortion. This gives now

$$\mathrm{VaPo}\,(\mathbf{X}) = \sum_{i=1}^{p} \begin{pmatrix} l_{i,0} \\ \vdots \\ l_{i,n} \end{pmatrix}^T \begin{pmatrix} \mathcal{U}_i \\ \vdots \\ \mathcal{U}_i \end{pmatrix}. \tag{3.57}$$

This can also be written as

$$\mathrm{VaPo}\,(\mathbf{X}) = \sum_{i=1}^{p} l_i\, \mathcal{U}_i, \tag{3.58}$$

with

$$l_i = \sum_{t=0}^{n} l_{i,t}. \tag{3.59}$$

The VaPo protected against insurance technical risks is given by

$$\mathrm{VaPo}^{prot}\,(\mathbf{X}) = \sum_{i=1}^{p} \begin{pmatrix} l^*_{i,0} \\ \vdots \\ l^*_{i,n} \end{pmatrix}^T \begin{pmatrix} \mathcal{U}_i \\ \vdots \\ \mathcal{U}_i \end{pmatrix}, \tag{3.60}$$

or equivalently

$$\mathrm{VaPo}^{prot}\,(\mathbf{X}) = \sum_{i=1}^{p} l^*_i\, \mathcal{U}_i, \tag{3.61}$$

with

$$l^*_i = \sum_{t=0}^{n} l^*_{i,t}. \tag{3.62}$$

Remark. Observe that (3.57) and (3.58) provide two representations for $\mathrm{VaPo}\,(\mathbf{X})$. Firstly, we have the **cash flow representation**, which corresponds to Valuation Scheme A in Section 3.2. That is, (3.57) implies

$$\mathrm{VaPo}\,(\mathbf{X}) = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}^T \begin{pmatrix} \sum_{i=1}^{p} l_{i,0}\, \mathcal{U}_i \\ \vdots \\ \sum_{i=1}^{p} l_{i,n}\, \mathcal{U}_i \end{pmatrix}. \tag{3.63}$$

Secondly, we have the **instrument representation** (3.58) which corresponds to Valuation Scheme B in Section 3.2.

Analogously, we have the two representations (3.60) and (3.61) for the VaPo protected against insurance technical risks $\mathrm{VaPo}^{prot}\,(\mathbf{X})$.

For many purposes the instrument representation (3.58) of the VaPo suffices. Sometimes, however, it may be necessary to work with the cash flow representation (3.63), see for example Section 3.7 below.

Applying an accounting principle $\mathcal{A}_0$ to the VaPo (or equivalently to the financial instruments $\mathcal{U}_i$) gives then a monetary value for the basic reserves at time 0.

Remark. It is important to see that the valuation portfolio construction in (3.55) is seen from time 0. If the cash flows have no insurance technical risks (as in Section 3.3) there are no deviations in $\Lambda_i(\mathbf{X})$ over time, which means that $l_{i,k}$ is constant in time. But if we have insurance technical risks involved, then

$$l_{i,k}^{(m)} = E\left[\Lambda_i(\mathbf{X}_k)\middle|\mathcal{T}_m\right], \tag{3.64}$$

$$l_{i,k}^{*,m} = \frac{1}{\varphi_m^{(\mathcal{T})}}\, E\left[\varphi_k^{(\mathcal{T})}\, \Lambda_i(\mathbf{X}_k)\middle|\mathcal{T}_m\right] \tag{3.65}$$

are functions of time (see also Chapter 6). This then leads to time dependent valuation portfolios

$$\mathrm{VaPo}_{(m)}(\mathbf{X}) \quad \text{and} \quad \mathrm{VaPo}_{(m)}^{prot}(\mathbf{X}). \tag{3.66}$$

We then also need to study the changes in these valuation portfolios over time, i.e.

$$\mathrm{VaPo}_{(m)}(\mathbf{X}) - \mathrm{VaPo}_{(m-1)}(\mathbf{X}) \tag{3.67}$$

and

$$\mathrm{VaPo}_{(m)}^{prot}(\mathbf{X}) - \mathrm{VaPo}_{(m-1)}^{prot}(\mathbf{X}), \tag{3.68}$$

which considers the update of information $\mathcal{T}_{m-1} \mapsto \mathcal{T}_m$ and is similar to the claims development result in non-life insurance, see for example Merz-Wüthrich [MW08] and Salzmann-Wüthrich [SW10].

3.7 Approximate valuation portfolio

In Section 3.2 we have constructed the VaPo for a rather simple example. We have considered a small homogeneous portfolio and its liabilities were easily described by financial instruments. In practice the situation is often more complicated. Life insurance companies have high-dimensional portfolios which usually involve embedded options and guarantees as well as management decisions. I.e. the valuation portfolio becomes path dependent and the determination of the liability cash flows and the appropriate financial instruments is not straightforward. In such situations one often tries to approximate the VaPo by a financial portfolio. Here, we will define the approximate VaPo (denoted by VaPo^{approx}) which plays the role of a replicating portfolio.

Let us choose a filtered probability space $(\Omega, \mathcal{F}_n, P, \mathbb{F})$ and assume that we have an insurance liability cash flow $\mathbf{X} \in L^2_{n+1}(P, \mathbb{F})$.

In order to construct an approximate VaPo we choose a set of basic tradable financial instruments $\mathcal{U}_1, \ldots, \mathcal{U}_q$ from which we believe that they can replicate the liabilities in an appropriate way and for which we can *easily* describe their price processes

$$U_t^{(i)} = \mathcal{A}_t(\mathcal{U}_i), \qquad \text{for } t = 0, \ldots, n, \tag{3.69}$$

i.e. we want to choose q financial instruments for which we have a good understanding.

We now want to approximate the cash flow representation (3.63) of

$$\mathrm{VaPo}\,(\mathbf{X}) = \sum_{k=0}^{n} \mathrm{VaPo}\,(\mathbf{X}_k)\,. \tag{3.70}$$

That is, for all single cash flows X_k, $k = 0, \ldots, n$, our goal is to choose $\mathbf{y}_k \in \mathbb{R}^q$ such that

$$\mathrm{VaPo}\,(\mathbf{Y}_k) = \sum_{i=1}^{q} y_{i,k}\, \mathcal{U}_i \tag{3.71}$$

approximates $\mathrm{VaPo}\,(\mathbf{X}_k)$. Or in vector notation, we choose $\mathbf{y} \in \mathbb{R}^{q\times(n+1)}$ such that

$$\mathrm{VaPo}\,(\mathbf{Y}) = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}^T \begin{pmatrix} \sum_{i=1}^q y_{i,0}\, \mathcal{U}_i \\ \vdots \\ \sum_{i=1}^q y_{i,n}\, \mathcal{U}_i \end{pmatrix} \tag{3.72}$$

approximates $\mathrm{VaPo}\,(\mathbf{X})$, see (3.63). That is, our aim is to choose $\mathbf{y} \in \mathbb{R}^{q\times(n+1)}$ such that $\mathbf{X}$ and $\mathbf{Y}$ are “close”. Of course, close will depend on some distance function.

If there is no insurance technical risk and if $\mathcal{U}_1, \ldots, \mathcal{U}_q$ is a complete financial basis for the liabilities we can achieve

$$\mathbf{X} = \mathbf{Y} \qquad P\text{-a.s.} \tag{3.73}$$

In general, we are not able to achieve (3.73) nor is it possible to evaluate the random vectors $\mathbf{X}$ and $\mathbf{Y}$ for all sample points $\omega \in \Omega$. Therefore, one then chooses a finite set of so-called scenarios $\Omega_K = \{\omega_1, \ldots, \omega_K\} \subset \Omega$ and one evaluates the random vectors $\mathbf{X}$ and $\mathbf{Y}$ in these scenarios. We introduce a distance function

$$\mathrm{dist}\,(\mathbf{X}(\cdot), \mathbf{Y}(\cdot), \Omega_K) \in \mathbb{R}, \tag{3.74}$$

then the approximate valuation portfolio is given by

$$\mathbf{y}^* = \arg \min_{\mathbf{y}\in\mathbb{R}^{q\times(n+1)}} \mathrm{dist}\,(\mathbf{X}(\cdot), \mathbf{Y}(\cdot), \Omega_K)\,, \tag{3.75}$$

and for $k = 0, \ldots, n$ we obtain

$$\mathrm{VaPo}^{approx}\left(\mathbf{X}_k\right)=\sum_{i=1}^{q} y_{i,k}^{*}\, \mathcal{U}_i, \tag{3.76}$$

or, respectively,

$$\mathrm{VaPo}^{approx}\left(\mathbf{X}\right)=\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}^{T} \begin{pmatrix}\sum_{i=1}^{q} y_{i,0}^{*}\, \mathcal{U}_i\\ \vdots\\ \sum_{i=1}^{q} y_{i,n}^{*}\, \mathcal{U}_i\end{pmatrix} \tag{3.77}$$

Remark. It is important to realize that the approximate valuation portfolio $\mathbf{y}^*$ depends on the choice of (a) the financial instruments $\mathcal{U}_1,\ldots,\mathcal{U}_q$, (b) the choice of the scenarios Ω_K, and (c) the choice of the distance function. Based on the purpose of the approximate valuation portfolio (e.g. profit testing, solvency, extremal behaviour) these choices will vary and there is *no* obvious best choice.

Example 3.3 (Cash flow matching).

We assume that we want to match the entire cash flow $\mathbf{X}$ as good as possible and we use the L^2-distance measure. Assume that there are positive deterministic weight functions $\chi_t : \Omega_K \to \mathbb{R}_+$ given for $t=0,\ldots,n$. Our distance function is defined by

$$\mathrm{dist}\left(\mathbf{X}(\cdot),\mathbf{Y}(\cdot),\Omega_K\right)=\sum_{k=1}^{K}\sum_{t=0}^{n}\chi_t(\omega_k)\left(X_t(\omega_k)-Y_t(\omega_k)\right)^2. \tag{3.78}$$

For $\chi_t(\cdot)$ we can make different choices. Often one wants to account for time values, therefore one chooses the financial deflator $\boldsymbol{\varphi}^{(\mathcal{G})}$ (see Assumption 2.15) and a normalized positive deterministic weight function $p : \Omega_K \to \mathbb{R}_+$ with $\sum_{k=1}^{K} p(\omega_k)=1$ and defines for $t=0,\ldots,n$

$$\chi_t(\omega_k)=p(\omega_k)\ \left(\varphi_t^{(\mathcal{G})}(\omega_k)\right)^2. \tag{3.79}$$

The distance function is then rewritten as

$$\begin{aligned}\mathrm{dist}\left(\mathbf{X}(\cdot),\mathbf{Y}(\cdot),\Omega_K\right)&=\sum_{k=1}^{K}p(\omega_k)\sum_{t=0}^{n}\left(\varphi_t^{(\mathcal{G})}(\omega_k)\right)^2\left(X_t(\omega_k)-Y_t(\omega_k)\right)^2\\ &=E_K\left[\sum_{t=0}^{n}\left(\varphi_t^{(\mathcal{G})}X_t-\varphi_t^{(\mathcal{G})}Y_t\right)^2\right],\end{aligned} \tag{3.80}$$

where E_K denotes the expected value under the discrete probability measure P_K which assigns probability weight $p(\omega_k)$ to the scenarios in Ω_K.

The distance function defined in (3.80) tries to match pointwise in time the values of the cash flows $\mathbf{X}$ and $\mathbf{Y}$ as good as possible. Other approaches often work under equivalent probability measures (risk neutral measures or forward measures) so that the discount factors become measurable at the beginning of the corresponding periods.

□

Exercise 3.4.

Calculate the approximate valuation portfolio explicitly under distance function (3.80).

Hint: Note that we have a quadratic form in (3.80). Set the gradient equal to zero and calculate the Hessian matrix (see Ingersoll [Ing87], formula (37) on page 8).

□

Example 3.5 (Time value matching).

We assume that we want to match the time value of $\mathbf{X}$ as good as possible and we use the L^2-distance measure. For a positive deterministic weight function χ_t similar to (3.79) we define the distance function

$$\begin{aligned}\operatorname{dist}\left(\mathbf{X}(\cdot), \mathbf{Y}(\cdot), \Omega_K\right) &= \sum_{k=1}^{K} p(\omega_k) \left\{ \sum_{t=0}^{n} \varphi_t^{(\mathcal{G})}(\omega_k) \left(X_t(\omega_k) - Y_t(\omega_k)\right) \right\}^2 \\ &= E_K\left[\left(\sum_{t=0}^{n} \varphi_t^{(\mathcal{G})} X_t - \varphi_t^{(\mathcal{G})} Y_t \right)^2 \right], \end{aligned} \tag{3.81}$$

where E_K denotes the expected value under the discrete probability measure P_K which assigns probability weight $p(\omega_k)$ to the scenarios in Ω_K.

The distance function defined in (3.81) tries to match time value of entire cash flows $\mathbf{X}$ and $\mathbf{Y}$ as good as possible. Note that the difference is, that we match the entire time value of $\mathbf{X}$ in (3.81) whereas in (3.80) we match cash flow X_k individually in k.

□

Exercise 3.6.

Calculate the approximate valuation portfolio explicitly under distance function (3.81).

Hint: Note that we have a quadratic form in (3.81). Set the gradient equal to zero and calculate the Hessian matrix (see Ingersoll [Ing87], formula (37) on page 8).

□

3.8 Conclusions on Chapter 3

We have decomposed the cash flow $\mathbf{X}$ in a two-step procedure:

1. Choose a multidimensional vector space whose basis consists of financial instruments $\mathcal{U}_1, \ldots, \mathcal{U}_p$.
2. Express the cash flow $\mathbf{X}$ as a vector in this vector space. The number of each unit is determined by the expected number of units (where the expectation is calculated with possibly distorted probabilities).

Calculating the monetary value of the valuation portfolio is then the third step where we use an accounting principle to give values to the vectors in the multidimensional vector space.

We should mention that we have constructed our VaPo for a very basic example. In practice the VaPo construction is much more difficult because, for example, (a) modelling embedded options and guarantees can become very difficult, see Section 3.7; (b) often one has not the necessary information on single policies in the portfolio (e.g. collective policies). Moreover, in practice one faces a lot of problems about data storing and data management since the volume of the data can become very large.

Finally, we mention that we can also construct the VaPo if the financial instruments do not exist on the financial market, e.g. a 41-years zero coupon bond. The VaPo construction still works. However, calculating the monetary value of the VaPo is not straightforward if the instruments do not exist on the financial market.

3.9 Examples

In this section we give a numerical example to the deterministic Example 3.1 (endowment insurance policy). Note that the mathematical details for the evaluation of the accounting principles are given in Chapter 4, below.

For the deterministic mortality table we choose Table 3.1.

time	survival	death
50	$l_{50} = 1'000$	
51	$l_{51} = 996$	$d_{50} = 4$
52	$l_{52} = 991$	$d_{51} = 5$
53	$l_{53} = 986$	$d_{52} = 5$
54	$l_{54} = 981$	$d_{53} = 5$
55	$l_{55} = 975$	$d_{54} = 6$

Table 3.1. Deterministic mortality table, portfolio of 1'000 insured lives

Example 3.7 (Equity-linked life insurance).

We choose an equity-linked life insurance product. Assume that $(I_s)_s$ denotes the price process of the equity index $\mathbf{I}$ (see (4.9)) in the economic world $\mathcal{E}$. That is, we choose an accounting principle $\mathcal{E}$ that corresponds to financial market prices, moreover $\mathcal{E}_s$ denotes these market prices at time s, henceforth

$$I_s = \mathcal{E}_s\left(\mathbf{I}\right) = \mathcal{E}\left(\mathbf{I} \middle| \mathcal{G}_s\right), \tag{3.82}$$

and that $Z_s^{(t)} = P(s,t)$, $s = 0, \ldots, t$, denotes the price process of the zero coupon bond paying 1 at time t. I.e.

$$Z_s^{(t)} = Q_s\left[\mathbf{Z}^{(t)}\right] = Q\left[\mathbf{Z}^{(t)}\middle|\mathcal{G}_s\right] = \mathcal{E}_s\left(Z^{(t)}\right) = \mathcal{E}\left(Z^{(t)}\middle|\mathcal{G}_s\right), \tag{3.83}$$

where $\mathbf{Z}^{(t)}$ is the cash flow of the zero coupon bond $Z^{(t)}$ (see (4.10)). Assume that the zero coupon bond yield curves $R(s,t)$ (continuously-compounded spot rates) at time $s \leq t$ are given by

$$Z_s^{(t)} = \exp\left\{-(t-s)\; R(s,t)\right\} \quad \Longleftrightarrow \quad R(s,t) = -\frac{1}{t-s}\; \log Z_s^{(t)}. \tag{3.84}$$

Considering historical data we observe (source of zero coupon bond yield curves given by the Schweizerische Nationalbank [SNB]): see Table 3.2.

		$R(s,t)$				
s	$\ln(I_s/I_{s-1})$	$t-s=1$	$t-s=2$	$t-s=3$	$t-s=4$	$t-s=5$
1996	12.99%	1.94%	2.42%	2.79%	3.12%	3.42%
1997	13.35%	1.82%	1.92%	2.20%	2.48%	2.74%
1998	22.11%	1.71%	1.81%	1.95%	2.10%	2.27%
1999	5.41%	2.21%	2.06%	2.21%	2.31%	2.42%
2000	2.02%	3.37%	3.52%	3.53%	3.56%	3.60%
2001	8.60%	2.00%	2.85%	2.90%	2.96%	3.02%
2002	-12.41%	0.69%	1.84%	2.14%	2.38%	2.57%
2003	-14.83%	0.58%	0.79%	1.14%	1.46%	1.72%
2004	15.87%	0.99%	1.11%	1.42%	1.70%	1.94%
2005	1.83%	1.41%	1.14%	1.32%	1.48%	1.62%
average	5.49%	1.67%	1.95%	2.16%	2.35%	2.53%

Table 3.2. Equity index and yield curve of the zero coupon bond

We assume that our endowment insurance policy starts in year 2000, i.e. we identify the starting point at age $x = 50$ with the year $t_0 = 2000$.

Assume that the guaranteed interest rate is $i = 2\%$.

To adopt the option pricing formula to the case of non-constant interest rates we transform our price process I_s by a change of numeraire (see also Subsection 4.3.2) and consider for $t_0 \leq s \leq t$

$$\widetilde{I}_s = \frac{I_s}{Z_s^{(t)}} \qquad \text{for fixed } t, \tag{3.85}$$

that is, we consider the t-forward risk neutral measure for the zero-coupon bond numeraire $Z_s^{(t)}$, see for example Section 2.5 in Brigo-Mercurio [BM06].

Now we need to choose a stochastic model for the price process $\widetilde{I}_s$: In order to apply classical financial mathematics we switch to a continuous time model. We assume that, under the t-forward risk neutral measure, $\widetilde{I}_s$ is a martingale satisfying the following stochastic differential equation

$$d\widetilde{I}_s = \sigma \ \widetilde{I}_s \ dW_s, \tag{3.86}$$

where W_s is a standard Brownian motion under the t-forward risk neutral measure. Hence using Ito calculus, $\widetilde{I}_s$ can be rewritten as follows (see e.g. Subsection 3.4.3 in Lamberton-Lapeyre [LL91])

$$\widetilde{I}_s = \widetilde{I_{t_0}} \ \exp\left\{-\frac{\sigma^2}{2} \ (s - t_0) + \sigma \ W_{s-t_0}\right\}. \tag{3.87}$$

Using the general option pricing formula for European put options (see e.g. Section 9.4 in Elliott-Kopp [EK99]) we obtain the price process

$$\mathcal{E}_s\left(\mathrm{Put}^{(t)}\left(\mathbf{I}, (1+i)^{t-t_0}\right)\right) = K_s^{(t)} \ \Phi\left(-d_2(s,t)\right) - I_s \ \Phi\left(-d_1(s,t)\right), \tag{3.88}$$

with Φ standard Gaussian distribution and

$$K_s^{(t)} = (1+i)^{t-t_0} \ Z_s^{(t)}, \tag{3.89}$$

$$d_1(s,t) = \frac{\log\left(I_s/K_s^{(t)}\right) + \sigma^2(t-s)/2}{\sigma\sqrt{t-s}}, \tag{3.90}$$

$$d_2(s,t) = d_1(s,t) - \sigma\sqrt{t-s}. \tag{3.91}$$

Remark. For $Z_s^{(t)} = \exp\{-r \ (t-s)\}$ with $r > 0$ constant, (3.88) is the well-known Black-Scholes formula.

We choose I_s and $Z_s^{(t)}$ according to Table 3.2 with $I_{t_0} = 1$ and $\sigma = 15\%$ and obtain the following prices for the put options (observe that in year $t_0 = 2000$ we have a rather high yield $R(t_0, t)$, which gives a low price for our put option): see Table 3.3.

	$t-s=1$	$t-s=2$	$t-s=3$	$t-s=4$	$t-s=5$
$s=2000$	0.053	0.069	0.080	0.088	0.093
$s=2001$	0.034	0.051	0.066	0.076	
$s=2002$	0.117	0.131	0.144		
$s=2003$	0.249	0.267			
$s=2004$	0.140				

Table 3.3. Prices put options $\mathcal{E}_s(\mathrm{Put}^{(t)}(\mathbf{I}, (1+i)^{t-t_0}))$

Now we calculate the monetary value of the valuation portfolio of $\mathbf{X}$: Assume that the survival and death benefit (before index-linking) equal 100'000. Hence we require (premium equivalence principle)

$$\mathcal{E}_{t_0}\left(\mathrm{VaPo}\left(\mathbf{X}\right)\right) = Q_{t_0}\left[\mathbf{X}\right] \overset{(!)}{=} 0, \tag{3.92}$$

which gives the market-consistent pure risk premium $\Pi = 21'667$ (per policy).

Now we consider the valuation portfolios at different times $t_0 \le s \le t-1$. Denote by $\mathbf{X}_{(s+1)} = (0, \ldots, X_{s+1}, \ldots, X_t)$ the cash flow (outstanding liabilities) after time s.

$$\begin{aligned} \mathcal{E}_s^{(+)} &= \mathcal{E}_s\left(\mathrm{VaPo}\left(\mathbf{X}_{(s+1)}\right) - l_{50+s-t_0}\ \Pi\ Z^{(s)}\right) \\ &= \mathcal{E}_s\left(\mathrm{VaPo}\left(\mathbf{X}_{(s+1)}\right)\right) - l_{50+s-t_0}\ \Pi = Q_s\left[\mathbf{X}_{(s+1)}\right] - l_{50+s-t_0}\ \Pi, \end{aligned} \tag{3.93}$$

is the monetary value before the premium $l_{50+s-t_0}\ \Pi$ has been paid at time s, and

$$\mathcal{E}_s^{(-)} = \mathcal{E}_s\left(\mathrm{VaPo}\left(\mathbf{X}_{(s+1)}\right)\right) = Q_s\left[\mathbf{X}_{(s+1)}\right], \tag{3.94}$$

is the monetary value after the premium $l_{50+s-t_0}\ \Pi$ has been paid at time s. Of course $\mathcal{E}_{t_0}^{(+)} = \mathcal{E}_{t_0}\left(\mathrm{VaPo}\left(\mathbf{X}\right)\right) = 0$ (premium equivalence principle). This gives the following results for the monetary values of the valuation portfolios: see Table 3.4.

	$\mathcal{E}_s^{(+)}$	$\mathcal{E}_s^{(-)}$
$s = 2000$	0	21'666'637
$s = 2001$	26'370'714	47'950'684
$s = 2002$	32'423'186	53'894'823
$s = 2003$	39'619'061	60'982'365
$s = 2004$	74'244'766	95'499'737

Table 3.4. Development of the monetary values of the valuation portfolios

For the valuation portfolio protected against insurance technical risks, we proceed as follows: we define p_t and q_t as in (3.20). Moreover we choose $q_t^* = 1.5 \cdot q_t$ (first order life table). Hence we consider the premium for the yearly renewable term Π_s^R defined in (3.38) for our accounting principle $\mathcal{E}_{t_0}$. This gives the following monetary reinsurance loadings at time t_0: see Table 3.5.

	Π_s^R
$s = 2000$	167'885
$s = 2001$	162'340
$s = 2002$	115'180
$s = 2003$	68'723
$s = 2004$	27'818

Table 3.5. monetary yearly renewable terms premium

□

Example 3.8 (Wage index).

In non-life insurance the products are rather linked to other indices like the inflation index, wage index, the consumer price index or a medical expenses index. As index we choose the wage index (source Schweizerische Nationalbank [SNB]): see Table 3.6.

		$R(s,t)$				
s	$\frac{I_s}{I_{s-1}} - 1$	$t-s=1$	$t-s=2$	$t-s=3$	$t-s=4$	$t-s=5$
1996	1.30%	1.94%	2.42%	2.79%	3.12%	3.42%
1997	1.26%	1.82%	1.92%	2.20%	2.48%	2.74%
1998	0.47%	1.71%	1.81%	1.95%	2.10%	2.27%
1999	0.69%	2.21%	2.06%	2.21%	2.31%	2.42%
2000	0.29%	3.37%	3.52%	3.53%	3.56%	3.60%
2001	1.26%	2.00%	2.85%	2.90%	2.96%	3.02%
2002	2.48%	0.69%	1.84%	2.14%	2.38%	2.57%
2003	1.79%	0.58%	0.79%	1.14%	1.46%	1.72%
2004	1.40%	0.99%	1.11%	1.42%	1.70%	1.94%
2005	0.93%	1.41%	1.14%	1.32%	1.48%	1.62%
average	1.19%	1.67%	1.95%	2.16%	2.35%	2.53%

Table 3.6. Wage inflation index and yield curve of the zero coupon bond

This time we choose as minimal guaranteed interest rate of $i = 1.5\%$. For the volatility we choose $\sigma = 1\%$. This implies that the market-consistent pure risk premium Π equals $\Pi = 21'624$ (per policy) and the prices for the put options can be found in Table 3.7.

	$t-s=1$	$t-s=2$	$t-s=3$	$t-s=4$	$t-s=5$
$s = 2000$	$1.16 \cdot 10^{-4}$	$8.26 \cdot 10^{-6}$	$8.42 \cdot 10^{-7}$	$7.36 \cdot 10^{-8}$	$4.74 \cdot 10^{-9}$
$s = 2001$	$2.82 \cdot 10^{-3}$	$2.28 \cdot 10^{-4}$	$6.14 \cdot 10^{-5}$	$1.39 \cdot 10^{-5}$	
$s = 2002$	$4.60 \cdot 10^{-3}$	$1.21 \cdot 10^{-3}$	$4.75 \cdot 10^{-4}$		
$s = 2003$	$3.72 \cdot 10^{-3}$	$8.27 \cdot 10^{-3}$			
$s = 2004$	$2.43 \cdot 10^{-3}$				

Table 3.7. Put option prices $\mathcal{E}_s(\mathrm{Put}^{(t)}(\mathbf{I}, (1+i)^{t-t_0}))$

Observe that the premium Π and the put prices are smaller in the wage index example than in the equity-linked example. This comes from the fact that the choice of σ is much smaller in the second example.

The monetary values of the valuation portfolios are provided in Table 3.8.

	$\mathcal{E}_s^{(+)}$	$\mathcal{E}_s^{(-)}$
$s = 2000$	0	21'624'505
$s = 2001$	18'723'288	40'261'295
$s = 2002$	39'780'582	61'210'467
$s = 2003$	61'740'997	83'062'759
$s = 2004$	83'857'251	105'070'890

Table 3.8. Development of the monetary values of the valuation portfolios

And the reinsurance loadings are given in Table 3.9.

	Π_s^R
$s = 2000$	157'404
$s = 2001$	145'186
$s = 2002$	95'278
$s = 2003$	46'890
$s = 2004$	0.0014

Table 3.9. Monetary yearly renewable terms premium

The reinsurance premium looks rather small compared to the pure risk premium $l_t\ \Pi\ Z_{t_0}^{(s)}$. This comes from the fact that σ is rather small, that the minimal guarantee $i = 1.5\%$ is rather low compared to the yield $R(t_0, \cdot)$ in year $t_0 = 2000$, and from the fact the randomness of D_t is rather small compared to the total volume l_t.

□